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On the stationary Navier–Stokes equations in exterior domains

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ABSTRACT

This paper is concerned with the existence and uniqueness questions on weak solutions of the stationary Navier–Stokes equations in an exterior domain Ω in $\mathbb{R}^3$, where the external force is given by $\operatorname{div} F$ with $F = F(x) = (F^i_j(x))_{i,j=1,2,3}$. First, we prove the existence and uniqueness of a weak solution for $F \in L_{3/2,\infty}(\Omega) \cap L_{p,q}(\Omega)$ with $3/2 < p < 3$ and $1 \le q \le \infty$ provided $\|F\|_{L_{3/2,\infty}(\Omega)}$ is sufficiently small. Here $L_{p,q}(\Omega)$ denotes the well-known Lorentz space. We next show that weak solutions satisfying the energy inequality are unique for $F \in L_{3/2,\infty}(\Omega) \cap L_2(\Omega)$ under the same smallness condition on $\|F\|_{L_{3/2,\infty}(\Omega)}$. This result provides a complete answer to the uniqueness question of weak solutions satisfying the energy inequality, the existence of which was proved by Leray in 1933. Finally, we establish the existence of weak solutions for data F in a very large class, for instance, in $L_{3/2}(\Omega) + L_2(\Omega)$, which generalizes Leray's existence result.

0. Introduction

Let Ω be an exterior domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. In this paper, we consider the following problem for the stationary Navier–Stokes equations:

$$\begin{cases} -\Delta v + \operatorname{div}(v \otimes v) + \nabla \pi = \operatorname{div} F & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ v(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{NS}$$

Here $v = v(x) = (v^1(x), v^2(x), v^3(x))$ and $\pi = \pi(x)$ denote the unknown velocity vector and unknown pressure of a viscous incompressible fluid at point $x \in \Omega$, respectively, while $F = F(x) = (F^i_j(x))_{i,j=1,2,3}$ is the given tensor with $\operatorname{div} F = \left(\sum_{j=1}^3 \frac{\partial F^1_j}{\partial x_j}, \sum_{j=1}^3 \frac{\partial F^2_j}{\partial x_j}, \sum_{j=1}^3 \frac{\partial F^3_j}{\partial x_j} \right)$ denoting the external force.

The purpose of the paper is to establish *almost optimal* existence and uniqueness results for weak solutions of the Navier–Stokes problem (NS) in exterior domains. The existence of weak solutions v with the finite Dirichlet integral (i.e., $\nabla v \in L_2(\Omega)$) was already obtained by Leray [1] for arbitrary $F \in L_2(\Omega)$ using the special nonlinear structure $\operatorname{div}(v \otimes v) = v \cdot \nabla v$ with $\operatorname{div} v = 0$ in (NS). On the other hand, from the viewpoint of scaling invariance, the space $L_3(\Omega)$ plays an essential role in investigating various properties of solutions v of (NS). It should be noted that if v solves (NS) in $\mathbb{R}^3$, then so does v_λ for all $\lambda > 0$, where $v_\lambda(x) \equiv \lambda v(\lambda x)$. A Banach space X with the norm $\|\cdot\|_X$ is called scaling invariant if it holds $\|v_\lambda\|_X = \|v\|_X$ for all $\lambda > 0$. It is well known that such a scaling invariant space X is a suitable one in which we find solutions of (NS). For instance, the space $L_3(\Omega)$ is a typical one with such a scaling invariant property.

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Unfortunately, since the linearized problem of (NS), i.e., the Stokes problem is not uniquely solvable in $L_3(\Omega)$ (see [2–7]), it is difficult to construct the solution v in $L_3(\Omega)$ even for small data F in $L_{3/2}(\Omega)$.

To overcome this difficulty, Galdi–Simader [8] constructed the solution v having the property $\sup_{x\in\Omega}(1+|x|)|v(x)| < \infty$ provided that $\sup_{x\in\Omega}(1+|x|^2)|F(x)|$ is sufficiently small. Then, Borchers–Miyakawa [9] and Novotny–Padula [10] refined their result by introducing the space $X(\Omega)$ with the norm $\|v\|_{X(\Omega)} \equiv \sup_{x\in\Omega}(|x||v(x)| + |x|^2|\nabla v(x)|)$, and proved the existence of solutions in $X(\Omega)$ if $\sup_{x\in\Omega}(|x|^2|F(x)| + |x|^3|\nabla F(x)|)$ is sufficiently small. Notice that the space $X(\mathbb{R}^3)$ is also a scaling invariant class. Later on, Kozono–Yamazaki [7] considered the Lorentz space $L_{p,q}(\Omega)$, and established the existence and uniqueness results for the Stokes equations. In particular, the scaling invariant space $L_{3,\infty}(\Omega)$ plays an essential role in the solvability of (NS). They showed that if F is sufficiently small in $L_{3/2,\infty}(\Omega)$, then there exists at least one solution v in $L_{3,\infty}(\Omega)$ of (NS) with $\nabla v \in L_{3/2,\infty}(\Omega)$. The uniqueness of such a solution is obtained under the additional assumption that all possible solutions are small in $L_{3,\infty}(\Omega)$.

In this paper, we shall study the existence and uniqueness of weak solutions of (NS). First of all, we are interested in the question whether the uniqueness of weak solutions in $L_{3,\infty}(\Omega)$ does hold under the only smallness condition on the size of F in $L_{3/2,\infty}(\Omega)$. Although Kozono–Yamazaki [7] showed existence of small weak solutions v in $L_{3,\infty}(\Omega)$ for small F in $L_{3/2,\infty}(\Omega)$, it is still an open problem whether or not there is another weak solution in $L_{3,\infty}(\Omega)$. To show the uniqueness it seems necessary to assume smallness of all possible weak solutions themselves in $L_{3,\infty}(\Omega)$. However, up to the present, we do not know whether the $L_{3,\infty}$-norm for *every* weak solution is controllable only in terms of the $L_{3/2,\infty}$-norm of F. In case of a bounded domain Ω, Galdi–Simader–Sohr [11] introduced the notion of very weak solutions, and showed existence and uniqueness of small solutions in $L^3(\Omega)$ provided F is sufficiently small. Their arguments are also applicable to prove that there exists a unique weak solution $u \in L_3(\Omega)$ with $\nabla u \in L_{3/2}(\Omega)$ for small F in $L_{3/2}(\Omega)$. For the proof of uniqueness, they made use of the duality argument where the solvability of the adjoint equations can be reduced to the investigation of the perturbed Stokes operator with the linear convective term depending on the weak solution v itself. More precisely, for arbitrary two weak solutions v_1 and v_2 in $L^3(\Omega)$ with $\|v_2\|_{L_3(\Omega)}$ sufficiently small, they considered an operator $\mathcal{L}_p$ for $1 < p < \infty$ defined by

$$\mathcal{L}_p \equiv A_p + B_p \quad \text{with } A_p = -P_p\Delta, \qquad B_p v = P_p(v_1 \cdot \nabla v + v_2 \cdot (\nabla v)^t)$$

where A_p and P_p denote the well-known Stokes operator and Helmholtz projection, respectively. It was shown in [11] that the invertibility of $\mathcal{L}_p$ follows from the assumption that v_1 and v_2 belong to $L_3(\Omega)$ with $\|v_2\|_{L_3(\Omega)}$ sufficiently small. Recently, Kim [12] refined their method and made it clear that the integrability such as $v_1, v_2 \in L_3(\Omega)$ enables us to treat B_p as the perturbation of A_p with its A_p-bound zero and that smallness hypothesis on v_2 in $L_3(\Omega)$ yields the fact that $0 \in \rho(\mathcal{L}_p)$, where $\rho(\mathcal{L}_p)$ denotes the resolvent set of $\mathcal{L}_p$.

On the other hand, in our exterior domain Ω, it is necessary to deal with v_1 and v_2 in $L_{3,\infty}(\Omega)$, which causes difficulty to handle B_p as a compact perturbation of A_p. To overcome such an inconvenience, we impose an additional integrability on F in $L_{3/2,\infty}(\Omega) \cap L_{p,q}(\Omega)$ for some supercritical exponent $3/2 < p < 3$ with $1 \le q \le \infty$, which enables us to gain the regularity of the solution v of the equations $\mathcal{L}_p v = P_p(\operatorname{div} F)$. As a result, we can show the unique solvability of (NS) for $F \in L_{3/2,\infty}(\Omega) \cap L_{p,q}(\Omega)$ with $3/2 < p < 3$ and $1 \le q \le \infty$ provided $\|F\|_{L_{3/2,\infty}(\Omega)}$ is sufficiently small. It should be noted that smallness of F is necessary only in the critical space $L_{3/2,\infty}(\Omega)$.

The same uniqueness question arises also in the weak solutions with the finite Dirichlet integral. It was shown by Leray [1] that for every $F \in L_2(\Omega)$, there is at least one weak solution v with $\nabla v \in L_2(\Omega)$ satisfying the energy inequality

$$\|v\|^2_{L_2(\Omega)} \le -\int_\Omega F \cdot \nabla v dx.$$

However, the uniqueness of such a solution remains open even for small data F in $L_2(\Omega)$. Compared with exterior domains, every weak solution in *bounded* domains fulfills the energy identity and is unique provided the L_2-norm of F is sufficiently small because it necessarily belongs to the L_3-space whose norm can be controlled by means of the L_2-norm of F. This argument cannot be adapted to prove the uniqueness of weak solution of (NS) in exterior domains for which Poincaré's inequality does not hold. In our second theorem, we shall show that the uniqueness of weak solutions satisfying the energy inequality is obtained only under the smallness hypothesis of F in the $L_{3/2,\infty}$-norm although $F \in L_{3/2,\infty} \cap L_2(\Omega)$. This result provides a complete answer to the uniqueness question of weak solutions satisfying the energy inequality.

Our final result is concerned with the existence of weak solutions of (NS) for data F in a larger class than that of Leray [1]. In particular, we shall show the existence of at least one weak solution for all arbitrary data F in $L_{3/2,q}(\Omega) + L_2(\Omega)$ with $1 \le q < \infty$. This result follows adapting the proof of an existence result in [12] for very weak solutions of (NS) in bounded domains.

The outline of the paper is as follows. In Section 1, we shall state three main results in the paper. Section 2 is devoted to recalling some preliminary facts on Sobolev inequalities, Hölder inequalities and Stokes equations in Lorentz spaces. In Section 3, we then prove an existence result which plays key roles in proving all the main results. Detailed proofs of the main results will be provided in Section 4.

1. Results

Throughout this paper, we shall freely identify the space of scalar functions with that of vector or tensor functions. For $1 < p < \infty$ and $1 \le q \le \infty$, $L_p(\Omega)$ and $L_{p,q}(\Omega)$ denote the usual Lebesgue and Lorentz spaces over Ω with norms $\|\cdot\|_p$

and $\|\cdot\|_{p,q}$, respectively. We also denote by $(\cdot,\cdot)$ the duality pairing between $L_{p,q}(\Omega)$ and $L_{p',q'}(\Omega)$, where $1/p+1/p'=1$ and $1/q+1/q'=1$. The closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\nabla\cdot\|_p$ is then denoted by $\dot H_p^1(\Omega)$. Finally, by real interpolation, we define $\dot H^1_{p,q}(\Omega)$ by $\dot H^1_{p,q}(\Omega)\equiv(\dot H^1_{p_0}(\Omega),\dot H^1_{p_1}(\Omega))_{\theta,q}$, where $1<p_0<p<p_1<\infty$ and $0<\theta<1$ satisfy $1/p=(1-\theta)/p_0+\theta/p_1$. Note that for $1\le q<\infty$, $C_0^\infty(\Omega)$ is dense in both $L_{p,q}(\Omega)$ and $\dot H^1_{p,q}(\Omega)$; see [13] for instance. On the other hand, it can be shown (see Lemma 2.1) that if $1<p<3$ and $w\in\dot H^1_{p,q}(\Omega)$, then $w(x)\to 0$ as $|x|\to\infty$ and $w=0$ on $\partial\Omega$ in some weak sense. Hence it makes sense to define weak solutions of (NS) as follows:

Definition 1.1. Suppose that $F\in L_{p,q}(\Omega)$, $3/2\le p<3$ *and* $1\le q\le\infty$. *Then*

$$\{v,\pi\}\in\dot H^1_{p,q}(\Omega)\times L_{p,q}(\Omega)$$

is called a *weak solution* or simply a *solution* of (NS) if there holds

$$\begin{cases}(\nabla v,\nabla\varphi)-(v\otimes v,\nabla\varphi)-(\pi,\operatorname{div}\varphi)=-(F,\nabla\varphi) & \text{for all } \varphi\in C_0^\infty(\Omega),\\ \operatorname{div} v=0 & \text{in } \Omega.\end{cases}\tag{1.1}$$

The first main result in the paper is the unique solvability of (NS) in supercritical solution spaces under the only smallness condition on $\|F\|_{3/2,\infty}$. To state it in a concise way, let us introduce the following function spaces: for $3/2\le p<3$ and $1\le q\le\infty$, we define

$$V_{p,q}=\dot H^1_{3/2,\infty}(\Omega)\cap\dot H^1_{p,q}(\Omega)\quad\text{and}\quad \Pi_{p,q}=L_{3/2,\infty}(\Omega)\cap L_{p,q}(\Omega).$$

Both $V_{p,q}$ and $\Pi_{p,q}$ are Banach spaces equipped with the natural norms

$$\|v\|_{V_{p,q}}=\|\nabla v\|_{3/2,\infty}+\|\nabla v\|_{p,q}\quad\text{and}\quad\|\pi\|_{\Pi_{p,q}}=\|\pi\|_{3/2,\infty}+\|\pi\|_{p,q},$$

respectively. Note that $V_{3/2,\infty}\times\Pi_{3/2,\infty}=\dot H^1_{3/2,\infty}(\Omega)\times L_{3/2,\infty}(\Omega)$ is a critical solution space for (NS), while $V_{p,q}\times\Pi_{p,q}$ is supercritical if $p>3/2$.

Theorem 1.1. *For $3/2<p<3$ and $1\le q\le\infty$, there is a small positive constant $\delta=\delta(\Omega,p,q)$ such that if $F\in\Pi_{p,q}$ satisfies $\|F\|_{3/2,\infty}\le\delta$, then there exists a unique weak solution $\{v,\pi\}\in V_{p,q}\times\Pi_{p,q}$ of (NS). Moreover, we have*

$$\|v\|_{3,\infty}+\|\nabla v\|_{3/2,\infty}+\|\pi\|_{3/2,\infty}\le C\|F\|_{3/2,\infty}\tag{1.2}$$

and

$$\|v\|_{p^*,q}+\|\nabla v\|_{p,q}+\|\pi\|_{p,q}\le C'\|F\|_{p,q}$$

for some positive constants $C=C(\Omega)$ and $C'=C'(\Omega,p,q)$, where $p^*=3p/(3-p)$ is the Sobolev exponent to p.

Remark 1.1. (1) The existence assertion of Theorem 1.1 is essentially due to Kozono–Yamazaki [7]. In fact, they proved the existence of a weak solution $\{v,\pi\}\in V_{p,\infty}\times\Pi_{p,\infty}$ of (NS) for $3/2\le p<3$ under the same smallness condition that $\|F\|_{3/2,\infty}\le\delta(\Omega,p)$. Hence the uniqueness of weak solutions in the supercritical class is one of the main achievements of Theorem 1.1. However it still remains open to prove the uniqueness of weak solutions of (NS) in the critical class $V_{3/2,\infty}\times\Pi_{3/2,\infty}$.

(2) The weak solution $\{v,\pi\}$ obtained by Theorem 1.1 belongs to a supercritical class $V_{p,\infty}\times\Pi_{p,\infty}$ with some $p>3/2$. By real interpolation, there is $p\in(3/2,3)$ such that $\{v,\pi\}\in V_{p,p}\times\Pi_{p,p}$. Hence a standard bootstrap argument allows us to deduce the further regularity property of $\{v,\pi\}$. On the other hand, a complete regularity property of a weak solution in $V_{3/2,\infty}\times\Pi_{3/2,\infty}$ can also be deduced from the crucial estimate (1.2) under some smallness condition on $\|F\|_{3/2,\infty}$. See Remark 3.1(2) for details.

Weak solutions in the energy class $\dot H^1_2(\Omega)\times L_2(\Omega)$ have been of particular importance in the theory of the Navier–Stokes equations since Leray's fundamental existence theory in [1]. In fact, Leray established the existence of at least one weak solution $\{v,\pi\}\in\dot H^1_2(\Omega)\times L_{2,\mathrm{loc}}(\Omega)$ of (NS) satisfying the energy inequality

$$\|\nabla v\|_2^2\le-(F,\nabla v).$$

Here F is an arbitrary large tensor in $L_2(\Omega)$. The uniqueness of Leray's weak solutions under the smallness condition on $\|F\|_{3/2,\infty}$ is the second main result in the paper.

Theorem 1.2. *There is a small positive constant $\delta'=\delta'(\Omega)<\delta(\Omega,2,2)$ such that if $F\in L_{3/2,\infty}(\Omega)\cap L_2(\Omega)$ satisfies $\|F\|_{3/2,\infty}\le\delta'$, then there exists a unique weak solution $\{v,\pi\}\in\dot H^1_2(\Omega)\times L_{2,\mathrm{loc}}(\Omega)$ of (NS) satisfying the energy inequality.*

Remark 1.2. (1) In [8] (see also [14, Section IX.9]), Galdi–Simader obtained a uniqueness result for Leray's weak solutions of (NS) under the stronger assumption that $\sup_{x\in\Omega}(1+|x|^2)|F(x)|$ is sufficiently small. Theorem 1.2 seems to be an optimal extension of their uniqueness result.

(2) Theorem 1.2 is indeed a corollary of Theorem 1.1 and a uniqueness criterion due to Kozono–Yamazaki in [15]. Moreover, from the proof of Theorem 1.2, we deduce that the weak solution $\{v, \pi\}$ has the additional regularity $\{v, \pi\} \in V_{2,2} \times \Pi_{2,2}$ and thus satisfies the energy identity

$$\|\nabla v\|_2^2 = -(F, \nabla v).$$

Finally, we extend Leray's existence result to more general data F. To state our result concisely, let $\hat{L}_{3/2,\infty}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $L_{3/2,\infty}(\Omega)$. Then $\hat{L}_{3/2,\infty}(\Omega)$ is a proper closed subspace of $L_{3/2,\infty}(\Omega)$ since $C_0^\infty(\Omega)$ is not dense in $L_{3/2,\infty}(\Omega)$. Note however that $L_{3/2,q}(\Omega) \subset \hat{L}_{3/2,\infty}(\Omega)$ for any $1 \le q < \infty$. Let us also define

$$\hat{\Pi}_{p,q} = \hat{L}_{3/2,\infty}(\Omega) \cap L_{p,q}(\Omega) \quad \text{for } 3/2 \le p < 3,\ 1 \le q \le \infty.$$

Theorem 1.3. *Suppose that either* (i) $\{p, q\} = \{3/2, \infty\}$ *or* (ii) $3/2 < p < 3$, $1 \le q \le \infty$. *Then for every* $F \in \hat{\Pi}_{p,q} + L_2(\Omega)$, *there exists at least one weak solution* $\{v, \pi\}$ *of* (NS) *with* $v \in V_{p,q} + \dot{H}_2^1(\Omega)$ *and* $\pi \in \Pi_{p,q} + L_{2,\mathrm{loc}}(\Omega)$.

Remark 1.3. (1) An analogous existence result was obtained in [12] for weak solutions in $\dot{H}^1_{3/2}(\Omega)$ of (NS) in bounded domains.
(2) Theorem 1.3 obviously generalizes Leray's existence result in [1] because $L_2(\Omega) \hookrightarrow L_{3/2}(\Omega) + L_2(\Omega) \hookrightarrow \hat{\Pi}_{3/2,\infty} + L_2(\Omega)$.

2. Preliminaries

2.1. Some basic inequalities

First of all, we recall three basic lemmas, proofs of which are outlined here for the sake of readers' convenience.

Lemma 2.1. *Let* $1 < p < 3$ *and* $1 \le q \le \infty$. *Then for every* $w \in \dot{H}^1_{p,q}(\Omega)$, *we have*

$$w \in L_{p^*,q}(\Omega), \qquad \nabla w \in L_{p,q}(\Omega), \qquad w|_{\partial\Omega} = 0 \quad \text{and} \quad \|w\|_{p^*,q} \le C\|\nabla w\|_{p,q}$$

for some constant $C = C(p, q) > 0$, *where* $p^* = 3p/(3-p)$ *is the Sobolev exponent to* p.

Proof. The lemma easily follows by real interpolation from the well-known Sobolev embedding theorem and trace theorem in $\dot{H}^1_p(\Omega)$. See also [7, Lemma 2.1]. □

Lemma 2.2. *Suppose that* $f \in L_{p_1,q_1}(\Omega)$ *and* $g \in L_{p_2,q_2}(\Omega)$, *where* $1 < p_1, p_2 < \infty$, $1 \le q_1, q_2 \le \infty$ *and* $1/p = 1/p_1 + 1/p_2$.
(1) *If* $p > 1$, *then* $fg \in L_{p,q}(\Omega)$, *where* $q = \min(q_1, q_2)$, *and*

$$\|fg\|_{p,q} \le C\|f\|_{p_1,q_1}\|g\|_{p_2,q_2}$$

for some constant $C = C(p_1, q_1, p_2, q_2) > 0$.
(2) *If* $p \ge 1$ *and* $1/q_1 + 1/q_2 \ge 1$, *then* $fg \in L_{p,1}(\Omega)$ *and*

$$\|fg\|_{p,1} \le C\|f\|_{p_1,q_1}\|g\|_{p_2,q_2},$$

for some constant $C = C(p_1, q_1, p_2, q_2) > 0$.

Proof. In the case when $p > 1$ in (1) and (2), the lemma was proved by Kozono–Yamazaki [15] applying the real interpolation theory. The remaining case is just the Hölder inequality in Lorentz spaces. □

Lemma 2.3. *Suppose that* $1 < p < 3$, $1 \le q \le \infty$ *and* $v \in L_{3,\infty}(\Omega)$. *Then for every* $w \in \dot{H}^1_{p,q}(\Omega)$, *we have*

$$v \otimes w \in L_{p,q}(\Omega) \quad \text{and} \quad \|v \otimes w\|_{p,q} \le C\|v\|_{3,\infty}\|\nabla w\|_{p,q}$$

for some constant $C = C(p, q) > 0$. *Thus, if* $\operatorname{div} v = 0$ *in* Ω *and* $w \in \dot{H}^1_2(\Omega)$, *then*

$$(v \otimes w, \nabla w) = 0.$$

Proof. The lemma can be deduced easily from Lemmas 2.1 and 2.2 and the following simple observation:

$$(v \otimes w, \nabla w) = \int_\Omega v \otimes w : \nabla w \, dx = \int_\Omega v \cdot \nabla \left(\frac{1}{2}|w|^2 \right) dx = 0$$

for any $w \in C_0^\infty(\Omega)$. □

Remark 2.1. An immediate consequence of Lemma 2.3 is the fact that if $\{v, \pi\} \in \dot{H}^1_2(\Omega) \times L_2(\Omega)$ is a weak solution of (NS) with the additional property $v \in L_{3,\infty}(\Omega)$, then it always satisfies the energy identity

$$\|\nabla v\|_2^2 = -(F, \nabla v).$$

This is easily proved by taking $\varphi = v$ in the weak formulation (1.1) of (NS).

2.2. The Stokes equations

Next, let us consider the following problem for the stationary Stokes equations:

$$\begin{cases} -\Delta v + \nabla\pi = \operatorname{div} F & \text{in } \Omega, \\ \operatorname{div} v = g & \text{in } \Omega, \\ \quad v = v_* & \text{on } \partial\Omega, \\ v(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{S}$$

In the case when Ω is the whole space $\mathbb{R}^3$, a complete L^p-theory of the Stokes equations can be deduced for any $1 < p < \infty$ from the Calderon–Zygmund theory of singular integrals using the volume potential associated with the fundamental solution $\{U, q\}$, where $U = \{U_j^i\}$, $q = \{q_j\}$, $U_j^i(x) = -\frac{1}{8\pi}\left(\frac{\delta_{ij}}{|x|} + \frac{x_i x_j}{|x|^3}\right)$ and $q_j = \frac{x_j}{4\pi|x|^3}$ for $i, j = 1, 2, 3$. See Galdi's book [2] for more details (see also [7, Lemmas 2.4 and 2.5] for a different approach). A similar L^p-result for the Stokes equations in bounded domains was proved by Cattabriga [16] also for any $1 < p < \infty$. However, it is well known that such an L^p-theory holds for our exterior problem (S) if and only if $3/2 < p < 3$ (see [2,3,5]). An obvious counter-example for the critical exponent $p = 3/2$ is provided by the fundamental solution $\{U, q\}$ itself. Note that $|\nabla U(x)| = O\left(|x|^{-2}\right)$ as $|x| \to \infty$ and so $\|\nabla U\|_{L_{3/2}(\Omega)} = \infty$, where $\Omega = \{x \in \mathbb{R}^3 : |x| > 1\}$, but each $\{U_j, q_j\}$ is a solution of (S) with smooth data, i.e., $F = 0$, $g = 0$ and $v_* \in C^\infty(\partial\Omega)$. By duality, the other critical exponent $p = 3$ should be also excluded for the unique solvability in $H_p^1(\Omega)$ of the exterior problem (S).

To circumvent the difficulty of solving (S) uniquely in critical Lebesgue spaces, Kozono–Yamazaki [7] introduced more general classes of function spaces, that is, Lorentz spaces $L_{p,q}(\Omega)$ and Lorentz–Sobolev spaces $\dot{H}_{p,q}^1(\Omega)$, and then proved the existence of unique solutions of (S) in a critical space. Some of the main results in [7] may be reformulated as follows.

Theorem 2.1. (1) *Let $v_* \equiv 0$. Suppose that $\{p, q\}$ satisfies one of the following conditions* (i), (ii) *or* (iii).

(i) $p = 3/2$, $q = \infty$;
(ii) $3/2 < p < 3$, $1 \le q \le \infty$;
(iii) $p = 3$, $q = 1$.

Then for every $\{F, g\} \in L_{p,q}(\Omega) \times L_{p,q}(\Omega)$, there is a unique weak solution $\{v, \pi\} \in \dot{H}_{p,q}^1(\Omega) \times L_{p,q}(\Omega)$ of (S). *Moreover, there exists a constant $C = C(\Omega, p, q) > 0$ such that*

$$\|v\|_{p^*,q} + \|\nabla v\|_{p,q} + \|\pi\|_{p,q} \le C\left(\|F\|_{p,q} + \|g\|_{p,q}\right). \tag{2.1}$$

(2) *If $\{F, g\} \in L_{p_1,q_1}(\Omega) \times L_{p_1,q_1}(\Omega)$ with $3/2 < p_1 < \infty$ and $q \le q_1 \le \infty$ in addition, then the solution $\{v, \pi\}$ has the additional regularity*

$$\nabla v \in L_{p_1,q_1}(\Omega) \quad \textit{and} \quad \pi \in L_{p_1,q_1}(\Omega).$$

In particular, if $3/2 < p_1 < 3$, then we also have $v \in \dot{H}_{p_1,q_1}^1(\Omega)$.

Remark 2.2. One important feature of Theorem 2.1 is to show the existence of solutions having the same decay property at infinity as the fundamental solution $\{U, q\}$. Note that $|U| \in L_{3,\infty}(\Omega)$ and $|\nabla U| \in L_{3/2,\infty}(\Omega)$. An existence result allowing such a solution was also obtained by Galdi–Simader [8] (see also [2]) but under the stronger assumption that $\sup_{x\in\Omega}\left(1 + |x|^2\right)|F(x)| < \infty$.

3. A key existence result

In this section, we prove the following result which plays essential roles in the proofs of the main theorems in the paper.

Theorem 3.1. *Suppose that either* (i) $\{p, q\} = \{3/2, \infty\}$ *or* (ii) $3/2 < p < 3$, $1 \le q \le \infty$. *Then there are positive constants $\delta_0 = \delta_0(\Omega, p, q)$, $C_0 = C_0(\Omega)$ and $C_0' = C_0'(\Omega, p, q)$ such that if $F \in \Pi_{p,q}$ satisfies $\|F\|_{3/2,\infty} \le \delta_0$, then there exists at least one weak solution $\{v, \pi\} \in V_{p,q} \times \Pi_{p,q}$ of* (NS) *satisfying the estimates*

$$\|v\|_{3,\infty} + \|\nabla v\|_{3/2,\infty} + \|\pi\|_{3/2,\infty} \le C_0\|F\|_{3/2,\infty} \tag{3.2}$$

and

$$\|v\|_{p^*,q} + \|\nabla v\|_{p,q} + \|\pi\|_{p,q} \le C_0'\|F\|_{p,q}. \tag{3.3}$$

Proof. Let $v \in V_{p,q}$ be fixed. Then since $v \in \dot{H}_{3/2,\infty}^1(\Omega)$ obviously, it follows from Lemmas 2.1 and 2.3 that

$$v \otimes v \in L_{3/2,\infty}(\Omega) \quad \text{and} \quad \|v \otimes v\|_{3/2,\infty} \le C\|\nabla v\|_{3/2,\infty}^2 \tag{3.4}$$

for some constant $C > 0$. Hence by Theorem 2.1(1), there exists a unique $\overline{v} = \mathcal{T}(v) \in \dot{H}^1_{3/2,\infty}(\Omega)$ such that for some unique $\overline{\pi} \in L_{3/2,\infty}(\Omega)$, the pair $\{\overline{v}, \overline{\pi}\}$ is a weak solution of the following Stokes problem:

$$\begin{cases} -\Delta \overline{v} + \nabla \overline{\pi} = \operatorname{div}(F - v \otimes v) & \text{in } \Omega, \\ \operatorname{div} \overline{v} = 0 & \text{in } \Omega, \\ \qquad \overline{v} = 0 & \text{on } \partial\Omega, \\ \overline{v}(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{3.5}$$

In fact, it turns out that $\overline{v} \in V_{p,q}$. To show this, we may assume that $3/2 < p < 3$. Then by Lemmas 2.1 and 2.3 again, we have

$$v \otimes v \in L_{p,q}(\Omega) \quad \text{and} \quad \|v \otimes v\|_{p,q} \le C \|\nabla v\|_{3/2,\infty} \|\nabla v\|_{p,q} \tag{3.6}$$

for some $C = C(p, q) > 0$. From (3.4) and (3.6), we deduce that

$$G(v) \equiv F - v \otimes v \in \Pi_{p,q},$$

which implies, in particular, that $G(v) \in \Pi_{p,\infty}$. Hence it follows from Theorem 2.1(2) that $\overline{v} \in \dot{H}^1_{p,\infty}(\Omega)$. On the other hand, since $G(v) \in L_{p,q}(\Omega)$ and since $3/2 < p < 3$, we deduce from Theorem 2.1(1) that the Stokes problem (3.5) has a unique solution $\{\tilde{v}, \tilde{\pi}\}$ in $\dot{H}^1_{p,q}(\Omega) \times L_{p,q}(\Omega)$ which may be different from $\{\overline{v}, \overline{\pi}\}$. However, both $\{\tilde{v}, \tilde{\pi}\}$ and $\{\overline{v}, \overline{\pi}\}$ are solutions of the Stokes problem (3.5) in the common space $\dot{H}^1_{p,\infty}(\Omega) \times L_{p,\infty}(\Omega)$ and thus, by Theorem 2.1(1) again, we obtain

$$\mathcal{T}(v) = \overline{v} = \tilde{v} \in \dot{H}^1_{3/2,\infty}(\Omega) \cap \dot{H}^1_{p,q}(\Omega) = V_{p,q}.$$

We have shown that the mapping $\mathcal{T}$, defined by $\mathcal{T}(v) = \overline{v}$, is an operator on the Banach space $V_{p,q}$. Moreover, it follows from Theorem 2.1(1), (3.4) and (3.6) that for all $v, v_1, v_2 \in V_{p,q}$, we have

$$\|\nabla \mathcal{T}(v)\|_{3/2,\infty} \le C\|F - v \otimes v\|_{3/2,\infty} \le C^* \|F\|_{3/2,\infty} + C^* \|\nabla v\|^2_{3/2,\infty},$$

$$\|\nabla \mathcal{T}(v)\|_{p,q} \le C\|F - v \otimes v\|_{p,q} \le C^*_{p,q} \|F\|_{p,q} + C^*_{p,q} \|\nabla v\|_{3/2,\infty} \|\nabla v\|_{p,q}$$

and

$$\begin{aligned} \|\mathcal{T}(v_1) - \mathcal{T}(v_2)\|_{V_{p,q}} &\le C \|v_1 \otimes v_1 - v_2 \otimes v_2\|_{\Pi_{p,q}} \\ &\le C^*_{p,q} \left(\|\nabla v_1\|_{3/2,\infty} + \|\nabla v_2\|_{3/2,\infty} \right) \|v_1 - v_2\|_{V_{p,q}} \end{aligned}$$

for some constants $C^* = C^*(\Omega)$ and $C^*_{p,q} = C^*_{p,q}(\Omega, p, q)$ with $C^*_{p,q} \ge C^* > 1$.

Without loss of generality, we may now assume that $\|F\|_{3/2,\infty} > 0$ and $\|F\|_{p,q} > 0$; otherwise, $F = 0$ and so we can take $\{v, \pi\} = \{0, 0\}$ as a solution. Furthermore, let us suppose that

$$\|F\|_{3/2,\infty} < \delta_0 \equiv \frac{1}{8} \left(C^*_{p,q}\right)^{-2}, \tag{3.7}$$

and let B be the closed set of all $v \in V_{p,q}$ such that

$$\|\nabla v\|_{3/2,\infty} \le 2C^* \|F\|_{3/2,\infty} \quad \text{and} \quad \|\nabla v\|_{p,q} \le 2C^*_{p,q} \|F\|_{p,q}.$$

Then for all $v, v_1, v_2 \in B$, we obtain

$$\|\nabla \mathcal{T}(v)\|_{3/2,\infty} \le C^* \|F\|_{3/2,\infty} + C^* \left(2C^* \|F\|_{3/2,\infty}\right)^2 \le 2C^* \|F\|_{3/2,\infty},$$

$$\|\nabla \mathcal{T}(v)\|_{p,q} \le C^*_{p,q} \|F\|_{p,q} + C^*_{p,q} \left(2C^* \|F\|_{3/2,\infty}\right) \left(2C^*_{p,q} \|F\|_{p,q}\right) \le 2C^*_{p,q} \|F\|_{p,q}$$

and

$$\begin{aligned} \|\mathcal{T}(v_1) - \mathcal{T}(v_2)\|_{V_{p,q}} &\le C^*_{p,q} \left(4C^* \|F\|_{3/2,\infty}\right) \|v_1 - v_2\|_{V_{p,q}} \\ &\le \frac{1}{2} \|v_1 - v_2\|_{V_{p,q}}. \end{aligned}$$

Therefore, $\mathcal{T}$ is a contraction on the complete metric space B and thus has a fixed point v in B by the Banach fixed point theorem.

To complete the proof, it remains to derive the estimates (3.2) and (3.3) for the solution $\{v, \pi\}$ of (NS), where π is the pressure associated with v. To do so, we can argue as before using Lemma 2.1, Theorem 2.1(1), (3.4) and (3.6) and (3.7) together with the fact that $v = \mathcal{T}(v) \in B$. Indeed, we have

$$\begin{aligned}\|v\|_{3,\infty} + \|\nabla v\|_{3/2,\infty} + \|\pi\|_{3/2,\infty} &\le C\|F - v \otimes v\|_{3/2,\infty} \\ &\le C\|F\|_{3/2,\infty} + C\|\nabla v\|_{3/2,\infty}^2 \\ &\le C\|F\|_{3/2,\infty} + C\left(2C^*\|F\|_{3/2,\infty}\right)^2 \\ &\le C_0\|F\|_{3/2,\infty}\end{aligned}$$

for some $C_0 = C_0(\Omega) > 0$, which proves (3.2). The proof of (3.3) is similar and omitted. This completes the proof of Theorem 3.1. $\square$

Remark 3.1. (1) The existence assertion of Theorem 3.1 is essentially due to Kozono–Yamazaki [7]. Our new achievement in this paper is to derive the important estimate (3.2) for the solution $\{v, \pi\}$ from which we conclude that $\|v\|_{3,\infty}$ can be arbitrarily small as $\|F\|_{3/2,\infty}$ becomes small. This fact plays a key role in proving almost optimal uniqueness results for (NS) stated in the first two main theorems as well as regularity results below.

(2) Let $\{v, \pi\}$ be a weak solution in $V_{3/2,\infty} \times \Pi_{3/2,\infty}$ of (NS) satisfying the estimate (3.2). Then from Theorems 3.1 and 2.1(2) and an interior regularity result in [17], we can deduce the following regularity properties of $\{v, \pi\}$:

(i) For $3/2 < p < \infty$ and $1 \le q \le \infty$, there is a positive constant $\delta_0' = \delta_0'(\Omega, p, q)$ such that if $\|F\|_{3/2,\infty} \le \delta_0'$ and $F \in \Pi_{p,q}$, then $\nabla v \in L_{p,q}(\Omega)$ and $\pi \in L_{p,q}(\Omega)$. In particular, if $3/2 < p < 3$, then $v \in V_{p,q}$.

(ii) For $1 < p < \infty$, there is a positive constant $\delta_0'' = \delta_0''(\Omega, p)$ such that if $\|F\|_{3/2,\infty} \le \delta_0''$ and $F \in H^m_{p,loc}(\Omega)$ for some integer $m \ge 0$, then $v \in H^{m+2}_{p,\mathrm{loc}}(\Omega)$ and $\pi \in H^{m+1}_{p,\mathrm{loc}}(\Omega)$.

Both regularity results are nontrivial at all because of the critical regularity of $\{v, \pi\}$ and the proofs are all based on the estimate (3.2).

(3) To prove the existence and regularity of a weak solution $\{v, \pi\} \in \dot{H}^1_{3/2,\infty}(\Omega) \times L_{3/2,\infty}(\Omega)$ of the exterior Navier–Stokes problem, it seems to be indispensable to impose some smallness condition on the data F. On the contrary, in case when Ω is a bounded domain in $\mathbb{R}^3$, the existence and complete regularity of solutions in $\dot{H}^1_{3/2}(\Omega) \times L_{3/2}(\Omega)$ of (NS) were proved recently by Kim [12] without any smallness condition. The uniqueness was proved earlier by Galdi–Simader–Sohr in [11] under the smallness condition on $\|F\|_{3/2}$.

4. Proofs of main theorems

4.1. Proof of Theorem 1.1

Let $3/2 < p < 3$ and $1 \le q \le \infty$, and suppose that $F \in \Pi_{p,q}$ and $\|F\|_{3/2,\infty} < \delta_0$, where $\delta_0 = \delta_0(\Omega, p, q)$ is the same constant as in Theorem 3.1. Then by Theorem 3.1, there exists at least one solution $\{v_1, \pi_1\} \in V_{p,q} \times \Pi_{p,q}$ of (NS) satisfying the estimates (3.2) and (3.3). This proves the existence assertion of the theorem, in particular. Hence to complete the proof, it remains to prove the uniqueness. Suppose that $\{v_2, \pi_2\} \in V_{p,q} \times \Pi_{p,q}$ is a solution of (NS) which is possibly different from $\{v_1, \pi_1\}$, and let us define $\{v, \pi\} \in V_{p,q} \times \Pi_{p,q}$ by $\{v, \pi\} = \{v_1 - v_2, \pi_1 - \pi_2\}$. Then $\{v, \pi\}$ is a solution in $V_{p,q} \times \Pi_{p,q}$ of the following Stokes problem:

$$\begin{cases} -\Delta v + \nabla \pi = \operatorname{div} G & \text{in } \Omega, \\ \operatorname{div} v = g & \text{in } \Omega, \\ \qquad v = 0 & \text{on } \partial\Omega, \\ v(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{4.1}$$

where $G = v \otimes v_1 + v_2 \otimes v$. Assume for the moment that $\{v, \pi\}$ has the following additional regularity

$$\{v, \pi\} \in \dot{H}^1_2(\Omega) \times L_2(\Omega). \tag{4.2}$$

Then by virtue of Lemma 2.3, we can take $\varphi = v$ in the weak formulation of (4.1) to obtain

$$\|\nabla v\|_2^2 = -(G, \nabla v) = -(v \otimes v_1, \nabla v) \le C\|v_1\|_{3,\infty}\|\nabla v\|_2^2$$

for some $C > 0$. Since $\{v_1, \pi_1\}$ satisfies the estimate (3.2), we thus obtain

$$\|\nabla v\|_2^2 \le C_0\|F\|_{3/2,\infty}\|\nabla v\|_2^2$$

for some $C_0 = C_0(\Omega) > 0$. Therefore, assuming that

$$\|F\|_{3/2,\infty} < \delta \equiv \min(\delta_0, 1/2C_0),$$

we conclude that $\|\nabla v\|_2^2 = 0$ and so $\{v_1, \pi_1\} = \{v_2, \pi_2\}$ in Ω.

Therefore, to complete the proof of Theorem 1.1, we have only to prove (4.2). This can be shown by a bootstrap argument based on Theorem 2.1(2). First of all, noting that

$$v, v_1, v_2 \in V_{p,q} \hookrightarrow V_{p,\infty} \hookrightarrow L_{3,\infty}(\Omega) \cap L_{p^*,\infty}(\Omega),$$

we deduce from Lemma 2.2 that $G = v \otimes v_1 + v_2 \otimes v \in L_{3/2,\infty}(\Omega) \cap L_{r_1,\infty}(\Omega)$, where $r_1 = p^*/2 > 3/2$. Suppose that $r_1 < 3$. Then since $3/2 < r_1 < 3$, it follows from Theorem 2.1(2), Lemmas 2.1 and 2.2 that

$$v \in V_{r_1,\infty} \hookrightarrow L_{3,\infty}(\Omega) \cap L_{r_1^*,\infty}(\Omega) \quad \text{and} \quad G \in L_{3/2,\infty}(\Omega) \cap L_{r_2,\infty}(\Omega),$$

where

$$\frac{1}{r_2} = \frac{1}{p^*} + \frac{1}{r_1^*} = \frac{1}{r_1} + \left(\frac{1}{p^*} - \frac{1}{3}\right) < \frac{1}{r_1}.$$

Similarly, if $r_2 < 3$, then we have

$$v \in V_{r_2,\infty} \quad \text{and} \quad G \in L_{3/2,\infty}(\Omega) \cap L_{r_3,\infty}(\Omega),$$

where

$$\frac{1}{r_3} = \frac{1}{p^*} + \frac{1}{r_2^*} = \frac{1}{r_2} + \left(\frac{1}{p^*} - \frac{1}{3}\right) < \frac{1}{r_2}.$$

Hence by a simple induction, we conclude that

$$v \in V_{r_j,\infty} \quad \text{and} \quad G \in L_{3/2,\infty}(\Omega) \cap L_{r_{j+1},\infty}(\Omega)$$

for all j with $r_j < 3$, where $\{r_j\}$ is a sequence defined recursively by

$$r_1 = \frac{p^*}{2} \quad \text{and} \quad \frac{1}{r_{j+1}} = \frac{1}{p^*} + \frac{1}{r_j^*} = \frac{1}{r_j} + \left(\frac{1}{p^*} - \frac{1}{3}\right) \quad (j \geq 1).$$

Since $3 < p^* < \infty$, it follows that $0 < 1/r_1 < 2/3$ and $1/r_j > 1/r_{j+1} > 1/r_j - 1/3$ for all $j \geq 1$. Hence there exists the smallest $j = j_0 \geq 1$ such that $0 < 1/r_j \leq 1/3$ or equivalently $r_j \geq 3$. By definition of j_0, we deduce that $G \in L_{3/2,\infty}(\Omega) \cap L_{r_{j_0},\infty}(\Omega)$ and $r_{j_0} \geq 3$, which implies that $G \in L_{r,\infty}(\Omega)$ for all $3/2 < r < 3$. Hence by Theorem 2.1(2), we have

$$\{v, \pi\} \in \dot{H}^1_{r,\infty}(\Omega) \times L_{r,\infty}(\Omega) \quad \text{for all } 3/2 < r < 3,$$

from which (4.2) follows by real interpolation. This completes the proof of Theorem 1.1. □

4.2. *Proof of Theorem 1.2*

In fact, Theorem 1.2 is an immediate corollary of Theorem 3.1 (or Theorem 1.1) and the following uniqueness criterion for (NS) due to Kozono–Yamazaki [15].

Theorem 4.1. *There is an absolute constant $\delta'' > 0$ such that if $\{v, \pi\}$ is a weak solution in $\dot{H}^1_2(\Omega) \times L_{2,\mathrm{loc}}(\Omega)$ of (NS) with the additional property*

$$v \in L_{3,\infty}(\Omega) \quad \text{and} \quad \|v\|_{3,\infty} < \delta'', \tag{4.3}$$

then $\{v, \pi\} = \{v', \pi'\}$ for any weak solution $\{v', \pi'\}$ in $\dot{H}^1_2(\Omega) \times L_{2,\mathrm{loc}}(\Omega)$ of (NS) satisfying the energy inequality.

Proof of Theorem 1.2. Suppose that $F \in \Pi_{2,2}$ and $\|F\|_{3/2,\infty} < \delta_0$, where $\delta_0 = \delta_0(\Omega, 2, 2) > 0$. Then by Theorem 3.1, there exists at least one solution $\{v, \pi\} \in V_{2,2} \times \Pi_{2,2}$ of (NS) such that

$$\|v\|_{3,\infty} \leq C_0 \|F\|_{3/2,\infty}$$

for some constant $C_0 = C_0(\Omega) > 0$. Obviously, $\{v, \pi\}$ is a weak solution in $\dot{H}^1_2(\Omega) \times L_2(\Omega)$ of (NS). Assume now that

$$\|F\|_{3/2,\infty} < \delta' \equiv \min(\delta_0, \delta''/C_0).$$

Then $\|v\|_{3,\infty} < \delta''$ and the proof is complete by Theorem 4.1. □

4.3. *Proof of Theorem 1.3*

Let us first consider the following problem for the perturbed Navier–Stokes equations:

$$\begin{cases} -\Delta v + \operatorname{div}(v \otimes v + v_1 \otimes v + v \otimes v_2) + \nabla \pi = \operatorname{div} F & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ \qquad v = 0 & \text{on } \partial\Omega, \\ v(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{PNS}$$

where v_1, v_2 are given vector fields in Ω.

Proposition 4.1. *There is an absolute constant $\delta_1 > 0$ such that if $v_1, v_2 \in L_{3,\infty}(\Omega)$, $\operatorname{div} v_1 = 0$ in Ω and $\|v_2\|_{3,\infty} < \delta_1$, then for every $F \in L_2(\Omega)$ there exists at least one weak solution $\{v, \pi\} \in \dot{H}_2^1(\Omega) \times L_{2,\mathrm{loc}}(\Omega)$ of (PNS) satisfying the estimate*

$$\|\nabla v\|_2 \le 2\|F\|_2. \tag{4.4}$$

Proof. Choose a large number $R_0 > 0$ so that $\mathbb{R}^3 \setminus \Omega$ is contained in the open ball $B_{R_0} = \{x \in \mathbb{R}^3 : |x| < R_0\}$. For a fixed $R > R_0$, we consider the following problem in the bounded domain $\Omega_R = \Omega \cap B_R$:

$$\begin{cases} -\Delta v + \operatorname{div}(v \otimes v + v_1 \otimes v + v \otimes v_2) + \nabla \pi = \operatorname{div} F & \text{in } \Omega_R, \\ \operatorname{div} v = 0 & \text{in } \Omega_R, \\ \quad v = 0 & \text{on } \partial\Omega_R. \end{cases} \tag{4.5}$$

We will first derive the a priori estimate (4.4). Let $\{v, \pi\} \in \dot{H}_2^1(\Omega_R) \times L_2(\Omega_R)$ be a weak solution of (4.5). Then since Ω_R is bounded and $\operatorname{div} v = \operatorname{div} v_1 = 0$ in Ω, it follows from Lemmas 2.1 and 2.3 that $v \in L_6(\Omega_R) \hookrightarrow L_{3,\infty}(\Omega_R)$ and

$$\begin{aligned} \int_{\Omega_R} |\nabla v|^2 \, dx &= \int_{\Omega_R} (v \otimes v_2 - F) : \nabla v \, dx \\ &\le C\|v_2\|_{3,\infty} \left(\int_{\Omega_R} |\nabla v|^2 \, dx \right) + \|F\|_2 \left(\int_{\Omega_R} |\nabla v|^2 \, dx \right)^{1/2} \end{aligned}$$

for some absolute constant $C > 0$. Hence assuming that

$$\|v_2\|_{3,\infty} < \delta_1 \equiv \frac{1}{2C},$$

we deduce that

$$\int_{\Omega_R} |\nabla v|^2 \, dx \le 4\|F\|_2^2. \tag{4.6}$$

By a standard method based on this a priori estimate, we can prove the existence of at least one weak solution $\{v_R, \pi_R\} \in \dot{H}_2^1(\Omega_R) \times L_2(\Omega_R)$ of (4.5) satisfying the estimate (4.6). Extending v_R to Ω by zero outside Ω_R, we observe that v_R belongs to $\dot{H}_2^1(\Omega)$ and satisfies the uniform estimate (4.4). Hence there exists a sequence R_j, with $R_j \to \infty$, such that v_{R_j} converges weakly in $\dot{H}_2^1(\Omega)$ to some $v \in \dot{H}_2^1(\Omega)$ satisfying (4.4) as well.

Fix any $\varphi \in C_0^\infty(\Omega)$ with $\operatorname{div} \varphi = 0$ in Ω. Then since $\{v_R, \pi_R\}$ is a weak solution of (4.5), there holds

$$\left(\nabla v_{R_j} - v_{R_j} \otimes v_{R_j} - v_1 \otimes v_{R_j} - v_{R_j} \otimes v_2, \nabla \varphi\right) = -(F, \nabla \varphi)$$

for all sufficiently large j. Letting $j \to \infty$, we have

$$(\nabla v - v \otimes v - v_1 \otimes v - v \otimes v_2, \nabla \varphi) = -(F, \nabla \varphi).$$

Therefore, by a well-known result (see [2] or [18] e.g.), we conclude that $\{v, \pi\}$ is a weak solution of (PNS) for some scalar $\pi \in L_{2,\mathrm{loc}}(\Omega)$. This completes the proof of Proposition 4.1. □

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let ε be a fixed positive number with $\varepsilon \le \delta_0$, where $\delta_0 = \delta_0(\Omega, p, q)$ is the same constant as in Theorem 3.1. Then since

$$F \in \hat{\Pi}_{p,q} + L_2(\Omega) \quad \text{and} \quad \hat{\Pi}_{p,q} = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{3/2,\infty}} \cap L_{p,q}(\Omega),$$

there are $F_1 \in \hat{\Pi}_{p,q}$, $F_2 \in L_2(\Omega)$ and $F_1^\varepsilon \in C_0^\infty(\Omega)$ such that $F = F_1 + F_2$ and such that $\|F_1 - F_1^\varepsilon\|_{3/2,\infty} < \varepsilon$. Note that $F_1 - F_1^\varepsilon \in \hat{\Pi}_{p,q} \hookrightarrow \Pi_{p,q}$. Hence by Theorem 3.1, there exists at least one solution $\{v_\varepsilon, \pi_\varepsilon\} \in V_{p,q} \times \Pi_{p,q}$ of (NS) with F replaced by $F_1 - F_1^\varepsilon$ such that

$$\|v_\varepsilon\|_{3,\infty} \le C_0 \|F_1 - F_1^\varepsilon\|_{3/2,\infty} < C_0\, \varepsilon \tag{4.7}$$

for some $C_0 = C_0(\Omega) > 0$. It is easy to show that $\{v, \pi\} = \{v_\varepsilon + \overline{v}, \pi_\varepsilon + \overline{\pi}\}$ is the desired solution of the original problem (NS) if $\{\overline{v}, \overline{\pi}\}$ is a weak solution in $\dot{H}_2^1(\Omega) \times L_{2,\mathrm{loc}}(\Omega)$ of the following perturbed problem:

$$\begin{cases} -\Delta \overline{v} + \operatorname{div}(\overline{v} \otimes \overline{v} + v_\varepsilon \otimes \overline{v} + \overline{v} \otimes v_\varepsilon) + \nabla \overline{\pi} = \operatorname{div} F_2^\varepsilon & \text{in } \Omega, \\ \operatorname{div} \overline{v} = 0 & \text{in } \Omega, \\ \quad \overline{v} = 0 & \text{on } \partial\Omega, \\ \overline{v}(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{4.8}$$

where $F_2^\varepsilon = F_1^\varepsilon + F_2 \in L_2(\Omega)$. Now, let us define

$$\varepsilon = \min\left(\delta_0, \delta_1/C_0\right).$$

Then it follows from (4.7) that $\|v_\varepsilon\|_{3,\infty} < \delta_1$. Hence by Proposition 4.1, there exists at least one weak solution $\{\overline{v}, \overline{\pi}\} \in \dot{H}_2^1(\Omega) \times L_{2,\mathrm{loc}}(\Omega)$ of (4.8). This completes the proof of Theorem 1.3. □

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References

[1] J. Leray, Étude de diverses équations intégrales non linéaires et de quelques problémes que pose l'Hydrodynamique, J. Math. Pures Appl. 9 (1933) 1–82.
[2] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes Equations. Vol. 1, Linearized Steady Problems, Springer, Berlin, 1994.
[3] G. Galdi, C.G. Simader, Existence, uniqueness and L^q-estimates for the Stokes problem in exterior domains, Arch. Ration. Mech. Anal. 112 (1990) 291–318.
[4] H. Kozono, H. Sohr, New a priori estimates for the Stokes equations in exterior domains, Indiana Univ. Math. J. 41 (1991) 1–27.
[5] H. Kozono, H. Sohr, On a new class of generalized solutions for the Stokes equations in exterior domains, Ann. Sc. Norm. Super. Pisa 19 (1992) 155–181.
[6] H. Kozono, H. Sohr, M. Yamazaki, Representation formula, net force and energy relation to the stationary Navier–Stokes equations in 3-dimensional exterior domains, Kyushu J. Math. 51 (1997) 239–260.
[7] H. Kozono, M. Yamazaki, Exterior problem for the stationary Navier–Stokes equations in the Lorentz space, Math. Ann. 310 (1998) 279–305.
[8] G. Galdi, C.G. Simader, New estimates for the steady-state Stokes problem in exterior domains with application to the Navier–Stokes problem, Differential Integral Equations 7 (1994) 847–861.
[9] W. Borchers, T. Miyakawa, On stability of exterior stationary Navier–Stokes flows, Acta Math. 174 (1995) 311–382.
[10] A. Novotny, M. Padula, Note on decay of solutions of steady Navier–Stokes equations in 3-D exterior domains, Differential Integral Equations 8 (1995) 1833–1842.
[11] G. Galdi, C.G. Simader, H. Sohr, A class of solutions to stationary Stokes and Navier–Stokes equations with boundary data in $W^{-1/q,q}$, Math. Ann. 331 (2005) 41–74.
[12] H. Kim, Existence and regularity of very weak solutions of the stationary Navier–Stokes equations, Arch. Ration. Mech. Anal. 193 (2009) 117–152.
[13] J. Bergh, J. Löfström, Interpolation Spaces, Springer-Verlag, Berlin, Heidelberg, New York, 1976.
[14] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes equations. Vol. 2, Nonlinear Steady Problems, Springer, Berlin, 1994.
[15] H. Kozono, M. Yamazaki, Uniqueness criterion of weak solutions to the stationary Navier–Stokes equations in exterior domains, Nonlinear Anal. 38 (1999) 959–970.
[16] L. Cattabriga, Su un problema al contorno relativo al sistema di equazioni di Stokes, Rend. Semin. Mat. Univ. Padova 31 (1961) 308–340.
[17] H. Kim, H. Kozono, A removable isolated singularity theorem for the stationary Navier–Stokes equations, J. Differential Equations 220 (2006) 68–84.
[18] H. Sohr, The Navier–Stokes Equations. An Elementary Functional Analytic Approach, Birkhäuser, 2001.