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Non-commutative Orlicz spaces associated to a modular on τ-measurable operators

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ABSTRACT

In this paper, we consider non-commutative Orlicz spaces as modular spaces and show that they are complete with respect to their modular. We prove some convergence theorems for τ-measurable operators and deal with uniform convexity of non-commutative Orlicz spaces.

1. Introduction

The theory of modulars on linear spaces and the corresponding theory of modular linear spaces were founded by Nakano [1] and were intensively developed by his mathematical school: Amemiya, Koshi, Shimogaki, Yamamuro [2,3] and others. Further and the most complete development of these theories are due to Orlicz, Mazur, Musielak, Luxemburg, Turpin [4] and their collaborators. At present the theory of modulars and modular spaces is extensively applied, in particular, in the study of various Orlicz spaces [5] and interpolation theory [6], which in their turn have broad applications [4,7]. The importance for applications consists in the richness of the structure of modular function spaces, that – besides being Banach spaces (or F-spaces in a more general setting) – are equipped with modular equivalent of norm or metric notions.

This article is devoted to a study of some properties of non-commutative Orlicz spaces. Non-commutative Orlicz spaces can be defined either in an algebraic way [8] or via Banach function spaces [9,10]. Al-Rashed and Zegarliński [11] established the theory of non-commutative Orlicz spaces associated to a non-commutative Orlicz functional. Their non-commutative Orlicz functional is related to those introduced by [12] where the author used a specific Young function $\varphi(x) = \cosh(x) - 1$, which has a particular importance in quantum information geometry. Recently, they investigated a theory associated with a faithful normal state on a semi-finite von Neumann algebra [13]. Some further results to non-commutative Orlicz spaces are due to Muratov [14]. We consider another approach based on the concept of modular function spaces. Using the generalized singular value function of a τ-measurable operator, we define a modular on the collection of all τ-measurable operators. This modular function defines a corresponding modular space, which is called the non-commutative Orlicz space.

The organization of the paper is as follows. In the second section we provide some necessary preliminaries related to the theory of τ-measurable operators affiliated with a von Neumann algebra and the classical theory of modular spaces. In Section 3 we introduce a definition of non-commutative Orlicz spaces associated to a modular on τ-measurable operators

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and give several equivalent norms on such spaces. We show that the non-commutative Orlicz space is complete with respect to the modular. Finally, in Section 4 we prove that the non-commutative Orlicz space $L^{\varphi}(\widetilde{\mathfrak{M}}, \tau)$ is uniformly convex if the Orlicz function φ is uniformly convex and satisfies the Δ_2-condition.

2. Preliminaries

In this section, we collect some basic facts and introduce some notation related to τ-measurable operators and modular function spaces. We denote by $\mathfrak{M}$ a semi-finite von Neumann algebra on a Hilbert space $\mathfrak{H}$, with a fixed faithful and normal semi-finite trace τ. For standard facts concerning von Neumann algebras, we refer the reader to [15,16]. The identity in $\mathfrak{M}$ is denoted by $\mathbf{1}$ and we denote by $\mathcal{P}(\mathfrak{M})$ the complete lattice of all self-adjoint projections in $\mathfrak{M}$. A linear operator $x : \mathcal{D}(x) \to \mathfrak{H}$ with domain $\mathcal{D}(x) \subseteq \mathfrak{H}$ is called affiliated with $\mathfrak{M}$, if $ux = xu$ for all unitaries u in the commutant $\mathfrak{M}'$ of $\mathfrak{M}$. This is denoted by $x\eta\mathfrak{M}$. Note that the equality $ux = xu$ involves the equality of the domains of the operators ux and xu, that is, $\mathcal{D}(x) = u^{-1}(\mathcal{D}(x))$. If x is in the algebra $\mathcal{B}(\mathfrak{H})$ of all bounded linear operators on the Hilbert space $\mathfrak{H}$, then x is affiliated with $\mathfrak{M}$ if and only if $x \in \mathfrak{M}$. If x is a self-adjoint operator in $\mathcal{B}(\mathfrak{H})$ affiliated with $\mathfrak{M}$, then the spectral projection $e^x(B)$ is an element of $\mathfrak{M}$ for any Borel set $B \subseteq \mathbb{R}$.

A closed and densely defined operator x, affiliated with $\mathfrak{M}$, is called τ-measurable if and only if there exists a number $\lambda \geq 0$ such that

$$\tau\left(e^{|x|}(\lambda, \infty)\right) < \infty.$$

The collection of all τ-measurable operators is denoted by $\widetilde{\mathfrak{M}}$. With the sum and product defined as the respective closure of the algebraic sum and product, it is well known that $\widetilde{\mathfrak{M}}$ is a $*$-algebra [17]. Given $0 < \varepsilon, \delta \in \mathbb{R}$, we define $\mathcal{V}(\varepsilon, \delta)$ to be the set of all $x \in \widetilde{\mathfrak{M}}$ for which there exists $p \in \mathcal{P}(\mathfrak{M})$ such that $\|xp\|_{\mathcal{B}(\mathfrak{H})} \leq \varepsilon$ and $\tau(\mathbf{1} - p) \leq \delta$. An alternative description of this set is given by

$$\mathcal{V}(\varepsilon, \delta) = \left\{x \in \widetilde{\mathfrak{M}} \quad : \tau\left(e^{|x|}(\varepsilon, \infty)\right) < \delta\right\}.$$

The collection $\{\mathcal{V}(\varepsilon, \delta)\}_{\varepsilon,\delta>0}$ is a neighborhood base at 0 for a vector space topology τ_m on $\widetilde{\mathfrak{M}}$. For $x \in \widetilde{\mathfrak{M}}$, the generalized singular value function $\mu(x; \cdot) = \mu(|x|; \cdot)$ is defined by

$$\mu(x; t) = \inf\left\{\lambda \geq 0 : \quad \tau\left(e^{|x|}(\lambda, \infty)\right) \leq t\right\}, \quad t \geq 0.$$

It follows directly that the generalized singular value function $\mu(x)$ is a decreasing right-continuous function on the positive half-line $[0, \infty)$. Moreover,

$$\mu(uxv) \leq \|u\|\|v\|\mu(x)$$

for all $u, v \in \mathfrak{M}$, and $x \in \widetilde{\mathfrak{M}}$ as well as

$$\mu(f(x)) = f(\mu(x))$$

whenever $0 \leq x \in \widetilde{\mathfrak{M}}$ and f is an increasing continuous function on $[0, \infty)$, which is satisfying $f(0) = 0$. The space $\widetilde{\mathfrak{M}}$ is a partially ordered vector space under the ordering $x \geq 0$ defined by $\langle x\xi, \xi\rangle \geq 0$, $\xi \in \mathcal{D}(x)$. If $0 \leq x_\alpha \uparrow x$ holds in $\widetilde{\mathfrak{M}}$, then $\sup \mu(x_\alpha; t)\uparrow_\alpha \mu(x; t)$ for each $t \geq 0$. The trace τ is extended to the positive cone of $\widetilde{\mathfrak{M}}$ as a non-negative extended real-valued functional which is positively homogeneous, additive, unitarily invariant and normal. Furthermore,

$$\tau(x^*x) = \tau(xx^*)$$

for all $x \in \widetilde{\mathfrak{M}}$ and

$$\tau(f(x)) = \int_0^\infty f(\mu(x; t))dt \tag{2.1}$$

whenever $0 \leq x \in \widetilde{\mathfrak{M}}$ and f is a non-negative Borel function, which is bounded on a neighborhood of 0 and satisfies $f(0) = 0$. The generalized singular value functions are the analog (and, actually, generalization) of the decreasing rearrangements of functions in the classical setting. In the following proposition, we list some properties of the rearrangement mapping $\mu(\cdot; t)$.

Proposition 2.1. *Let x, y and z be τ-measurable operators.*

(i) *the map $t \in (0, \infty) \mapsto \mu(x; t)$ is non-increasing and continuous from the right. Moreover,*

$$\lim_{t\downarrow 0} \mu(x; t) = \|x\| \in [0, \infty].$$

(ii) $\mu(x; t) = \mu(|x|; t) = \mu(x^*; t)$.
(iii) $\mu(x; t) \leq \mu(y; t),\ t > 0$, *if* $0 \leq x \leq y$.
(iv) $\mu(x + y; t + s) \leq \mu(x; t) + \mu(y; s),\ t, s > 0$.
(v) $\mu(zxy; t) \leq \|z\|\|y\|\mu(x; t),\ t > 0$.
(vi) $\mu(xy; t + s) \leq \mu(x; t)\mu(y; s)\, t, s > 0$.

For further details and proofs, we refer the reader to [18,19,9,10]. In the following, we recall some basic concepts about modular spaces and Orlicz spaces.

Definition 2.2. Let $\mathfrak{X}$ be an arbitrary vector space.

(a) A functional $\rho : \mathfrak{X} \to [0, \infty]$ is called a modular if for arbitrary $x, y \in \mathfrak{X}$,

(i) $\rho(x) = 0$ if and only if $x = 0$,

(ii) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,

(iii) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \ge 0$,

(b) if (iii) is replaced by

(iii)′ $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \ge 0$, then we say that ρ is a convex modular.

A modular ρ defines a corresponding modular space, i.e., the vector space $\mathfrak{X}_\rho$ given by

$$\mathfrak{X}_\rho = \{x \in \mathfrak{X} : \quad \rho(\lambda x) \to 0 \quad \text{as } \lambda \to 0\}.$$

If ρ is a convex modular, the modular space $\mathfrak{X}_\rho$ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf\left\{\lambda > 0 : \quad \rho\left(\frac{x}{\lambda}\right) \le 1\right\}.$$

A convex function φ defined on the interval $[0, \infty)$, nondecreasing and continuous for $\alpha \ge 0$ and such that $\varphi(0) = 0$, $\varphi(\alpha) > 0$ for $\alpha > 0$, $\varphi(\alpha) \to \infty$ as $\alpha \to \infty$, is called an Orlicz function. We say that an Orlicz function φ is an $\mathcal{N}$-function if it satisfies the conditions $\frac{\varphi(\alpha)}{\alpha} \to 0$ as $\alpha \to 0^+$ and $\frac{\varphi(\alpha)}{\alpha} \to \infty$ as $\alpha \to \infty$.

Let φ be an Orlicz function and let (Ω, Σ, ν) be a measure space. Let us consider the space $L^0(\nu)$ consisting of all measurable real-valued functions on Ω. Define for every $f \in L^0(\nu)$ the Orlicz modular $\rho_\varphi(f)$ by the formula

$$\rho_\varphi(f) = \int_\Omega \varphi(|f(x)|)d\nu(x).$$

The associated modular function space with respect to this modular is called an Orlicz space, and will be denoted by $L^\varphi(\Omega, \nu)$ or briefly L^φ. In other words,

$$L^\varphi = \{f \in L^0(\nu) : \quad \rho_\varphi(\lambda f) \to 0 \quad \text{as } \lambda \to 0\}$$

or equivalently as

$$L^\varphi = \{f \in L^0(\nu) : \quad \rho_\varphi(\lambda f) < \infty \quad \text{for some } \lambda > 0\}.$$

It is well known that $(L^\varphi, \|.\|_{\rho_\varphi})$ is a Banach space, where $\|.\|_{\rho_\varphi}$ is the Luxemburg norm. The basic fact relating one structure to another is that $\|f\|_{\rho_\varphi} \to 0$ if and only if $\rho_\varphi(\lambda f_n) \to 0$ for every $\lambda > 0$. The geometry of the space $(L^\varphi, \|.\|_{\rho_\varphi})$ is rather well known [20,21,4,22,23]. It turns out, however, that some basic geometrical properties of $(L^\varphi, \|.\|_{\rho_\varphi})$ such as reflexivity, strict convexity and uniform convexity can hold only if φ satisfies the Δ_2-condition, i.e., there exists $k > 0$ such that $\varphi(2\alpha) \le k\varphi(\alpha)$ for all $\alpha > 0$. We should recall that the Δ_2-condition is equivalent to the fact that $\rho_\varphi(f) < \infty$ for any $f \in L^\varphi$. Also, φ satisfies the Δ_2-condition if and only if it follows from $\rho_\varphi(f_n) \to 0$ that $\|f_n\|_{\rho_\varphi} \to 0$. For more information on the theory of Orlicz spaces we refer the reader to [4,23].

3. Non-commutative Orlicz spaces

In this section, we define a modular functional on $\widetilde{\mathfrak{M}}$. To this end we assume $\Omega = [0, \infty)$, with ν a Lebesgue measure on Ω. Let $L^\varphi(\Omega, \nu)$ be an Orlicz space on $\Omega = [0, \infty)$ with respect to the modular functional

$$\rho_\varphi(f) = \int_0^\infty \varphi(|f(t)|)d\nu(t)$$

for every $f \in L^\varphi$.

We now define the functional $\widetilde{\rho}_\varphi$ on $\widetilde{\mathfrak{M}}$ as follows:

$$\widetilde{\rho}_\varphi : \widetilde{\mathfrak{M}} \to [0, \infty]$$
$$\widetilde{\rho}_\varphi(x) = \tau(\varphi(|x|)).$$

It follows from [19, Corollary 2.8] that $\widetilde{\rho}_\varphi(x) = \int_0^\infty \varphi(\mu(|x|; t))\,dt$. In the following we show that $\widetilde{\rho}_\varphi$ is a convex modular functional. It is sufficient to show that $\widetilde{\rho}_\varphi$ satisfies the following condition

$$\widetilde{\rho}_\varphi(\alpha x + \beta y) \le \alpha\widetilde{\rho}_\varphi(x) + \beta\widetilde{\rho}_\varphi(y) \tag{3.1}$$

if $\alpha + \beta = 1$ and $\alpha, \beta \ge 0$.

Proposition 3.1. *$\widetilde{\rho}_\varphi$ is a convex modular functional on $\widetilde{\mathfrak{M}}$.*

Proof. This proposition is an immediate consequence of [19, Proposition 4.6 (ii)] if we choose $a = \sqrt{\alpha}\mathbf{1}$ and $b = \sqrt{\beta}\mathbf{1}$, where α and β are positive scalars with $\alpha + \beta = 1$. □

Now, we can define the non-commutative Orlicz space as follows,

$$L^\varphi(\widetilde{\mathfrak{M}}, \tau) = \left\{x \in \widetilde{\mathfrak{M}} : \quad \widetilde{\rho}_\varphi(\lambda x) \to 0 \quad \text{as } \lambda \to 0\right\}$$

or equivalently

$$L^\varphi(\widetilde{\mathfrak{M}}, \tau) = \left\{x \in \widetilde{\mathfrak{M}} : \quad \widetilde{\rho}_\varphi(\lambda x) < \infty \quad \text{for some } \lambda > 0\right\}.$$

The vector space $L^\varphi(\widetilde{\mathfrak{M}}, \tau)$ can be equipped with the Luxemburg norm defined by

$$\begin{aligned}
\|x\|_{\widetilde{\rho}_\varphi} &= \inf\left\{\lambda > 0 : \quad \widetilde{\rho}_\varphi\left(\frac{x}{\lambda}\right) \le 1\right\} \\
&= \inf\left\{\lambda > 0 : \quad \tau\left(\varphi\left(\frac{|x|}{\lambda}\right)\right) \le 1\right\} \\
&= \inf\left\{\lambda > 0 : \quad \int_0^\infty \varphi\left(\frac{1}{\lambda}\mu(|x|; t)\right) dt\right\} \\
&= \|\mu(|x|; t)\|_{\rho_\varphi}.
\end{aligned}$$

Similar to the commutative case one can define the Amemiya and Orlicz norms as follows:

$$\begin{aligned}
\|x\|^A_{\widetilde{\rho}_\varphi} &= \inf \frac{1}{\lambda}\left\{1 + \widetilde{\rho}_\varphi(\lambda x) : \quad \lambda > 0\right\} \\
&= \inf \frac{1}{\lambda}\left\{1 + \rho_\varphi\left(\lambda \mu(x; t)\right) : \quad \lambda > 0\right\} \\
&= \|\mu(x; t)\|^A_{\rho_\varphi}
\end{aligned}$$

and

$$\|x\|^O_{\widetilde{\rho}_\varphi} = \|\mu(x; t)\|^O_{\rho_\varphi}.$$

Moreover, we have also the following relations between these norms

$$\|x\|^O_{\widetilde{\rho}_\varphi} = \|x\|^A_{\widetilde{\rho}_\varphi}$$

and

$$\|x\|_{\widetilde{\rho}_\varphi} \le \|x\|^O_{\widetilde{\rho}_\varphi} \le 2\|x\|_{\widetilde{\rho}_\varphi}$$

for all $x \in L^\varphi(\widetilde{\mathfrak{M}}, \tau)$ [24]. It follows from [9, Theorem 4.5] that the non-commutative Orlicz space $L^\varphi(\widetilde{\mathfrak{M}}, \tau)$ is a Banach space under any one of these norms. One can define another norm on $L^\varphi(\widetilde{\mathfrak{M}}, \tau)$ as follows

$$\begin{aligned}
\|x\|^{\widetilde{O}}_{\widetilde{\rho}_\varphi} &= \sup\left\{\tau(|xy|) : y \in L^{\varphi^*}(\widetilde{\mathfrak{M}}, \tau) \quad \text{and} \quad \widetilde{\rho}_{\varphi^*}(y) \le 1\right\} \\
&= \sup\left\{\int_0^\infty \mu(|xy|; t) : y \in L^{\varphi^*}(\widetilde{\mathfrak{M}}, \tau) \quad \text{and} \quad \widetilde{\rho}_{\varphi^*}(y) \le 1\right\},
\end{aligned}$$

where the function $\varphi^* : [0, \infty) \to [0, \infty]$ defined by

$$\varphi^*(u) = \sup\{uv - \varphi(v) : v \ge 0\},$$

is called the complementary function to φ in the sense of Young (see [4,24]). It is known that the Orlicz space $L^\varphi(0, \infty)$ equipped with the Orlicz norm is the Köthe dual of $L^{\varphi^*}(0, \infty)$ equipped with the Luxemburg norm. It follows from Theorem 5.6 of [25] that the norms $\|x\|^O_{\widetilde{\rho}_\varphi}$ and $\|x\|^{\widetilde{O}}_{\widetilde{\rho}_\varphi}$ are equal.

Theorem 3.2. *Let φ be an $\mathcal{N}$-function. Then*

$$\|x\|^O_{\widetilde{\rho}_\varphi} = \|x\|^{\widetilde{O}}_{\widetilde{\rho}_\varphi} \tag{3.2}$$

for any $x \in L^\varphi(\widetilde{\mathfrak{M}}, \tau)$.

By the same token the Hölder inequality is now an easy consequence of [25, Theorem 5.6].

Theorem 3.3 (*Hölder inequality*)**.** *Let φ be an $\mathcal{N}$-function, φ^* complementary to φ in the sense of Young, $x \in L^{\varphi}(\widetilde{\mathfrak{M}}, \tau)$ and $y \in L^{\varphi^*}(\widetilde{\mathfrak{M}}, \tau)$. Then*

$$\tau(|xy|) \le \|x\|^{O}_{\widetilde{\rho}_{\varphi}} \|y\|_{\widetilde{\rho}_{\varphi^*}} \tag{3.3}$$

and

$$\tau(|xy|) \le \|x\|_{\widetilde{\rho}_{\varphi}} \|y\|^{O}_{\widetilde{\rho}_{\varphi^*}}. \tag{3.4}$$

We have the following properties of the non-commutative Orlicz modular $\widetilde{\rho}_{\varphi}$, which is similar to the Orlicz modular ρ_{φ}.

Proposition 3.4. *Let $x \in L^{\varphi}(\widetilde{\mathfrak{M}}, \tau)$.*

(i) *If $\|x\|_{\widetilde{\rho}_{\varphi}} > 1$, then $\widetilde{\rho}_{\varphi}(x) \ge \|x\|_{\widetilde{\rho}_{\varphi}}$.*
(ii) *If $x \neq 0$, then $\widetilde{\rho}_{\varphi}\left(\frac{x}{\|x\|_{\widetilde{\rho}_{\phi}}}\right) \le 1$.*

Proof. (i) Suppose that $\|x\|_{\widetilde{\rho}_{\varphi}} > 1$ and choose $0 < \varepsilon < 1$ such that $(1-\varepsilon)\|x\|_{\widetilde{\rho}_{\varphi}} > 1$. Then

$$\begin{aligned} 1 < \widetilde{\rho}_{\varphi}\left(\frac{x}{(1-\varepsilon)\|x\|_{\widetilde{\rho}\,\varphi}}\right) &\le \frac{1}{(1-\varepsilon)\|x\|_{\widetilde{\rho}\,\varphi}}\,\widetilde{\rho}(x) + \left(1 - \frac{1}{(1-\varepsilon)\|x\|_{\widetilde{\rho}\,\varphi}}\right)\widetilde{\rho}_{\varphi}(0) \\ &= \frac{1}{(1-\varepsilon)\|x\|_{\widetilde{\rho}}}\widetilde{\rho}_{\varphi}(x), \end{aligned}$$

which implies that

$$(1-\varepsilon)\|x\|_{\widetilde{\rho}_{\varphi}} < \widetilde{\rho}_{\varphi}(x).$$

Letting $\varepsilon \to 0$, the result follows.

(ii) Let $\varepsilon > 0$ be given. Then there exists $\lambda_{\varepsilon} > 0$ such that $\widetilde{\rho}_{\varphi}\left(\frac{x}{\lambda_{\varepsilon}}\right) \le 1$ and $\lambda_{\varepsilon} < \varepsilon + \|x\|_{\widetilde{\rho}_{\varphi}}$ by the definition of Luxemburg norm. This ensures that $\widetilde{\rho}_{\varphi}\left(\frac{x}{\|x\|_{\widetilde{\rho}_{\varphi}}+\varepsilon}\right) < 1$. Since $0 < \varepsilon < 1$ is arbitrary, we get $\widetilde{\rho}_{\varphi}\left(\frac{x}{\|x\|_{\widetilde{\rho}_{\varphi}}}\right) \le 1$. □

We recall that the function φ satisfies the Δ_2-condition, if there exists $k > 0$ such that $\varphi(2\alpha) \le k\varphi(\alpha)$ for all $\alpha > 0$. We then obtain the following properties of a non-commutative Orlicz modular $\widetilde{\rho}_{\varphi}$ when φ satisfies the Δ_2-condition. To prove the next proposition we need the following lemma.

Lemma 3.5 ([26,27])**.** *Let x, y be τ-measurable operators. Then there exist partial isometries u, v in $\mathfrak{M}$ such that*

$$|x+y| \le u|x|u^* + v|y|v^*.$$

Proposition 3.6. *If φ satisfies the Δ_2-condition, then*

(i) *$\widetilde{\rho}_{\varphi}\left(\frac{x}{\|x\|_{\widetilde{\rho}_{\varphi}}}\right) = 1$ for each $x \in L^{\varphi}(\widetilde{\mathfrak{M}}, \tau)$;*
(ii) *$\|x\|_{\widetilde{\rho}_{\varphi}} = 1$ if and only if $\widetilde{\rho}_{\varphi}(x) = 1$;*
(iii) *$\widetilde{\rho}_{\varphi}(x_n) \to 0$ if and only if $\|x_n\|_{\widetilde{\rho}_{\varphi}} \to 0$;*
(iv) *$\widetilde{\rho}_{\varphi}(x+y) \le k\left(\widetilde{\rho}_{\varphi}(x) + \widetilde{\rho}_{\varphi}(y)\right)$ for all x and y in $\widetilde{\mathfrak{M}}$, with $k > 0$ satisfying $\varphi(2\alpha) \le k\varphi(\alpha)$ for any $\alpha > 0$.*

Proof. (i) Let $x \in L^{\varphi}(\widetilde{\mathfrak{M}}, \tau)$. We have

$$\widetilde{\rho}_{\varphi}\left(\frac{x}{\|x\|_{\widetilde{\rho}_{\varphi}}}\right) = \tau\left(\varphi\left(\frac{x}{\|x\|_{\widetilde{\rho}_{\varphi}}}\right)\right) = \int_0^{\infty} \varphi\left(\frac{1}{\|\mu\|_{\rho_{\varphi}}}\mu(|x|;t)\right) = \rho_{\varphi}\left(\frac{\mu(|x|)}{\|\mu(|x|)\|_{\rho_{\varphi}}}\right) = 1.$$

The last equality follows from [20, Proposition 0].

(ii) If $\|x\|_{\widetilde{\rho}_{\varphi}} = 1$, then it is obvious that $\widetilde{\rho}_{\varphi}(x) = 1$. Conversely, let $\widetilde{\rho}_{\varphi}(x) = 1$. Then $1 = \widetilde{\rho}_{\varphi}(x) = \rho_{\varphi}(\mu(|x|))$. Remark 8.15.(3) of [4] implies that $\|\mu(|x|)\|_{\rho_{\varphi}} = 1$, so we have $\|x\|_{\widetilde{\rho}_{\varphi}} = 1$.

(iii) It follows immediately from Theorem 8.14 of [4].

(iv) By Lemma 3.5 there exist partial isometries u, v in $\widetilde{\mathfrak{M}}$ such that

$$|x+y| \le u|x|u^* + v|y|v^*.$$

$$\begin{aligned}
\widetilde{\rho}(x+y) &= \tau\left(\varphi(|x+y|)\right) = \int_0^\infty \varphi\left(\mu(|x+y|;t)\right) dt \\
&\le \int_0^\infty \varphi\big(\mu(u|x|u^* + v|y|v^*;t)\big)\, dt \\
&\le \int_0^\infty \varphi\left(\mu\left(|x|;\frac{t}{2}\right) + \mu\left(|y|;\frac{t}{2}\right)\right) dt \\
&= \int_0^\infty \varphi\left(2\frac{\mu\left(|x|;\frac{t}{2}\right) + \mu\left(|y|;\frac{t}{2}\right)}{2}\right) dt \\
&\le k\int_0^\infty \varphi\left(\frac{\mu\left(|x|;\frac{t}{2}\right) + \mu\left(|y|;\frac{t}{2}\right)}{2}\right) dt \\
&\le \frac{k}{2}\left[\int_0^\infty \varphi\left(\mu\left(|x|;\frac{t}{2}\right)\right) dt + \int_0^\infty \varphi\left(\mu\left(|y|;\frac{t}{2}\right)\right) dt\right] \\
&= k\left[\int_0^\infty \varphi\left(\mu\left(|x|;t\right)\right) dt + \int_0^\infty \varphi\left(\mu(|y|;t)\right) dt\right] \\
&= k\left[\tau\left(\varphi(|x|)\right) + \tau\left(\varphi(|y|)\right)\right] \\
&= k\left[\widetilde{\rho}_\varphi(x) + \widetilde{\rho}_\varphi(y)\right]. \quad \square
\end{aligned}$$

Definition 3.7. Let $\{x_n\}$ and x be in $\widetilde{\mathfrak{M}}$. Then

(i) the sequence $\{x_n\}$ is $\widetilde{\rho}_\varphi$-convergent to x and write $x_n \xrightarrow{\widetilde{\rho}_\varphi} x$ if $\widetilde{\rho}_\varphi(x_n - x) \to 0$.

(ii) A sequence of τ-measurable operators $\{x_n\}$ with $x_n \in L^\varphi(\widetilde{\mathfrak{M}}, \tau)$ is $\widetilde{\rho}_\varphi$-Cauchy if $\widetilde{\rho}_\varphi(x_n - x_m) \to 0$ as $n, m \to \infty$.

(iii) A non-commutative Orlicz modular $\widetilde{\rho}_\varphi$ is complete if any $\widetilde{\rho}_\varphi$-Cauchy sequence is $\widetilde{\rho}_\varphi$-convergent to an element of $L^\varphi(\widetilde{\mathfrak{M}}, \tau)$.

To prove that the non-commutative Orlicz space $L^\varphi(\widetilde{\mathfrak{M}}, \tau)$ is $\widetilde{\rho}_\varphi$-complete, we need the non-commutative extension of Fatou's lemma.

Lemma 3.8 (*Fatou's Lemma*)**.** *Let $\{x_n\}$ be a sequence of τ-measurable operators converging to x in the measure topology. Then*

$$\widetilde{\rho}_\varphi(x) \le \liminf_{n\to\infty} \widetilde{\rho}_\varphi(x_n).$$

Proof. It is a fairly direct consequence of [19, Lemma 3.4] and the usual Fatou Lemma. $\square$

Theorem 3.9. *The non-commutative Orlicz space $L^\varphi(\widetilde{\mathfrak{M}}, \tau)$ is $\widetilde{\rho}_\varphi$-complete.*

Proof. Let $\{x_n\}$ be a $\widetilde{\rho}_\varphi$-Cauchy sequence in $L^\varphi(\widetilde{\mathfrak{M}}, \tau)$ and let $\varepsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that

$$\widetilde{\rho}_\varphi(x_n - x_m) < \varphi(\varepsilon) \quad (n, m \ge n_0).$$

By the definition of $\widetilde{\rho}_\varphi$ we have

$$\int_0^\infty \varphi\left(\mu(|x_n - x_m|;t)\right) dt < \varphi(\varepsilon) \quad (n, m \ge n_0).$$

It follows from the nonnegativity of $\mu(x;t)$ and convexity of φ that

$$\begin{aligned}
\varphi\left(\int_0^1 \mu(|x_n - x_m|;t)dt\right) &\le \int_0^1 \varphi\left(\mu(|x_n - x_m|;t)dt\right) \\
&\le \int_0^\infty \varphi\left(\mu(|x_n - x_m|;t)dt\right) \\
&< \varphi(\varepsilon) \quad (n, m \ge n_0).
\end{aligned}$$

Thus

$$\varphi\left(\int_0^1 \mu(|x_n - x_m|;t)dt\right) < \varphi(\varepsilon) \quad (n, m \ge n_0).$$

Hence

$$\int_0^1 \mu(|x_n - x_m|; t)dt < \varepsilon \quad (n, m \geq n_0) \tag{3.5}$$

since the function φ is increasing. Moreover, the quantity $\int_0^1 \mu(|.|; t)dt$ is the norm for the Banach function space $(L^1 + L^\infty)(\widetilde{\mathfrak{M}}, \tau)$. It follows from (3.5) that $\{x_n\}$ is a Cauchy sequence in $(L^1+L^\infty)(\widetilde{\mathfrak{M}}, \tau)$. Since $(L^1+L^\infty)(\widetilde{\mathfrak{M}}, \tau)$ injects continuously into $\widetilde{\mathfrak{M}}$ [9], it follows that $\{x_n\}$ is Cauchy in the measure topology (τ_m). So there exists $x \in \widetilde{\mathfrak{M}}$ such that $x_n \xrightarrow{\tau_m} x$. By Fatou Lemma, we obtain

$$\widetilde{\rho}_\varphi(x) \leq \liminf_{n\to\infty} \widetilde{\rho}_\varphi(x_n).$$

For n sufficiently large there holds

$$\widetilde{\rho}_\varphi(x - x_n) \leq \liminf_{m\to\infty} \widetilde{\rho}_\varphi(x_m - x_n) < \varphi(\varepsilon).$$

Hence the sequence $\{x_n\}$ is $\widetilde{\rho}_\varphi$-convergent to x. We have $x \in L^\varphi(\widetilde{\mathfrak{M}}, \tau)$, since $\widetilde{\rho}_\varphi(x - x_n) < \infty$ and $x - x_n \in L^\varphi(\widetilde{\mathfrak{M}}, \tau)$. □

In the following, we prove some convergence theorems for τ-measurable operators with respect to the modular $\widetilde{\rho}_\varphi$.

Proposition 3.10 (*Monotone Convergence Theorem*). *Let $\{x_n\}$ be a sequence of τ-measurable operators such that $|x_n| \leq |x|$ for some $x \in \widetilde{\mathfrak{M}}$. Assume $|x_n| \nearrow |x|$ in the measure topology. Then*

$$\widetilde{\rho}_\varphi(x) = \lim_{n\to\infty} \widetilde{\rho}_\varphi(x_n). \tag{3.6}$$

Proof. By Proposition 1.7 of [25], the monotonicity of the sequence $\{|x_n|\}$ implies monotonicity of the generalized singular value functions $\{\mu(x_n; .)\}$ and $|x_n| \nearrow |x|$ leads to $\{\mu(x_n; .)\} \nearrow \{\mu(x; .)\}$, a.e. Therefore we have

$$\widetilde{\rho}_\varphi(x_n) = \tau(\varphi(|x_n|)) = \int_0^\infty \varphi\left(\mu(x_n; t)\right) dt \leq \int_0^\infty \varphi\left(\mu(x; t)\right) dt = \widetilde{\rho}_\varphi(x).$$

Hence the above inequality and Lemma 3.8 imply (3.6). □

Note that in Proposition 3.10 one can replace the assumption $|x_n| \leq |x|$ by $\mu(x_n; t) \leq \mu(x; t)$.

Theorem 3.11 (*Dominated Convergence Theorem*). *Let φ satisfy the Δ_2-condition. Let $\{x_n\}$ be a sequence of τ-measurable operators converging to x in the measure topology. Assume that there exist τ-measurable operators y_n, $n = 1, 2, \ldots$ and y in $L^\varphi(\widetilde{\mathfrak{M}}, \tau)$, satisfying the following conditions:*

(i) $|x_n| \leq |y_n|$,

(ii) $\widetilde{\rho}_\varphi(y) = \lim_{n\to\infty} \widetilde{\rho}_\varphi(y_n)$,

(iii) *the sequence $\{y_n\}$ converges to y in the measure topology.*

Then x_n and x are in $L^\varphi(\widetilde{\mathfrak{M}}, \tau)$ and

$$\lim_{n\to\infty} \|x_n - x\|_{\widetilde{\rho}_\varphi} = 0.$$

Proof. Obviously (i) and Proposition 2.1 imply that $\mu(x_n; t) \leq \mu(y_n; t)$. Therefore for the Orlicz function φ, we have

$$\int_0^\infty \varphi\left(\mu(\lambda x_n; t)\right) dt \leq \int_0^\infty \varphi\left(\mu(\lambda y_n; t)\right) dt \quad (\lambda > 0)$$

whence

$$\widetilde{\rho}_\varphi(\lambda x_n) \leq \widetilde{\rho}_\varphi(\lambda y_n) \quad (\lambda > 0).$$

This ensures that $x_n \in L^\varphi(\widetilde{\mathfrak{M}}, \tau)$, since $y_n \in L^\varphi(\widetilde{\mathfrak{M}}, \tau)$. We can apply Fatou's lemma for the modular $\widetilde{\rho}_\varphi$ to get

$$\widetilde{\rho}_\varphi(\lambda x) \leq \liminf_{n\to\infty} \widetilde{\rho}_\varphi(\lambda x_n) \leq \liminf_{n\to\infty} \widetilde{\rho}_\varphi(\lambda y_n) = \widetilde{\rho}_\varphi(\lambda y) \quad (\lambda > 0). \tag{3.7}$$

Inequality (3.7) implies that $x \in L^\varphi(\widetilde{\mathfrak{M}}, \tau)$. By [4, Theorem 8.13] we have

$$\widetilde{\rho}_\varphi(x_n - x) < \infty \quad \text{and} \quad \widetilde{\rho}_\varphi(y) < \infty,$$

since the Orlicz function φ satisfies the Δ_2-condition. By Proposition 2.1 (iii) we get

$$\mu(x_n - x; t) \leq \mu\left(x; \frac{t}{2}\right) + \mu\left(x_n; \frac{t}{2}\right) \leq \mu\left(x; \frac{t}{2}\right) + \mu\left(y_n; \frac{t}{2}\right).$$

Hence

$$\varphi\left[\mu(x_n - x; t)\right] \leq \varphi\left[\mu\left(x; \frac{t}{2}\right) + \mu\left(y_n; \frac{t}{2}\right)\right].$$

By our assumption φ is a convex function and satisfies the Δ_2-condition. So

$$\varphi(\alpha+\beta) = \varphi\left(\frac{2\alpha}{2} + \frac{2\beta}{2}\right) \leq \frac{1}{2}\left(\varphi(2\alpha) + \varphi(2\beta)\right) \leq \frac{k}{2}\left(\varphi(\alpha) + \varphi(\beta)\right).$$

Hence

$$\varphi\left[\mu(x_n - x; t)\right] \leq \frac{k}{2}\left[\varphi\left(\mu\left(x; \frac{t}{2}\right)\right) + \varphi\left(\mu\left(y_n; \frac{t}{2}\right)\right)\right].$$

It follows from Lemmata 3.1 and 3.4 of [19] that

$$\lim_{n\to\infty} \mu(x_n - x; t) = 0 \quad \text{and} \quad \mu\left(y; \frac{t}{2}\right) \leq \liminf_{n\to\infty} \mu\left(y_n; \frac{t}{2}\right).$$

Hence the non-negative function

$$\frac{k}{2}\left[\varphi\left(\mu\left(x; \frac{t}{2}\right)\right) + \varphi\left(\mu\left(y_n; \frac{t}{2}\right)\right)\right] - \varphi\left[\mu(x_n - x; t)\right] \geq 0$$

satisfies

$$\begin{aligned}
&\liminf_{n\to\infty}\left\{\frac{k}{2}\left[\varphi\left(\mu\left(x; \frac{t}{2}\right)\right) + \varphi\left(\mu\left(y_n; \frac{t}{2}\right)\right)\right] - \varphi\left[\mu(x_n - x; t)\right]\right\} \\
&\quad \geq \frac{k}{2}\left[\varphi\left(\mu\left(x; \frac{t}{2}\right)\right) + \varphi\left(\mu\left(y; \frac{t}{2}\right)\right)\right].
\end{aligned}$$

The usual Fatou lemma therefore implies that

$$\begin{aligned}
&\int_0^\infty \frac{k}{2}\left[\varphi\left(\mu\left(x; \frac{t}{2}\right)\right) + \varphi\left(\mu\left(y; \frac{t}{2}\right)\right)\right] dt \\
&\quad \leq \int_0^\infty \liminf_{n\to\infty}\left\{\frac{k}{2}\left[\varphi\left(\mu\left(x; \frac{t}{2}\right)\right) + \varphi\left(\mu\left(y_n; \frac{t}{2}\right)\right)\right] - \varphi\left[\mu(x_n - x; t)\right]\right\} dt \\
&\quad \leq \liminf_{n\to\infty} \int_0^\infty \left\{\frac{k}{2}\left[\varphi\left(\mu\left(x; \frac{t}{2}\right)\right) + \varphi\left(\mu\left(y_n; \frac{t}{2}\right)\right)\right] - \varphi\left[\mu(x_n - x; t)\right]\right\} dt.
\end{aligned}$$

By assumption (ii) of the hypothesis and the fact that every integral is finite, we obtain

$$-\limsup_{n\to\infty} \int_0^\infty \varphi\left[\mu(x_n - x; t)\right] dt \geq 0,$$

that is,

$$\lim_{n\to\infty} \widetilde{\rho}_\varphi(x_n - x) = 0.$$

Since the Orlicz function φ satisfies the Δ_2-condition, we get

$$\lim_{n\to\infty} \|x_n - x\|_{\widetilde{\rho}_\varphi} = 0. \quad \square$$

Note that Theorem 3.11 generalizes [19, Theorem 3.6] and [13, Theorem 4.7]. In other words, the usual dominated convergence theorem follows from Theorem 3.11 if $\varphi(t) = t^p$.

Corollary 3.12. *Let $\{x_n\}$ be a sequence of τ-measurable operators converging to x in the measure topology. Assume that there exist τ-measurable operators y_n, $n = 1, 2, \ldots$ and y in $L^p(\widetilde{\mathfrak{M}}, \tau)$, $0 < p < \infty$, satisfying the following conditions:*

(i) $|x_n| \leq |y_n|$,

(ii) $\|y\|_p = \lim_{n\to\infty} \|y_n\|_p$,

(iii) *the sequence $\{y_n\}$ converges to y in the measure topology. Then x_n and x are in $L^p(\widetilde{\mathfrak{M}}, \tau)$ and*

$$\lim_{n\to\infty} \|x_n - x\|_p = 0.$$

Remark 3.13. Assume the hypotheses of Theorem 3.11 to be valid. It follows from $x_n \xrightarrow{\tau_m} x$ that $\mu(x_n) \to \mu(x)$ almost everywhere. Hence $\varphi(\mu(x_n)) \to \varphi(\mu(x))$. We can apply the usual dominated convergence theorem to get

$$\lim_{n\to\infty} \widetilde{\rho}_\varphi(x_n) = \widetilde{\rho}_\varphi(x).$$

Corollary 3.14. *Let φ be an Orlicz function satisfying the Δ_2-condition. Let $\{x_n\}$ be a sequence of τ-measurable operators such that $x_n \to x$ in the measure topology. If there exists $y \in \widetilde{\mathfrak{M}}$ such that $\widetilde{\rho}_\varphi(y) = \tau(\varphi(y)) < \infty$ (or $y \in L^\varphi(\widetilde{\mathfrak{M}}, \tau)$) and $|x_n| \le y$ for $n = 1, 2, \ldots,$ then*

$$\lim_{n\to\infty} \widetilde{\rho}_\varphi(x_n) = \widetilde{\rho}_\varphi(x).$$

4. Uniform convexity of non-commutative Orlicz spaces

An Orlicz function φ is called uniformly convex if there exists a function σ taking the interval $(0, 1)$ into itself such that for every $\alpha > 0$, $0 < \varepsilon < 1$ and $0 \le \delta \le \varepsilon$ there holds the inequality

$$\varphi\left(\frac{1+\delta}{2}\alpha\right) \le (1-\sigma(\varepsilon))\,\frac{\varphi(\alpha)+\varphi(\delta\alpha)}{2}.$$

A convex modular ρ on vector space $\mathfrak{X}$ will be called uniformly convex, if for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that for all $x, y \in \mathfrak{X}$, the conditions $\rho(x) = \rho(y) = 1$ and $\rho(x-y) > \varepsilon$ imply $\rho(\frac{x+y}{2}) < 1-\delta$.

Definition 4.1. A Banach space $\mathcal{X}$ is called uniformly convex, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in \mathcal{X}$ satisfying $\|x\| = \|y\| = 1$, the inequality $\|x-y\| > \varepsilon$ implies $\|\frac{x+y}{2}\| < 1-\delta$. It is known that Banach spaces $L^p(\Omega, \nu)$ when $1 < p < \infty$ are uniformly convex spaces. It is well-known that every uniformly convex Banach space is reflexive.

In the paper [28], the authors showed that if $E(0, \infty)$ is a separable rearrangement invariant Banach function space with uniformly convex norm, then its non-commutative analog $E(\widetilde{\mathfrak{M}}, \tau)$ also has a uniformly convex norm. It follows from [28, Theorem 3.1] that the non-commutative Orlicz space $L^\varphi(\widetilde{\mathfrak{M}}, \tau)$ under norm $\|.\|_{\widetilde{\rho}_\varphi}$ is uniformly convex, if φ is uniformly convex and satisfies the Δ_2-condition. By using some ideas from [20], in the next theorem we describe the relationship between uniform convexity of φ and uniform convexity of the modular $\widetilde{\rho}_\varphi$.

Theorem 4.2. *Let φ be uniformly convex and satisfy the Δ_2-condition. Then uniform convexity of φ forces uniform convexity of the modular $\widetilde{\rho}_\varphi$ and the associated Luxemburg norm $\|.\|_{\widetilde{\rho}_\varphi}$.*

Proof. First, we prove the modular

$$\widetilde{\rho}_\varphi(x) = \tau\left(\varphi(|x|)\right) = \int_0^\infty \varphi\left(\mu(|x|; t)\right) dt$$

to be uniformly convex. Without loss of generality we may suppose that $0 < \varepsilon < 1$. Let $\widetilde{\rho}_\varphi(x) = \widetilde{\rho}_\varphi(y) = 1$ and $\widetilde{\rho}_\varphi(x-y) > \varepsilon$. There exist partial isometries $u, v \in \widetilde{\mathfrak{M}}$ such that

$$|x+y| \le u|x|u^* + v|y|v^*.$$

By uniform convexity of φ, we have

$$\varphi\left(\frac{\alpha+\beta}{2}\right) \le (1-\sigma(1-\varepsilon))\,\frac{\varphi(\alpha)+\varphi(\beta)}{2} \quad (\alpha, \beta \ge 0).$$

Hence by using [19, Theorem 4.4] we obtain

$$\begin{aligned}
1-\widetilde{\rho}_\varphi\left(\frac{x+y}{2}\right) &= \frac{\widetilde{\rho}_\varphi(x)+\widetilde{\rho}_\varphi(y)}{2} - \widetilde{\rho}_\varphi\left(\frac{x+y}{2}\right)\\
&= \frac{1}{2}\left[\tau\left(\varphi(|x|)\right)+\tau\left(\varphi(|y|)\right)\right] - \tau\left(\varphi\left(\left|\frac{x+y}{2}\right|\right)\right)\\
&\ge \frac{1}{2}\left[\tau\left(\varphi(|x|)\right)+\tau\left(\varphi(|y|)\right)\right] - \tau\left(\varphi\left(\frac{u|x|u^*}{2}+\frac{v|y|v^*}{2}\right)\right)\\
&= \frac{1}{2}\left[\tau\left(\varphi(|x|)\right)+\tau\left(\varphi(|y|)\right)\right] - \int_0^\infty \varphi\left[\mu\left(\frac{u|x|u^*}{2}+\frac{u|y|u^*}{2}; t\right)\right] dt\\
&\ge \frac{1}{2}\left[\tau\left(\varphi(|x|)\right)+\tau\left(\varphi(|y|)\right)\right] - \int_0^\infty \varphi\left[\mu\left(\frac{u|x|u^*}{2}; t\right)+\mu\left(\frac{u|y|u^*}{2}; t\right)\right] dt
\end{aligned}$$

$$\begin{aligned}&\geq \frac{1}{2}\left(\int_0^\infty \varphi(\mu(|x|;t))dt + \int_0^\infty \varphi(\mu(|y|;t))dt\right)\\ &- \frac{1-\sigma(1-\varepsilon)}{2}\left(\int_0^\infty \varphi(\mu(|x|;t))dt + \int_0^\infty \varphi(\mu(|y|;t))dt\right)\\ &= \frac{\sigma(1-\varepsilon)}{2}\left[\tau\left(\varphi(|x|)\right) + \tau\left(\varphi(|y|)\right)\right].\end{aligned}$$

In other words

$$\begin{aligned}\frac{1}{2}k\left(\frac{2}{\sigma(1-\varepsilon)}\right)\left(1-\widetilde{\rho}_\varphi\left(\frac{x+y}{2}\right)\right) &\geq \frac{1}{2}k\left(\widetilde{\rho}_\varphi(x)+\widetilde{\rho}_\varphi(y)\right)\\ &\geq \frac{1}{2}\left(\widetilde{\rho}_\varphi(2x)+\widetilde{\rho}_\varphi(2y)\right) = \widetilde{\rho}_\varphi\left(\frac{2x}{2}+\frac{2(-y)}{2}\right)\\ &= \widetilde{\rho}_\varphi(x-y) > \varepsilon,\end{aligned}$$

with $k > 0$ satisfying $\varphi(2\alpha) \leq k\varphi(\alpha)$ for any $\alpha > 0$. By the preceding inequality, we have

$$\frac{k}{\sigma(1-\varepsilon)} - \frac{k}{\sigma(1-\varepsilon)}\widetilde{\rho}_\varphi\left(\frac{x+y}{2}\right) > \varepsilon.$$

Therefore

$$\widetilde{\rho}_\varphi\left(\frac{x+y}{2}\right) < \left(\frac{k}{\sigma(1-\varepsilon)} - \varepsilon\right)\frac{\sigma(1-\varepsilon)}{k} = 1 - \frac{\varepsilon\sigma(1-\varepsilon)}{k}.$$

Hence uniform convexity of $\widetilde{\rho}_\varphi$ follows with $\zeta(\varepsilon) = \frac{\varepsilon\sigma(1-\varepsilon)}{k}$.

Now, let $\|x\|_{\widetilde{\rho}_\varphi} = \|y\|_{\widetilde{\rho}_\varphi} = 1$ and $\|x-y\|_{\widetilde{\rho}_\varphi} > \varepsilon$. There exists a $\eta > 0$ dependent on $\varepsilon > 0$ such that $\widetilde{\rho}_\varphi(x-y) > \eta$. By Proposition 3.6 we have $\widetilde{\rho}_\varphi(x) = \widetilde{\rho}_\varphi(y) = 1$. Since $\widetilde{\rho}_\varphi$ is uniformly convex, $\widetilde{\rho}_\varphi\left(\frac{x+y}{2}\right) < 1 - \zeta(\eta)$. It follows that there is $\delta > 0$ dependent only on ε such that $\|\frac{x+y}{2}\|_{\widetilde{\rho}_\varphi} < 1 - \delta$.

In the converse case there would exist a number $\xi > 0$ and a sequence of operators z_n in $L^\varphi(\widetilde{\mathfrak{M}}, \tau)$ such that $\widetilde{\rho}_\varphi(z_n) < 1-\xi$ and $\|z_n\|_{\widetilde{\rho}} \uparrow 1$. We put $\theta_n = \|z_n\|_{\widetilde{\rho}_\varphi}^{-1}$. Then $\|\theta_n z_n\|_{\widetilde{\rho}_\varphi} = 1$, with $1 \leq \theta_n < 2$ for sufficiently large n. It follows from $\|\theta_n z_n\|_{\widetilde{\rho}_\varphi} = 1$ that $\widetilde{\rho}_\varphi(\theta_n z_n) = 1$. Hence

$$\begin{aligned}1 = \widetilde{\rho}_\varphi(\theta_n z_n) &= \widetilde{\rho}_\varphi\left((\theta_n-1)2z_n + (2-\theta_n)z_n\right)\\ &\leq (\theta_n-1)\widetilde{\rho}_\varphi(2z_n) + (2-\theta_n)\widetilde{\rho}_\varphi(z_n)\\ &\leq (\theta_n-1)k\widetilde{\rho}_\varphi(z_n) + (2-\theta_n)\widetilde{\rho}_\varphi(z_n)\end{aligned}$$

for sufficiently large n. Taking $n \to \infty$ in the above inequality and applying the inequality $\widetilde{\rho}_\varphi(z_n) < 1-\xi$, we obtain $1 < 1.(1-\xi)$, a contradiction. □

Corollary 5.16 of [25] presents conditions under which reflexivity of the classical space $E(0,\infty)$ implies reflexivity of its non-commutative analog $E(\widetilde{\mathfrak{M}}, \tau)$. Via Corollary 5.16 of [25] we now obtain the following consequence.

Corollary 4.3. *Under the assumptions of Theorem 4.2, the non-commutative Orlicz space $L^\varphi(\widetilde{\mathfrak{M}}, \tau)$ equipped with norm $\|.\|_{\widetilde{\rho}_\varphi}$ is reflexive.*

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