J. Math. Anal. Appl. 387 (2012) 103–113

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Dense 2-generator subsemigroups of 2×2 matrices

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ARTICLE INFO

Article history:
Received 28 April 2011
Available online 30 August 2011
Submitted by T. Ransford

Keywords:
Dense subsemigroups
Linear fractional transformations

ABSTRACT

We construct explicit examples of 2-generator dense subsemigroups of $GL(2,\mathbb{R})$ and $SL(2,\mathbb{R})$. In the complex case, we prove the existence of uncountably many 2-generator dense subsemigroups of $GL(2,\mathbb{C})$. We also show that the semigroup of real linear fractional transformations on a proper subinterval of the real line does not admit any 2-generator dense subsemigroups, and then we construct a 3-parameter family of examples of 3-generator dense subsemigroups.

1. Introduction

Let $\mathcal{F}$ denote the semigroup of real linear fractional transformations from $(0,\infty)$ to $(0,\infty)$ i.e., maps of the form

$$f(x)=\frac{ax+b}{cx+d};\quad a,b,c,d\geqslant 0 \text{ and } ad-bc\neq 0.$$

The action of $\mathcal{F}$ on $(0,\infty)$ is naturally related to the action of $GL(2,\mathbb{R}_+)$ on $\mathbb{R}_+^2$ via the homomorphism below (here $GL(2,\mathbb{R}_+)$ denotes the semigroup of invertible 2×2 matrices with nonnegative entries):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{ax+b}{cx+d}. \tag{1.1}$$

Following [2], we define the following notions regarding semigroup actions on topological spaces.

Definition 1.1. Let X be a topological space and G be a semigroup acting by continuous functions on X. The action of G on X is called

(a) *hypercyclic*, if there exists an element $x\in X$ whose G-orbit, defined by $\{g(x)\colon\ g\in G\}$, is dense in X;
(b) *topologically transitive*, if for every pair of nonempty open subsets U and V of X, there exists $g\in G$ so that $g(U)\cap V\neq\emptyset$;
(c) *topologically k-transitive*, if the induced action of G on X^k (Cartesian product) is topologically transitive. Topological 2-transitivity is called *weak topological mixing*.

For an introduction to topological transitivity and mixing properties of operators and semigroups, see [1,5]. If G acts by open continuous maps on a Baire space X, then topological transitivity and hypercyclicity are equivalent [1]. Hence, in our settings (the action of $\mathcal{F}$ on $(0,\infty)$ and the action of $GL(2,\mathbb{R}_+)$ on $\mathbb{R}_+^2$), hypercyclicity and topological transitivity are equivalent. On the other hand, a subsemigroup of $GL(2,\mathbb{R}_+)$ is weakly topologically mixing, if and only if it is dense in $GL(2,\mathbb{R}_+)$ (see Proposition 3.6), hence our interest in finding minimally generated dense subsemigroups of $\mathcal{F}$ and $GL(2,\mathbb{R}_+)$.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.08.063

Here, the topology of $\mathcal{F}$ is the topology of point-wise convergence: a sequence $f_i \in \mathcal{F}$, $i \in \mathbb{N}$, is said to be convergent to $f \in \mathcal{F}$ if for every $x > 0$, we have $f_i(x) \to f(x)$ as $i \to \infty$. A subset $\mathcal{S}$ of $\mathcal{F}$ is called *dense*, if for every $f \in \mathcal{F}$ there exists a sequence $f_i \in \mathcal{S}$, $i \in \mathbb{N}$, such that $f_i \to f$ as $i \to \infty$.

In addition to studying dense subsemigroups of $GL(2, \mathbb{R}_+)$ and $\mathcal{F}$, we will also study the related problem below, which was first asked in [8].

Problem. What is the least number of generators that can generate a dense subsemigroup of the set of $n \times n$ matrices?

Here, the set of $n \times n$ matrices is given its standard topology via identifying $n \times n$ matrices with the n^2-dimensional Euclidean space.

Hypercyclicity. G. Costakis et al. [4] showed that there exists a hypercyclic pair of 2×2 real matrices. More generally, $(A_1, \ldots, A_k)$ is called a hypercyclic k-tuple of commuting $n \times n$ matrices $A_1, \ldots, A_k$, if there exists an n-vector whose orbit under the action of the semigroup generated by the A_i's, $1 \leqslant i \leqslant k$, is dense in the n-dimensional space. In [6], Feldman proved that there exists a hypercyclic semigroup generated by $n+1$ diagonalizable matrices in dimension n. Costakis et al. [4] showed that one can find a hypercyclic abelian semigroup of n matrices in dimension n.

It is also worth mentioning that, in every dimension n (real or complex), one can construct a pair of non-commuting matrices (A, B) such that the orbit of almost every n-vector under the action of the semigroup generated by A and B is dense [8]. In this paper, we prove a stronger result in dimension 2: we show that, in both real and complex cases, there exists a pair of 2×2 matrices so that the semigroup generated by the pair is dense in the set of all 2×2 matrices (see Examples 3.4 and 5.2).

Dense subgroups. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let $SL(2, \mathbb{K})$ denote the group of 2×2 matrices with determinant 1 in the field $\mathbb{K}$. A subgroup of $SL(2, \mathbb{K})$ is called elementary if the commutator of every two elements of infinite order in the subgroup has trace 2. Also, a subgroup of $SL(2, \mathbb{K})$ is called discrete if no sequence of distinct elements in the subgroup converges. Jørgensen [10] studied the non-elementary subgroups of $SL(2, \mathbb{K})$, and proved the following facts among other results:

(i) *The complex case*: a non-elementary subgroup of $SL(2, \mathbb{C})$ is discrete if and only if each of its subgroups generated by two elements is discrete.
(ii) *The real case*: a subgroup of $SL(2, \mathbb{R})$ is discrete if and only if each subgroup generated by one element is discrete.

Jørgensen also proved that every dense subgroup of $SL(2, \mathbb{R})$ has a dense subgroup generated by two elements. In particular, one concludes that dense 2-generator subgroups of $SL(2, \mathbb{R})$ exist. In this paper, we construct an explicit example of a dense 2-generator subsemigroup of $SL(2, \mathbb{R})$ (see Proposition 4.1).

Here, we mention two statements, which resemble Jørgensen's result on the existence of 2-generator dense subgroups. X. Wang [16] has shown that every dense subgroup of the group of orientation preserving Möbius transformations on S^n has a dense subgroup that is generated by at most n elements, $n \geqslant 2$. A similar statement about the group $U(n, 1)$ was obtained by W. Cao [3]. In each of these settings, one can ask for minimally generated dense subsemigroups.

Dense subsemigroups of $\boldsymbol{S(X)}$. Let X be a locally compact Hausdorff space, and let $S(X)$ denote the semigroup of all continuous maps from X to X. Then $S(X)$ is a topological semigroup with the compact-open topology. Several authors have studied dense subsemigroups of $S(X)$. Here, we mention the results of S. Subbiah on the existence of minimally generated dense subsemigroups of $S(\mathbb{R}^n)$. In [14], Subbiah showed that there exist three continuous self-maps of $\mathbb{R}^n$ that generate a dense subsemigroup of $S(\mathbb{R}^n)$ in the compact-open topology. In other words, there exist three continuous self-maps of $\mathbb{R}^n$ such that every continuous self-map of $\mathbb{R}^n$ can be approximated on any compact set by a suitable composition of the three self-maps. It was conjectured in [12] that three is the least number of generators of a dense subsemigroup of $S(\mathbb{R}^n)$. However, Subbiah later showed that the conjecture is false at least when $n = 1$ i.e., $S(\mathbb{R})$ contains a subsemigroup that is generated by two elements [15].

Description of results. For clarity, we divide the results of this paper into three parts.

(i) *The proper-interval case.* The minimum number of generators that can generate a hypercyclic subsemigroup of $\mathcal{F}$ is 2 (see [7], where a complete list of 2-generator hypercyclic subsemigroups of $\mathcal{F}$ is given). It turns out that we need at least three elements in $\mathcal{F}$ to generate a dense subsemigroup of $\mathcal{F}$. We present a three-parameter family of examples of 3-generator dense subsemigroups of $\mathcal{F}$ (see Theorem 2.4). We also show that $GL(2, \mathbb{R}_+)$ does not have any 2-generator dense subsemigroup, however it has 3-generator dense subsemigroups.
(ii) *The full-real-line case.* The semigroup generated by only one matrix in $GL(2, \mathbb{R})$ can never be dense or even hypercyclic (this can be seen by looking at the Jordan normal form of the matrix; see [11,13]). In Section 3, we construct an explicit example of a pair of real matrices that generates a dense subsemigroup of $GL(2, \mathbb{R})$ (see Example 3.4).

Let $\bar{\mathcal{F}}$ denote the set of all real linear fractional transformations on $\mathbb{R}$ (i.e., maps of the form $(ax+b)/(cx+d)$ with $ad-bc \neq 0$ and no restriction on the signs of a, b, c, d). The existence of 2-generator dense subsemigroups of $GL(2,\mathbb{R})$ implies the existence of 2-generator dense subsemigroups of $\bar{\mathcal{F}}$ (see Corollary 3.5).
Moreover, we show that $SL(2,\mathbb{R})$ has 2-generator dense subsemigroups by constructing an explicit example (see Proposition 4.1).

(iii) *The complex case.* Finally, in Section 5, we prove the existence of uncountably many 2-generator dense subsemigroups of 2×2 complex matrices (see Example 5.2).

2. Dense subsemigroups of $\mathcal{F}$

Let

$$R(x) = 1 + \frac{a}{x} \quad \text{and} \quad S(x) = \frac{x}{b},$$

where $a > 0$ and $b > 1$. Let Λ be the semigroup of real linear fractional transformations generated by R and S i.e.,

$$\Lambda = \left\{ R^{m_1} S^{n_1} \ldots R^{m_k} S^{n_k}\colon\ \forall i,\ m_i, n_i \geqslant 0,\ k \geqslant 1 \right\},$$

and let $\bar{\Lambda}$ be the closure of Λ in the topology of point-wise convergence on $\mathcal{F}$.

Lemma 2.1. *Suppose $a > 0$ and $b > 1$. Then, for every $(m_1, \ldots, m_{k+1}) \in \mathbb{Z}^{k+1}$, $k \geqslant 0$, we have*

$$\frac{b^{m_{k+1}} x}{(b^{m_1} + \cdots + b^{m_k})x/a + 1} \in \bar{\Lambda}.$$

Proof. Proof is by induction on k. For $k = 0$, we need to show that for every $m_1 \in \mathbb{Z}$, we have $b^{m_1}x \in \bar{\Lambda}$. For positive integers m and n, one calculates

$$S^m R S^n R(x) = \frac{b^{-m}(a + x + ab^n x)}{x + a}.$$

Let l be a fixed integer, and set $n = l + m$. Then as $m \to \infty$, we have $S^m R S^n R(x) \to b^l x/(1 + x/a)$, and so $f_l(x) = b^l x/(1 + x/a) \in \bar{\Lambda}$ for all $l \in \mathbb{Z}$. Next, we let $l \to \infty$ to get $b^{m_1}x = \lim f_l S^{l-m_1}(x) \in \bar{\Lambda}$, which proves the basis of the induction.

Now, suppose that the assertion of the lemma is true for $k \geqslant 0$, and let $(m_1, \ldots, m_{k+2}) \in \mathbb{Z}^{k+2}$. From the inductive hypothesis, we conclude that $g(x) = b^{m_{k+1}}x/(dx + 1) \in \bar{\Lambda}$, where $d = (b^{m_1} + \cdots + b^{m_k})/a$. Then, for $l = m_{k+2} - m_{k+1}$, it follows that

$$f_l g(x) = \frac{b^{m_{k+2}} x}{(d + b^{m_{k+1}}/a)x + 1} \in \bar{\Lambda},$$

and the inductive step is completed. $\square$

Given $s > 0$, there exists a sequence $\{m_i\}_{i=1}^{\infty}$ of integers so that $sa = \sum_{i=1}^{\infty} b^{m_i}$. It follows from Lemma 2.1 that

$$T_s(x) = \frac{x}{sx + 1} = \lim_{k \to \infty} \frac{x}{(b^{m_1} + \cdots + b^{m_k})x/a + 1} \in \bar{\Lambda}. \tag{2.1}$$

On the other hand,

$$S^m R S^m(x) = \frac{1}{b^m} + \frac{a}{x} \to \frac{a}{x},$$

as $m \to \infty$. Hence,

$$I(x) = \frac{a}{x} \in \bar{\Lambda}. \tag{2.2}$$

Lemma 2.2. *Suppose that $\alpha, \beta, \gamma \geqslant 0$ and $0 \leqslant \alpha - \beta\gamma \leqslant \min(1, \alpha^2)$. Then*

$$F(\alpha, \beta, \gamma)(x) = \frac{\alpha x + \beta}{\gamma x + 1} \in \bar{\Lambda}.$$

Proof. It is sufficient to consider the case where $\alpha, \beta, \gamma > 0$ and $0 < \alpha - \beta\gamma \leqslant \min(1, \alpha^2)$. For $u, v, w \geqslant 0$, it follows from (2.1) and (2.2) that

$$T_u I T_{v/a} I T_w(x) = \frac{(1+vw)x + v}{(u+w+uvw)x + 1 + uv} \in \bar{\Lambda}.$$

Now, given α, β, γ, we set

$$u = \frac{-\sqrt{d}+1}{\beta}, \qquad v = \frac{\beta}{\sqrt{d}}, \qquad w = \frac{-\sqrt{d}+\alpha}{\beta}, \tag{2.3}$$

where $d = \alpha - \beta\gamma$. The conditions on α, β, γ given in the lemma guarantee that $u, v, w \geqslant 0$. These choices of u, v, w are made so that $T_u I T_{v/a} I T_w = F(\alpha, \beta, \gamma)$, and the proof is completed. □

For $f(x) = (\alpha x + \beta)/(\gamma x + \delta) \in \mathcal{F}$ with $\delta \neq 0$, let

$$\det(f) = \frac{1}{\delta^2}(\alpha\delta - \beta\gamma), \qquad \sigma(f) = \frac{\alpha^2}{\delta^2}.$$

Let $\mathcal{F}_+ = \{f \in \mathcal{F}\colon \det(f) > 0\}$, and for $k \in \mathbb{Z}$, let

$$\mathcal{U}_k = \bigl\{f \in \mathcal{F}_+\colon \det(f) \leqslant \min\bigl(b^k, b^{-k}\sigma(f)\bigr)\bigr\}. \tag{2.4}$$

Theorem 2.3. $\bar{\Lambda} \cap \mathcal{F}_+ = \bigcup_{k\in\mathbb{Z}} \mathcal{U}_k$.

Proof. Since $\mathcal{U}_0 \subseteq \bar{\Lambda}$ (by Lemma 2.2) and $S^{-1}(x) = bx \in \bar{\Lambda}$ by Lemma 2.1, it follows that $\mathcal{U}_k = S^{-k}\mathcal{U}_0 \subseteq \bar{\Lambda}$, and so $\mathcal{U} = \bigcup_{k\in\mathbb{Z}} \mathcal{U}_k \subseteq \bar{\Lambda} \cap \mathcal{F}_+$. It is left to show that $\bar{\Lambda} \cap \mathcal{F}_+ \subseteq \mathcal{U}$. First, we show that $\mathcal{U}$ is a semigroup under composition. To see this, let $f(x) = (\alpha x + \beta)/(\gamma x + 1) \in \mathcal{U}_k$ and $g(x) = (ux + v)/(wx + 1) \in \mathcal{U}_l$ for some $k, l \in \mathbb{Z}$. Then

$$fg(x) = \frac{(\alpha u + \beta w)x + (\alpha v + \beta)}{(\gamma u + w)x + (\gamma v + 1)}.$$

One verifies that

$$0 < \det(fg) = \frac{(\alpha - \beta\gamma)(u - vw)}{(\gamma v + 1)^2} \leqslant \min\left(b^{k+l}, b^{-k-l}\left(\frac{\alpha u + \beta w}{\gamma v + 1}\right)^2\right),$$

and so $fg \in \mathcal{U}$ i.e., $\mathcal{U}$ is a semigroup.

Next, a simple calculation shows that for every nonnegative integer k, we have $S^k(x) = F(b^{-k}, 0, 0) \in \mathcal{U}_1$ and $RS^kR = F(b^k + 1/a, 1, 1/a) \in \mathcal{U}_k$. Now, every function f in $\Lambda \cap \mathcal{F}_+$ can be factored into terms of the form RS^kR and S^k, and since $\mathcal{U}$ is a semigroup, it follows that $\Lambda \cap \mathcal{F}_+ \subseteq \mathcal{U}$. Since $\mathcal{U}$ is closed in $\mathcal{F}$, we conclude that $\bar{\Lambda} \cap \mathcal{F}_+ \subseteq \mathcal{U}$, and the proof is completed. □

An implication of Theorem 2.3 is that the closure of the semigroup of maps generated by $R(x) = 1 + a/x$ and $S(x) = x/b$ for $a > 0$ and $b > 1$ depends only on b, since $\bigcup_{k\in\mathbb{Z}} \mathcal{U}_k$ is determined entirely by b.

In the next theorem, we show that 3-generator dense subsemigroups of $\mathcal{F}$ exist.

Theorem 2.4. *Let $a, c > 0$ and $b > 1$ so that $\ln c/\ln b \notin \mathbb{Q}$. Then the semigroup generated by $1 + a/x$, x/b, and x/c is dense in $\mathcal{F}$.*

Proof. Let $\mathcal{U}$ be defined as in (2.4). Suppose that $\alpha, \beta, \gamma > 0$ so that $0 \leqslant \alpha - \beta\gamma$. Since $\ln c/\ln b \notin \mathbb{Q}$, it follows that there exist a sequence $\{k_i\}_{i=1}^{\infty}$ of integers and a sequence $\{l_i\}_{i=1}^{\infty}$ of positive integers so that $b^{k_i}c^{-l_i} \to \alpha$. Then, we have

$$\lim_{i\to\infty} \min\bigl(b^{k_i}c^{-l_i}, b^{-k_i}c^{l_i}\alpha^2\bigr) = \alpha,$$

and so for i large enough, we have

$$\alpha - \beta\gamma \leqslant \min\bigl(b^{k_i}c^{-l_i}, b^{-k_i}c^{l_i}\alpha^2\bigr),$$

which in turn implies that

$$0 \leqslant c^{l_i}\alpha - \bigl(c^{l_i}\beta\bigr)\gamma \leqslant \min\bigl(b^{k_i}, b^{-k_i}\bigl(c^{l_i}\alpha\bigr)^2\bigr).$$

By Theorem 2.3, we conclude that $F(c^{l_i}\alpha, c^{l_i}\beta, \gamma) \in \bar{\Lambda}$, and so $F(\alpha, \beta, \gamma) = c^{-l_i}F(c^{l_i}\alpha, c^{l_i}\beta, \gamma) \in \bar{\Lambda}$ as well. The case of $\beta = 0$ or $\gamma = 0$ follows by using a limiting process.

By composing $F(\alpha, \beta, \gamma)$ with a/x, we deduce that $F(u, v, w) \in \bar{\Lambda}$ for all $u, w \geqslant 0$ and $v > 0$. The case of $v = 0$ can be dealt with by using another limiting process. □

3. Dense subsemigroups of $GL(2,\mathbb{R})$

In this section, we prove the existence of 3-generator dense subsemigroups of $GL(2,\mathbb{R}_+)$ and the existence of 2-generator dense subsemigroups of $GL(2,\mathbb{R})$.

Proposition 3.1. *Suppose that $a, c > 0$ and $b > 1$ so that $\ln c/\ln b \notin \mathbb{Q}$. Then the semigroup generated by the matrices*

$$A = \begin{pmatrix} 1/a & a \\ 1/a & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1/b & 0 \\ 0 & b \end{pmatrix}, \qquad C = \begin{pmatrix} 1/c & 0 \\ 0 & c \end{pmatrix},$$

is dense in the semigroup of real matrices with nonnegative entries and $\det = \pm 1$.

Proof. Let $X = [q, r; s, t]$ be a 2×2 matrix with nonnegative entries and $\det(X) = \pm 1$. We show that there exists a sequence $D_i \in \langle A, B, C\rangle$ so that $D_i \to X$ as $i \to \infty$. Without loss of generality, we can assume that $t \neq 0$. The linear fractional maps associated with A, B, and C are $1 + a^2/x$, x/b^2, and x/c^2. By Theorem 2.4, these three linear fractional maps generate a dense subsemigroup of $\mathcal{F}$. It follows that, for each $i \geqslant 1$, there exists a matrix $D_i = [\alpha_i, \beta_i; \gamma_i, \delta_i] \in \langle A, B, C\rangle$ so that

$$\left|\frac{\alpha_i}{\delta_i} - \frac{q}{t}\right| + \left|\frac{\beta_i}{\delta_i} - \frac{r}{t}\right| + \left|\frac{\gamma_i}{\delta_i} - \frac{s}{t}\right| < \frac{1}{i}. \tag{3.1}$$

If i is large enough, we deduce from (3.1) that $\det(X) = \det(D_i)$. Hence, there exists λ depending only on X so that

$$\left|\frac{1}{\delta_i^2} - \frac{1}{t^2}\right| = \left|\frac{\alpha_i\delta_i - \beta_i\gamma_i}{\delta_i^2} - \frac{qt - rs}{t^2}\right| < \frac{\lambda}{i}.$$

Therefore, $\delta_i \to t$ as $i \to \infty$, and consequently $\alpha_i \to q$, $\beta_i \to r$, and $\gamma_i \to s$ by (3.1). In other words, $D_i \to X$ as $i \to \infty$, and the lemma follows. □

Corollary 3.2. *Suppose that $a > 0$ and $b > 1 > c > 0$ so that $\ln c/\ln b \notin \mathbb{Q}$. Then the semigroup generated by the matrices*

$$\begin{pmatrix} 1 & a \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \qquad \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.2}$$

is dense in $GL(2,\mathbb{R}_+)$.

Proof. Let $\mathcal{S}$ denote the closure of the semigroup generated by these three matrices. We first show that $dI_{2\times 2} \in \mathcal{S}$ for every $d \geqslant 0$. Choose sequences of positive integers k_i, l_i so that $b^{k_i}c^{l_i} \to d/a$. Then

$$\begin{pmatrix} 0 & d \\ 1 & 0 \end{pmatrix} = \lim_{i\to\infty} \begin{pmatrix} c^{l_i} & b^{k_i}c^{l_i}a \\ 1 & 0 \end{pmatrix} = \lim_{i\to\infty} \begin{pmatrix} c^{l_i} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & b^{k_i} \end{pmatrix} \in \mathcal{S},$$

and so $dI_{2\times 2} = [0, d; 1, 0]^2 \in \mathcal{S}$. Next, let X be any 2×2 matrix with nonnegative entries and $\mu = \det(X) \neq 0$. Let $\hat{F} = F/\sqrt{\det(F)}$ for an invertible matrix, and let $\hat{\mathcal{S}} = \{\hat{F},\ F \in \mathcal{S}\}$. By Proposition 3.1, there exists $D_i \in \hat{\mathcal{S}}$ so that $D_i \to \hat{X}$ as $i \to \infty$. Choose d_i so that $d_i D_i \in \mathcal{S}$. Since $(\sqrt{\mu}/d_i)I_{2\times 2} \in \mathcal{S}$, we have

$$X = \sqrt{\mu}\hat{X} = \lim_{i\to\infty} (\sqrt{\mu}/d_i)I_{2\times 2}(d_i D_i) \in \mathcal{S},$$

and so $\mathcal{S}$ contains every 2×2 matrix with nonnegative entries. □

The following corollary is an immediate consequence of Corollary 3.2.

Corollary 3.3. *Suppose that $a > 0$ and $b > 1 > c > 0$ so that $\ln c/\ln b \notin \mathbb{Q}$. Then the semigroup generated by the matrices*

$$\begin{pmatrix} 1 & a \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & -b \end{pmatrix}, \qquad \begin{pmatrix} -c & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.3}$$

is dense in $GL(2,\mathbb{R})$.

Proof. Let $\mathcal{T}$ denote the closure of the semigroup of matrices generated by these three matrices. Then $\mathcal{T}$ contains the closure of the semigroup generated by $[1, a; 1, 0]$, $[1, 0; 0, b^2]$, and $[c^2, 0; 0, 1]$, which contains all real 2×2 matrices with nonnegative entries by Corollary 3.2. It follows that every matrix belongs to $\mathcal{T}$, since every matrix can be written as a suitable product of $[1, 0; 0, -b]$, $[-c, 0; 0, 1]$, and a matrix with nonnegative entries. □

Now, we construct an explicit example of a pair of 2×2 matrices that generates a dense semigroup in the set of 2×2 matrices in the real case.

Example 3.4. The semigroup of matrices generated by

$$A = \begin{pmatrix} 1 & 1/2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -8/3 \end{pmatrix}, \tag{3.4}$$

is dense in the set of 2×2 real matrices.

Proof. One verifies that $ABA^3BA = [-2/9, 0; 0, 1] = C$, and so $\langle A, B\rangle = \langle A, B, C\rangle$, which is dense in the set of 2×2 real matrices by Corollary 3.3. $\square$

Recall that $\bar{\mathcal{F}}$ denotes the set of all real linear fractional maps on the extended real line $R \cup \{\infty\}$ i.e., maps of the form $(ax+b)/(cx+d)$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc \neq 0$.

Corollary 3.5. *There exists a* 2*-generator dense subsemigroup of $\bar{\mathcal{F}}$. In fact, the maps* $1 + 1/(2x)$ *and* $-3x/8$ *generate a dense subsemigroup of $\bar{\mathcal{F}}$.*

Our last result in this section is regarding the weak topological mixing property. In the sequel, let $M_{2\times 2}$ denote the set of all 2×2 matrices with real entries.

Proposition 3.6. *Let G be a subsemigroup of $M_{2\times 2}$. Then the following statements are equivalent.*

(i) *The action of G on $\mathbb{R}^2$ is weakly topologically mixing.*
(ii) *The action of G on $M_{2\times 2}$ is hypercyclic.*
(iii) *The set G is dense in $M_{2\times 2}$.*

In particular, if A and B are defined as in (3.4)*, then the action of $\langle A, B\rangle$ on $\mathbb{R}^2$ is weakly topologically mixing.*

Proof. As we mentioned in the introduction, if G acts by open continuous maps on a Baire space (which is the case here), topological transitivity and hypercyclicity are equivalent i.e., (i) and (ii) are equivalent.

Next, we show that (ii) and (iii) are equivalent. If the action of S is hypercyclic, then there exists a matrix X whose G-orbit $\{gX\colon\ g \in G\}$ is dense in $M_{2\times 2}$. Clearly X needs to be invertible, and so the map $C \mapsto CX^{-1}$ is a homeomorphism on $M_{2\times 2}$, which maps the G-orbit of X to G. Since the G-orbit of X is dense, it follows that the set G is dense as well i.e., (ii) implies (iii). If (iii) holds, then the G-orbit of the identity matrix (being G itself) is dense in $M_{2\times 2}$, and so the action of G is hypercyclic. $\square$

There is an algebraic obstruction to 3-transitivity for the action of $M_{2\times 2}$ on $\mathbb{R}^2$ [7, Proposition 5.5], and so Proposition 3.6 is optimal, since it is stating that there exists a 2-generator weakly topologically mixing subsemigroup of $M_{2\times 2}$. We conjecture that in any dimension there exists a pair of matrices that generate a weakly topologically mixing subsemigroup.

4. Dense subsemigroups of $SL(2, \mathbb{R})$

The existence of 2-generator dense subgroups of $SL(2, \mathbb{R})$ has been observed by Jørgensen [10]. In this section, we prove the existence of 2-generator dense subsemigroups of $SL(2, \mathbb{R})$.

Let $\bar{\mathcal{F}}_+$ denote the set of real linear fractional transformations of the form $(ax+b)/(cx+d)$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. We repeat the same line of arguments presented in Sections 2 and 3, namely Lemmas 2.1 and 2.2 and Theorems 2.3 and 2.4, and then Proposition 3.1, Corollary 3.2, and finally Example 3.4. The arguments in each step must be modified to suit the current setting, however these modifications are straightforward and the details are omitted.

Proposition 4.1. *The matrices*

$$A = \begin{pmatrix} 1/\sqrt{2} & -\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} \sqrt{2/3} & 0 \\ 0 & \sqrt{3/2} \end{pmatrix}, \tag{4.1}$$

generate a dense subsemigroup of $SL(2, \mathbb{R})$.

Proof. Suppose $a < 0$ and $b > 1$, and let $R(x) = 1 + a/x$ and $S(x) = x/b$. Also let $\bar{\Lambda}$ denote the closure of the semigroup generated by R and S in the weak point-wise topology on $\bar{\mathcal{F}}_+$. Proof of Lemma 2.1 (and the discussion thereafter) can be modified to show that $T_s(x) = x/(sx+1) \in \bar{\Lambda}$ for every $s \leqslant 0$. An argument similar to the proof of Lemma 2.2 implies that $(\alpha x + \beta)/(\gamma x + 1) \in \bar{\Lambda}$ as long as $\beta, \gamma \leqslant 0 \leqslant \alpha$ and $0 \leqslant \alpha - \beta\gamma \leqslant \min(1, \alpha^2)$. By composing with $b^l x \in \bar{\Lambda}$ for $l \in \mathbb{Z}$, we conclude that

$$\left\{\frac{\alpha x+\beta}{\gamma x+\delta}\colon \beta,\gamma\leqslant 0<\alpha\delta-\beta\gamma\leqslant \min\bigl(b^k, b^{-k}\alpha^2\delta^{-2}\bigr)\right\}\subseteq \bar{\Lambda}.$$

Proof of Theorem 2.4 can be modified to show that if $a<0<c<1<b$ so that $\ln c/\ln b\notin\mathbb{Q}$, then the closure of the semigroup generated by $1+a/x$, x/b, and x/c contains all maps of the form $(\alpha x+\beta)/(\gamma x+1)$ with $\alpha-\beta\gamma>0$ and $\beta,\gamma\leqslant 0\leqslant\alpha$. To remove the condition $\beta,\gamma\leqslant 0\leqslant\alpha$, one needs to compose with $a/x\in\bar{\Lambda}$ repeatedly, and so $\langle 1+a/x, x/b, x/c\rangle$ is dense in $\bar{\mathcal{F}}_+$.

Next, the proof of Proposition 3.1 shows that for $a<0<c<1<b$ with $\ln c/\ln b\notin\mathbb{Q}$, the semigroup generated by $A=[1/a,-a;1/a,0]$, $B=[1/b,0;0,b]$, and $C=[1/c,0;0,c]$ is dense in $SL(2,\mathbb{R})$. Now, we set $a=-\sqrt{2}$, $b=\sqrt{3/2}$, and $c=\sqrt{1/2}$. One verifies that $ABA^3BA=C$ i.e., $\langle A,B,C\rangle=\langle A,B\rangle$. It follows that $\langle A,B\rangle$ is dense in $SL(2,\mathbb{R})$. □

Recall that $PSL(2,\mathbb{R})=SL(2,\mathbb{R})/\{\pm I_{2\times 2}\}$ is the group of orientation-preserving isometries of $\mathbb{H}^2$, the hyperbolic plane. Let $P\in\mathbb{H}^2$ and $X=(X_1,X_2)$ be an orthonormal frame at P. Similarly, let $Q\in\mathbb{H}^2$ and let $Y=(Y_1,Y_2)$ be an orthonormal frame at Y having the same orientation as X. Then there exists an isometry τ such that $\tau(P)=Q$ and $\tau(X_i)=Y_i$ for $i=1,2$. Hence, we have the following corollary.

Corollary 4.2. *There exists a pair of orientation-preserving isometries of $\mathbb{H}^2$ so that for every $P\in\mathbb{H}^2$ and an orthonormal frame X at P, the orbit of (P,X) under the action of the semigroup of maps induced by the pair is dense in $OF(\mathbb{H}^2)$, the orthonormal frame bundle of the hyperbolic plane.*

5. Dense subsemigroups of $GL(2,\mathbb{C})$

In this section, we consider the set of 2×2 complex matrices and prove a result analogous to Corollary 3.3 in the complex case. At the end of this section, we prove the existence of examples of 2-generator dense subsemigroups of 2×2 complex matrices. In the sequel, $i=\sqrt{-1}$.

Corollary 5.1. *Let $a,b,c,u\in\mathbb{C}$ such that the following conditions hold.*

(i) $a,b,c,u\neq 0$.
(ii) $b=ri$ *with* $r>1>|c|$.
(iii) *The three numbers* 1, $\ln|c|/\ln|b|$, $\arg(c)/2\pi$ *are rationally independent.*

Then the semigroup generated by the matrices

$$A=\begin{pmatrix} u & a\\ 1 & 0\end{pmatrix},\qquad B=\begin{pmatrix} 1 & 0\\ 0 & b\end{pmatrix},\qquad C=\begin{pmatrix} c & 0\\ 0 & 1\end{pmatrix},\tag{5.1}$$

is dense in the set of 2×2 complex matrices.

Proof. The argument presented in the proof of Lemma 2.1 works in the complex case for $R(x)=u+a/x$ and $S(x)=x/b$, as long as $a,u\neq 0$ and $|b|>1$. For every complex number s there exists a sequence $\{m_j\}_{j=1}^{\infty}$ of integers so that $sa=\sum_{j=1}^{k} b^{m_j}$. To see this, we note that every positive real number can be written as a series with terms of the form b^{4k}, $k\in\mathbb{Z}$, while every negative real number can be written as a series with the terms of the form b^{2k}, $k\in\mathbb{Z}$. Similarly, every purely imaginary number ti can be written as a series with terms of the form b^{4k+1}, if $t>0$, or terms of the form b^{4k+3} if $t<0$. It then follows from Eq. (2.1) that $T_s(x)=x/(sx+1)\in\bar{\Lambda}$ for all $s\in\mathbb{C}$. Now, the proof of Lemma 2.2 can be used to show that $(\alpha x+\beta)/(\gamma x+1)\in\bar{\Lambda}$ for every $\alpha,\beta,\gamma\in\mathbb{C}$ (in the complex case, Eqs. (2.3) are always solvable if $\beta,\alpha-\beta\gamma\neq 0$. The cases where $\beta=0$ or $\alpha-\beta\gamma=0$ can be dealt with by taking limits).

So far, we have shown that the semigroup generated by $R(x)=u+a/x$ and $S(x)=x/b$ is dense in the set of Möbius transformations (which is isometric to $SL(2,\mathbb{C})$). The proof of Corollary 3.2 can be used to show that the semigroup generated by A, B, and C is dense, if we show that the set $\langle b,c\rangle=\{b^m c^n\colon m,n\in\mathbb{N}\}$ is dense in $\mathbb{C}$. Let z be an arbitrary nonzero complex number. It follows from condition (iii) and the multidimensional Kronecker's approximation theorem [9, §23.6] that for any $\epsilon>0$ there exist positive integers m,n and an integer L so that

$$\left|n\left(\frac{\arg(c)}{2\pi}\right)-\left(\frac{\arg(z)}{2\pi}\right)+L\right|<\epsilon,\tag{5.2}$$

$$\left|n\left(\frac{\ln|c|}{\ln|b|^4}\right)-\left(\frac{\ln|z|}{\ln|b|^4}\right)+m\right|<\epsilon.\tag{5.3}$$

It follows from the inequalities (5.2) and (5.3) that $|\ln|c^n b^{4m}|-\ln|z||<\epsilon|b|^4$ and $|\arg(c^n b^{4m})-\arg(z)+2\pi L|<2\pi\epsilon$. Since ϵ was arbitrary, we conclude that $\langle b,c\rangle$ is dense in $\mathbb{C}$, and the proof is completed. □

We are now ready to prove the existence of examples of 2-generator dense subsemigroups of complex 2×2 matrices. Recall that a set F is called *cocountable* in E if $E \setminus F$ is countable.

Example 5.2. For $r > 3$, let

$$b = ri, \tag{5.4}$$

$$u = -\left(\frac{1}{2b}\right)^{1/5} \left(b^2 + 2b + 8(b+4)\sqrt{b^2+4}\right)^{1/5}, \tag{5.5}$$

$$a = u^2\left(-b - 2 + \sqrt{4+b^2}\right)/(2b), \tag{5.6}$$

$$c = \frac{1}{2}\left(b^2 - b + 2 + (-b+1)\sqrt{b^2+4}\right). \tag{5.7}$$

Then there exists a cocountable subset $F \subseteq (3, \infty)$ so that, for every $r \in F$, the semigroup generated by the matrices $A = [u, a; 1, 0]$ and $B = [1, 0; 0, b]$ is dense in the set of 2×2 complex matrices.

Proof. We have selected a, b, c, and u so that $ABA^3BA = C$ i.e., $\langle A, B, C\rangle = \langle A, B\rangle$. Thus, we only need to verify the conditions of Corollary 5.1 for a, b, c, and u. Clearly $a, u \neq 0$ and $|b| = r > 1$. By direct computation, we have

$$|c|^2 = \frac{1}{2}\left(r^4 - 3r^2 + r\sqrt{r^2-4} - r^3\sqrt{r^2-4}\right) \in (0, 1),$$

for all $r > 3$.

Now, $f(r) = \arg(c)/2\pi$ and $g(r) = \ln|c|/\ln|b|$ are both analytic functions of $r \in (3, \infty)$. Let $\mathcal{H}$ denote the set of $r > 3$ so that 1, $f(r)$, $g(r)$ are rationally dependent. We need to show that $\mathcal{H}$ is a countable set. On the contrary, suppose that $\mathcal{H}$ is uncountable. For each $r \in \mathcal{H}$, there exists a triplet of integers $(l(r), m(r), n(r)) \neq (0, 0, 0)$ so that

$$l(r) + m(r) f(r) + n(r) g(r) = 0.$$

The function $r \mapsto (l(r), m(r), n(r))$ maps the uncountable set $\mathcal{H}$ to the countable set $\mathbb{Z}^3 \setminus \{(0, 0, 0)\}$. It follows that there exist uncountably many values of r that are mapped to the same triplet $(l, m, n) \neq (0, 0, 0)$, and so the equation

$$H(r) = l + mf(r) + ng(r) = 0,$$

has uncountably many solutions for $r > 3$. Since f and g are analytic functions of r, it follows that H is an analytic function of r, and so $H(r) \equiv 0$ for all $r > 3$. On the other hand, as $r \to \infty$, one shows that $f(r) \to 1/4$ and $g(r) \to -1$, and so $l + m/4 - n = 0$. If $n \neq 0$, then

$$h(r) = \frac{g(r) + 1}{1/4 - f(r)} = \frac{m}{n},$$

but it is straightforward to check that $h(r)$ is not constant (alternatively, one can check that $\lim_{r\to\infty} h(r) = 0$, which gives $m = 0$, and then because g is not a constant function, it follows that $l = n = 0$). Hence $n = 0$, which in turn implies that $l = m = 0$, since f is not a constant function. This is a contradiction, and the proof is completed. $\square$

Recall that the Möbius group $SL(2, \mathbb{C})$ is isomorphic to the group of orientation-preserving isometries of $\mathbb{H}^3$, the three-dimensional hyperbolic space. As we noted in the proof of Corollary 5.1, the semigroup generated by the maps $R(x) = 1 + a/x$ and $S(x) = x/b$ is dense in $SL(2, \mathbb{C})$ as long as $a \neq 0$ and $b = ri$ for $r > 1$. Let $P \in \mathbb{H}^3$ and $X = (X_1, X_2, X_3)$ be an orthonormal frame at P. Similarly, let $Q \in \mathbb{H}^3$ and $Y = (Y_1, Y_2, Y_3)$ be an orthonormal frame at Q having the same orientation as X. Then, there exists an isometry τ such that $\tau(P) = Q$ and $D\tau(X_i) = Y_i$ for $i = 1, 2, 3$, where $D\tau$ is the derivative of τ at P. And so we have the following corollary.

Corollary 5.3. *There exists a pair of orientation-preserving isometries on $\mathbb{H}^3$ so that for every $P \in \mathbb{H}^3$ and an orthonormal frame X at P, the orbit of (P, X) under the action of the semigroup of maps induced by the pair is dense in $OF(\mathbb{H}^3)$, the orthonormal frame bundle of the hyperbolic 3-space.*

6. Non-existence of 2-generator dense subsemigroups of $\mathcal{F}$

In Section 2, we obtained 3-generator dense subsemigroups for $\mathcal{F}$. In this section, we show that $\mathcal{F}$ has no 2-generator dense subsemigroups. The proper subinterval of $\mathbb{R}$ under consideration is $(0, \infty)$; however, using a conjugation by a linear fractional map, the results in this section are valid on any proper subinterval of $\mathbb{R}$. Given a pair of functions $f, g \in \mathcal{F}$ and a positive real number x, the orbit of x under the action of $\langle f, g\rangle$ (the semigroup generated by f and g) is the set

$$\{f^{m_1}g^{n_1}\dots f^{m_k}g^{n_k}(x)\colon \forall i,\ m_i, n_i \geqslant 0;\ k \geqslant 0\}.$$

The induced action of $f \in \mathcal{F}$ on $(0,\infty)^2$ is defined by

$$f(x, y) = \bigl(f(x), f(y)\bigr).$$

We use the same character to denote f and its induced action on $(0,\infty)^2$. Let $\hat{\mathcal{F}}$ denote the set of real linear fractional maps from $(0,1)$ to $(0,1)$. The conjugation $\theta : (0,\infty) \to (0,1)$, defined by $\theta(x) = 1/(x+1)$, gives a one-to-one correspondence between $\hat{\mathcal{F}}$ and $\mathcal{F}$. In particular, $\langle f, g\rangle$ is dense in $\mathcal{F}$ if and only if $\langle \hat f, \hat g\rangle$ is dense in $\hat{\mathcal{F}}$, where $\hat f = \theta f \theta^{-1}$ and $\hat g = \theta g \theta^{-1}$.

We prove by contradiction that there are no 2-generator dense subsemigroups in $\mathcal{F}$. Suppose that $\langle f, g\rangle$ is dense in $\mathcal{F}$. Then $\langle f, g\rangle$ must have dense orbits in $(0,\infty)$, and so by the results in [7], there exist $a, b, c \geqslant 0$ so that one of the following occurs (up to order and a conjugation by a map of the form ux^v with $u > 0$ and $v \in \{1, -1\}$).

(I) (i) $a, b \geqslant 1$, $c \geqslant 0$, $b > 1$ if $c = 0$, and

$$f(x) = \frac{x}{x+a}, \qquad g(x) = bx + c. \tag{6.1}$$

(ii) $a, b > 1$, $\ln a/\ln b$ is irrational, and

$$f(x) = \frac{x}{a}, \qquad g(x) = bx. \tag{6.2}$$

(II) (i) $0 \leqslant c \leqslant 1$, $a > 0$, $b \geqslant 1$, $b > 1$ if $c = 0$, and

$$f(x) = \frac{a}{x+a}, \qquad g(x) = bx + c. \tag{6.3}$$

(ii) $a, b \geqslant 1$ and

$$f(x) = \frac{a}{x}, \qquad g(x) = bx + 1. \tag{6.4}$$

(III) $0 \leqslant c \leqslant 1$, $a > 0$, $b \geqslant 1$, $ab \leqslant 1$ if $c = 0$, and

$$f(x) = \frac{a}{x+a}, \qquad g(x) = c + \frac{ab}{x}. \tag{6.5}$$

Through the following sequence of lemmas (Lemmas 6.1–6.5), we eliminate all cases.

Lemma 6.1. *Suppose that case* (I) *occurs so that f and g are given by* (6.1) *or* (6.2)*. Then the induced action of the semigroup $\langle f, g\rangle$ has no dense orbits in $(0,\infty)^2$. In particular, $\langle f, g\rangle$ is not dense in $\mathcal{F}$ in these cases.*

Proof. In case (I), the functions f and g are both increasing, and so the induced action of $\langle f, g\rangle$ on $(0,\infty)^2$ preserves the regions $\{(x, y)\colon 0 < x \leqslant y\}$ and $\{(x, y)\colon x \geqslant y > 0\}$. $\square$

Lemma 6.2. *Suppose that case* (III) *occurs so that f and g are given by* (6.5)*. Then the induced action of the semigroup $\langle f, g\rangle$ has no dense orbits in $(0,\infty)^2$. In particular, $\langle f, g\rangle$ is not dense in $\mathcal{F}$ in this case.*

Proof. One can easily check that the region $\{(x, y)\colon k^{-1} \leqslant y/x \leqslant k\}$ is invariant under the induced actions of f and g for every positive k. $\square$

Lemma 6.3. *Suppose that the subcase* (ii) *of case* (II) *occurs so that f and g are given by* (6.4)*. Then the induced action of the semigroup $\langle f, g\rangle$ has no dense orbits in $(0,\infty)^2$. In particular, $\langle f, g\rangle$ is not dense in $\mathcal{F}$ in this case.*

Proof. By conjugating f and g via $\theta(x) = 1/(x+1)$, one gets

$$\hat f(x) = \frac{1-x}{ax-x+1} \quad \text{and} \quad \hat g(x) = \frac{x}{2x-bx+b}.$$

We have

$$\operatorname{Im}(\hat g) = [0, 1/2], \qquad \operatorname{Im}(\hat f \hat g) = \bigl[1/(a+1), 1\bigr],$$

where $\mathrm{Im}(\hat{g})$ denotes the image of the function $\hat{g}$ in $[0,1]$. Since $(\hat{f})^2 = \mathrm{Id}$ (the identity map), it follows that if $\hat{h} \in \langle \hat{f}, \hat{g} \rangle$ and $\hat{h} \neq \hat{f}, \mathrm{Id}$, then $\mathrm{Im}(\hat{h}) \subseteq \mathrm{Im}(\hat{g})$ or $\mathrm{Im}(\hat{h}) \subseteq \mathrm{Im}(\hat{f}\hat{g})$. It follows that for every $(x, y) \in [0,1]^2$ and $\hat{h} \in \langle \hat{f}, \hat{g} \rangle$ with $\hat{h} \neq \hat{f}, \mathrm{Id}$, we have

$$\left|\hat{h}(x) - \hat{h}(y)\right| \leqslant \frac{a}{a+1}.$$

Hence, the open set $\{(x, y) \in [0,1]^2\colon\ |x - y| > a/(a+1)\}$ cannot contain more than two elements of each orbit. However the orbit of the point $(0,1)$ is dense under the action of $\hat{\mathcal{F}}$, hence the orbit of $(0,1)$ is dense under the action of any dense subsemigroup. Since $\langle \hat{f}, \hat{g} \rangle$ has no dense orbits, we conclude that it is not dense in $\hat{\mathcal{F}}$. $\square$

Lemma 6.4. *Suppose that the subcase* (i) *of case* (II) *occurs so that f and g are given by* (6.3) *and $c \neq 0$. Then the induced action of the semigroup $\langle f, g \rangle$ has no dense orbits in $(0,\infty)^2$. In particular, $\langle f, g \rangle$ is not dense in $\mathcal{F}$ in this case.*

Proof. The conjugation via $\theta(x) = 1/(x+1)$ gives the maps

$$\hat{f} = \frac{ax - x + 1}{2ax - x + 1} \quad \text{and} \quad \hat{g} = \theta g \theta^{-1}(x) = \frac{x}{cx - bx + x + 1}.$$

We have

$$\mathrm{Im}(\hat{f}) = [1/2, 1] \quad \text{and} \quad \mathrm{Im}(\hat{g}) = \big[0, 1/(c+1)\big].$$

It follows that for every $(x, y) \in [0,1]^2$ and $\hat{h} \in \langle \hat{f}, \hat{g} \rangle$ with $\hat{h} \neq \mathrm{Id}$, we have

$$\left|\hat{h}(x) - \hat{h}(y)\right| \leqslant \max\left(\frac{1}{2}, \frac{1}{c+1}\right).$$

If $c \neq 0$, then the open set $\{(x, y) \in [0,1]^2\colon\ |x - y| > \max(1/2, 1/(c+1))\}$ cannot contain more than one element of each orbit i.e., the orbits are not dense, and so $\langle f, g \rangle$ is not dense in $\mathcal{F}$ in this case either. $\square$

Lemma 6.5. *Suppose that the subcase* (i) *of case* (II) *occurs so that f and g are given by* (6.3) *and $c = 0$. Then the induced action of the semigroup $\langle f, g \rangle$ has no dense orbits in $(0,\infty)^2$. In particular, $\langle f, g \rangle$ is not dense in $\mathcal{F}$ in this case.*

Proof. Using the conjugation $x \mapsto 1/x$, we obtain $\hat{f}(x) = 1 + a/x$ and $\hat{g}(x) = x/b$. Then the lemma follows from Theorem 2.3, since $\mathcal{U}$ does not include every $f \in \mathcal{F}_+$; for example the map $b^{1/3}x$ does not belong to $\mathcal{U}_k$ for any $k \in \mathbb{Z}$, and so by Theorem 2.3, it does not belong to $\bar{\Lambda}$. $\square$

Lemmas 6.1 through 6.5 imply our main result in this section.

Theorem 6.6. *The set $\mathcal{F}$ does not have a* 2*-generator dense subsemigroup.*

7. Orbit closures

In Section 6, we showed that there are no 2-generator dense subsemigroups of $\mathcal{F}$. In this section, we study the induced action of the semigroup generated by $R(x) = 1 + a/x$ and $S(x) = x/b$ on $(0,\infty)^2$, and show that it has no dense orbits in $(0,\infty)^2$.

Theorem 7.2 below describes the orbit closure of $(x, y) \in (0,\infty)^2$ under the action of $\langle R, S \rangle$. It is more appropriate to give a geometric description of the orbit closures. Given a point $A = (x, y) \in (0,\infty)^2$, there exists a unique hyperbola tangential to the line $y = x$ at the origin that connects the origin to A. We denote this hyperbolic segment by $H(x, y)$. Also, we denote the infinite half-line in $(0,\infty)^2$ with slope 1 starting at (x, y) by $L(x, y)$. Finally, let $\Omega(x, y)$ denote the closed region bounded by $H(x, y)$, $L(x, y)$, $H(a/x, a/y)$, and $L(a/x, a/y)$. If $x = y$, then this region degenerates to the half-line $y = x$, and so in this case we set $\Omega(x, x) = \{(t, t);\ t \geqslant 0\}$. In the sequel, $\bar{\Lambda}$ denotes the closure of the semigroup generated by $R(x) = 1 + a/x$ and $S(x) = x/b$, where $a > 0$ and $b > 1$. We begin with the following lemma.

Lemma 7.1. *Let $(x, y) \in (0,\infty)^2$. Then for every $(u, v) \in \Omega(x, y)$, there exists $f \in \bar{\Lambda}$ so that $f(x, y) = (u, v)$.*

Proof. Since $\Omega(x, y)$ is invariant under the map $I(x) = a/x$ and $I(x) \in \bar{\Lambda}$, without loss of generality, we assume that $x \geqslant y$ and $u \geqslant v$. If $x = y$ or $u = v$, the claim follows from the fact that the orbits of $\langle R, S \rangle$ on $[0,\infty)$ are all dense (and that $(0,0)$ belongs to every orbit closure). Thus, suppose that $x > y$ and $u > v$. Since $(u, v) \in \Omega(x, y)$, we have

$$v \geqslant \max\left(u - x + y, \frac{uxy}{ux - uy + xy}\right). \tag{7.1}$$

It follows from Lemma 2.2 (by setting $\alpha = 1$) that maps of the form $f(x) = (x+\beta)/(\gamma x+1)$ belong to $\bar{\Lambda}$, where $\beta, \gamma \geqslant 0$ and $\beta\gamma \leqslant 1$. We choose β and γ so that $f(x) = u$ and $f(y) = v$. In fact, we need to have

$$\beta = \frac{xy(-u+v) + uv(x-y)}{ux - vy}, \qquad \gamma = \frac{x-y-u+v}{ux-vy}.$$

The conditions $\beta, \gamma \geqslant 0$ are deduced directly from inequalities $u > v$ and (7.1). The condition $\beta\gamma \leqslant 1$ is equivalent to

$$(ux - vy)^2 - \big(xy(-u+v) + uv(x-y)\big)(x-y-u+v) \geqslant 0,$$

which can be factorized as $(u-v)(v+x)(x-y)(u+y) > 0$, which is true, since $x > y$ and $u > v$. $\square$

Theorem 7.2. *Let $R(x) = 1 + a/x$ and $S(x) = x/b$, where $a > 0$ and $b > 1$. Then for any $(x, y) \in (0,\infty)^2$, the closure of the orbit of (x, y) under the action of $\langle R, S\rangle$ is given by*

$$\bigcup_{k\in\mathbb{Z}} \Omega\big(b^k x, b^k y\big).$$

Proof. Lemma 7.1 and the fact that $b^k x \in \bar{\Lambda}$, for all $k \in \mathbb{Z}$, imply that the set $\bar{\Omega} = \bigcup_{k\in\mathbb{Z}} \Omega(b^k x, b^k y)$ is included in the orbit closure of (x, y). To show that the orbit closure is included in $\bar{\Omega}$, it is sufficient to show that $\bar{\Omega}$ is invariant under R and S. The set $\bar{\Omega}$ is clearly invariant under S. Moreover, we have $R(x) = 1 + a/x = M \circ I(x)$, where $M(x) = x + 1$ and $I(x) = a/x$. Since $\bar{\Omega}$ is invariant under both I and M, we see that it is invariant under R as well, and the proof is completed. $\square$

Theorem 7.2 shows that the orbits of $\langle R, S\rangle$ on $(0,\infty)$ are never dense, since for example the points $(t, 0)$ for $t > 0$ do not belong to $\Omega(b^k x, b^k y)$ for any $k \in \mathbb{Z}$ and any $(x, y) \in (0,\infty)^2$. However, in the finite-interval case, dense orbits exist. To see this, we use the conjugation $\theta(x) = 1/(x + 1)$ to move to the interval $[0, 1]$, and denote the conjugated maps using the hat notation.

Proposition 7.3. *The orbit of $(x, y) \in [0, 1]^2$ under the action of the semigroup $\langle \hat{R}, \hat{S}\rangle$ is dense in $[0, 1]^2$ if and only if (x, y) belongs to the perimeter of the square $[0, 1]^2$ except the vertices $(0, 0)$ and $(1, 1)$.*

Proof. The claim that none of the orbits starting from an interior point are dense follows from Theorem 7.2. The orbits starting from $(0, 0)$ and $(1, 1)$ are clearly not dense. Since the point $(0, 1)$ belongs to the orbit of every point on the perimeter of $[0, 1]^2$ except $(0, 0)$ and $(1, 1)$, it is sufficient to prove that the orbit of $(0, 1)$ is dense. Let $\bar{O}$ denote the closure of the orbit of $(0, 1)$ in $[0, 1]^2$. Let u be an arbitrary nonnegative number. It follows from (2.1), after conjugating by θ, that

$$\hat{T}_u(x) = \frac{u(1-x) + x}{(u+1)(1-x) + x}$$

belongs to the closure of $\langle \hat{R}, \hat{S}\rangle$. It follows that $(f(0), f(1)) = (u/(u+1), 1) \in \bar{O}$, which implies that the segment $[0, 1] \times \{1\}$ is a subset of $\bar{O}$. By applying $\hat{R} = (1-x)/(2-2x+ax)$ to this segment, we obtain $[0, 1/2] \times \{0\} \subseteq \bar{O}$. By applying $\hat{S}$ to the segment repeatedly, we get $[0, 1] \times \{0\} \subseteq \bar{O}$. It follows that $\hat{T}_u([0, 1] \times \{0\}) = [u/(u+1), 1] \times \{u/(u+1)\} \subseteq \bar{O}$ for any nonnegative u. And so $\Delta = \{(x, y) \in [0, 1]^2\colon\ x \geqslant y\} \subseteq \bar{O}$. By applying $\hat{R}$ to Δ, we get $\{(x, y) \in [0, 1]^2\colon\ y \leqslant 1/2\} \subseteq \bar{O}$, and by applying $\hat{S}$ repeatedly to this latter set, we conclude that $[0, 1]^2 \subseteq \bar{O}$. $\square$

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