



Products of functions in BMO and \mathcal{H}^1 spaces on spaces of homogeneous type

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ABSTRACT

We give an extension to certain RD -space \mathcal{X} , i.e., space of homogeneous type in the sense of Coifman and Weiss, which has the reverse doubling property, of the definition and various properties of the product of functions in $BMO(\mathcal{X})$ and $\mathcal{H}^1(\mathcal{X})$, and functions in Lipschitz space $\Lambda_{\frac{1}{p}-1}(\mathcal{X})$ and $\mathcal{H}^p(\mathcal{X})$ for $p \in (\frac{n}{n+\theta}, 1]$, where n and θ denote respectively the “dimension” and the order of \mathcal{X} .

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1. Introduction

It is well known that $BMO(\mathbb{R}^n)$ is the dual space of $\mathcal{H}^1(\mathbb{R}^n)$ and that multiplication by $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is a bounded operator on $BMO(\mathbb{R}^n)$. Those facts allow Bonami, Iwaniec, Jones and Zinsmeister, to define in [2] a product $b \times h$ of $b \in BMO(\mathbb{R}^n)$ and $h \in \mathcal{H}^1(\mathbb{R}^n)$ as a distribution, operating on a test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ by the rule

$$(b \times h, \varphi) := (b\varphi, h). \quad (1)$$

They proved that such distributions are sums of a function in $L^1(\mathbb{R}^n)$ and a distribution in a Hardy–Orlicz space $\mathcal{H}^p(\mathbb{R}^n, \nu)$ where

$$\wp(t) = \frac{t}{\log(e+t)} \quad \text{and} \quad d\nu(x) = \frac{dx}{\log(e+|x|)}.$$

The idea of defining the above product is motivated among other things by the fact that for $1 < p < \infty$, the product fg of $f \in L^p(\mathbb{R}^n)$ and g in the dual space $L^{p'}(\mathbb{R}^n)$ of $L^p(\mathbb{R}^n)$ is integrable (consequently is a distribution). The Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ being the right substitute of $L^1(\mathbb{R}^n)$ in many problems, it seems natural to look at its product with its dual space $BMO(\mathbb{R}^n)$. Following of the idea in [2], Bonami and Feuto proved in [1] that for h in the Hardy space $\mathcal{H}^p(\mathbb{R}^n)$ ($0 < p < 1$), the Hardy–Orlicz space is replaced by $\mathcal{H}^p(\mathbb{R}^n)$ provided b belongs to the inhomogeneous Lipschitz space $\Lambda_{n(\frac{1}{p}-1)}(\mathbb{R}^n)$.

The space of homogeneous type introduced by Coifman and Weiss in [4] being the right space for generalized results stated in the Euclidean spaces, we give here the analogous of those results in this context. For this purpose, we consider a space of homogeneous type (\mathcal{X}, d, μ) (see Section 2 for more explanations about this space) in which all annuli are not empty, i.e., $B(x, R) \setminus B(x, r) \neq \emptyset$ for all $x \in \mathcal{X}$ and $0 < r < R < \infty$, where $B(x, r) = \{y \in \mathcal{X} : d(x, y) < r\}$ is the ball centered

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at x and with radius r . According to [14], the doubling measure μ then satisfies the reverse doubling property: there exist two positive constants κ and c_μ depending only on μ , such that

$$\frac{\mu(B)}{\mu(\tilde{B})} \geq c_\mu \left(\frac{r(B)}{r(\tilde{B})} \right)^\kappa \quad \text{for all balls } \tilde{B} \subset B, \quad (2)$$

where $r(B)$ denotes the radius of the ball B . This reverse doubling condition yields that $\mu(\mathcal{X}) = \infty$. Using the doubling condition (11) and the reverse condition (2), we have that

$$c_\mu \lambda^\kappa \mu(B(x, r)) \leq \mu(B(x, \lambda r)) \leq C_\mu \lambda^\eta \mu(B(x, r)) \quad (3)$$

for all $x \in \mathcal{X}$, $r > 0$ and $\lambda \geq 1$. We will refer to η as the dimension of the space. We will also assume that there exists a positive non-decreasing function φ defined on $[0, \infty)$ such that for all $x \in \mathcal{X}$ and $r > 0$,

$$\mu(B(x, r)) \sim \varphi(r).^1 \quad (4)$$

Notice that (2), (3) and (4) imply that

$$r^\eta \lesssim^2 \varphi(r) \lesssim r^\kappa \quad \text{if } 0 < r < 1, \quad (5)$$

and

$$r^\kappa \lesssim \varphi(r) \lesssim r^\eta \quad \text{if } 1 \leq r. \quad (6)$$

These spaces are particular cases of the spaces of homogeneous type named *RD-spaces* in [6] (see also [5,11,15]). An example of such spaces is obtained by considering a Lie group X with polynomial growth equipped with a left Haar measure μ and the Carnot–Carathéodory metric d associated with a Hörmander system of left invariant vector fields (see [6,7,9,12]).

We use the maximal characterization of Hardy spaces in space of homogeneous type developed by Grafakos, Liu and Yang in [5]. It was proved that this maximal characterization of $\mathcal{H}^p(\mathcal{X}, d, \mu)$ agrees with the atomic characterization of Coifman and Weiss in [3] if $p \in (\frac{\eta}{\eta+\theta}, 1]$, where θ is as in relation (12). We recall that for $p \in (0, 1]$ and $q \in [1, \infty] \cap (p, \infty]$, a function $a \in L^q(\mathcal{X}, d, \mu)$ is called a (p, q) -atom if the following conditions are fulfilled:

- (a1) a is supported in a ball B ;
- (a2) $\|a\|_{L^q(\mathcal{X}, d, \mu)} \leq [\mu(B)]^{\frac{1}{q} - \frac{1}{p}}$ if $q < \infty$ and $\|a\|_{L^\infty(\mathcal{X}, d, \mu)} \leq \mu(B)^{-\frac{1}{p}}$ if $q = \infty$;
- (a3) $\int_{\mathcal{X}} a(x) d\mu(x) = 0$.

It was proved in Corollary 4.19 of [5] that for $p \in (\frac{\eta}{\eta+\theta}, 1]$ and $q \in (p, \infty] \cap [1, \infty]$, $f \in \mathcal{H}^p(\mathcal{X}, d, \mu)$ if and only if there exist a sequence $(a_i)_{i \in \mathbb{N}}$ of (p, q) -atoms, each a_i supported in a ball B_i , and a sequence $(\lambda_i)_{i \in \mathbb{N}}$ of scalars such that

$$f = \sum_{i=1}^{\infty} \lambda_i a_i \quad \text{and} \quad \sum_{i=1}^{\infty} |\lambda_i|^p < \infty, \quad (7)$$

where the first series is considered in the sense of distribution as defined in [5], and $\|f\|_{\mathcal{H}^p(\mathcal{X})} \sim \inf\{(\sum_{i \in \mathbb{N}} |\lambda_i|^p)^{\frac{1}{p}}\}$, the infimum being taken over all the decompositions of f as above and $\|f\|_{\mathcal{H}^p(\mathcal{X})}$ as in (13). For $b \in \text{BMO}(\mathcal{X}, d, \mu)$ and $h \in \mathcal{H}^1(\mathcal{X}, d, \mu)$ as in (7), the series $\sum_{i=1}^{\infty} \lambda_i (b - b_{B_i}) a_i$ and $\sum_{i=1}^{\infty} \lambda_i b_{B_i} a_i$ converge in the sense of distribution as we can see in the proof of Theorem 1.1. Thus we define the product of $b \times h$ as the sum of both series, i.e., we put

$$b \times h := \sum_{i=1}^{\infty} \lambda_i (b - b_{B_i}) a_i + \sum_{i=1}^{\infty} \lambda_i b_{B_i} a_i.$$

Our main result can be stated as follows.

Theorem 1.1. For $h \in \mathcal{H}^1(\mathcal{X}, d, \mu)$ and $b \in \text{BMO}(\mathcal{X}, d, \mu)$, the product $b \times h$ can be given a meaning in the sense of distributions. Moreover, if x_0 is a fixed element of \mathcal{X} then we have the inclusion

$$b \times h \in L^1(\mathcal{X}, d, \mu) + \mathcal{H}^0(\mathcal{X}, d, \nu), \quad (8)$$

¹ Hereafter we propose the following abbreviation $A \sim B$ for the inequalities $C^{-1}A \leq B \leq CB$, where C is a positive constant independent of the main parameters.

² $A \lesssim B$ means the ratio A/B is bounded away from zero by a constant independent of the relevant variables in A and B .

where

$$d\nu(x) = \frac{d\mu(x)}{\log(e + d(x_0, x))}.$$

This result is a generalization of Theorem A of [2], while in Theorem 4.1 we obtain that Hardy–Orlicz class is replaced by the classical weight Hardy space $\mathcal{H}^p(\mathcal{X}, d, \tau)$ ($d\tau(x) = w(x)d\mu(x)$ for some appropriate weight) when $h \in \mathcal{H}^p(\mathcal{X}, d, \mu)$ and $b \in \Lambda_{\frac{1}{p}-1}(\mathcal{X}, d, \mu)$. This result is new even in the Euclidean case, since in [1] there was only a remark on the possibility of such estimates.

Section 2 is devoted to notation and definitions. We recall in this paragraph the definition of spaces of homogeneous type and the grand maximal characterization of Hardy space as introduced in [5]. In Section 3, we give a prerequisite on Hardy–Orlicz space and prove some lemmas which we need for the proof of the main result. We prove our main result in the last section, as well as its extensions.

Throughout the paper, C denotes positive constants that are independent of the main parameters involved, with values which may differ from line to line.

2. Notation and definitions

A quasimetric d on a set \mathcal{X} is a function $d : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$, which satisfies

- (i) $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all x, y in \mathcal{X} ;
- (iii) there exists a finite constant $K_0 \geq 1$ such that for all x, y, z in \mathcal{X} ,

$$d(x, y) \leq K_0(d(x, z) + d(z, y)). \quad (9)$$

The set \mathcal{X} equipped with a quasimetric d is called quasimetric space.

Let μ be a positive Borel measure on (\mathcal{X}, d) such that all balls defined by d have finite and positive measures. We say that the triple (\mathcal{X}, d, μ) is a space of homogeneous type if there exists a constant $C \geq 1$ such that for all $x \in \mathcal{X}$ and $r > 0$, we have

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)). \quad (10)$$

This property is known as the doubling property. If C_0 is the smallest constant for which (10) holds, then by iterating (10), we have

$$\frac{\mu(B)}{\mu(\tilde{B})} \leq C_\mu \left(\frac{r(B)}{r(\tilde{B})} \right)^n \quad \text{for all balls } \tilde{B} \subset B, \quad (11)$$

where $n = \log_2(C_0)$ and $C_\mu = C_0(2K_0)^n$.

Notice that from the reverse doubling property, $\mu(\{x\}) = 0$ for all $x \in \mathcal{X}$. We also have that

$$\mu(B(x, r + d(x, y))) \sim \mu(B(y, r)) + \mu(B(y, d(y, x)))$$

for $x, y \in \mathcal{X}$ and $r > 0$.

In this paper, $\mathcal{X} = (\mathcal{X}, d, \mu)$ is a space of homogeneous type in which relations (2) and (4) are satisfied. We also assume (see [8]) that there exist two constants $A'_0 > 0$ and $0 < \theta \leq 1$ such that

$$|d(x, z) - d(y, z)| \leq A'_0 d(x, y)^\theta [d(x, z) + d(y, z)]^{1-\theta}. \quad (12)$$

The space is saying to be of order θ . We will refer to the constants $K_0, C_0, n, \kappa, C_\mu, c_\mu, A'_0$ and θ mentioned above, as the constants of the space. We will not mention the measure and the quasimetric when talking about the space (\mathcal{X}, d, μ) . But if we use another measure than μ , this will be mentioned explicitly. The following abbreviation for the measure of balls will be also used: $V_r(x) = \mu(B(x, r))$ and $V(x, y) = \mu(B(x, d(x, y)))$ for all $x, y \in \mathcal{X}$ and $r > 0$.

Definition 2.1. (See [5].) Let $x_0 \in \mathcal{X}$, $r > 0$, $0 < \beta \leq 1$ and $\gamma > 0$. A complex-valued function φ on \mathcal{X} is called a test function of type (x_0, r, β, γ) if the following hold:

- (i) $|\varphi(x)| \leq C \frac{1}{\mu(B(x, r+d(x, x_0)))} \left(\frac{r}{r+d(x_0, x)} \right)^\gamma$ for all $x \in \mathcal{X}$;
- (ii) $|\varphi(x) - \varphi(y)| \leq C \left(\frac{d(x, y)}{r+d(x_0, x)} \right)^\beta \frac{1}{\mu(B(x, r+d(x, x_0)))} \left(\frac{r}{r+d(x_0, x)} \right)^\gamma$ for all x, y in \mathcal{X} satisfying $d(x, y) \leq \frac{r+d(x_0, x)}{2K_0}$.

We denote by $\mathcal{G}(x_0, r, \beta, \gamma)$ the set of all test functions of type (x_0, r, β, γ) , equipped with the norm

$$\|\varphi\|_{\mathcal{G}(x_0, r, \beta, \gamma)} = \inf\{C: \text{(i) and (ii) hold}\}.$$

In what follows, we fix an element x_0 in \mathcal{X} and put $\mathcal{G}(\beta, \gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)$. It is easy to prove that $\mathcal{G}(x_1, r, \beta, \gamma) = \mathcal{G}(\beta, \gamma)$ with equivalent norms for all $x_1 \in \mathcal{X}$ and $r > 0$. Furthermore, it is easy to check that $\mathcal{G}(\beta, \gamma)$ is a Banach space.

For a given $\epsilon \in (0, \theta]$ and $\beta, \gamma \in (0, \epsilon]$, $\mathcal{G}_0^\epsilon(\beta, \gamma)$ denotes the completion of $\mathcal{G}(\epsilon, \epsilon)$ in $\mathcal{G}(\beta, \gamma)$. Equip $\mathcal{G}_0^\epsilon(\beta, \gamma)$ with the norm $\|\varphi\|_{\mathcal{G}_0^\epsilon(\beta, \gamma)} = \|\varphi\|_{\mathcal{G}(\beta, \gamma)}$, and denote $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$ its dual space; that is the set of linear functionals f from $\mathcal{G}_0^\epsilon(\beta, \gamma)$ to \mathbb{C} with the property that there exists a constant $C > 0$ such that for all $\varphi \in \mathcal{G}_0^\epsilon(\beta, \gamma)$, $|\langle f, \varphi \rangle| \leq C \|\varphi\|_{\mathcal{G}(\beta, \gamma)}$. This dual space will be referred to as a distribution space.

For $f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))'$, the grand maximal function f^* of f in the sense of Grafakos, Liu and Yang [5] is defined for $x \in \mathcal{X}$ by

$$f^*(x) = \sup \{ |\langle f, \varphi \rangle| : \varphi \in \mathcal{G}_0^\epsilon(\beta, \gamma), \|\varphi\|_{\mathcal{G}(x, r, \beta, \gamma)} \leq 1 \text{ for some } r > 0 \}.$$

The corresponding Hardy space $\mathcal{H}^p(\mathcal{X})$ is defined for $p \in (0, \infty]$ to be the set of $h \in (\mathcal{G}_0^\epsilon(\beta, \gamma))'$ for which

$$\|h\|_{\mathcal{H}^p(\mathcal{X})} := \|h^*\|_{L^p(\mathcal{X})} < \infty. \quad (13)$$

It is proved in Proposition 3.15 and Theorem 4.17 of [5] that for $\epsilon \in (0, \theta]$ and $p \in (\frac{n}{n+\epsilon}, 1]$, the definition of $\mathcal{H}^p(\mathcal{X})$ as stated above is independent of the choice of the underlying space of distribution, i.e., if $f \in (\mathcal{G}_0^\epsilon(\beta_1, \gamma_1))'$ with

$$n(1/p - 1) < \beta_1, \gamma_1 < \epsilon \quad (14)$$

and $\|f\|_{\mathcal{H}^p(\mathcal{X})} < \infty$, then $f \in (\mathcal{G}_0^\epsilon(\beta_2, \gamma_2))'$ for every β_2 and γ_2 satisfying (14).

In the rest of the paper, $0 < \epsilon \leq \theta$ is fixed and $p \in (\frac{n}{n+\epsilon}, 1]$. We also fix the underlying space of distribution $\mathcal{G}_0^\epsilon(\beta, \gamma)'$ with β and γ as in (14).

As mentioned in the introduction, the dual space of $\mathcal{H}^1(\mathcal{X})$ is $\text{BMO}(\mathcal{X})$ (the space of bounded mean oscillation functions), defined as the set of locally integrable functions b satisfying

$$\frac{1}{\mu(B)} \int_B |b(x) - b_B| d\mu(x) \leq A \quad \text{for all balls } B, \quad (15)$$

where $b_B = \frac{1}{\mu(B)} \int_B b(x) d\mu(x)$, and A a non-negative constant depending only on b and the space constant. We put

$$\|b\|_{\text{BMO}(\mathcal{X})} = \sup_{B: \text{ball}} \frac{1}{\mu(B)} \int_B |b(x) - b_B| d\mu(x)$$

and

$$\|b\|_{\text{BMO}^+(\mathcal{X})} = \|b\|_{\text{BMO}(\mathcal{X})} + |f_{\mathbb{B}}|,$$

where \mathbb{B} is the ball center at x_0 and with radius 1. When the measure of \mathcal{X} is finite, $(\text{BMO}(\mathcal{X}), \|\cdot\|_{\text{BMO}})$ is a Banach space. The set of equivalence classes of functions under the relation “ b_1 and b_2 in $\text{BMO}(\mathcal{X})$ are equivalent if and only if $b_1 - b_2$ is constant” which we still denote by $\text{BMO}(\mathcal{X})$ equipped with $\|\cdot\|_{\text{BMO}(\mathcal{X})}$ is a Banach space.

As proved in [3], we have that for every $1 \leq q < \infty$,

$$\|b\|_{\text{BMO}(\mathcal{X})} \lesssim \sup_{B: \text{balls}} \left(\frac{1}{\mu(B)} \int_B |b - b_B|^q d\mu \right)^{\frac{1}{q}} \lesssim \|b\|_{\text{BMO}(\mathcal{X})} \quad (16)$$

for all b in $\text{BMO}(\mathcal{X})$, where the supremum is taken over all balls of \mathcal{X} .

We also have, by the doubling condition of the measure μ , that for $b \in \text{BMO}(\mathcal{X})$, and B a ball in (\mathcal{X}, d) ,

$$|b_B - b_{2^k B}| \leq C(1+k) \|b\|_{\text{BMO}(\mathcal{X})} \quad \text{for all non-negative integer } k.$$

Theorem B of [3] (see also Theorem 5.3 of [6]) stated that for $\frac{n}{n+\epsilon} < p < 1$, the dual space of Hardy space $\mathcal{H}^p(\mathcal{X})$ is the Lipschitz space $\Lambda_{\frac{1}{p-1}}(\mathcal{X})$. We recall that for $0 < \gamma$, the Lipschitz space $\Lambda_\gamma(\mathcal{X})$ is the set of those functions f on \mathcal{X} for which

$$|f(x) - f(y)| \leq A \mu(B)^\gamma, \quad (17)$$

where B is any ball containing both x and y and A is a non-negative constant depending only on f .

We can see that this definition of Lipschitz recovers the Euclidean case only when $0 < \gamma < \frac{1}{n}$. In fact, unless γ is sufficiently small, it can happen that the only functions satisfying (17) are the constants. But, as shown in [4] there are situations where these spaces are not trivial. However, we are going to consider only $0 < \gamma < \frac{\epsilon}{n}$, since it is the range in which the atomic definition of Hardy coincides with the maximal function characterization. Let put

$$\|f\|_{\Lambda_\gamma(\mathcal{X})} = \inf \{ A : (17) \text{ holds} \}.$$

Then $\|\cdot\|_{\Lambda_\gamma(\mathcal{X})}$ is a norm on the set of equivalence classes of functions under the relation “ b_1 and b_2 in $\Lambda_\gamma(\mathcal{X})$ are equivalent if and only if $b_1 - b_2$ is constant”, which we still denote $\Lambda_\gamma(\mathcal{X})$.

3. A prerequisite about Orlicz spaces

Let

$$\wp(t) = \frac{t}{\log(e+t)} \quad \text{for all } t > 0.$$

A μ -measurable function $f : \mathcal{X} \rightarrow \mathbb{R}$ is said to belong to the Orlicz space $L^\wp(\mathcal{X})$ if

$$\|f\|_{L^\wp(\mathcal{X})} := \inf \left\{ k > 0 : \int_{\mathcal{X}} \wp(k^{-1}|f(x)|) d\mu(x) \leq 1 \right\} < \infty.$$

It is easy to see that $L^1(\mathcal{X}) \subset L^\wp(\mathcal{X})$. More precisely, we have

$$\|f\|_{L^\wp(\mathcal{X})} \leq \|f\|_{L^1(\mathcal{X})}.$$

We are going to recall some results involving Orlicz spaces mentioned in [2], which are also valid in the context of space of homogeneous type.

(i) If $\text{Exp} L(\mathcal{X})$ is the Orlicz space associated to the Orlicz function $t \mapsto e^t - 1$ and $L \log L(\mathcal{X})$ the one associated to $t \mapsto t \log(e+t)$ then we have the following Hölder type inequality

$$\|fg\|_{L^\wp(\mathcal{X})} \leq 4\|f\|_{L^1(\mathcal{X})}\|g\|_{\text{Exp} L(\mathcal{X})}$$

for all $f \in L^1(\mathcal{X})$ and $g \in \text{Exp} L(\mathcal{X})$ by using the elementary inequality

$$\frac{ab}{\log(e+ab)} \leq a + e^b - 1 \quad \text{for all } a, b \geq 0. \quad (18)$$

We also have the duality between $\text{Exp} L(\mathcal{X})$ and $L \log L(\mathcal{X})$, that is,

$$\|fg\|_{L^1(\mathcal{X})} \leq 2\|f\|_{L \log L(\mathcal{X})}\|g\|_{\text{Exp} L(\mathcal{X})},$$

using the following inequalities

$$ab \leq a \log(1+a) + e^b - 1 \quad \text{for all } a, b \geq 0. \quad (19)$$

(ii) Since the Orlicz function \wp we consider is not convex, the triangular inequality does not hold for $\|\cdot\|_{L^\wp(\mathcal{X})}$. But we have the following substitute

$$\|f+g\|_{L^\wp(\mathcal{X})} \leq 4\|f\|_{L^\wp(\mathcal{X})} + 4\|g\|_{L^\wp(\mathcal{X})}, \quad (20)$$

for $f, g \in L^\wp(\mathcal{X})$. This relation remains valid if we replace the measure μ by any absolutely continuous one compared to μ .

(iii) $L^\wp(\mathcal{X})$ equipped with the metric

$$\mathfrak{d}(f, g) := \inf \left\{ \delta > 0 : \int_{\mathcal{X}} \wp(\delta^{-1}|f(x) - g(x)|) d\mu(x) \leq \delta \right\}$$

is a complete linear metric space.

(iv) If $\mathfrak{d}(f, g) \leq 1$, then

$$\|f - g\|_{L^\wp} \leq \mathfrak{d}(f, g) \leq 1. \quad (21)$$

(v) A sequence $(f_n)_{n>0}$ converge in $L^\wp(\mathcal{X})$ to f if and only if $\lim_{n \rightarrow \infty} \|f_n - f\|_{L^\wp} = 0$.

We define the Hardy–Orlicz space $\mathcal{H}^\wp(\mathcal{X})$ to be the subset of $\mathcal{G}_0^\epsilon(\beta, \gamma)'$ consists of distributions f such that $f^* \in L^\wp(\mathcal{X})$, and we put

$$\|f\|_{\mathcal{H}^\wp(\mathcal{X})} := \|f^*\|_{L^\wp(\mathcal{X})}.$$

In [13], it was proved that this characterization of Hardy–Orlicz spaces coincide with some atomic characterization.

Lemma 3.1. *Let \mathfrak{b} be in $\text{BMO}(\mathcal{X})$. There exists a constant C such that for every $(1, q)$ -atom \mathfrak{a} supported in a ball B ,*

$$\|(\mathfrak{b} - \mathfrak{b}_B)\mathfrak{a}^*\|_{L^1(\mathcal{X})} \leq C\|\mathfrak{b}\|_{\text{BMO}(\mathcal{X})}.$$

Proof. Let $b \in \text{BMO}(\mathcal{X})$ and a be a $(1, q)$ -atom supported in $B = B(x_0, R)$. We have

$$\| (b - b_B) a^* \|_{L^1(\mathcal{X})} = \int_{B(x_0, 2K_0 R)} |b(z) - b_B| a^*(z) d\mu(z) + \int_{B^c(x_0, 2K_0 R)} |b(z) - b_B| a^*(z) d\mu(z), \quad (22)$$

where $B^c(x_0, 2K_0 R) = \mathcal{X} \setminus B(x_0, 2K_0 R)$. Furthermore, we have

$$a^*(z) \leq C \mathcal{M}a(z) \quad \text{for all } z \in \mathcal{X}, \quad (23)$$

where $\mathcal{M}a(z) = \sup_{B \ni z} \frac{1}{\mu(B)} \int_B |a(x)| d\mu(x)$ denotes the Hardy–Littlewood maximal function of a , according to Proposition 3.9 of [5]. We also have

$$a^*(z) \leq C \left(\frac{R}{d(z, x_0)} \right)^\beta \frac{1}{\mu(B(z, d(z, x_0)))} \quad \text{for all } z \notin B(x_0, 2K_0 R), \quad (24)$$

as it is shown in the proof of Lemma 4.4 of [5]. If we take (23) into first term of the sums (22) and use the Hölder inequality with $1 < q < \infty$, we then have

$$\int_{B(x_0, 2K_0 R)} |b(z) - b_B| a^*(z) d\mu(z) \leq \left(\int_{B(x_0, 2K_0 R)} |b(z) - b_B|^{q'} d\mu(z) \right)^{\frac{1}{q'}} \left(\int_{\mathcal{X}} \mathcal{M}a(z)^q d\mu(z) \right)^{\frac{1}{q}}.$$

Since the Hardy–Littlewood maximal operator \mathcal{M} is bounded in $L^q(\mathcal{X})$, there exists a positive constant C such that

$$\int_{B(x_0, 2K_0 R)} |b(z) - b_B| a^*(z) d\mu(z) \leq C \|b\|_{\text{BMO}(\mathcal{X})}$$

according to relation (16).

On the other hand if we take (24) in the second term of (22) we have

$$\begin{aligned} & \int_{B^c(x_0, 2K_0 R)} |b(z) - b_B| a^*(z) d\mu(z) \\ & \leq C \sum_{k=1}^{\infty} \int_{(2K_0)^{k+1} B \setminus (2K_0)^k B} \left(\frac{R}{d(z, x_0)} \right)^\beta \frac{|b(z) - b_B|}{\mu(B(z, d(z, x_0)))} d\mu(z) \\ & \leq C \sum_{k=1}^{\infty} (2K_0)^{-k\beta} \left(\frac{1}{\mu((2K_0)^{k+1} B)} \int_{(2K_0)^{k+1} B} |b(z) - b_{(2K_0)^{k+1} B}| d\mu(z) + |b_{(2K_0)^{k+1} B} - b_B| \right), \end{aligned}$$

where the second inequality comes from the fact that $\mu(B(z, d(z, x_0))) \sim \mu(B(x_0, d(z, x_0)))$. Since the series $\sum_{k=1}^{\infty} (2K_0)^{-k\beta}$ converges, we also have that there exists a constant C , independent of b and a , such that

$$\int_{B^c(x_0, 2K_0 R)} |b(z) - b_B| a^*(z) d\mu(z) \leq C \|b\|_{\text{BMO}(\mathcal{X})},$$

which completes the proof. \square

It is well known that the John–Nirenberg inequality is valid in the context of spaces of homogeneous type (see [10]). This inequality states that there exist positive constants K_1 and K_2 such that for any $b \in \text{BMO}(\mathcal{X})$ with $\|b\|_{\text{BMO}(\mathcal{X})} \neq 0$ and any ball $B \subset \mathcal{X}$, we have

$$\mu(\{x \in B : |b(x) - b_B| > \lambda\}) \leq K_1 \exp\left(-\frac{K_2 \lambda}{\|b\|_{\text{BMO}(\mathcal{X})}}\right) \mu(B) \quad \text{for all } \lambda > 0. \quad (25)$$

An immediate consequence of this inequality is that there is a positive constant K_3 , depending only on the space constants, such that

$$\frac{1}{\mu(B)} \int_B \exp\left(\frac{|b(x) - b_B|}{K_3 \|b\|_{\text{BMO}(\mathcal{X})}}\right) d\mu(x) \leq 2 \quad (26)$$

for all balls B in \mathcal{X} .

Notice that we can choose K_3 as big as we like.

Lemma 3.2. Let \mathbb{B} be the ball centered at x_0 with radius 1. There exists a positive constant K_4 such that for any $b \in \text{BMO}(\mathcal{X})$ with $\|b\|_{\text{BMO}(\mathcal{X})} \neq 0$,

$$\int_X \frac{e^{\frac{|b(x)-b_{\mathbb{B}}|}{K_4 \|b\|_{\text{BMO}(\mathcal{X})}} - 1}}{(1+d(x_0, x))^{2n}} d\mu(x) \leq 1.$$

Proof. Let $b \in \text{BMO}(\mathcal{X})$ with $\|b\|_{\text{BMO}(\mathcal{X})} \neq 0$. We have

$$\int_X \frac{e^{\frac{|b(x)-b_{\mathbb{B}}|}{K_3 \|b\|_{\text{BMO}(\mathcal{X})}} - 1}}{(1+d(x_0, x))^{2n}} d\mu(x) = \int_{\mathbb{B}} \frac{e^{\frac{|b(x)-b_{\mathbb{B}}|}{K_3 \|b\|_{\text{BMO}(\mathcal{X})}} - 1}}{(1+d(x_0, x))^{2n}} d\mu(x) + \int_{\mathbb{B}^c} \frac{e^{\frac{|b(x)-b_{\mathbb{B}}|}{K_3 \|b\|_{\text{BMO}(\mathcal{X})}} - 1}}{(1+d(x_0, x))^{2n}} d\mu(x),$$

where $\mathbb{B}^c = \mathcal{X} \setminus \mathbb{B}$. The first term in the right-hand side is less than $\mu(\mathbb{B})$. For the second term, we have

$$\begin{aligned} \int_{\mathbb{B}^c} \frac{e^{\frac{|b(x)-b_{\mathbb{B}}|}{K_3 \|b\|_{\text{BMO}(\mathcal{X})}} - 1}}{(1+d(x_0, x))^{2n}} d\mu(x) &= \sum_{k=0}^{\infty} \int_{2^k \leq d(x_0, x) < 2^{k+1}} \frac{e^{\frac{|b(x)-b_{\mathbb{B}}|}{K_3 \|b\|_{\text{BMO}(\mathcal{X})}} - 1}}{(1+d(x_0, x))^{2n}} d\mu(x) \\ &\leq \sum_{k=0}^{\infty} 2^{-2nk} \int_{B(x_0, 2^{k+1})} (e^{\frac{|b(x)-b_{\mathbb{B}}|}{K_3 \|b\|_{\text{BMO}(\mathcal{X})}} - 1}) d\mu(x). \end{aligned}$$

Using the fact that $|b_{\mathbb{B}} - b_{B(x_0, 2^{k+1})}| \leq \log(2^{\frac{C_0(k+1)}{\log 2}}) \|b\|_{\text{BMO}(\mathcal{X})}$ and $\mu(B(x_0, 2^{k+1})) \leq 2^{(k+1)\log_2 C_0} \mu(\mathbb{B})$, we have the term we are estimated less than

$$C\mu(\mathbb{B}) \sum_{k=0}^{\infty} 2^{(-n + \frac{C_0}{K_3 \log 2})k}.$$

Take $K_3 > \frac{C_0}{n \log 2}$. Then the series (24) converges. Therefore,

$$\int_X \frac{e^{\frac{|b(x)-b_{\mathbb{B}}|}{K_3 \|b\|_{\text{BMO}(\mathcal{X})}} - 1}}{(1+d(x_0, x))^{2n}} d\mu(x) \leq C\mu(\mathbb{B}).$$

The result follows from taking $K_4 = \max(K_3, CK_3\mu(\mathbb{B}))$. \square

Let us introduce the following measures

$$d\nu := \frac{d\mu(x)}{\log(e + d(x_0, x))} \quad \text{and} \quad d\sigma(x) := \frac{d\mu(x)}{(1 + d(x_0, x))^{2n}},$$

where n is the dimension of \mathcal{X} . It follows from the above lemma that for $b \in \text{BMO}(\mathcal{X})$, we have

$$\|b - b_{\mathbb{B}}\|_{\text{ExpL}(\mathcal{X}, \sigma)} \leq C \|b\|_{\text{BMO}(\mathcal{X})}. \quad (27)$$

We can also see that for a ν -measurable function f , we have

$$\|f\|_{L^{\varphi}(\mathcal{X}, \nu)} \leq \|f\|_{L^1(\mathcal{X})}. \quad (28)$$

The next result is the analogous of Lemma 3.2 of [2] in the context of spaces of homogeneous type, and its proof is just an adaptation of the one give in that paper.

Lemma 3.3. Let $f \in \text{ExpL}(\mathcal{X}, \sigma)$. Then for $g \in L^1(\mathcal{X})$, we have $g \cdot f \in L^{\varphi}(\mathcal{X}, \nu)$ and

$$\|g \cdot f\|_{L^{\varphi}(\mathcal{X}, \nu)} \leq C \|g\|_{L^1(\mathcal{X})} \|f\|_{\text{ExpL}(\mathcal{X}, \sigma)}. \quad (29)$$

If, moreover, $f \in \text{BMO}(\mathcal{X})$, then

$$\|g \cdot f\|_{L^{\varphi}(\mathcal{X}, \nu)} \leq C \|g\|_{L^1(\mathcal{X})} \|f\|_{\text{BMO}^+(\mathcal{X})}. \quad (30)$$

Proof. Let $f \in \text{ExpL}(\mathcal{X}, \sigma)$ and $g \in L^1(\mathcal{X})$. If $\|g\|_{L^1(\mathcal{X})} = 0$ or $\|f\|_{\text{ExpL}(\mathcal{X}, \sigma)} = 0$ then there is nothing to prove. Thus we assume that $\|g\|_{L^1(\mathcal{X})} \|f\|_{\text{ExpL}(\mathcal{X}, \sigma)} \neq 0$. Let us put $A = 8(n+1)\|g\|_{L^1(\mathcal{X})}$ and $B = 8(n+1)\|f\|_{\text{ExpL}(\mathcal{X}, \sigma)}$. We are going to prove that the constant C is $64(n+1)^2$. For this, it suffices to prove that

$$\int_{\mathcal{X}} \frac{\frac{1}{AB} |fg| d\mu(x)}{\log(e + \frac{1}{AB} |fg|) \log(e + d(x_0, x))} \leq 1.$$

For this purpose, we will use the following elementary inequality:

$$2(n+1) \log(e + d(x_0, x)) > \log(e + (1 + d(x_0, x))^{2n}) \quad \text{for all } x \in \mathcal{X}, \quad (31)$$

and for all $a, b > 0$,

$$\log(e + a) \log(e + b) > \frac{1}{2} \log(e + ab). \quad (32)$$

It comes from the relation (31) that

$$\frac{\frac{1}{AB} |fg|}{\log(e + \frac{1}{AB} |fg|) \log(e + d(x_0, x))} \leq \frac{\frac{2(n+1)}{AB} |fg|}{\log(e + \frac{1}{AB} |fg|) \log(e + (1 + d(x_0, x))^{2n})}$$

so that applying relation (32) to the left-hand side of the inequality, yields

$$\begin{aligned} \frac{\frac{1}{AB} |fg|}{\log(e + \frac{1}{AB} |fg|) \log(e + d(x_0, x))} &\leq \frac{\frac{4(n+1)}{AB} |fg|}{\log(e + \frac{1}{AB} |fg|) (1 + d(x_0, x))^{2n}} \\ &\leq 4(n+1) \frac{|g|}{A} + \frac{4(n+1)(e^{\frac{|f|}{B}} - 1)}{(1 + d(x_0, x))^{2n}}, \end{aligned}$$

according to relation (18). Taking the integral of both sides, we obtain inequality (29), since

$$\frac{4(n+1)(e^{\frac{|f|}{B}} - 1)}{(1 + d(x_0, x))^{2n}} \leq \frac{1}{2} \frac{(e^{8(n+1)\frac{|f|}{B}} - 1)}{(1 + d(x_0, x))^{2n}} = \frac{1}{2} \frac{(e^{\frac{|f|}{\|f\|_{\text{ExpL}(\mathcal{X}, \sigma)}}} - 1)}{(1 + d(x_0, x))^{2n}},$$

and

$$4(n+1) \frac{|g|}{A} = \frac{1}{2} \frac{|g|}{\|g\|_{L^1(\mathcal{X})}}.$$

The inequality (30) is also trivial if $\|f\|_{\text{BMO}(\mathcal{X})} = 0$. Thus we assume that f is not constant almost everywhere and we put $f \cdot g = (f - f_{\mathbb{B}}) \cdot g + f_{\mathbb{B}} \cdot g$, so that using relation (20), relation (29) and (27), we have

$$\begin{aligned} \|f \cdot g\|_{L^p(\mathcal{X}, \nu)} &\leq C(\|(f - f_{\mathbb{B}}) \cdot g\|_{L^p(\mathcal{X}, \nu)} + \|f_{\mathbb{B}} \cdot g\|_{L^p(\mathcal{X}, \nu)}) \\ &\leq C(\|f - f_{\mathbb{B}}\|_{\text{ExpL}(\mathcal{X}, \sigma)} \|g\|_{L^1(\mathcal{X})} + \|f_{\mathbb{B}}\| \|g\|_{L^1(\mathcal{X})}) \\ &\leq C\|g\|_{L^1(\mathcal{X}, \nu)} \|f\|_{\text{BMO}^+(\mathcal{X})}, \end{aligned}$$

which completes our proof. \square

4. Proof of our main result

Proof of Theorem 1.1. Let $b \in \text{BMO}(\mathcal{X})$ and $h = \sum_{i=1}^{\infty} \lambda_i a_i \in \mathcal{H}^1(\mathcal{X})$, where $(a_i)_{i \geq 1}$ is a sequence of $(1, \infty)$ -atoms, with a_i supported in the ball B_i , and $(\lambda_i)_{i \geq 1}$ a sequence of scalars such that $\sum_{i=1}^{\infty} |\lambda_i| < \infty$. To prove our theorem, it is enough to show that the series

$$\sum_{i=1}^{\infty} \lambda_i (b - b_{B_i}) a_i \quad \text{and} \quad \sum_{i=1}^{\infty} \lambda_i b_{B_i} a_i$$

are convergent in $L^1(\mathcal{X})$ and $\mathcal{H}^p(\mathcal{X}, \nu)$ respectively, since the product $b \times h$ by definition is the sum of both series.

The convergence of the first series in $L^1(\mathcal{X})$ is immediate, since for all index i , we have

$$\|\lambda_i (b - b_{B_i}) a_i\|_{L^1(\mathcal{X})} \leq |\lambda_i| \|b\|_{\text{BMO}(\mathcal{X})} \quad \text{and} \quad \sum_{i=1}^{\infty} |\lambda_i| < \infty,$$

according to Lemma 3.1. For the second series, we consider the partial sum

$$S_k^\ell := \sum_{i=k}^{\ell} \lambda_i \mathbf{a}_i \mathbf{b}_{B_i} \quad \text{for } k < \ell. \quad (33)$$

Our series converges in $\mathcal{H}^p(\mathcal{X}, \nu)$ if and only if $\lim_{k \rightarrow \infty} \|(S_k^\ell)^*\|_{L^p(\mathcal{X}, \nu)} = 0$. But we have

$$(S_k^\ell)^* \leq \sum_{i=k}^{\ell} |\lambda_i| (\mathbf{a}_i \mathbf{b}_{B_i})^* \leq \sum_{i=k}^{\ell} |\lambda_i| |\mathbf{b} - \mathbf{b}_{B_i}| (\mathbf{a}_i)^* + \left(\sum_{i=k}^{\ell} |\lambda_i| (\mathbf{a}_i)^* \right) |\mathbf{b}|,$$

so that

$$\begin{aligned} \|(S_k^\ell)^*\|_{L^p(\mathcal{X}, \nu)} &\leq C \left[\left\| \sum_{j=k}^{\ell} |\lambda_j| |\mathbf{b} - \mathbf{b}_{B_j}| (\mathbf{a}_j)^* \right\|_{L^p(\mathcal{X}, \nu)} + \left\| \left(\sum_{i=k}^{\ell} |\lambda_i| (\mathbf{a}_i)^* \right) |\mathbf{b}| \right\|_{L^p(\mathcal{X}, \nu)} \right] \\ &\leq C \left[\left\| \sum_{i=k}^{\ell} |\lambda_i| |\mathbf{b} - \mathbf{b}_{B_i}| (\mathbf{a}_i)^* \right\|_{L^1(\mathcal{X})} + \left\| \left(\sum_{i=k}^{\ell} |\lambda_i| (\mathbf{a}_i)^* \right) |\mathbf{b}| \right\|_{L^p(\mathcal{X}, \nu)} \right] \\ &\leq C \|\mathbf{b}\|_{\text{BMO}^+(\mathcal{X})} \sum_{i=k}^{\ell} |\lambda_i|, \end{aligned}$$

where the last inequality comes from Lemmas 3.1 and 3.3. It comes out that,

$$\lim_{k \rightarrow \infty} \|(S_k^\ell)^*\|_{L^p(\mathcal{X}, \nu)} \leq C \|\mathbf{b}\|_{\text{BMO}^+(\mathcal{X})} \lim_{k \rightarrow \infty} \sum_{i=k}^{\ell} |\lambda_i| = 0,$$

since $\sum_{i=1}^{\infty} |\lambda_i| < \infty$. \square

Let us consider now the Hardy space $\mathcal{H}^p(\mathcal{X})$ with $p < 1$. We have the following result.

Theorem 4.1. Let $\frac{n}{n+1} < p < 1$. For $f \in \Lambda_{\frac{1}{p}-1}(\mathcal{X})$ and $g \in \mathcal{H}^p(\mathcal{X})$, we can give a meaning to the product $f \times g$ as a distribution. Moreover, we have the inclusion

$$f \times g \in L^1(\mathcal{X}) + \mathcal{H}^p(\mathcal{X}, d, \tau), \quad \text{where } d\tau(x) = \frac{d\mu(x)}{(2K_0^2 + K_0 d(x_0, x))^{(1-p)n}}.$$

Proof. Let $f \in \Lambda_{\frac{1}{p}-1}(\mathcal{X})$ and $g \in \mathcal{H}^p(\mathcal{X})$. We assume that g has the following atomic decomposition

$$g = \sum_{i=1}^{\infty} \lambda_i \mathbf{a}_i,$$

where \mathbf{a}_i 's are atoms supported respectively in the balls B_i . All we have to prove is that the series

$$\sum_{i=0}^{\infty} \lambda_i (f - f_{B_i}) \mathbf{a}_i \quad (34)$$

and

$$\sum_{i=0}^{\infty} \lambda_i f_{B_i} \mathbf{a}_i \quad (35)$$

converge respectively in $L^1(\mathcal{X})$ and in $\mathcal{H}^p(\mathcal{X}, d, \tau)$. Arguing as in the previous theorem, we have that series (34) converges normally in $L^1(\mathcal{X})$. It remains to prove that (35) converge in $\mathcal{H}^p(\mathcal{X}, d, \tau)$. As in Theorem 1.1, we have

$$(S_k^\ell)^* \leq \sum_{i=k}^{\ell} |\lambda_i| (\mathbf{a}_i f_{B_i})^* \leq \sum_{i=k}^{\ell} |\lambda_i| |f - f_{B_i}| (\mathbf{a}_i)^* + \left(\sum_{i=k}^{\ell} |\lambda_i| (\mathbf{a}_i)^* \right) |f|, \quad (36)$$

where $S_k^\ell = \sum_{i=k}^{\ell} \lambda_i \mathbf{a}_i \mathbf{b}_{B_i}$ for $k < \ell$. We claim that Lemma 3.1 remains true if we replace the space $\text{BMO}(\mathcal{X})$ by $\Lambda_{\frac{1}{p}-1}(\mathcal{X})$ and the $(1, q)$ -atoms by (p, q) -atoms with $q \geq 1$, i.e., for $f \in \Lambda_{\frac{1}{p}-1}(\mathcal{X})$ and a (p, q) -atom supported in the ball B ,

$$\|(f - f_B)\mathfrak{a}^*\|_{L^1(\mathcal{X})} \leq C\|f\|_{\Lambda_{\frac{1}{p}-1}(\mathcal{X})}.$$

In fact, by the definition of Lipschitz space $\Lambda_{\frac{1}{p}-1}(\mathcal{X})$, we have

$$\int_{B(x_0, 2K_0R)} |f(z) - f_B|\mathfrak{a}^*(z) d\mu(z) \leq C\|f\|_{\Lambda_{\frac{1}{p}-1}(\mathcal{X})}.$$

In other respect,

$$\mathfrak{a}^*(z) \leq C\mu(B(x_0, R))^{1-\frac{1}{p}} \left(\frac{R}{d(z, x_0)}\right)^\beta \frac{1}{\mu(B(z, d(z, x_0)))}$$

for all $z \notin B(x_0, 2K_0R)$ according to Lemma 4.4 of [5].

Arguing as in the proof of Lemma 3.1, we have that $\sum |\lambda_i| |f - f_{B_i}|(\mathfrak{a}_i)^*$ converges in $L^1(\mathcal{X})$. The proof of the theorem will be complete if we establish that for any ball B of radius 1, we have that for $f \in \Lambda_{\frac{1}{p}-1}(\mathcal{X})$ and $\psi \in L^p(B)$

$$\int_B (|f(x)\psi(x)|)^p d\tau(x) \leq C\|f\|_{\Lambda_{\frac{1}{p}-1}^+(\mathcal{X})}^p \int_B |\psi(x)|^p d\mu(x), \quad (37)$$

where $\|f\|_{\Lambda_{\frac{1}{p}-1}^+(\mathcal{X})}^p = \|f\|_{\Lambda_{\frac{1}{p}-1}(\mathcal{X})}^p + \max(|f(x_0)|, 1)^p$. Following the method in [1], we have

$$\begin{aligned} \int_B \frac{|f(x)\psi(x)|^p}{(2K_0^2 + K_0d(x_0, x))^{\mathfrak{n}(1-p)}} d\mu(x) &\leq \int_{B \cap \{|f| \leq 1\}} |\psi(x)|^p d\mu(x) \\ &+ \int_{B \cap \{|f| > 1\}} |f(x)|^p \frac{|\psi(x)|^p}{(2K_0^2 + K_0d(x_0, x))^{\mathfrak{n}(1-p)}} d\mu(x). \end{aligned}$$

Furthermore,

$$\begin{aligned} \int_{B \cap \{|f| > 1\}} |f(x)|^p \frac{|\psi(x)|^p}{(2K_0^2 + K_0d(x_0, x))^{\mathfrak{n}(1-p)}} d\mu(x) &\leq \int_{B \cap \{|f| > 1\}} |f(x) - f(x_0)|^p \frac{|\psi(x)|^p}{(2K_0^2 + K_0d(x_0, x))^{\mathfrak{n}(1-p)}} d\mu(x) \\ &+ |f(x_0)|^p \int_{B \cap \{|f| > 1\}} \frac{|\psi(x)|^p}{(2K_0^2 + K_0d(x_0, x))^{\mathfrak{n}(1-p)}} d\mu(x). \end{aligned}$$

Since $B \subset B(x_0, 2K_0^2 + K_0d(x, x_0))$ for all x in the ball B of radius 1, it comes from the definition of the Lipschitz space $\Lambda_{\frac{1}{p}-1}(\mathcal{X})$ that the first term in the right-hand side of the above inequality is less or equal to

$$\|f\|_{\Lambda_{\frac{1}{p}-1}(\mathcal{X})}^p \int_B \frac{\mu(B(x_0, 2K_0^2 + K_0d(x_0, x)))^{1-p}}{(2K_0^2 + K_0d(x_0, x))^{\mathfrak{n}(1-p)}} |\psi(x)|^p d\mu(x).$$

But, from (4) and (6), we have that $\mu(B(x_0, 2K_0^2 + K_0d(x_0, x))) \lesssim (2K_0^2 + K_0d(x_0, x))^{\mathfrak{n}}$.

Thus

$$\int_{B \cap \{|f| > 1\}} |f(x)|^p \frac{|\psi(x)|^p}{(2K_0^2 + K_0d(x_0, x))^{\mathfrak{n}(1-p)}} d\mu \lesssim (\|f\|_{\Lambda_{\frac{1}{p}-1}(\mathcal{X})}^p + |f(x_0)|^p) \int_B |\psi(x)|^p d\mu(x).$$

The result follows from covering the whole space by almost disjoint balls of radius 1. \square

Remark 4.2. Let $\frac{\mathfrak{n}}{\mathfrak{n}+\epsilon} < p < 1$ and $\gamma := \frac{1}{p} - 1$. Then, for $h \in \mathcal{H}^p(\mathcal{X})$ and $f \in \Lambda_\gamma(\mathcal{X}) \cap L^\infty(\mathcal{X})$, the product $h \times f$ can be given a meaning in the sense of distributions. Moreover, we have the inclusion

$$h \times f \in L^1(\mathcal{X}) + \mathcal{H}^p(\mathcal{X}). \quad (38)$$

Proof. Let $\mathfrak{h} \in \mathcal{H}^p(\mathcal{X})$ be as in (7), where the atoms involved are (p, ∞) -atoms, and $f \in \Lambda_\gamma(\mathcal{X})$. From Theorem 4.1, we deduce that

$$\sum_{i=1}^{\infty} \lambda_i (f - f_{B_i}) \alpha_i$$

converges in $L^1(\mathcal{X})$. For the series $\sum_{i=1}^{\infty} \lambda_i f_{B_i} \alpha_i$, we just have to remark that the functions $\{\frac{1}{\|f\|_{L^\infty(\mathcal{X})}} f_{B_j} \alpha_i\}_{i=1}^{\infty}$ are (p, ∞) -atoms. In fact,

- (i) $\text{supp } f_{B_i} \alpha_i \subset B_i$, since $\text{supp } \alpha_i \subset B_i$,
- (ii) $\int_{\mathcal{X}} f_{B_i} \alpha_i(x) dx = 0$,
- (iii) $|f_{B_i} \alpha_i(x)| \leq \|f\|_{L^\infty(\mathcal{X})} \mu(B_i)^{-\frac{1}{p}}$,

and this finishes the proof, since $\sum_{i=1}^{\infty} |\lambda_i|^p < \infty$. \square

Remark 4.3. In the case $\mu(X) < \infty$, all our results remain valid, provided that we consider the constant function $\mu(\mathcal{X})^{-\frac{1}{p}}$ as an atom, and put

$$\|b\|_{\text{BMO}(\mathcal{X})} = \sup_{B: \text{ball}} \frac{1}{\mu(B)} \int_B |b(x) - b_B| d\mu(x) + \|b\|_{L^1(\mathcal{X})}$$

and

$$\|f\|_{A_\gamma(\mathcal{X})} = \sup \left\{ \frac{|f(x) - f(y)|}{\mu(B)}, \text{ for all ball } B \ni x, y \right\} + \left| \int_{\mathcal{X}} f(x) d\mu(x) \right|.$$

In this case the reverse doubling condition (2), need to be satisfied just for small balls.

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