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Complex interpolation of Besov spaces and Triebel–Lizorkin spaces with variable exponents

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ABSTRACT

This paper concerns the complex interpolation of Besov spaces and Triebel–Lizorkin spaces with variable exponents.

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1. Introduction

From the viewpoint of harmonic analysis, more and more attention has been paid to function spaces of variable exponents during this decade. A breakthrough in this field is the boundedness of the Hardy–Littlewood maximal operator $\mathcal{M}$ obtained by Diening [3]. The first concrete application arose from a model of electrorheological fluids in [16]. Recently many researchers are oriented to more and more applications to PDE and the function spaces are improved accordingly [5]. This paper concerns the complex interpolation of Besov spaces and Triebel–Lizorkin spaces with variable exponents. Now we fix some notations.

Denote by $\mathcal{P}_0(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot):\mathbb{R}^n\to(0,\infty)$ such that

$$0<p_-=\operatorname*{ess\,inf}_{x\in\mathbb{R}^n}p(x),\qquad \operatorname*{ess\,sup}_{x\in\mathbb{R}^n}p(x)=p_+<\infty. \tag{1}$$

For $p\in\mathcal{P}_0(\mathbb{R}^n)$, we let $L^{p(\cdot)}(\mathbb{R}^n)$ be the set of measurable functions f on $\mathbb{R}^n$ such that for some $\lambda>0$,

$$\int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)}\mathrm{d}x\leqslant 1.$$

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The infimum of such λ will be denoted by $\|f\|_{L^{p(\cdot)}}$. The set $L^{p(\cdot)}(\mathbb{R}^n)$ becomes a quasi Banach function space when equipped with the Luxemburg–Nakano norm $\|f\|_{L^{p(\cdot)}}$ [10,12,13]. More precisely,

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0: \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leqslant 1\right\}.$$

To ensure that the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, we postulate the following standard conditions: There exists a constant $C_{\log}(p)$ such that

$$\big|p(x) - p(y)\big| \leqslant \frac{C_{\log}(p)}{\log(e + |x-y|^{-1})} \quad \big(x, y \in \mathbb{R}^n,\ x \neq y\big) \tag{2}$$

and there exist constants $C_{\log}(p)$ and p_∞ such that

$$\big|p(x) - p_\infty\big| \leqslant \frac{C_{\log}(p)}{\log(e + |x|)} \quad \big(x \in \mathbb{R}^n\big). \tag{3}$$

We write $C^{\log}(\mathbb{R}^n)$ as the set of all real-valued functions $p: \mathbb{R}^n \to \mathbb{R}$ satisfying (2) and (3). Define the class $\mathcal{B}_0(\mathbb{R}^n)$ of all $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ such that $\mathcal{M}$ is bounded on $L^{\alpha p(\cdot)}(\mathbb{R}^n)$ for some positive number α.

We now define Triebel–Lizorkin spaces of variable integrability. We use $\mathcal{F}$ and $\mathcal{F}^{-1}$ to denote the Fourier transform and its inverse respectively. Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ be a function satisfying $\chi_{B(0,1/2)} \leqslant \mathcal{F}\Phi \leqslant \chi_{B(0,1)}$, where $B(0,r) = \{x \in \mathbb{R}^n: |x| < r\}$. Set $\Phi_j(x) = 2^{nj}\Phi(2^j x)$ for $j \in \mathbb{N}_0 = \mathbb{Z} \cup \{0\}$. If we define $\theta_j = \Phi_j - \Phi_{j-1}$ for $j \in \mathbb{N}$ and $\theta_0 = \Phi_0$, then we have $\sum_{j=0}^\infty \theta_j \equiv 1$. To describe the vector-valued norm, for $p, q \in \mathcal{B}_0(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$, we write $L^{p(\cdot)}(\ell^{q(\cdot)})$ to denote the space consisting of all sequences $\{g_j\}_0^\infty$ of measurable functions on $\mathbb{R}^n$ such that

$$\big\|\{g_j\}_0^\infty\big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} = \Big\|\big\|\{g_j\}_0^\infty\big\|_{\ell^{q(\cdot)}}\Big\|_{L^{p(\cdot)}} = \left\|\left(\sum_{j=0}^\infty |g_j(\cdot)|^{q(\cdot)}\right)^{\frac{1}{q(\cdot)}}\right\|_{L^{p(\cdot)}} < \infty.$$

Definition 1.1. Let $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{B}_0(\mathbb{R}^n)$ and $s(\cdot) \in C^{\log}$. Then define the variable exponent Triebel–Lizorkin space $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ as the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} = \|\{2^{s(\cdot)j}\theta_j * f\}_0^\infty\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} < \infty$.

As is established in [4], the definition of $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$ does not depend on the starting function Φ.

Next, we shall describe the complex interpolation functor introduced by Triebel [19]. Let $A = \{z \in \mathbb{C}: 0 < \mathrm{Re}\ z < 1\}$ be a strip in the complex plane and denote by $\bar{A}$ its closure. Then an $\mathcal{S}'(\mathbb{R}^n)$-valued mapping f on $\bar{A}$ is said to be analytic on A, if it satisfies the following:

1. The function $(\mathcal{F}^{-1}\varphi \cdot \mathcal{F}f)(x,z)$ of $(x,z) \in \mathbb{R}^n \times \bar{A}$ is a uniformly continuous and bounded function for all $\varphi \in C_c^\infty(\mathbb{R}^n)$.
2. For any $x \in \mathbb{R}^n$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$, the function $(\mathcal{F}^{-1}\varphi \cdot \mathcal{F}f)(x,z)$ of $z \in A$ is analytic.

We recall that $(\mathcal{F}^{-1}\varphi\mathcal{F}f)(\cdot, z)$ is an entire analytic function on $\mathbb{R}^n$ if $z \in \bar{A}$ is fixed.

Definition 1.2. Let $p_0(\cdot), p_1(\cdot), q_0(\cdot), q_1(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{B}_0(\mathbb{R}^n)$ and $s_0(\cdot), s_1(\cdot) \in C^{\log}$. The space $F(F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)})$ consists of all $\mathcal{S}'(\mathbb{R}^n)$-analytic functions $f(z)$ on A such that

$$\big\|f(z)\big\|_{F(F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)})} = \sup\Big\{\big\|f(\ell + \mathrm{i}t)\big\|_{F^{s_\ell(\cdot)}_{p_\ell(\cdot),q_\ell(\cdot)}}: \ell = 0, 1,\ t \in \mathbb{R}\Big\} < \infty.$$

We introduce the complex interpolation spaces for variable Triebel–Lizorkin spaces. For exhaustive details of complex interpolation, we refer to [2]. But in the present paper, we follow [19].

Definition 1.3. Let $\theta \in (0,1)$, $p_0(\cdot), p_1(\cdot), q_0(\cdot), q_1(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{B}_0(\mathbb{R}^n)$ and $s_0(\cdot), s_1(\cdot) \in C^{\log}$. The space $(F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)})_\theta$ consists of all $g \in \mathcal{S}'(\mathbb{R}^n)$ which is realized as $g = f(\theta)$ for some $f(z) \in F(F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)})$. The norm is defined by

$$\|g\|_{(F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)})_\theta} = \inf\Big\{\big\|f(z)\big\|_{F(F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)})}: g = f(\theta),\ f(z) \in F\big(F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}\big)\Big\}.$$

With the above definitions in mind, we formulate one of our main results in this paper.

Theorem 1.4. *Let $p_0(\cdot), p_1(\cdot), q_0(\cdot), q_1(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{B}_0(\mathbb{R}^n)$ and $s_0(\cdot), s_1(\cdot) \in C^{\log}$. For $\theta \in (0,1)$, define $p(\cdot), q(\cdot), s(\cdot)$ by the equations*

$$\frac{1}{p(\cdot)} = \frac{1-\theta}{p_0(\cdot)} + \frac{\theta}{p_1(\cdot)}, \qquad \frac{1}{q(\cdot)} = \frac{1-\theta}{q_0(\cdot)} + \frac{\theta}{q_1(\cdot)}, \qquad s(\cdot) = (1-\theta)s_0(\cdot) + \theta s_1(\cdot). \tag{4}$$

Then we have

$$\left(F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}\right)_\theta \approx F^{s(\cdot)}_{p(\cdot),q(\cdot)} \tag{5}$$

with norm equivalence.

A direct consequence of Theorem 1.4 is the complex interpolation results for local Hardy spaces with variable exponent in [4,11]. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Denote by $\mathcal{M}_{\mathrm{loc}} f$ the local grand maximal operator given by

$$\mathcal{M}_{\mathrm{loc}} f(x) \equiv \sup\left\{ \left| t^{-n}\psi\left(t^{-1}\cdot\right) * f(x)\right| \colon 0 < t < 1, \sum_{|\alpha|\leqslant N} \sup_{x\in\mathbb{R}^n} \left(1+|x|\right)^N \left|\partial^\alpha \psi(x)\right| < \infty \right\},$$

where we choose and fix a large integer N.

Definition 1.5. One defines $h^{p(\cdot)}$ as the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the quasi-norm $\|f\|_{h^{p(\cdot)}} = \|\mathcal{M}_{\mathrm{loc}} f\|_{L^{p(\cdot)}}$ is finite.

In [11] we proved that $h^{p(\cdot)}$ and $F^0_{p(\cdot)2}$ are mutually equivalent. Therefore, we obtain the following corollary.

Corollary 1.6. *Let $0 < \theta < 1$ and $p_0(\cdot), p_1(\cdot) \in \mathcal{B}_0(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$. Define $p(\cdot)$ by*

$$\frac{1}{p(\cdot)} = \frac{1-\theta}{p_0(\cdot)} + \frac{\theta}{p_1(\cdot)}.$$

Then

$$\left(h^{p_0(\cdot)}, h^{p_1(\cdot)}\right)_\theta \approx h^{p(\cdot)}$$

with norm equivalence.

The definition and formulation of our result for Besov spaces is analogous. Let $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. The space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is the collection of all sequences $\{g_j\}_0^\infty$ of measurable functions on $\mathbb{R}^n$ such that

$$\left\|\{g_j\}_{j=0}^\infty\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf\left\{ \mu > 0 \colon \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\left\{ \frac{f_j}{\mu} \right\}_{j=0}^\infty \right) \leqslant 1 \right\} < \infty,$$

where

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\{f_j\}_{j=0}^\infty\right) = \sum_{j=0}^\infty \inf\left\{ \lambda_j \colon \int_{\mathbb{R}^n} \left(\frac{|f_j(x)|}{\lambda_j^{\frac{1}{q(x)}}} \right)^{p(x)} \mathrm{d}x \leqslant 1 \right\}.$$

Since we assume that $q_+ < \infty$,

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\{f_j\}_{j=0}^\infty\right) = \sum_{j=0}^\infty \left\| |f_j|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \tag{6}$$

holds.

Definition 1.7. Let $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$ and $s(\cdot) \in C^{\log}(\mathbb{R}^n)$. The Besov space with variable exponents $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is the collection of $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} = \left\| \left\{ 2^{js(\cdot)} \theta_j * f \right\}_0^\infty \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

We have a counterpart of Definitions 1.2 and 1.3 for Besov spaces.

Definition 1.8. Let $p_0(\cdot), p_1(\cdot), q_0(\cdot), q_1(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{B}_0(\mathbb{R}^n)$ and $s_0(\cdot), s_1(\cdot) \in C^{\log}$. The space $F(B^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, B^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)})$ consists of all $\mathcal{S}'(\mathbb{R}^n)$-analytic functions $f(z)$ on A such that

$$\big\| f(z) \big\|_{F(B^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, B^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)})} = \sup\Big\{ \big\| f(\ell + it) \big\|_{B^{s_\ell(\cdot)}_{p_\ell(\cdot),q_\ell(\cdot)}} : \ell = 0, 1,\ t \in \mathbb{R} \Big\} < \infty.$$

Definition 1.9. Let $\theta \in (0,1)$, $p_0(\cdot), p_1(\cdot), q_0(\cdot), q_1(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathcal{B}_0(\mathbb{R}^n)$ and $s_0(\cdot), s_1(\cdot) \in C^{\log}$. The space $(B^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, B^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)})_\theta$ consists of all $g \in \mathcal{S}'(\mathbb{R}^n)$ which is realized as $g = f(\theta)$ for some $f(z) \in F(B^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, B^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)})$. The norm is defined by

$$\|g\|_{(B^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, B^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)})_\theta} = \inf\Big\{ \big\| f(z) \big\|_{F(B^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, B^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)})} : g = f(\theta),\ f(z) \in F\big(B^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, B^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}\big) \Big\}.$$

We remark that Xu considered the case when q is fixed [21,22].

In [1], the following proposition is proven:

Proposition 1.10. *(See [1, Lemma 4.7].) Let $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < p_- \leqslant p_+ < \infty$ and $1 < q_- \leqslant q_+ < \infty$. Then the inequality*

$$\big\| \{\eta_{k,m} * f_k\}_{k=0}^\infty \big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant c \big\| \{f_k\}_0^\infty \big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$$

holds for every sequence $\{f_k\}_{k=0}^\infty$ of L^1_{loc}-functions and $m > 2n$.

Therefore, the definition of $B^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$ does not depend on the choices of admissible θ. Our second main result reads

Theorem 1.11 *(Complex interpolation for variable function Besov spaces). Under the same conditions as Theorem* 1.4*, we have*

$$\big(B^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, B^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}\big)_\theta \approx B^{s(\cdot)}_{p(\cdot),q(\cdot)}. \tag{7}$$

The proof of Theorem 1.4 does not seem to parallel to the proof of [19, Theorem 2.4.7]. Indeed, the definition of $f(z)$ in [19, p. 71, (13)] does not seem to work in the setting of variable Lebesgue spaces. If we define $f(z)$ as is described in [19, p. 71, (13)], then, in order that the proof works, we need to assume that p_0/p_1 is constant. To circumvent this difficulty, we first use the atomic decomposition technique as is described in Proposition 2.6. Another difficulty is that we need to "freeze" p_0 and p_1 despite the atomic decomposition. To this end, we need to utilize the maximal estimate Lemma 2.5. Keeping these techniques in mind, we carefully prove Theorem 1.4.

Here we describe how we organized this paper. In Section 2 we collect some tools needed for the proof of Theorem 1.4. We prove Theorem 1.4 in Section 3. Finally in Section 4 we prove Theorem 1.11.

2. Preliminaries

To go into the details of the proof, we collect some elementary facts on variable Lebesgue spaces and Besov/Triebel–Lizorkin spaces of variable exponents.

2.1. More about the variable Lebesgue spaces

We denote by $\mathcal{P}(\mathbb{R}^n)$ the set of bounded measurable functions $p(\cdot)$ on $\mathbb{R}^n$ with range in $(1,\infty)$. Note that Pick gave an example of $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that the Hardy–Littlewood maximal operator $\mathcal{M}$ is not bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (see [15]), which shows that the condition $p(x) > 1$ is not sufficient for the boundedness of $\mathcal{M}$ unlike the classical Lebesgue spaces. Therefore, we need to define the class $\mathcal{B}(\mathbb{R}^n)$ of all functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ for which the $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. To develop the theory of function spaces with $p_- < 1$, we need to relax the condition of $\mathcal{B}(\mathbb{R}^n)$ and we recall the class $\mathcal{B}_0(\mathbb{R}^n)$ of all $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ such that $\mathcal{M}$ is bounded on $L^{\alpha p(\cdot)}(\mathbb{R}^n)$ for some positive number α.

We need a Hölder inequality generalized to variable Lebesgue spaces. For the proof we refer to [9,18].

Proposition 2.1 *(Generalized Hölder inequality). Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and*

$$p'(\cdot) = \frac{p(\cdot)}{p(\cdot) - 1}.$$

Then

$$\int_{\mathbb{R}^n} \big| f(x) g(x) \big| \, \mathrm{d}x \leqslant \left(1 + \frac{1}{p_-} - \frac{1}{p_+} \right) \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}}$$

for every $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$.

The next proposition concerns the generalized Minkowski inequality, which is an immediate consequence of [18, Theorems 3.5 and 3.7, Corollary in Section 3.6].

Proposition 2.2 *(Generalized Minkowski inequality). Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then*

$$\left\| \int_{\mathbb{R}^n} f(\cdot, y)\, \mathrm{d}y \right\|_{L^{p(\cdot)}} \leqslant \left(1 + \frac{1}{p_-} - \frac{1}{p_+}\right) \int_{\mathbb{R}^n} \left\| f(\cdot, y) \right\|_{L^{p(\cdot)}} \mathrm{d}y.$$

The next proposition is shown in [5].

Proposition 2.3. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $f \in L^{p(\cdot)}(\mathbb{R}^n)$. Then*

$$\begin{aligned} &\min\left\{ \left(\int_{\mathbb{R}^n} \left| f(x) \right|^{p(x)} \mathrm{d}x \right)^{\frac{1}{p_-}}, \left(\int_{\mathbb{R}^n} \left| f(x) \right|^{p(x)} \mathrm{d}x \right)^{\frac{1}{p_+}} \right\} \\ &\quad \leqslant \| f \|_{L^{p(\cdot)}} \leqslant \max\left\{ \left(\int_{\mathbb{R}^n} \left| f(x) \right|^{p(x)} \mathrm{d}x \right)^{\frac{1}{p_-}}, \left(\int_{\mathbb{R}^n} \left| f(x) \right|^{p(x)} \mathrm{d}x \right)^{\frac{1}{p_+}} \right\}. \end{aligned}$$

Given a function f, we take $p_{\min}(f)$ and $p_{\max}(f)$ so that

$$\left(\int_{\mathbb{R}^n} \left| f(x) \right|^{p(x)} \mathrm{d}x \right)^{\frac{1}{p_{\min}(f)}} \leqslant \| f \|_{L^{p(\cdot)}} \leqslant \left(\int_{\mathbb{R}^n} \left| f(x) \right|^{p(x)} \mathrm{d}x \right)^{\frac{1}{p_{\max}(f)}}. \tag{8}$$

For later consideration, we use the following simple estimate:

Proposition 2.4. *Let $p(\cdot)$ be a measurable function on $\mathbb{R}^n$ with range in $[\alpha, \beta]$, where $\alpha > 0$. Let $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. If $p(\cdot)q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ and $g(\cdot) \in L^{p(\cdot)q(\cdot)}(\mathbb{R}^n)$, then*

$$\left\| \left| g(\cdot) \right|^{p(\cdot)} \right\|_{L^{q(\cdot)}} \leqslant \max\left(\left\| g(\cdot) \right\|_{L^{p(\cdot)q(\cdot)}}^{\beta}, \left\| g(\cdot) \right\|_{L^{p(\cdot)q(\cdot)}}^{\alpha} \right).$$

Proof. Let $p(\cdot)$ be a measurable function on $\mathbb{R}^n$ with range in $[\alpha, \beta]$. Let $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ and $g(\cdot) \in L^{p(\cdot)q(\cdot)}$. If $\| g(\cdot) \|_{L^{p(\cdot)q(\cdot)}} \geqslant 1$, then we have

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\frac{|g(x)|^{p(x)}}{\| g(\cdot) \|_{L^{p(\cdot)q(\cdot)}}^{\beta}} \right)^{q(x)} \mathrm{d}x &= \int_{\mathbb{R}^n} \left(\frac{|g(x)|}{\| g(\cdot) \|_{L^{p(\cdot)q(\cdot)}}^{\frac{\beta}{p(x)}}} \right)^{p(x)q(x)} \mathrm{d}x \\ &\leqslant \int_{\mathbb{R}^n} \left(\frac{|g(x)|}{\| g(\cdot) \|_{L^{p(\cdot)q(\cdot)}}} \right)^{p(x)q(x)} \mathrm{d}x \leqslant 1 \end{aligned}$$

by $1 \leqslant \beta / p(x)$ for almost all $x \in \mathbb{R}^n$. A similar argument yields $\| g(\cdot) \|_{L^{p(\cdot)q(\cdot)}} < 1$. □

2.2. Boundedness of maximal operators

We depend upon the following convolution inequality obtained in [4]. To formulate it precisely, we use the notation:

$$\eta_{\nu,m}(x) = \frac{2^{\nu n}}{(1 + 2^{\nu}|x|)^m} \quad \left(x \in \mathbb{R}^n\right).$$

Lemma 2.5. *(See [4, Theorem 3.2].) Let $m > n$. Assume that $p(\cdot), q(\cdot) \in \mathcal{B}_0(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ fulfill $p_-, q_- > 1$. Then we have*

$$\| \eta_{\nu,m} * f_{\nu} \|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \lesssim \| f_{\nu} \|_{L^{p(\cdot)}(\ell^{q(\cdot)})}$$

for all sequences of measurable functions $\{f_j\}_{j=1}^{\infty}$.

As we can see an example from the paper [4], we cannot obtain a similar result if we replace the constant u with variable functions.

2.3. Some facts on variable exponent function spaces

Diening et al. [4] showed that the decomposition of the space $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ into molecules or atoms and that the lifting property under suitable assumptions on $p(\cdot)$, $q(\cdot)$ and $\alpha(\cdot)$ are available. In particular, denoting $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, we state these results for the $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ case:

Proposition 2.6. *Let $p(\cdot), q(\cdot) \in \mathcal{B}_0(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ and $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $s \gg 0$. If $f \in F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, then there exist a collection of functions $\{a_{\nu,m}\}_{(\nu,m)\in\mathbb{N}_0\times\mathbb{Z}^n}$ and a sequence of coefficients $\{\lambda_{\nu,m}\}_{(\nu,m)\in\mathbb{N}_0\times\mathbb{Z}^n}$ with the following conditions.*

1. *The coefficients $\{\lambda_{\nu,m}\}_{(\nu,m)\in\mathbb{N}_0\times\mathbb{Z}^n}$ satisfy*
$$\left\| \left\{ 2^{\nu s(\cdot)} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}| \chi_{Q_{\nu,m}}(\cdot) \right\}_{\nu=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \lesssim \|f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}}.$$
Here and below $Q_{\nu,m} = 2^{-\nu}m + [0, 2^{-\nu}]^n$ for $m \in \mathbb{Z}^n$ and $\chi_{Q_{\nu,m}}$ denotes the characteristic function on $Q_{\nu,m}$.
2. *The function $a_{\nu,m}$ is supported on $2^{-\nu}m + [-2^{-\nu+2}, 2^{-\nu+2}]^n$ and satisfies*
$$\left\| \partial^{\alpha} a_{\nu,m} \right\|_{\infty} \leqslant 1$$
for all α with $|\alpha| \leqslant [s_- + 1]_+$.
3. *The function f can be decomposed as follows*:
$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m},$$
where the convergence is in $\mathcal{S}'(\mathbb{R}^n)$.

Proposition 2.7. *Let $p(\cdot), q(\cdot) \in \mathcal{B}_0(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ and $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $s \gg 0$. Suppose that we are given a collection of functions $\{a_{\nu,m}\}_{(\nu,m)\in\mathbb{N}_0\times\mathbb{Z}^n}$ and a sequence of coefficients $\{\lambda_{\nu,m}\}_{(\nu,m)\in\mathbb{N}_0\times\mathbb{Z}^n}$:*

1. *The coefficients $\{\lambda_{\nu,m}\}_{(\nu,m)\in\mathbb{N}_0\times\mathbb{Z}^n}$ satisfy*
$$\left\| \left\{ 2^{\nu s(\cdot)} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}| \chi_{Q_{\nu,m}}(\cdot) \right\}_{\nu=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} < \infty.$$
2. *The function $a_{\nu,m}$ is supported on $2^{-\nu}m + [-2^{-\nu+2}, 2^{-\nu+2}]^n$ and satisfies*
$$\left\| \partial^{\alpha} a_{\nu,m} \right\|_{\infty} \leqslant 1$$
for all α with $|\alpha| \leqslant [s_- + 1]_+$.

Then
$$f := \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m},$$
converges in $\mathcal{S}'(\mathbb{R}^n)$ and satisfies
$$\|f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim \left\| \left\{ 2^{\nu s(\cdot)} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}| \chi_{Q_{\nu,m}}(\cdot) \right\}_{\nu=0}^{\infty} \right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})}.$$

For the $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ case, we have a counterpart:

Proposition 2.8. *Let $p(\cdot), q(\cdot) \in \mathcal{B}_0(\mathbb{R}^n) \cap C^{\log}(\mathbb{R}^n)$ and $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $s \gg 1$. If $f \in B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, then there exist a collection of functions $\{a_{\nu,m}\}_{(\nu,m)\in\mathbb{N}_0\times\mathbb{Z}^n}$ and a sequence of coefficients $\{\lambda_{\nu,m}\}_{(\nu,m)\in\mathbb{N}_0\times\mathbb{Z}^n}$ with the following conditions.*

1. *The coefficients $\{\lambda_{\nu,m}\}_{(\nu,m)\in\mathbb{N}_0\times\mathbb{Z}^n}$ satisfy*
$$\left\| \left\{ 2^{\nu s(\cdot)} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}| \chi_{Q_{\nu,m}}(\cdot) \right\}_{\nu=0}^{\infty} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \lesssim \|f\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}}.$$
Here and below $Q_{\nu,m} = 2^{-\nu}m + [0, 2^{-\nu}]^n$ for $m \in \mathbb{Z}^n$ and $\chi_{Q_{\nu,m}}$ denotes the characteristic function on $Q_{\nu,m}$.

2. *The function $a_{\nu,m}$ is supported on $2^{-\nu}m+[-2^{-\nu+2},2^{-\nu+2}]^n$ and satisfies*

$$\left\|\partial^\alpha a_{\nu,m}\right\|_\infty \leqslant 1$$

for all α with $|\alpha|\leqslant K$.

3. *The function f can be decomposed as follows*:

$$f=\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu,m}a_{\nu,m},$$

where the convergence is in $\mathcal{S}'(\mathbb{R}^n)$.

Proposition 2.9. *Let $p(\cdot),q(\cdot)\in\mathcal{B}_0(\mathbb{R}^n)\cap C^{\log}(\mathbb{R}^n)$ and $s(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $s\gg 1$. Suppose that we are given a collection of functions $\{a_{\nu,m}\}_{(\nu,m)\in\mathbb{N}_0\times\mathbb{Z}^n}$ and a sequence of coefficients $\{\lambda_{\nu,m}\}_{(\nu,m)\in\mathbb{N}_0\times\mathbb{Z}^n}$:*

1. *The coefficients $\{\lambda_{\nu,m}\}_{(\nu,m)\in\mathbb{N}_0\times\mathbb{Z}^n}$ satisfy*

$$\left\|\left\{2^{\nu s(\cdot)}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu,m}|\chi_{Q_{\nu,m}}(\cdot)\right\}_{\nu=0}^{\infty}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}<\infty.$$

2. *The function $a_{\nu,m}$ is supported on $2^{-\nu}m+[-2^{-\nu+2},2^{-\nu+2}]^n$ and satisfies*

$$\left\|\partial^\alpha a_{\nu,m}\right\|_\infty \leqslant 1$$

for all α with $|\alpha|\leqslant K$.

Then

$$f:=\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu,m}a_{\nu,m},$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ and satisfies

$$\|f\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}}\lesssim\left\|\left\{2^{\nu s(\cdot)}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu,m}|\chi_{Q_{\nu,m}}(\cdot)\right\}_{\nu=0}^{\infty}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

It seems that the precise value of K in Propositions 2.8 and 2.9 is not investigated in [22]. A related decomposition result can be found in [8]. Also we remark that an approach that seems systematic and contains [22] appeared in [17,20].

Propositions 2.8 and 2.9 are followed from a molecule decomposition for $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ (Propositions 2.13 and 2.14) and arguments as in [4]. Hence we describe the molecule decomposition and outline of the proof. Therefore, we introduce several terms from [4]. Let $\mathcal{D}$ be the collection of dyadic cubes in $\mathbb{R}^n$ and denote by $\mathcal{D}^+$ the subcollection of those dyadic cubes with side-length at most 1. Let $\mathcal{D}_\nu=\{\mathcal{Q}\in\mathcal{D}:\ \ell(\mathcal{Q})=2^{-\nu}\}$. For a cube $\mathcal{Q}$ let $\ell(\mathcal{Q})$ denote the side length of $\mathcal{Q}$ and $x_{\mathcal{Q}}$ the "lower left corner". For $c>0$, we let $c\mathcal{Q}$ denote the cube with the same center and orientation as $\mathcal{Q}$ but with side length $c\ell(\mathcal{Q})$. We describe the molecules defining by [4].

Definition 2.10. Let $\nu\in\mathbb{N}_0$, $k\in\mathbb{Z}$, $l\in\mathbb{N}_0$ and $M\geqslant n$. A function $m_{Q_{\nu,s}}$ is said to be a (k,l,M)-smooth molecule for $Q_{\nu,s}$ if it satisfies the following conditions for some $i>M$:

(M1) if $\nu>0$, then $\int x^\gamma m_{\nu,s}(x)\,dx=0$ for all $|\gamma|\leqslant k$; and
(M2) $|D^\gamma m_{\nu,s}(x)|\leqslant 2^{|\gamma|\nu}|Q_{\nu,s}|^{1/2}\eta_{\nu,i}(x+x_{Q_{\nu,s}})$ for all multi-indices $\gamma\in\mathbb{N}_0^n$ with $|\gamma|\leqslant l$.

The conditions (M1) and (M2) are called the *moment* and *decay conditions*, respectively.

Definition 2.11. Let K, $L:\mathbb{R}^n\to\mathbb{R}$ and $M>n$. The family $\{m_{\nu,s}\}_{(\nu,s)\in\mathbb{N}_0\times\mathbb{Z}^n}$ is said to be a *family of* (K,L,M)*-smooth molecules* if $m_{\nu,s}$ is $(\lfloor K^-_{Q_{\nu,s}}\rfloor,\lfloor L^-_{Q_{\nu,s}}\rfloor,M)$-smooth for every $Q_{\nu,s}\in\mathcal{D}^+$.

Definition 2.12. We say that $\{m_{\nu,s}\}_{(\nu,s)\in\mathbb{N}_0\times\mathbb{Z}^n}$ is a *family of smooth molecules for* $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if it is a family of $(N+\epsilon,s+1+\epsilon,M)$-smooth molecules, where

$$N(x)=\frac{n}{\min\{1,p(x),q(x)\}}-n-s(x),$$

for some constant $\epsilon>0$, and M is a sufficiently large constant.

The number M needs to be chosen sufficiently large, for instance

$$2\frac{n+C_{\log}(s)}{\min\{1,p_-,q_-\}}$$

will do.

The space $b^{s(\cdot)}_{p(\cdot),q(\cdot)}$ consists of all sequences of real numbers $\{s_{\nu,m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ such that

$$\left\|\{s_{\nu,m}\}_{(\nu,m)}\right\|_{b^{s(\cdot)}_{p(\cdot),q(\cdot)}}=\left\|\left\{2^{\nu s(\cdot)}\sum_{m\in\mathbb{Z}^n}|s_{\nu,m}|Q_{\nu,m}|^{\frac{-1}{2}}\chi_{Q_{\nu,m}}\right\}_{\nu=0}^{\infty}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}<\infty.$$

Proposition 2.13. *Let the functions $p(\cdot),q(\cdot)\in C^{\log}(\mathbb{R}^n)\cap\mathcal{B}_0$ and $s(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $s_->0$. Suppose that $\{m_{\nu,l}\}_{(\nu,l)\in\mathbb{N}_0\times\mathbb{Z}^n}$ is a family of smooth molecules for $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$, and that $\{s_{\nu,l}\}_{(\nu,l)\in\mathbb{N}_0\times\mathbb{Z}^n}\in b^{s(\cdot)}_{p(\cdot),q(\cdot)}$. Then $f=\sum_{\nu\geqslant0}\sum_{l\in\mathbb{Z}^n}s_{\nu,l}m_{\nu,l}$ converges in $\mathcal{S}'$ and*

$$\|f\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}}\lesssim\left\|\{s_{\nu,l}\}_{(\nu,l)}\right\|_{b^{s(\cdot)}_{p(\cdot),q(\cdot)}}. \tag{9}$$

Proof. We outline the proof.

Firstly, we assume that $f=\sum_{\nu\geqslant0}\sum_{l\in\mathbb{Z}^n}s_{\nu,l}m_{\nu,l}$ converges in $\mathcal{S}'$. Then (9) holds by using arguments as in [4, Lemma 6.3, Proposition 6.2].

Finally, we prove that $f=\sum_{\nu\geqslant0}\sum_{l\in\mathbb{Z}^n}s_{\nu,l}m_{\nu,l}$ converges in $\mathcal{S}'$ whenever $\{s_{\nu,l}\}_{(\nu,l)\in\mathbb{N}_0\times\mathbb{Z}^n}\in b^{s(\cdot)}_{p(\cdot),q(\cdot)}$. By using arguments as in [1], we see that the inclusion $b^{s(\cdot)}_{p(\cdot),q(\cdot)}\hookrightarrow b^{s_-}_{p(\cdot),q_+}$. This inclusion implies that the desired assertion holds because we can use the arguments as in [22]. □

Proposition 2.14. *Let the functions $p(\cdot),q(\cdot)\in C^{\log}(\mathbb{R}^n)\cap\mathcal{B}_0(\mathbb{R}^n)$ and $s(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $s_->0$. Then*

$$\|S_\varphi f\|_{b^{s(\cdot)}_{p(\cdot),q(\cdot)}}\lesssim\|f\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}}$$

for every $f\in B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Here S_φ is called φ-transform (see [4,6,7]*).*

Proof. This proof is analogous to [4]. □

Finally, we record a lifting property. This lemma was proved in [14].

Lemma 2.15 *(Lifting property). Let $\mathcal{B}^\sigma$ denote the Bessel potential operator $\mathcal{B}^\sigma=\mathcal{F}^{-1}(1+|\xi|^2)^{-\sigma/2}\mathcal{F}$ for $\sigma\in\mathbb{R}$. Let $p(\cdot),q(\cdot)\in\mathcal{B}_0(\mathbb{R}^n)\cap C^{\log}(\mathbb{R}^n)$, $s(\cdot)\in C^{\log}(\mathbb{R}^n)$ and $\sigma\in\mathbb{R}$.*

1. *The Bessel potential operator $\mathcal{B}^\sigma$ is an isomorphism between $F^{s(\cdot)}_{p(\cdot),q(\cdot)}$ and $F^{s(\cdot)+\sigma}_{p(\cdot),q(\cdot)}$.*
2. *The Bessel potential operator $\mathcal{B}^\sigma$ is an isomorphism between $B^{s(\cdot)}_{p(\cdot),q(\cdot)}$ and $B^{s(\cdot)+\sigma}_{p(\cdot),q(\cdot)}$.*

2.4. Some fundamental inequalities for complex interpolation

Finally, we need some fundamental inequalities for complex interpolation.

The first one concerns with the Poisson kernel, that is,

$$\mu_j(\theta,t)=\frac{\sin\pi\theta}{e^{\pi t}+e^{-\pi t}-2\cos(j\pi+\pi\theta)},$$

for $j=0,1$.

Lemma 2.16. *(See [19].) Let $A=\{z:\ 0<\operatorname{Re}z<1\}$, $\bar{A}=\{z:\ 0\leqslant\operatorname{Re}z\leqslant1\}$ and $r\in(0,\infty)$. Then there exist two positive functions $\mu_0(\theta,t)$ and $\mu_1(\theta,t)$ in $(0,1)\times\mathbb{R}$ such that*

$$\left|g(z)\right|^r\leqslant\left(\frac{1}{1-\theta}\int_{\mathbb{R}}\left|g(it)\right|^r\mu_0(\theta,t)\,dt\right)^{1-\theta}\left(\frac{1}{\theta}\int_{\mathbb{R}}\left|g(1+it)\right|^r\mu_1(\theta,t)\,dt\right)^{\theta}$$

with $\theta=\operatorname{Re}z$ for any analytic function $g(z)$ on A which is uniformly continuous and bounded on $\bar{A}$. Furthermore, if $\theta\in(0,1)$, then

$$\frac{1}{1-\theta}\int_{\mathbb{R}}\mu_0(\theta,t)\,dt=\frac{1}{\theta}\int_{\mathbb{R}}\mu_1(\theta,t)\,dt=1.$$

Lemma 2.17. *(See [19].) Let $\{\kappa_j\}_{j=1}^{\infty}$ be a positive sequence and $\alpha > 0$. Then we have*

$$\sum_{j=0}^{\infty} \kappa_j \left(\sum_{k=0}^{j} \kappa_k \right)^{\alpha-1} \leqslant \frac{1}{\min(\alpha, 1)} \left(\sum_{k=0}^{\infty} \kappa_k \right)^{\alpha}.$$

The proof of a similar assertion can be found in [19, p. 68, Lemma 1] and we essentially use this lemma in the form of [19, p. 68, Lemma 1]. Here for the sake of convenience, we include the proof.

Proof. If $\alpha \geqslant 1$, then the inequality is trivial. If $\alpha < 1$, then we have

$$\sum_{j=0}^{\infty} \kappa_j \left(\sum_{k=0}^{j} \kappa_k \right)^{\alpha-1} \leqslant {\kappa_0}^{\alpha-1} + \sum_{j=1}^{\infty} \int_{\sum_{k=0}^{j-1} \kappa_k}^{\sum_{k=0}^{j} \kappa_k} t^{\alpha-1} \, \mathrm{d}t = \int_{0}^{\sum_{k=0}^{\infty} \kappa_k} t^{\alpha-1} \, \mathrm{d}t = \frac{1}{\alpha} \left(\sum_{k=0}^{\infty} \kappa_k \right)^{\alpha}.$$

The lemma is therefore proved. □

3. Proof of Theorem 1.4

We outline the proof of Theorem 1.4. First we prove that

$$\left(F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)} \right)_{\theta} \subset F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \tag{10}$$

in Step 1 and then

$$\left(F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)} \right)_{\theta} \supset F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \tag{11}$$

in Step 2.

Proof. *Step 1.* Let $f(z) \in F(F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)})$. A normalization allows us to assume that $\|f(z)\|_{F(F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)})} = 1$. We put $g_k(x,z) = [\theta_k * f(\cdot, z)](x)$ for $k \in \mathbb{N}_0$. Fix an auxiliary parameter r so that $0 < r < \min(p_-, (q_0)_-, (q_1)_-)$. We write

$$G(x,\theta) = \left(\sum_{k=0}^{\infty} \left| 2^{ks(x)} g_k(x,\theta) \right|^{q(x)} \right)^{\frac{1}{q(x)}}, \qquad \mathrm{I} = \left\{ \int_{\mathbb{R}^n} G(x,\theta)^{p(x)} \, \mathrm{d}x \right\}^{\frac{1}{p_{\max}(G(\cdot,\theta))}}.$$

Then by virtue of (8) we have

$$\left\| f(\theta) \right\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant \left\{ \int_{\mathbb{R}^n} \left(\sum_{k=0}^{\infty} \left(\left| 2^{ks(x)} g_k(x,\theta) \right|^r \right)^{\frac{q(x)}{r}} \right)^{\frac{r}{q(x)} \cdot \frac{p(x)}{r}} \mathrm{d}x \right\}^{\frac{1}{p_{\max}(G(\cdot,\theta))}}. \tag{12}$$

Observe that the spaces involved $\ell^{q/r}$ and $L^{p(\cdot)/r}(\mathbb{R}^n)$ are Banach spaces. We use Lemma 2.16 with $2^{ks(x)} g_k(x,z)$ instead of $g(z)$ with $x \in \mathbb{R}^n$ fixed. Let

$$a_k(x) = \frac{1}{1-\theta} \int_{\mathbb{R}} \left| g_k(x, \mathrm{i}t) \right|^r \mu_0(\theta, t) \, \mathrm{d}t, \qquad b_k(x) = \frac{1}{\theta} \int_{\mathbb{R}} \left| g_k(x, 1 + \mathrm{i}t) \right|^r \mu_1(\theta, t) \, \mathrm{d}t.$$

Then (4) and the Hölder inequality yield

$$\left[\sum_{k=0}^{\infty} \left(\left| 2^{ks(x)} g_k(x,\theta) \right|^r \right)^{\frac{q(x)}{r}} \right]^{\frac{r}{q(x)}} \leqslant \left(\sum_{k=0}^{\infty} 2^{ks_0(x)q_0(x)} a_k(x)^{\frac{q_0(x)}{r}} \right)^{\frac{r}{q_0(x)}(1-\theta)} \left(\sum_{k=0}^{\infty} 2^{ks_1(x)q_1(x)} b_k(x)^{\frac{q_1(x)}{r}} \right)^{\frac{r}{q_1(x)}\theta}. \tag{13}$$

By the Minkowski inequality we have

$$\left\| \left\{ 2^{ks_0(x)r} a_k(x) \right\}_{k=0}^{\infty} \right\|_{\ell^{\frac{q_0(x)}{r}}} \leqslant \frac{1}{1-\theta} \int_{\mathbb{R}} \left(\sum_{k=0}^{\infty} 2^{ks_0(x)q_0(x)} \left| g_k(x, \mathrm{i}t) \right|^{q_0(x)} \right)^{\frac{1}{q_0(x)} \cdot r} \mu_0(\theta, t) \, \mathrm{d}t.$$

A similar estimate holds for the second factor on the right-hand side of (13). Hence, (12) and (13) give

$$\mathrm{I} \leqslant \left[\int_{\mathbb{R}^n} \left(\frac{1}{1-\theta} \int_{\mathbb{R}} \left(\sum_{k=0}^{\infty} 2^{k s_0(x) q_0(x)} \left| g_k(x, \mathrm{i}t) \right|^{q_0(x)} \right)^{\frac{1}{q_0(x)} \cdot r} \mu_0(\theta, t)\, \mathrm{d}t \right)^{(1-\theta)\frac{p(x)}{r}} \right.$$
$$\left. \times \left(\frac{1}{\theta} \int_{\mathbb{R}} \left(\sum_{k=0}^{\infty} 2^{k s_1(x) q_1(x)} \left| g_k(x, 1+\mathrm{i}t) \right|^{q_1(x)} \right)^{\frac{1}{q_1(x)} \cdot r} \mu_1(\theta, t)\, \mathrm{d}t \right)^{\theta \frac{p(x)}{r}} \mathrm{d}x \right]^{\frac{1}{p_{\max}(G(\cdot,\theta))}}. \tag{14}$$

By (4), we have

$$1 = \frac{1}{\frac{p_0(x)}{(1-\theta)p(x)}} + \frac{1}{\frac{p_1(x)}{\theta p(x)}}$$

and

$$\frac{p_0(\cdot)}{(1-\theta)p(\cdot)} \in \mathcal{P}\big(\mathbb{R}^n\big), \qquad \frac{p_1(\cdot)}{\theta p(\cdot)} \in \mathcal{P}\big(\mathbb{R}^n\big).$$

We use (4) and the generalized Hölder inequality (see Proposition 2.1) to obtain

$$\mathrm{I} \lesssim \left\| \left(\int_{\mathbb{R}} \left(\sum_{k=0}^{\infty} 2^{k s_0(x) q_0(x)} \left| g_k(\cdot, \mathrm{i}t) \right|^{q_0(x)} \right)^{\frac{1}{q_0(x)} \cdot r} \frac{\mu_0(\theta, t)}{1-\theta}\, \mathrm{d}t \right)^{(1-\theta)\frac{p(\cdot)}{r}} \right\|_{L^{\frac{p_0(\cdot)}{(1-\theta)p(\cdot)}}}^{\frac{1}{p_{\max}(G(\cdot,\theta))}}$$
$$\times \left\| \left(\int_{\mathbb{R}} \left(\sum_{k=0}^{\infty} 2^{k s_1(x) q_1(x)} \left| g_k(\cdot, 1+\mathrm{i}t) \right|^{q_1(x)} \right)^{\frac{1}{q_1(x)} \cdot r} \frac{\mu_1(\theta, t)}{\theta}\, \mathrm{d}t \right)^{\theta \frac{p(\cdot)}{r}} \right\|_{L^{\frac{p_1(\cdot)}{\theta p(\cdot)}}}^{\frac{1}{p_{\max}(G(\cdot,\theta))}}.$$

We write

$$J_l = \left\| \int_{\mathbb{R}} \left(\sum_{k=0}^{\infty} 2^{k s_l(x) q_l(x)} \left| g_k(\cdot, \mathrm{i}t) \right|^{q_l(x)} \right)^{\frac{1}{q_l(x)} \cdot r} \frac{\mu_l(\theta, t)}{(1-\theta)(1-l)+\theta l}\, \mathrm{d}t \right\|_{L^{\frac{p_l(\cdot)}{r}}}$$

for $l = 0, 1$ and define

$$\omega_l(\theta) = \begin{cases} p_+, & \text{if } J_l \geqslant 1, \\ p_-, & \text{otherwise.} \end{cases}$$

Then we have

$$\mathrm{I} \lesssim \left\| \int_{\mathbb{R}} \left(\sum_{k=0}^{\infty} 2^{k s_0(\cdot) q_0(\cdot)} \left| g_k(\cdot, \mathrm{i}t) \right|^{q_0(\cdot)} \right)^{\frac{1}{q_0(\cdot)} \cdot r} \frac{\mu_0(\theta, t)}{1-\theta}\, \mathrm{d}t \right\|_{L^{\frac{p_0(\cdot)}{r}}}^{\frac{(1-\theta)\omega_0(\theta)}{r p_{\max}(G(\cdot,\theta))}}$$
$$\times \left\| \int_{\mathbb{R}} \left(\sum_{k=0}^{\infty} 2^{k s_1(\cdot) q_1(\cdot)} \left| g_k(\cdot, 1+\mathrm{i}t) \right|^{q_1(\cdot)} \right)^{\frac{1}{q_1(\cdot)} \cdot r} \frac{\mu_1(\theta, t)}{\theta}\, \mathrm{d}t \right\|_{L^{\frac{p_1(\cdot)}{r}}}^{\frac{\theta\omega_1(\theta)}{r p_{\max}(G(\cdot,\theta))}}$$

by Proposition 2.4. If we invoke the generalized Minkowski inequality (see Proposition 2.2), Definition 1.3 and Lemma 2.16, then we have

$$\mathrm{I} \lesssim \left[\int_{\mathbb{R}} \left\| \left(\sum_{k=0}^{\infty} 2^{k s_0(\cdot) q_0(\cdot)} \left| g_k(\cdot, \mathrm{i}t) \right|^{q_0(\cdot)} \right)^{\frac{1}{q_0(\cdot)}} \right\|_{L^{p_0(\cdot)}}^{r} \frac{\mu_0(\theta, t)}{1-\theta}\, \mathrm{d}t \right]^{\frac{(1-\theta)\omega_0(\theta)}{r p_{\max}(G(\cdot,\theta))}}$$
$$\times \left[\int_{\mathbb{R}} \left\| \left(\sum_{k=0}^{\infty} 2^{k s_1(\cdot) q_1(\cdot)} \left| g_k(\cdot, 1+\mathrm{i}t) \right|^{q_1(\cdot)} \right)^{\frac{1}{q_1(\cdot)}} \right\|_{L^{p_1(\cdot)}}^{r} \frac{\mu_1(\theta, t)}{\theta}\, \mathrm{d}t \right]^{\frac{\theta\omega_1(\theta)}{r p_{\max}(G(\cdot,\theta))}}$$
$$= \left[\int_{\mathbb{R}} \left\| 2^{k s_0(\cdot)} \left| g_k(\cdot, \mathrm{i}t) \right| \right\|_{L^{p_0(\cdot)}(\ell^{q_0(\cdot)})}^{r} \frac{\mu_0(\theta, t)}{1-\theta}\, \mathrm{d}t \right]^{\frac{(1-\theta)\omega_0(\theta)}{r p_{\max}(G(\cdot,\theta))}}$$
$$\times \left[\int_{\mathbb{R}} \left\| 2^{k s_1(\cdot)} \left| g_k(\cdot, 1+\mathrm{i}t) \right| \right\|_{L^{p_1(\cdot)}(\ell^{q_1(\cdot)})}^{r} \frac{\mu_1(\theta, t)}{\theta}\, \mathrm{d}t \right]^{\frac{\theta\omega_1(\theta)}{r p_{\max}(G(\cdot,\theta))}}.$$

If we recall that f is normalized, then we obtain

$$\mathrm{I} \lesssim \sup_{t\in\mathbb{R}} \big\| 2^{ks_0(\cdot)} \big| g_k(\cdot, \mathrm{i}t) \big| \big\|_{L^{p_0(\cdot)}(\ell^{q_0(\cdot)})}^{\frac{(1-\theta)\omega_0(\theta)}{p_{\max}(G(\cdot,\theta))}} \sup_{t\in\mathbb{R}} \big\| 2^{ks_1(\cdot)} \big| g_k(\cdot, 1+\mathrm{i}t) \big| \big\|_{L^{p_1(\cdot)}(\ell^{q_1(\cdot)})}^{\frac{\theta\omega_1(\theta)}{p_{\max}(G(\cdot,\theta))}} \leqslant 1.$$

This proves (10).

Step 2. By the lifting property of function spaces (Lemma 2.15), we may assume that $(s_l)_- \gg 1$, $l = 0, 1$. Assuming s_- large enough by virtue of Proposition 2.6, we can decompose f as follows: $f = \sum_{\nu\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}$, where each $a_{\nu,m}$ satisfies $\operatorname{supp}(a_{\nu,m}) \subset 3Q_{\nu,m}$, $\|\partial^\alpha a_{\nu,m}\|_\infty \leqslant 2^{|\alpha|\nu}$ and the coefficient $\lambda = \{\lambda_{\nu,m}\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n}$ satisfies

$$\lambda_{\nu,m} \geqslant 0 \ \big(\nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\big), \quad \Big\| \Big\{ 2^{\nu s(\cdot)} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} \chi_{Q_{\nu,m}} \Big\}_{\nu=0}^\infty \Big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \lesssim \|f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}}. \tag{15}$$

It counts that we do not have to postulate the moment condition on $a_{\nu,m}$. By normalization, we assume that $\|f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}} = 1$.

For $\nu \in \mathbb{N}_0$ we define $\Lambda_\nu(x) := \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} \chi_{Q_{\nu,m}}(x)$. Then note that

$$\Big\| \Big\{ 2^{\nu s} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} \chi_{Q_{\nu,m}} \Big\}_{\nu=0}^\infty \Big\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \lesssim 1.$$

With this in mind, we abbreviate by writing $R_{\nu,m} = 2^{\nu s} \lambda_{\nu,m}$ below.

For each $x \in \mathbb{R}^n$ and $z \in \mathbb{C}^n$ with $\mathrm{Re}(z) \in [0, 1]$, we define

$$\begin{aligned}
\rho_1(x,z) &:= \frac{q(x)s(x)}{q_0(x)}(1-z) + \frac{q(x)s(x)}{q_1(x)} z - s_0(x)(1-z) - s_1(x) z,\\
\rho_2(x,z) &:= \frac{q(x)}{q_0(x)}(1-z) + \frac{q(x)}{q_1(x)} z,\\
\rho_3(x,z) &:= \left(\frac{p(x)}{q(x)p_0(x)} - \frac{1}{q_0(x)} \right)(1-z) + \left(\frac{p(x)}{q(x)p_1(x)} - \frac{1}{q_1(x)} \right) z.
\end{aligned}$$

Also, we define a holomorphic function $\Lambda_{\nu,m}$ by

$$\Lambda_{\nu,m}(z) = \left\{ \frac{1}{|Q_{\nu,m}|} \int_{Q_{\nu,m}} 2^{\nu \frac{\rho_1(y,z)}{K}} \lambda_{\nu,m}{}^{\frac{\rho_2(y,z)}{K}} \left(\sum_{j=0}^{\nu} \sum_{\tilde{m}\in\mathbb{Z}^n} R_{j,\tilde{m}}{}^{q(y)} \chi_{Q_{j,\tilde{m}}}(y) \right)^{\frac{\rho_3(y,z)}{K}} \mathrm{d}y \right\}^K$$

where K is a large integer. Abbreviate $\sum_{j=0}^{\nu} \sum_{\tilde{m}\in\mathbb{Z}^n} R_{j,\tilde{m}}{}^{q(y)} \chi_{Q_{j,\tilde{m}}}(y)$ to $S_\nu(y)$. Then, since we are assuming $s_l(\cdot) \in C^{\log}(\mathbb{R}^n)$, we have

$$\begin{aligned}
\big| 2^{\nu s_l(x)} \Lambda_{\nu,m}(l+\mathrm{i}t) \big| &\lesssim \left| \frac{1}{|Q_{\nu,m}|} \int_{Q_{\nu,m}} 2^{\frac{\nu}{K}\{s_l(x) - s_0(y)(1-l-\mathrm{i}t) - s_1(y)(l+\mathrm{i}t)\}} R_{\nu,m}{}^{\frac{\rho_2(y,l+\mathrm{i}t)}{K}} S_\nu(y)^{\frac{\rho_3(y,l+\mathrm{i}t)}{K}} \mathrm{d}y \right|^K\\
&\lesssim \left\{ \frac{1}{|Q_{\nu,m}|} \int_{Q_{\nu,m}} \big| 2^{\frac{\nu}{K}\{s_l(x) - s_0(y)(1-l-\mathrm{i}t) - s_1(y)(l+\mathrm{i}t)\}} R_{\nu,m}{}^{\frac{\rho_2(y,l+\mathrm{i}t)}{K}} S_\nu(y)^{\frac{\rho_3(y,l+\mathrm{i}t)}{K}} \big| \mathrm{d}y \right\}^K\\
&\lesssim \left\{ \eta_{\nu,M} * \left[\sum_{m\in\mathbb{Z}^n} R_{\nu,m}{}^{\frac{q(\cdot)}{Kq_l(\cdot)}} S_\nu{}^{\frac{\rho_3(\cdot,l)}{K}} \chi_{Q_{\nu,m}} \right](x) \right\}^K
\end{aligned}$$

for all $x \in Q_{\nu,m}$. Consequently, we obtain

$$\sum_{m\in\mathbb{Z}^n} \big| 2^{\nu s_l(x)} \Lambda_{\nu,m}(l+\mathrm{i}t) \big| \chi_{Q_{\nu,m}}(x) \lesssim \left\{ \eta_{\nu,M} * \left[\sum_{m\in\mathbb{Z}^n} R_{\nu,m}{}^{\frac{q(\cdot)}{Kq_l(\cdot)}} S_\nu{}^{\frac{\rho_3(\cdot,l)}{K}} \chi_{Q_{\nu,m}} \right](x) \right\}^K.$$

If we insert this pointwise estimate, then we have

$$\begin{aligned}
&\left\{ \sum_{\nu=0}^\infty \left(2^{\nu s_l(\cdot)} \sum_{m\in\mathbb{Z}^n} \big| \Lambda_{\nu,m}(l+\mathrm{i}t) \big| \chi_{Q_{\nu,m}} \right)^{q_l(\cdot)} \right\}^{\frac{1}{q_l(\cdot)}}\\
&\leqslant \left\{ \sum_{\nu=0}^\infty \left(\eta_{\nu,m} * \left[\sum_{m\in\mathbb{Z}^n} R_{\nu,m}{}^{\frac{q(\cdot)}{Kq_l(\cdot)}} S_\nu{}^{\frac{\rho_3(\cdot,l)}{K}} \chi_{Q_{\nu,m}} \right] \right)^{Kq_l(\cdot)} \right\}^{\frac{1}{q_l(\cdot)}}
\end{aligned}$$

and hence by virtue of Lemma 2.5

$$\begin{aligned}
&\left\| \left\{ \sum_{\nu=0}^{\infty} \left(2^{\nu s_l(\cdot)} \sum_{m\in\mathbb{Z}^n} \left|\Lambda_{\nu,m}(l+\mathrm{i}t)\right| \chi_{Q_{\nu,m}} \right)^{q_l(\cdot)} \right\}^{\frac{1}{q_l(\cdot)}} \right\|_{L^{p_l(\cdot)}} \\
&\lesssim \left\| \left(\sum_{\nu=0}^{\infty} \left[\sum_{m\in\mathbb{Z}^n} R_{\nu,m}{}^{\frac{q(\cdot)}{Kq_l(\cdot)}} S_\nu{}^{\frac{1}{Kq(\cdot)}\left(\frac{p(\cdot)}{p_l(\cdot)}-1\right)} \chi_{Q_{\nu,m}} \right]^{Kq_l(\cdot)} \right)^{\frac{1}{q_l(\cdot)}} \right\|_{L^{p_l(\cdot)}} \\
&= \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} R_{\nu,m}{}^{q(\cdot)} \left(\sum_{j=0}^{\nu} \sum_{\tilde{m}\in\mathbb{Z}^n} R_{j,\tilde{m}}{}^{q(\cdot)} \chi_{Q_{j,\tilde{m}}} \right)^{\frac{q_l(\cdot)p(\cdot)}{q(\cdot)p_l(\cdot)}-1} \chi_{Q_{\nu,m}} \right)^{\frac{1}{q_l(\cdot)}} \right\|_{L^{p_l(\cdot)}}.
\end{aligned}$$

If we write $S_\infty = \lim_{\nu\to\infty} S_\nu$ and use Lemma 2.17, then we have

$$\left\| \left\{ \sum_{\nu=0}^{\infty} \left(2^{\nu s_l(\cdot)} \sum_{m\in\mathbb{Z}^n} \left|\Lambda_{\nu,m}(l+\mathrm{i}t)\right| \chi_{Q_{\nu,m}} \right)^{q_l(\cdot)} \right\}^{\frac{1}{q_l(\cdot)}} \right\|_{L^{p_l(\cdot)}} \lesssim \left\| S_\infty{}^{\frac{p(\cdot)}{q(\cdot)p_l(\cdot)}} \right\|_{L^{p_l(\cdot)}}.$$

Since $\log p_0$ and $\log p_1$ are bounded, we are in the position of using Proposition 2.4 to obtain

$$\left\| S_\infty{}^{\frac{p(\cdot)}{q(\cdot)p_l(\cdot)}} \right\|_{L^{p_l(\cdot)}} = \left\| \left(\sum_{j=0}^{\infty} \sum_{\tilde{m}\in\mathbb{Z}^n} R_{j,\tilde{m}}{}^{q(\cdot)} \chi_{Q_{j,\tilde{m}}} \right)^{\frac{p(\cdot)}{q(\cdot)p_l(\cdot)}} \right\|_{L^{p_l(\cdot)}} \lesssim 1.$$

Consequently, we obtain

$$\left\| \left\{ \sum_{\nu=0}^{\infty} \left(2^{\nu s_l(\cdot)} \sum_{m\in\mathbb{Z}^n} \left|\Lambda_{\nu,m}(l+\mathrm{i}t)\right| \chi_{Q_{\nu,m}} \right)^{q_l(\cdot)} \right\}^{\frac{1}{q_l(\cdot)}} \right\|_{L^{p_l(\cdot)}} \lesssim 1. \tag{16}$$

Now we define

$$F(z,x) = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \Lambda_{\nu,m}(z) a_{\nu,m}(x).$$

If we combine (16) and Proposition 2.7, then we have

$$F(\theta) = f, \qquad \left\| F(l+\mathrm{i}t) \right\|_{F^{s_l(\cdot)}_{p_l(\cdot),q_l(\cdot)}} \lesssim 1.$$

Hence it follows from Definition 1.3 that

$$\|f\|_{(F^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, F^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)})_\theta} \lesssim \|f\|_{F^{s(\cdot)}_{p(\cdot),q(\cdot)}}.$$

Therefore, the proof of Theorem 1.4 is complete. □

4. Proof of Theorem 1.11

We outline the proof of Theorem 1.11. First we prove that

$$\left(B^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, B^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}\right)_\theta \subset B^{s(\cdot)}_{p(\cdot),q(\cdot)}\left(\mathbb{R}^n\right) \tag{17}$$

in Step 1 and then

$$\left(B^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, B^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)}\right)_\theta \supset B^{s(\cdot)}_{p(\cdot),q(\cdot)}\left(\mathbb{R}^n\right) \tag{18}$$

in Step 2.

Proof. *Step 1.* Let $f(z) \in F(B^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, B^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)})$ and $\|f(z)\|_{F(B^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, B^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)})} = 1$. Then we see that

$$\left\| 2^{ks_l(\cdot)} \left| g_k(\cdot, l+\mathrm{i}t) \right| \right\|_{L^{p_l(\cdot)}} \leqslant \left\| 2^{ks_l(\cdot)} \left| g_k(\cdot, l+\mathrm{i}t) \right| \right\|_{\ell^{q_l(\cdot)}(L^{p_l(\cdot)})} \leqslant 1$$

for any $t \in \mathbb{R}$, $k \in \mathbb{N}_0$ and $l = 0, 1$. Let r be chosen so that

$$0 < \sqrt{r} < \min\left\{1, (q_0)_-, (q_1)_-, \frac{(p_0)_-}{(q_0)_+}, \frac{(p_1)_-}{(q_1)_+}\right\}.$$

Then, for any $k \in \mathbb{N}_0$, we see that

$$\left\| \left(2^{ks_0(\cdot)r} a_k(\cdot)\right)^{\frac{q_0(\cdot)}{r}} \right\|_{L^{\frac{p_0(\cdot)}{q_0(\cdot)}}} \leqslant 1 \quad \text{and} \quad \left\| \left(2^{ks_1(\cdot)r} b_k(\cdot)\right)^{\frac{q_1(\cdot)}{r}} \right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}} \leqslant 1$$

by the Minkowski inequality and Proposition 2.4. Here, a_k and b_k are the same as in Step 1 of the proof of Theorem 1.4. Let

$$\Gamma_k = \max\left\{ \left\| \left(2^{ks_0(\cdot)r} a_k(\cdot)\right)^{\frac{q_0(\cdot)}{r}} \right\|_{L^{\frac{p_0(\cdot)}{q_0(\cdot)}}}, \left\| \left(2^{ks_1(\cdot)r} b_k(\cdot)\right)^{\frac{q_1(\cdot)}{r}} \right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}} \right\}.$$

Then,

$$\begin{aligned}
\int_{\mathbb{R}^n} \left(\frac{(2^{ks(x)r} |g_k(x,\theta)|^r)^{\frac{q(x)}{r}}}{\Gamma_k} \right)^{\frac{p(x)}{q(x)}} \mathrm{d}x &\leqslant \int_{\mathbb{R}^n} \left(\frac{(2^{k_0 s_0(x)r} a_k(x))^{\frac{q(x)(1-\theta)}{r}} (2^{k_0 s_1 r} b_k(x))^{\frac{q(x)\theta}{r}}}{\Gamma_k} \right)^{\frac{p(x)}{q(x)}} \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \left(\frac{(2^{k_0 s_0(x)r} a_k(x))^{\frac{1-\theta}{r}} (2^{k_0 s_1 r} b_k(x))^{\frac{\theta}{r}}}{\Gamma_k^{\frac{1-\theta}{q_0(x)}} \Gamma_k^{\frac{\theta}{q_1(x)}}} \right)^{p(x)} \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \left(\frac{2^{ks_0(x)r} a_k(x)}{\Gamma_k^{\frac{r}{q_0(x)}}} \right)^{\frac{p(x)(1-\theta)}{r}} \left(\frac{2^{ks_1(x)r} b_k(x)}{\Gamma_k^{\frac{r}{q_1(x)}}} \right)^{\frac{p(x)\theta}{r}} \mathrm{d}x.
\end{aligned}$$

By the Hölder inequality, we see that

$$\int_{\mathbb{R}^n} \left(\frac{(2^{ks(x)r} |g_k(x,\theta)|^r)^{\frac{q(x)}{r}}}{\Gamma_k} \right)^{\frac{p(x)}{q(x)}} \mathrm{d}x \lesssim 1.$$

Or equivalently,

$$\left\| \left(2^{ks(\cdot)} |g_k(\cdot,\theta)|\right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \lesssim \Gamma_k.$$

Inserting the definition of Γ, we have

$$\sum_{k=0}^{\infty} \left\| \left(2^{ks(\cdot)} |g_k(\cdot,\theta)|\right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \lesssim \sum_{k=0}^{\infty} \left\| \left(2^{ks_0(\cdot)r} a_k(\cdot)\right)^{\frac{q_0(\cdot)}{r}} \right\|_{L^{\frac{p_0(\cdot)}{q_0(\cdot)}}} + \sum_{k=0}^{\infty} \left\| \left(2^{ks_1(\cdot)r} b_k(\cdot)\right)^{\frac{q_1(\cdot)}{r}} \right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}.$$

By the Hölder inequality and the fact that $q_0(\cdot)/\sqrt{r} > 1$, we see that

$$\begin{aligned}
\sum_{k=0}^{\infty} \left\| \left(2^{ks_0(\cdot)r} a_k(\cdot)\right)^{\frac{q_0(\cdot)}{r}} \right\|_{L^{\frac{p_0(\cdot)}{q_0(\cdot)}}} &= \sum_{k=0}^{\infty} \left\| \left(\int_{\mathbb{R}^n} 2^{ks_0(\cdot)r} |g_k(\cdot, \mathrm{i}t)|^r \frac{\mu_0(\theta,t)}{1-\theta} \mathrm{d}t \right)^{q_0(\cdot)/\sqrt{r}} \right\|_{L^{\frac{p_0(\cdot)}{q_0(\cdot)\sqrt{r}}}}^{\frac{1}{\sqrt{r}}} \\
&\leqslant \sum_{k=0}^{\infty} \left\| \int_{\mathbb{R}^n} \left(2^{ks_0(\cdot)\sqrt{r}} |g_k(\cdot, \mathrm{i}t)|^{\sqrt{r}}\right)^{q_0(\cdot)} \frac{\mu_0(\theta,t)}{1-\theta} \mathrm{d}t \right\|_{L^{\frac{p_0(\cdot)}{q_0(\cdot)\sqrt{r}}}}^{\frac{1}{\sqrt{r}}}.
\end{aligned}$$

By the Minkowski inequality and the fact that $q_0(\cdot)\sqrt{r} < p_0(\cdot)$, we see that

$$\begin{aligned}
\sum_{k=0}^{\infty} \left\| \left(2^{ks_0(\cdot)r} a_k(\cdot)\right)^{\frac{q_0(\cdot)}{r}} \right\|_{L^{\frac{p_0(\cdot)}{q_0(\cdot)}}} &\leqslant \sum_{k=0}^{\infty} \left\| \int_{\mathbb{R}^n} \left(2^{ks_0(\cdot)\sqrt{r}} |g_k(\cdot, \mathrm{i}t)|^{\sqrt{r}}\right)^{q_0(\cdot)} \frac{\mu_0(\theta,t)}{1-\theta} \mathrm{d}t \right\|_{L^{\frac{p_0(\cdot)}{q_0(\cdot)\sqrt{r}}}}^{\frac{1}{\sqrt{r}}} \\
&\lesssim \sum_{k=0}^{\infty} \left(\int_{\mathbb{R}^n} \left\| 2^{ks_0(\cdot)q_0(\cdot)\sqrt{r}} |g_k(\cdot, \mathrm{i}t)|^{q_0(\cdot)\sqrt{r}} \right\|_{L^{\frac{p_0(\cdot)}{q_0(\cdot)\sqrt{r}}}} \frac{\mu_0(\theta,t)}{1-\theta} \mathrm{d}t \right)^{\frac{1}{\sqrt{r}}} \\
&\leqslant \left\{ \int_{\mathbb{R}^n} \left(\sum_{k=0}^{\infty} \left\| 2^{ks_0(\cdot)q_0(\cdot)\sqrt{r}} |g_k(\cdot, \mathrm{i}t)|^{q_0(\cdot)\sqrt{r}} \right\|_{L^{\frac{p_0(\cdot)}{q_0(\cdot)\sqrt{r}}}}^{\frac{1}{\sqrt{r}}} \right)^{\sqrt{r}} \frac{\mu_0(\theta,t)}{1-\theta} \mathrm{d}t \right\}^{\frac{1}{\sqrt{r}}} \\
&\leqslant 1.
\end{aligned}$$

A similar argument works for $g_k(\cdot, 1+it)$ and we obtain

$$\sum_{k=0}^{\infty} \left\| \left(2^{ks_1(\cdot)r} b_k(\cdot)\right)^{\frac{q_1(\cdot)}{r}} \right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}} \leqslant 1.$$

Putting together these observations, we obtain

$$\sum_{k=0}^{\infty} \left\| \left(2^{ks(\cdot)} \left|g_k(\cdot,\theta)\right|\right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \lesssim 1.$$

Hence, it follows from the definition of g_k that we have

$$\left\| f(\theta) \right\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} \lesssim 1.$$

This proves (17).

Step 2. We can decompose f as follows: $f = \sum_{\nu\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} a_{\nu,m}$, where each $a_{\nu,m}$ satisfies $\operatorname{supp}(a_{\nu,m}) \subset 3Q_{\nu,m}$, $\|\partial^\alpha a_{\nu,m}\|_\infty \leqslant 2^{|\alpha|\nu}$ and the coefficient $\lambda = \{\lambda_{\nu,m}\}_{\nu\in\mathbb{N}_0,\, m\in\mathbb{Z}^n}$ satisfies

$$\lambda_{\nu,m} \geqslant 0 \ \left(\nu \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\right), \quad \left\| \left\{ 2^{\nu s(\cdot)} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} \chi_{Q_{\nu,m}} \right\}_{\nu=0}^{\infty} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \lesssim \|f\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}}. \tag{19}$$

It counts that we do not have to postulate the moment condition on $a_{\nu,m}$. By normalization, we assume that $\|f\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}} = 1$. Let

$$\beta_\nu = \left\| \left(2^{\nu s(\cdot)} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu,m} \chi_{Q_{\nu,m}} \right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

We define a holomorphic function $\Lambda'_{\nu,m}$ by

$$\Lambda'_{\nu,m}(z) = \left\{ \frac{1}{|Q_{\nu,m}|} \int_{Q_{\nu,m}} 2^{\nu \frac{\rho_1(y,z)}{K}} \lambda_{\nu,m}^{\frac{\rho_2(y,z)}{K}} \left(\sum_{\tilde{m}\in\mathbb{Z}^n} \beta_\nu^{-1} R_{\nu,\tilde{m}}^{q(y)} \chi_{Q_{\nu,\tilde{m}}}(y) \right)^{\frac{\rho_3(y,z)}{K}} \mathrm{d}y \right\}^K,$$

where K is a large integer and ρ_1, ρ_2 and ρ_3 are the same as in Step 2 of the proof of Theorem 1.4. Abbreviate $\sum_{\tilde{m}\in\mathbb{Z}^n} R_{\nu,\tilde{m}}^{q(y)} \chi_{Q_{\nu,\tilde{m}}}(y)$ to $S'_\nu(y)$. By using Proposition 1.10 and the same argument of Step 2 in Section 3, we see that

$$\begin{aligned}
&\left\| \left\{ \sum_{m\in\mathbb{Z}^n} \left|2^{\nu s_l(\cdot)} \Lambda'_{\nu,m}(l+it)\right| \chi_{Q_{\nu,m}}(\cdot) \right\}_{\nu=0}^{\infty} \right\|_{\ell^{q_l(\cdot)}(L^{p_l(\cdot)})} \\
&\lesssim \left\| \left\{ \left(\eta_{\nu,M}(y) * \left[\sum_{m\in\mathbb{Z}^n} \beta_\nu^{\frac{-\rho_3(y,l)}{K}} R_{\nu,m}^{\frac{q(y)}{Kq_l(y)}} S_\nu'^{\frac{\rho_3(y,l)}{K}} \chi_{Q_{\nu,m}} \right](\cdot) \right)^K \right\}_{\nu=0}^{\infty} \right\|_{\ell^{q_l(\cdot)}(L^{p_l(\cdot)})} \\
&= \left\| \left\{ \eta_{\nu,M}(y) * \left[\sum_{m\in\mathbb{Z}^n} \beta_\nu^{\frac{-\rho_3(y,l)}{K}} R_{\nu,m}^{\frac{q(y)}{Kq_l(y)}} S_\nu'^{\frac{\rho_3(y,l)}{K}} \chi_{Q_{\nu,m}} \right](\cdot) \right\}_{\nu=0}^{\infty} \right\|_{\ell^{Kq_l(\cdot)}(L^{Kp_l(\cdot)})}^K \\
&\lesssim \left\| \left\{ \beta_\nu^{\frac{1}{q_l(\cdot)} - \frac{p(\cdot)}{q(\cdot)p_l(\cdot)}} \sum_{m\in\mathbb{Z}^n} R_{\nu,m}^{\frac{p(\cdot)}{p_l(\cdot)}} \chi_{Q_{\nu,m}} \right\}_{\nu=0}^{\infty} \right\|_{\ell^{q_l(\cdot)}(L^{p_l(\cdot)})}.
\end{aligned} \tag{20}$$

We estimate the right-hand side of (20). By the definition of β_ν we see that

$$\begin{aligned}
1 &= \int_{\mathbb{R}^n} \left\{ \frac{(\beta_\nu^{\frac{1}{q_l(\cdot)} - \frac{p(\cdot)}{q(\cdot)p_l(\cdot)}} \sum_{m\in\mathbb{Z}^n} R_{\nu,m}^{\frac{p(\cdot)}{p_l(\cdot)}} \chi_{Q_{\nu,m}})^{q_l(x)}}{\beta_\nu} \right\}^{\frac{p_l(x)}{q_l(x)}} \mathrm{d}x \\
&= \int_{\mathbb{R}^n} \left(\frac{\sum_{m\in\mathbb{Z}^n} R_{\nu,m}^{q(x)} \chi_{Q_{\nu,m}}}{\beta_\nu} \right)^{\frac{p(x)}{q(x)}} \mathrm{d}x.
\end{aligned}$$

This implies that

$$\left\| \left(\beta_\nu^{\frac{1}{q_l(\cdot)} - \frac{p(\cdot)}{q(\cdot)p_l(\cdot)}} \sum_{m\in\mathbb{Z}^n} R_{\nu,m}^{\frac{p(\cdot)}{p_l(\cdot)}} \chi_{Q_{\nu,m}} \right)^{q_l(\cdot)} \right\|_{L^{\frac{p_l(\cdot)}{q_l(\cdot)}}} \leqslant \beta_\nu. \tag{21}$$

Hence we have

$$\left\| \left\{ \sum_{m\in\mathbb{Z}^n} \left|2^{\nu s_l(\cdot)} \Lambda'_{\nu,m}(l+\mathrm{i}t)\right| \chi_{Q_{\nu,m}}(\cdot) \right\}_{\nu=0}^{\infty} \right\|_{\ell^{q_l(\cdot)}(L^{p_l(\cdot)})} \lesssim 1 \tag{22}$$

by (20) and (21). Now we define

$$F(z,x) = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \Lambda'_{\nu,m}(z) a_{\nu,m}(x).$$

If we combine (22) and Proposition 2.8, then we have

$$F(\theta) = f, \qquad \left\| F(l+\mathrm{i}t) \right\|_{B^{s_l(\cdot)}_{p_l(\cdot),q_l(\cdot)}} \lesssim 1.$$

Hence it follows from Definition 1.9 that

$$\|f\|_{(B^{s_0(\cdot)}_{p_0(\cdot),q_0(\cdot)}, B^{s_1(\cdot)}_{p_1(\cdot),q_1(\cdot)})_\theta} \lesssim \|f\|_{B^{s(\cdot)}_{p(\cdot),q(\cdot)}}.$$

This proves (18). Therefore, the proof of Theorem 1.11 is complete. □

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References

[1] Alexandre Almeida, Peter Hästö, Besov spaces with variable smoothness and integrability, J. Funct. Anal. 258 (5) (2010) 1628–1655.
[2] J. Bergh, J. Löfström, Interpolation Spaces: An Introduction, Springer-Verlag, Berlin/Heidelberg/New York, 1976.
[3] L. Diening, Maximal function on Musielak–Orilcz spaces and generalized Lebesgue spaces, Bull. Sci. Math. 129 (2005) 657–700.
[4] L. Diening, P. Hästö, S. Roudenko, Function spaces of variable smoothness and integrability, J. Funct. Anal. 256 (2009) 1731–1768.
[5] L. Diening, P. Harjulehto, P. Hästö, Y. Mizuta, T. Shimomura, Maximal functions in variable exponent spaces: Limiting cases of the exponent, Ann. Acad. Sci. Fenn. Math. 34 (2009) 503–522.
[6] M. Fraizer, B. Jawerth, Decomposition of Besov spaces, Indiana Univ. Math. J. 34 (1985) 777–799.
[7] M. Fraizer, B. Jawerth, A discrete transform and decompositions of distribution spaces, J. Funct. Anal. 93 (1990) 34–170.
[8] H. Kempka, Atomic, molecular and wavelet decomposition of 2-microlocal Besov and Triebel–Lizorkin spaces with variable integrability, Funct. Approx. Comment. Math. 43 (2) (2010) 171–208.
[9] O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. 41 (1991) 592–618.
[10] W. Luxenberg, Banach Function Spaces, Assen, 1955.
[11] E. Nakai, Y. Sawano, Hardy spaces with variable exponents and generalized Campanato spaces, in preparation.
[12] H. Nakano, Modulared Semi-Ordered Linear Spaces, Maruzen Co., Ltd., Tokyo, 1950, i+288 pp.
[13] H. Nakano, Topology of Linear Topological Spaces, Maruzen Co., Ltd., Tokyo, 1951, viii+281 pp.
[14] T. Noi, Fourier multiplier theorems for Besov and Triebel–Lizorkin spaces with variable exponents, submitted for publication.
[15] L. Pick, M. Růžička, An example of a space $L^{p(\cdot)}$ on which the Hardy–Littlewood maximal operator is not bounded, Expo. Math. 19 (2001) 369–371.
[16] K. Rajagopal, M. Růžička, On the modeling of electrorheological materials, Mech. Res. Comm. 23 (1996) 401–407.
[17] H. Rauhut, T. Ullrich, Generalized coorbit space theory and inhomogeneous function spaces of Besov–Lizorkin–Triebel type, J. Funct. Anal. 260 (11) (2011) 3299–3362.
[18] S.G. Samko, Differentiation and integration of variable order and the spaces $L^{p(x)}$, Contemp. Math. 212 (1998) 203–219.
[19] H. Triebel, Theory of Function Spaces, Birkhäuser, Basel, Boston, 1983.
[20] T. Ullrich, Continuous characterization of Besov–Lizorkin–Triebel spaces and new interpretations as coorbits, J. Funct. Space Appl., in press.
[21] J. Xu, Variable Besov and Triebel–Lizorkin spaces, Ann. Acad. Sci. Fenn. Math. 33 (2) (2008) 511–522.
[22] J. Xu, An atomic decomposition of variable Besov and Triebel–Lizorkin spaces, Armen. J. Math. 2 (1) (2009) 1–12 (English summary).