J. Math. Anal. Appl. 397 (2013) 644–657

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Self-adjoint domains, symplectic geometry, and limit-circle solutions

Siqin Yao [a], Jiong Sun [a], Anton Zettl [b,*]

[a] Math. Department, Inner Mongolia University, Hohhot, 010021, China

[b] Math. Department, Northern Illinois University, DeKalb, IL 60115, USA

ARTICLE INFO

Article history:
Received 29 September 2011
Available online 14 August 2012
Submitted by D. O'Regan

Keywords:
Self-adjoint differential operators
Symplectic geometry
Lagrangian subspaces
Limit-circle solutions
Boundary conditions

ABSTRACT

Everitt and Markus characterized the domains of self-adjoint operator realizations of very general even and odd order symmetric ordinary differential equations in terms of Lagrangian subspaces of symplectic spaces. Recently, for the even order case with real coefficients, Wang, Sun and Zettl constructed limit-circle (LC) solutions and Hao, Wang, Sun and Zettl characterized the self-adjoint domains in terms of LC solutions. These LC solutions are higher order analogues of the celebrated Titchmarsh–Weyl limit-circle solutions in the second-order case. This LC characterization has been used to obtain information about the discrete, continuous, and essential spectra of these operators. In this paper we investigate the connection between these two very different kinds of characterizations and thus add the methods of symplectic geometry to the techniques available for the investigation of the spectrum of self-adjoint operators.

1. Introduction

Given a symmetric (formally self-adjoint) differential expression M of even or odd order and a positive weight function w, the well known GKN Theorem (named after Glazman, Krein and Naimark) characterizes all self-adjoint realizations of the equation

$$My = \lambda w y \quad \text{on } J = (a, b), \quad -\infty \le a < b \le \infty, \tag{1.1}$$

in the Hilbert space $H = L^2(J, w)$. Eq. (1.1) generates a minimal operator $T_{\min}$ and a maximal operator $T_{\max}$ in H with domains $D_{\min}$ and $D_{\max}$, respectively.

Theorem 1 (*GKN*)**.** *Assume the deficiency indices d_+ and d_- of $T_{\min}$ are equal: $d_+ = d_- = d$. A linear submanifold $D(T)$ of $D_{\max}$ is the domain of a self-adjoint extension T of $T_{\min}$ if and only if there exist functions $w_1, w_2, \ldots, w_d$ in $D_{\max}$ satisfying the following conditions:*

(i) $w_1, w_2, \ldots, w_d$ *are linearly independent modulo* $D_{\min}$*;*
(ii) $[w_i, w_j](b) - [w_i, w_j](a) = 0,\ i, j = 1, \ldots, d$*;*
(iii) $D(T) = \{y \in D_{\max} : [y, w_j](b) - [y, w_j](a) = 0,\ j = 1, \ldots, d\}$.

Here $[\cdot, \cdot]$ denotes the Lagrange bracket associated with (1.1).

Note that the GKN characterization depends on maximal domain functions $w_j,\ j = 1, \ldots, d$. These functions depend on the coefficients of the differential equation and this dependence is implicit and complicated (see [1–5]).

* Corresponding author.
E-mail addresses: 89608930@qq.com (S. Yao), masun@imu.edu.cn (J. Sun), zettl@math.niu.edu (A. Zettl).

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doi:10.1016/j.jmaa.2012.07.066

In 1999 Everitt and Markus [6], using methods from symplectic algebra and geometry, characterized these self-adjoint domains in terms of Lagrangian subspaces of symplectic spaces. Below we refer to this as the GKN–EM or just the EM characterization.

In 2009 Wang et al. [7], using a method introduced by Sun [8], constructed LC solutions and used these to characterize the self-adjoint domains in terms of LC solutions (rather than maximal domain functions) for even order equations with real-valued coefficients and one regular endpoint. This was extended by Hao et al. in [9] to two singular endpoints.

Below we refer to these results in [7,9] as the LC characterization. In [10–12] (see also [13]), the LC characterization has been used to obtain information about the discrete, continuous, and essential spectra of these operators.

In this paper we investigate the connection between these EM and LC characterizations. We find a one-to-one correspondence between the two characterizations for each class of operators determined by strictly separated, totally coupled, and mixed boundary conditions. This adds methods from symplectic algebra and geometry to the arsenal of weapons available for the investigation of the spectrum of self-adjoint differential operators. As a first step in this direction we find a necessary symplectic geometry condition for a real number λ to be an eigenvalue of a self-adjoint realization of Eq. (1.1).

The organization of this paper is as follows: This Introduction is followed by a discussion of symmetric expressions in Section 2. Sections 3 and 4 discuss the LC and EM characterizations, Section 5 contains our results 'connecting' these two quite different characterizations and Section 6 uses the EM characterization to find a necessary condition – expressed in terms of symplectic geometry – for a real number λ to be an eigenvalue of some self-adjoint realization.

2. Symmetric expressions

In this section we briefly review quasi-differential symmetric expressions of even order with real coefficients. These generate symmetric differential operators and it is these operators and their self-adjoint extensions which are our primary interest in this paper. For a more comprehensive discussion of quasi-differential equations, the reader is referred to [14,15] in the scalar coefficient case and to [16–18] for the general case with matrix coefficients.

Let $J = (a, b)$ be an interval with $-\infty \le a < b \le \infty$ and let $n = 2k$ be a positive even integer. For a given set S, $M_n(S)$ denotes the set of $n \times n$ complex matrices with entries from S.

Let

$$\begin{aligned} Z_n(J,\mathbb{R}) := \{Q = (q_{rs})_{r,s=1}^n,\ & \\ & q_{r,r+1} \neq 0 \text{ a.e. on } J,\ q_{r,r+1}^{-1} \in L_{\text{loc}}(J,\mathbb{R}),\ 1 \le r \le n-1, \\ & q_{rs} = 0 \text{ a.e. on } J,\ 2 \le r+1 < s \le n; \\ & q_{rs} \in L_{\text{loc}}(J,\mathbb{R}),\ s \neq r+1,\ 1 \le r \le n-1\}. \end{aligned} \tag{2.1}$$

Let $Q \in Z_n(J, \mathbb{R})$. We define

$$V_0 := \{y : J \to \mathbb{C},\ y \text{ is measurable}\} \tag{2.2}$$

and

$$y^{[0]} := y \quad (y \in V_0). \tag{2.3}$$

Inductively, for $r = 1, \ldots, n$, we define

$$V_r = \{y \in V_{r-1} : y^{[r-1]} \in (AC_{\text{loc}}(J))\}, \tag{2.4}$$

$$y^{[r]} = q_{r,r+1}^{-1} \left\{ y^{[r-1]'} - \sum_{s=1}^{r} q_{rs} y^{[s-1]} \right\} \quad (y \in V_r), \tag{2.5}$$

where $q_{n,n+1} := 1$, and $AC_{\text{loc}}(J)$ denotes the set of complex-valued functions which are absolutely continuous on all compact subintervals of J. Finally we set

$$M\,y = M_Q\,y := (-1)^k y^{[n]}, \quad (y \in V_n). \tag{2.6}$$

The expression $M = M_Q$ is called the quasi-differential expression associated with Q. For V_n we also use the symbols $V(M)$ and $D(Q)$. The function $y^{[r]}$ $(0 \le r \le n)$ is called the rth quasi-derivative of y. Since the quasi-derivative depends on Q, we sometimes write $y_Q^{[r]}$ instead of $y^{[r]}$ to indicate this dependence.

We now define symmetric expressions.

Definition 1. Let $Q \in Z_n(J, \mathbb{R})$ and let $M = M_Q$ be defined as above. Assume that

$$Q = -E^{-1}Q^*E, \quad \text{where } E = E_n = ((-1)^r \delta_{r,n+1-s})_{r,s=1}^n. \tag{2.7}$$

Then $M = M_Q$ is called a symmetric differential expression.

Let $w \in L_{\mathrm{loc}}(J)$ be positive a.e. on J. For $Q \in Z_n(J, \mathbb{R})$ we study self-adjoint realizations of the equation

$$My = M_Q y = \lambda w y \quad \text{on } J = (a, b) \tag{2.8}$$

in the Hilbert space

$$H = L^2(J, w)$$

with its usual inner product

$$\langle y, v\rangle_w := \int_J y\, \overline{v}\, w.$$

The maximal and minimal operators associated with a symmetric expression Q and a positive weight function w in the Hilbert space H are defined as follows:

Definition 2. Assume $Q \in Z_n(J, \mathbb{R})$ satisfies (2.7) and let $M = M_Q$ be the associated symmetric expression. Let $w \in L_{\mathrm{loc}}(J, \mathbb{R})$ be positive a.e. on J. Define

$$\begin{aligned} &D_{\max} = \{y \in L^2(J, w) : y \in D(Q),\ w^{-1}My \in L^2(J, w)\},\\ &T_{\max}\, y = w^{-1}My, \quad y \in D_{\max}.\\ &T_{\min} = T^*_{\max},\\ &D_{\min} = D(T_{\min}). \end{aligned} \tag{2.9}$$

Lemma 1. *Let $T_{\min}$ and $T_{\max}$ be defined as above. Then $D_{\min}$ and $D_{\max}$ are dense in H, $T_{\min}$ and $T_{\max}$ are closed operators in H, $T^*_{\min} = T_{\max}$, $T_{\min} = T^*_{\max}$ and $T_{\min}$ is a symmetric operator in H.*

Proof. This is well known; see [15] or [16]. □

Lemma 2 (*Lagrange Identity*)**.** *Assume $Q \in Z_n(J, \mathbb{R})$ satisfies (2.7) and let $M = M_Q$ be the corresponding differential expression. Then for any $y, z \in D(Q)$ we have*

$$\bar{z}My - y\overline{Mz} = [y, z]', \tag{2.10}$$

where

$$[y, z] = (-1)^k \sum_{r=0}^{n-1} (-1)^{n+1-r}\, \bar{z}^{[n-r-1]}\, y^{[r]} = (-1)^k (Z^* E Y), \tag{2.11}$$

$$Y = \begin{pmatrix} y \\ y^{[1]} \\ \vdots \\ y^{[n-1]} \end{pmatrix}, \qquad Z = \begin{pmatrix} z \\ z^{[1]} \\ \vdots \\ z^{[n-1]} \end{pmatrix}. \tag{2.12}$$

Proof. See [15] or [16]. □

Lemma 3. *For any y, z in $D_{\max}$ we have*

$$\int_{c_1}^{c} \{\bar{z}My - y\overline{Mz}\} = [y, z](c) - [y, z](c_1), \tag{2.13}$$

where $c, c_1 \in J = (a, b)$.

Proof. This follows from (2.10) by integration. □

Lemma 4. *For any y, z in $D_{\max}$ we have*

$$\int_a^b \{\bar{z}My - y\overline{Mz}\} = [y, z](b) - [y, z](a).$$

Proof. This follows from (2.13) by taking limits as $c_1 \to a$, $c \to b$. That the limits exist and are finite can be seen from the definition of $D_{\max}$; see (2.9). □

Remark 1. Lemma 4 shows that for any y, z in $D_{\max}$, the Lagrange brackets $[y, z](b) = \lim_{c\to b}[y, z](c),\ c \in (a, b)$ and $[y, z](a) = \lim_{c_1\to a}[y, z](c_1),\ c_1 \in (a, b)$ exist and are finite. These finite limits play the roles of the quasi-derivatives $y^{[r]}(a), y^{[r]}(b)$ in the regular case; see Theorem 4.

Corollary 1. *If $My = \lambda wy$ and $Mz = \overline{\lambda}wz$ on some interval $(\alpha, \beta) \in (a, b)$, then $[y, z]$ is constant on (α, β).*

Proof. This follows directly from (2.10). □

Definition 3 (*Regular Endpoints*)**.** Let $Q \in Z_n(J, \mathbb{R}),\ J = (a, b)$. The expression $M = M_Q$ is said to be regular at a if for some $c, a < c < b$, we have

$$q^{-1}_{r,r+1} \in L(a, c), \quad r = 1, \dots, n-1;$$
$$q_{rs} \in L(a, c), \quad 1 \le r, s \le n,\ s \ne r+1.$$

There is a similar definition for the endpoint b.

Now we state an important lemma about the deficiency index d of the minimal operator $T_{\min}$.

Lemma 5. *Let M be an even order symmetric differential expression with real coefficients. Let $a \le \alpha < \beta \le b$. The number d of linearly independent solutions of*

$$My = \lambda wy \quad \text{on } (\alpha, \beta) \tag{2.14}$$

lying in $L^2((\alpha, \beta), w)$ is independent of $\lambda \in \mathbb{C}$, provided $\mathrm{Im}(\lambda) \ne 0$. If one endpoint of (α, β) is regular and the other is singular then the inequalities

$$k \le d \le 2k = n \tag{2.15}$$

hold; for $\lambda \in \mathbb{R}$, the number of linearly independent solutions of (2.14) lying in $L^2((\alpha, \beta), w)$ is less than or equal to d.

Let $c \in (a, b) = J$. If d_1 is the deficiency index on (a, c), d_2 is the deficiency index on (c, b) and d is the deficiency index on (a, b), then

$$d = d_1 + d_2 - n. \tag{2.16}$$

Proof. See [9,18]. □

Remark 2. With a suitable choice of M in Lemma 5, any value of d in the range given by (2.15) can be achieved.

The domain of the minimal operator can be characterized as follows:

Lemma 6. *The minimal domain $D_{\min}$ on (a, b) consists of all functions $y \in D_{\max} = D_{\max}(a, b)$ which satisfy*

$$[y, z](b) = 0 = [y, z](a), \tag{2.17}$$

for all $z \in D_{\max}$.

Proof. See [18, Theorem 3.11, p. 49]. □

3. The LC characterization

Von Neumann's formula [19, 8, p.30] is fundamental in the study of adjoints of symmetric operators in abstract Hilbert space, and in particular for the study of self-adjoint extensions of unbounded symmetric operators. In the case of ordinary differential operators the deficiency spaces N_λ and $N_{\overline{\lambda}}$ [19, p. 26] are finite dimensional and thus linear algebra can be used in these studies. If A is the minimal operator $T_{\min} = T_{\min}(M)$ generated by a symmetric differential expression M, then A^* is the maximal operator $T_{\max} = T_{\max}(M)$. In this section, for the benefit of the reader, we briefly review the recently established far reaching extension by Hao et al. [9] of the Von Neumann formula for very general even order symmetric differential expressions M on an interval (a, b) with two singular endpoints and arbitrary deficiency index d. In [7,9] this theorem plays a critical role in the construction of LC and LP solutions at each singular endpoint.

In Section 4 we review the EM characterization. This then puts us in a position to discuss the connection between the EM and LC characterizations in Section 5.

We start with the Wang et al. [7] (see also [9]) construction of LC solutions.

Theorem 2. *Let M be a symmetric differential expression of even order with real coefficients on (a, b) defined by Definition 1; let $c \in (a, b)$. Consider the equation*

$$My = \lambda w y. \tag{3.1}$$

Let d_1 denote the deficiency index of (3.1) on (a, c) and d_2 the deficiency index of (3.1) on (c, b). Assume that for some $\lambda = \lambda_1 \in \mathbb{R}$ (3.1) has d_1 linearly independent solutions on (a, c) which lie in $L^2((a, c), w)$ and that for some $\lambda = \lambda_2 \in \mathbb{R}$ (3.1) has d_2 linearly independent solutions on (c, b) which lie in $L^2((c, b), w)$. Then:

(1) *There exist d_1 linearly independent real-valued solutions $u_1, \ldots, u_{d_1}$ of (3.1) with $\lambda = \lambda_1$ on (a, c) which lie in $L^2((a, c), w)$.*
(2) *There exist d_2 linearly independent real-valued solutions $v_1, \ldots, v_{d_2}$ of (3.1) with $\lambda = \lambda_2$ on (c, b) which lie in $L^2((c, b), w)$.*
(3) *For $m_1 = 2d_1 - 2k$ the solutions $u_1, \ldots, u_{d_1}$ can be chosen such that the $m_1 \times m_1$ matrix $U = ([u_i, u_j](c)),\ 1 \le i, j \le m_1$, is given by*

$$U = (-1)^{k+1} E_{m_1}. \tag{3.2}$$

(4) *For $m_2 = 2d_2 - 2k$ the solutions $v_1, \ldots, v_{d_2}$ on (c, b) can be chosen such that the $m_2 \times m_2$ matrix $V = ([v_i, v_j](c)),\ 1 \le i, j \le m_2$, is given by*

$$V = (-1)^{k+1} E_{m_2}. \tag{3.3}$$

(5) *For every $y \in D_{\max}(a, b)$ we have*

$$[y, u_j](a) = 0, \quad \text{for } j = m_1 + 1, \ldots, d_1. \tag{3.4}$$

(6) *For every $y \in D_{\max}(a, b)$ we have*

$$[y, v_j](b) = 0, \quad \text{for } j = m_2 + 1, \ldots, d_2. \tag{3.5}$$

(7) *For $1 \le i, j \le d_1$, we have*

$$[u_i, u_j](a) = [u_i, u_j](c). \tag{3.6}$$

(8) *For $1 \le i, j \le d_2$, we have*

$$[v_i, v_j](b) = [v_i, v_j](c). \tag{3.7}$$

(9) *The solutions $u_1, \ldots, u_{d_1}$ can be extended to (a, b) such that the extended functions, also denoted by $u_1, \ldots, u_{d_1}$, satisfy $u_j \in D_{\max}(a, b)$ and u_j is identically zero in a left neighborhood of $b, j = 1, \ldots, d_1$.*
(10) *The solutions $v_1, \ldots, v_{d_2}$ can be extended to (a, b) such that the extended functions, also denoted by $v_1, \ldots, v_{d_2}$, satisfy $v_j \in D_{\max}(a, b)$ and v_j is identically zero in a right neighborhood of $a, j = 1, \ldots, d_2$.*

Proof. See Theorem 4.1 in [9]. □

Remark 3. This theorem assumes that there exist d linearly independent solutions in H for *some* real value of the spectral parameter λ. This is a weak additional hypothesis, since if it does not hold, then the essential spectrum covers the whole real line and any eigenvalue is embedded in the essential spectrum and its dependence on the boundary conditions in this case seems to be coincidental.

Definition 4 (*LC and LP Solutions*)**.** Let the hypotheses and notation of Theorem 2 hold. The solutions $u_1, \ldots, u_{m_1}$ and $v_1, \ldots, v_{m_2}$ are called LC solutions at the endpoints a and b, respectively. The solutions $u_{m_1+1}, \ldots, u_{d_1}$ and $v_{m_2+1}, \ldots, v_{d_2}$ are called LP solutions at a and b, respectively.

Next we discuss representations of the maximal domain [9]. We believe that these are of independent interest.

Theorem 3. *Let the notation and hypotheses of Theorem 2 hold. Then*

$$D_{\max}(a, b) = D_{\min}(a, b) \oplus \operatorname{span}\{u_1, \ldots, u_{m_1}\} \oplus \operatorname{span}\{v_1, \ldots, v_{m_2}\}. \tag{3.8}$$

Proof. See [9] for the proof. □

The next theorem characterizes all self-adjoint operator realizations of Eq. (3.1) in the space $L^2(J, w)$ where $J = (a, b)$ and w is a positive weight function.

Theorem 4 (*Hao, Sun, Wang and Zettl*). *Let the hypotheses and notation of Theorem 2 hold. Let $d = d_1 + d_2 - n$. Then d is the deficiency index of (3.1) on (a, b). Let $m_1 = 2d_1 - 2k$, and let $u_1, \dots, u_{m_1}$ be real-valued LC solutions of Eq. (3.1) with $\lambda = \lambda_1$ on (a, c) as constructed in Theorem 2; let $m_2 = 2d_2 - 2k$ and let $v_1, \dots, v_{m_2}$ be real-valued LC solutions of Eq. (3.1) with $\lambda = \lambda_2$ on (c, b) as constructed in Theorem 2. (See also Remark 4.) A linear submanifold $D(T)$ of $D_{\max}(a, b)$ is the domain of a self-adjoint extension T of $T_{\min}(a, b)$ if and only if there exists a complex $d \times m_1$ matrix A and a complex $d \times m_2$ matrix B such that the following three conditions hold:*

$$\operatorname{rank}(A : B) = d; \tag{3.9}$$

$$AE_{m_1}A^* = BE_{m_2}B^*; \tag{3.10}$$

$$D(T) = \left\{ y \in D_{\max} : A \begin{pmatrix} [y, u_1](a) \\ \vdots \\ [y, u_{m_1}](a) \end{pmatrix} + B \begin{pmatrix} [y, v_1](b) \\ \vdots \\ [y, v_{m_2}](b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}. \tag{3.11}$$

The Lagrange brackets in (3.11) have finite limits.

Theorem 4 is stated for the case where both endpoints are singular. It reduces to the case where one endpoint is regular or both are; see [9] for the specific forms of Theorem 4 for these cases.

Remark 4 (*LC and LP Solutions*). It is clear from Theorems 2 and 4 that the LC solutions contribute to the determination of the self-adjoint boundary conditions and the LP solutions do not. Nevertheless, the LP solutions play an important role in the study of the continuous spectrum (see [10]) and in the approximation of singular problems with regular ones. We plan to investigate these implications further in a subsequent paper.

4. The EM characterization

In this section we present the EM characterization. First, for the convenience of the reader, we recall some definitions on symplectic spaces. These are taken from the Everitt–Markus monograph [6] (see also [20]).

4.1. Complex symplectic spaces

Definition 5. A complex symplectic space S is a complex linear space, with a prescribed symplectic form $[:]$, namely a sesquilinear form

(i) $u, v \to [u : v], \quad S \times S \to \mathbb{C}, \quad$ so $[c_1u + c_2v : w] = c_1[u : w] + c_2[v : w]$,

which is skew-Hermitian,

(ii) $[u : v] = -\overline{[v : u]}, \quad$ so $[u : c_1v + c_2w] = \bar{c_1}[u : v] + \bar{c_2}[u : w]$,

and which is also non-degenerate,

(iii) $[u : S] = 0$ implies $u = 0$,

for all vectors $u, v, w \in S$ and complex scalars $c_1, c_2 \in \mathbb{C}$.

Definition 6. A linear submanifold L in the complex symplectic space S is called Lagrangian in the case where $[L : L] = 0$, that is,

$[u : v] = 0$ for all vectors $u, v \in L$.

Furthermore, a Lagrangian manifold $L \subset S$ is said to be complete in the case where

$u \in S$ and $[u : L] = 0$ imply $u \in L$.

Definition 7. In a complex symplectic space S, with symplectic form $[:]$, and finite dimension $D \geq 1$, define the following symplectic invariants of S:

$p = \max\{$complex dimension of linear subspace whereon $\Im[v : v] \geq 0\}$,

$q = \max\{$complex dimension of linear subspace whereon $\Im[v : v] \leq 0\}$.

(p, q) is called the signature of S, consisting of a pair of integers: the positivity index $p \geq 0$ and the negativity index $q \geq 0$.
In addition, we define the Lagrangian index Δ and the excess Ex of S:

$\Delta = \max\{$complex dimension of Lagrangian subspace of $S\}$,

$Ex = p - q$, excess of positivity over negativity index of S.

These symplectic invariants of S are each defined intrinsically in terms of the symplectic structure on S.

Definition 8. Let S be a complex symplectic space with symplectic form $[:]$. Then linear subspaces S_- and S_+ are symplectic ortho-complements in S, written as

$$S = S_- \oplus S_+,$$

for the case

$$\text{(i) } S = \operatorname{span}\{S_-, S_+\}, \qquad \text{(ii) } [S_- : S_+] = 0.$$

In these cases $S_- \cap S_+ = 0$, so S is the direct sum of S_- and S_+.

4.2. Complex symplectic spaces related to differential expressions

Next we define and investigate the structures of a complex symplectic space $\tilde{S}$, with its Lagrangian subspaces $\tilde{L}$, which arise in connection with boundary value problems associated with symmetric differential expressions M and Eq. (3.1):

$$My = \lambda w y \quad \text{on } J = (a, b), \ -\infty \le a < b \le \infty$$

in the Hilbert space $H = L^2(J, w)$ where $w \in L_{\mathrm{loc}}(J)$ is positive a.e. on J. Let $T_{\min},\ T_{\max},\ D_{\min},\ D_{\max}$ be defined as in (2.9).

Define the endpoint space by

$$\tilde{S} = D_{\max}/D_{\min};$$

then the endpoint space $\tilde{S}$ becomes a complex symplectic $(d^+ + d^-)$-space, with the symplectic product $[:]$ inherited from $D_{\max}$ as follows:

$$[\tilde{f} : \tilde{g}] = [f + D_{\min} : g + D_{\min}] := [f : g],$$

where the skew-Hermitian form $[f : g]$ is defined for $f, g \in D_{\max}$ by

$$[f : g] = \langle M(f), g\rangle - \langle f, M(g)\rangle = [f, g]_a^b.$$

Hence

$$D_{\min} = \{f \in D_{\max} : [f : D_{\max}] = 0\}.$$

Lemma 7. *The symplectic invariants of a complex vector space $\tilde{S}$ with the skew-Hermitian form $[:]$ defined above satisfy*

$$p = q = d, \qquad \dim \tilde{S} = 2d, \quad \text{and} \quad Ex = 0.$$

Here d is the deficiency index of $My = \lambda w y$ on (a, b). Furthermore, there exist complete Lagrangian subspaces $\tilde{L}$ of $\tilde{S}$.

Proof. From Proposition 1 in [6], we know that the symplectic invariants of a complex vector space $\tilde{S}$ are related to the deficiency indices $d^\pm$ of a symmetric differential expression M by $p = d^+,\ q = d^-$; also note that $d^+ = d^- = d$, so we have $p = q = d,\ \dim \tilde{S} = 2d$, and $Ex = 0$. Then from $Ex = 0$ and Theorem 2 in [20], there exist complete Lagrangian subspaces $\tilde{L}$ of $\tilde{S}$. □

The next theorem gives the Everitt–Markus characterization [6] of self-adjoint domains in terms of Lagrangian subspaces.

Theorem 5 (*GKN–EM*)**.** *Let M be a symmetric differential expression studied in Eq. (3.1), and let d denote the deficiency index of $My = \lambda w y$ on (a, b); then there exists a natural one-to-one correspondence between the set $\{T\}$ of all self-adjoint operators T on $D(T)$ generated by M and the set $\{\tilde{L}\}$ of all Lagrangian d-spaces $\tilde{L}$ in the complex symplectic $2d$-space $\tilde{S} = D_{\max}/D_{\min}$. Namely, take the correspondence $T \leftrightarrow L$ as given by the injective surjection which is defined in terms of the natural projection $\Psi : D_{\max} \to \tilde{S}$, according to*

$$\Psi D(T) = \tilde{L}, \quad \text{and} \quad D(T) = \Psi^{-1}\tilde{L}.$$

Hence we conclude that

$$f \in D(T) \quad \text{if and only if} \quad \tilde{f} \in \tilde{L},$$

or that $D(T)$ is precisely the pre-image of $\tilde{L}$ under the natural projection

$$\Psi : D(T)(\subset D_{\max}) \to \tilde{L} \subset \tilde{S}, \quad \text{that is}, \ D(T)/D_{\min} = \tilde{L}.$$

Proof. See [6] for a proof. □

5. The connection between the LC and EM characterizations

In this section we investigate the connection between the EM and LC characterizations discussed in Sections 3 and 4. Firstly, we give the specific structure of the complex symplectic spaces $\tilde{S}$ in terms of real-parameter square-integrable solutions of the differential equation (3.1). Secondly, we describe the self-adjoint domains and the complete Lagrangian subspaces $\tilde{L}$ of $\tilde{S}$. Thirdly, we study the structures of all the strictly separated, mixed and totally coupled complete Lagrangian subspaces.

5.1. Complex symplectic spaces and LC solutions

Consider a symmetric quasi-differential expression M of even order with real-valued coefficients on J, defined by Definition 1; let $\tilde{S} := D_{\max}/D_{\min}$ be the prescribed complex symplectic space. We now investigate the connection between the structure of this space and LC solutions.

Theorem 6. *Let the notation and hypotheses of Theorem 2 hold, and let $\tilde{S} = D_{\max}/D_{\min}$ be the complex vector space defined above. Then:*

(1) $\tilde{S} = \operatorname{span}\{\tilde{u}_1, \dots, \tilde{u}_{m_1},\ \tilde{v}_1, \dots, \tilde{v}_{m_2}\}$.
(2) *$\tilde{S}$ is the complexification of the unique real symplectic space $\mathbb{R}^{2d}$.*
(3) *$\tilde{S}$ is symplectically isomorphic to a complex symplectic space $\mathbb{C}^{2d}$.*
(4) *For the basis $\{\tilde{u}_1, \dots, \tilde{u}_{m_1},\ \tilde{v}_1, \dots, \tilde{v}_{m_2}\}$, let*

$$H = \begin{pmatrix} -U_{m_1\times m_1} & 0 \\ 0 & V_{m_2\times m_2} \end{pmatrix};$$

then H is a skew-Hermitian matrix, and for every $\tilde{f} = (f_1, \dots, f_{m_1}, \dot{f}_1, \dots, \dot{f}_{m_2})$ and $\tilde{g} = (g_1, \dots, g_{m_1}, \dot{g}_1, \dots, \dot{g}_{m_2})$ in $\tilde{S}$,

$$[\tilde{f} : \tilde{g}] = (f_1, \dots, f_{m_1}, \dot{f}_1, \dots, \dot{f}_{m_2}) H (g_1, \dots, g_{m_1}, \dot{g}_1, \dots, \dot{g}_{m_2})^*.$$

Here U is defined by (3.2) and V is defined by (3.3) in Theorem 2.

Proof. By Von Neumann's formula and Theorem 3, we obtain (1); from Theorem 1 in [20] and Lemma 7, we obtain (2); from example 1 in [20], (3) follows. Now, to prove (4): For every $\tilde{f},\ \tilde{g} \in \tilde{S}$,

$$\begin{aligned}
[\tilde{f} : \tilde{g}] &= [f_1 u_1 + \cdots + f_{m_1} u_{m_1} + \dot{f}_1 v_1 + \cdots + \dot{f}_{m_2} v_{m_2} : g_1 u_1 + \cdots + g_{m_1} u_{m_1} + \dot{g}_1 v_1 + \cdots + \dot{g}_{m_2} v_{m_2}] \\
&= (f_1, \dots, f_{m_1}, \dot{f}_1, \dots, \dot{f}_{m_2}) \cdot \begin{pmatrix} [u_1 : u_1] & \cdots & [u_1 : u_{m_1}] & [u_1 : v_1] & \cdots & [u_1 : v_{m_2}] \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ [u_{m_1} : u_1] & \cdots & [u_{m_1} : u_{m_1}] & [u_{m_1} : v_1] & \cdots & [u_{m_1} : v_{m_2}] \\ [v_1 : u_1] & \cdots & [v_1 : u_{m_1}] & [v_1 : v_1] & \cdots & [v_1 : v_{m_2}] \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ [v_{m_2} : u_1] & \cdots & [v_{m_2} : u_{m_1}] & [v_{m_2} : v_1] & \cdots & [v_{m_2} : v_{m_2}] \end{pmatrix} \\
&\quad \times (g_1, \dots, g_{m_1}, \dot{g}_1, \dots, \dot{g}_{m_2})^* \\
&= (f_1, \dots, f_{m_1}, \dot{f}_1, \dots, \dot{f}_{m_2}) \begin{pmatrix} -([u_i, u_j](a)) & 0 \\ 0 & ([v_i, v_j](b)) \end{pmatrix} (g_1, \dots, g_{m_1}, \dot{g}_1, \dots, \dot{g}_{m_2})^* \\
&= (f_1, \dots, f_{m_1}, \dot{f}_1, \dots, \dot{f}_{m_2}) \begin{pmatrix} -U_{m_1\times m_1} & 0 \\ 0 & V_{m_2\times m_2} \end{pmatrix} (g_1, \dots, g_{m_1}, \dot{g}_1, \dots, \dot{g}_{m_2})^* \\
&= (f_1, \dots, f_{m_1}, \dot{f}_1, \dots, \dot{f}_{m_2}) H (g_1, \dots, g_{m_1}, \dot{g}_1, \dots, \dot{g}_{m_2})^*.
\end{aligned}$$

It is easy to prove that H is a skew-Hermitian matrix. □

Theorem 7. *Let the notation and hypotheses of Theorem 2 hold, and let $\tilde{S} = D_{\max}/D_{\min}$ be the complex vector space defined above. Assume that*

$$\tilde{S}_- = \{\tilde{f} \in \tilde{S} : [f, v_1](b) = \cdots = [f, v_{m_2}](b) = 0\},$$
$$\tilde{S}_+ = \{\tilde{f} \in \tilde{S} : [f, u_1](a) = \cdots = [f, u_{m_1}](a) = 0\}.$$

Then

$$\tilde{S}_- = \operatorname{span}\{\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_{m_1}\},$$
$$\tilde{S}_+ = \operatorname{span}\{\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_{m_2}\},$$

and $\tilde{S} = \tilde{S}_- \oplus \tilde{S}_+$.

Proof. Now we prove that $\tilde{S}_- = \text{span}\{\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{m_1}\}$. For every $\tilde{f} \in \tilde{S}_-$, of course $\tilde{f} \in \tilde{S}$, so it can be written as

$$\tilde{f} = a_1\tilde{u}_1 + \cdots + a_m\tilde{u}_{m_1} + \dot{a}_1\tilde{v}_1 + \cdots + \dot{a}_{m_2}\tilde{v}_{m_2},$$

notice that $[f, v_1](b) = \cdots = [f, v_{m_2}](b) = 0$, then $\dot{a}_1 = \cdots = \dot{a}_{m_2} = 0$. Therefore

$$\tilde{S}_- \subseteq \text{span}\{\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{m_1}\}.$$

On the other hand, if $\tilde{f} \in \text{span}\{\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{m_1}, \}$, then $[f, v_j](b) = 0$. Therefore

$$\text{span}\{\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{m_1}\} \subseteq \tilde{S}_-.$$

So $\tilde{S}_- = \text{span}\{\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{m_1}\}$.

Similarly, we can prove $\tilde{S}_+ = \text{span}\{\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_{m_2}\}$.

For every $\tilde{f} = \sum_{i=1}^{m_1} a_i\tilde{u}_i \in \tilde{S}_-$ and $\tilde{g} = \sum_{j=1}^{m_2} b_j\tilde{v}_j \in \tilde{S}_+$, from part (4) in Theorem 6,

$$\begin{aligned}[\tilde{f} : \tilde{g}] &= (a_1, \ldots, a_{m_1}, 0, \ldots, 0)H(0, \ldots, 0, b_1, \ldots, b_{m_2})^* \\ &= (a_1, \ldots, a_{m_1}, 0, \ldots, 0)\begin{pmatrix} -U_{m_1\times m_1} & 0 \\ 0 & V_{m_2\times m_2}\end{pmatrix}(0, \ldots, 0, b_1, \ldots, b_{m_2})^* \\ &= 0,\end{aligned}$$

i.e. $[\tilde{S}_- : \tilde{S}_+] = 0$; then combining with part (1) in Theorem 6, we obtain $\tilde{S} = \tilde{S}_- \oplus \tilde{S}_+$. □

Since the symplectic form induced by the Lagrange bracket [:] is non-degenerate on $\tilde{S}_-$ and $\tilde{S}_+$, we have the following corollary:

Corollary 2. *Each of $\tilde{S}_-$ and $\tilde{S}_+$ is itself a complex symplectic space. Specifically, $\tilde{S}_-$ is symplectically isomorphic to $\mathbb{C}^{m_1}$, with [:] defined by the skew-Hermitian matrix $-U$, and $\tilde{S}_+$ is symplectically isomorphic to $\mathbb{C}^{m_2}$, with [:] defined by skew-Hermitian matrix V.*

Furthermore, let $D_\pm, p_\pm, q_\pm, \Delta_\pm, Ex_\pm$ denote the corresponding symplectic invariants for $\tilde{S}_\pm$; then

$$D_- = m_1, \qquad p_- = q_- = \Delta_- = \frac{m_1}{2}, \qquad Ex_- = 0,$$
$$D_+ = m_2, \qquad p_+ = q_+ = \Delta_+ = \frac{m_2}{2}, \qquad Ex_+ = 0.$$

Proof. From (4) in Theorem 6 and Theorem 1 in [20], the conclusions follow. □

5.2. The representation of complete Lagrangian subspaces

In this subsection, we will give the description of self-adjoint domains $D(T)$ in terms of complex symplectic geometry. Also, the specific form of the complete Lagrangian subspace $\tilde{L} = D(T)/D_{\min}$ is given.

Theorem 8. *Let the notation and hypotheses of Theorem 2 hold. Let $d = d_1 + d_2 - n$. Then d is the deficiency index of (1.1) on (a, b). A linear submanifold $D(T)$ of $D_{\max}$ is the domain of a self-adjoint extension T of $T_{\min}$ if and only if there exist $2d$ vectors $\alpha_i = (a_{i1}, \ldots, a_{im_1}) \in \mathbb{C}^{m_1}$, and $\beta_i = (b_{i1}, \ldots, b_{im_2}) \in \mathbb{C}^{m_2}$, $i = 1, \ldots, d$, such that*

$$\gamma_i = (-a_{i1}, \ldots, -a_{im_1}, b_{i1}, \ldots, b_{im_2}) \in \mathbb{C}^{2d} \text{ are linearly independent}; \tag{5.1}$$
$$[\gamma_i : \gamma_j] = 0; \tag{5.2}$$
$$D(T) = \{y \in D_{\max} : [y : w_i] = 0,\ w_i = \bar{\gamma}_i W,\ i = 1, \ldots, d\}, \tag{5.3}$$

where $\bar{\gamma}_i = (-\bar{a}_{i1}, \ldots, -\bar{a}_{im_1}, \bar{b}_{i1}, \ldots, \bar{b}_{im_2})$, $W = (u_1, \ldots, u_{m_1}, v_1, \ldots, v_{m_2})^T$.

Proof. From Theorem 4, $D(T)$ is the domain of a self-adjoint extension T of $T_{\min}$ if and only if there exists complex matrices $A_{d\times m_1}$ and $B_{d\times m_2}$ such that the conditions (3.9)–(3.11) hold. Denote each row vector of $A_{d\times m_1}$ and $B_{d\times m_2}$ by α_i *and* β_i, $i = 1, \ldots, d$; then there exist such complex matrices $A_{d\times m_1}$ and $B_{d\times m_2}$ if and only if there exist $2d$ vectors $\alpha_i = (a_{i1}, \ldots, a_{im_1}) \in \mathbb{C}^{m_1}$ and $\beta_i = (b_{i1}, \ldots, b_{im_2}) \in \mathbb{C}^{m_2}$, $i = 1, \ldots, d$. Now we prove that conditions (5.1)–(5.3) in this Theorem 8 are equivalent to conditions (3.9)–(3.11) in Theorem 4. First, let $\gamma_i = (-\alpha_i, \beta_i)$; then

$$(-\alpha_i, \beta_i)_{d\times 2d} = (A : B)\begin{pmatrix} -I_{m_1} & 0 \\ 0 & I_{m_2}\end{pmatrix},$$

where I_{m_1} and I_{m_2} are identity matrices.

So $\text{rank}(A : B) = d$ if and only if $\gamma_i \in \mathbb{C}^{2d},\ i = 1, \ldots, d$, are linearly independent.

Second, from $U = (-1)^{k+1}E_{m_1},\ V = (-1)^{k+1}E_{m_2}$, we have

$$AE_{m_1}A^* = BE_{m_2}B^* \Longleftrightarrow AUA^* = BVB^* \Longleftrightarrow (\alpha_i U\alpha_j^*) = (\beta_i V\beta_j^*).$$

Note that

$$\alpha_i U\alpha_j^* = -[\alpha_i : \alpha_j], \qquad \beta_i V\beta_j^* = [\beta_i : \beta_j],$$

and

$$[\gamma_i : \gamma_j] = \gamma_i H\gamma_j^* = [\alpha_i : \alpha_j] + [\beta_i : \beta_j].$$

So

$$AE_{m_1}A^* = BE_{m_2}B^* \quad \text{if and only if} \quad [\gamma_i : \gamma_j] = 0.$$

Third, from the symplectic form [:] in $\tilde{S}$, we have

$$A\begin{pmatrix} [y, u_1](a) \\ \vdots \\ [y, u_{m_1}](a) \end{pmatrix} + B\begin{pmatrix} [y, v_1](b) \\ \vdots \\ [y, v_{m_2}](b) \end{pmatrix} = \begin{pmatrix} [y : -\bar{\alpha}_1\upsilon] \\ \vdots \\ [y : -\bar{\alpha}_d\upsilon] \end{pmatrix} + \begin{pmatrix} [y : \bar{\beta}_1\vartheta] \\ \vdots \\ [y : \bar{\beta}_d\vartheta] \end{pmatrix},$$

where $\upsilon = (u_1, \ldots, u_{m_1})^T,\ \vartheta = (v_1, \ldots, v_{m_2})^T$. Let $w_i = -\bar{\alpha}_i\upsilon + \bar{\beta}_i\vartheta$; then $w_i = \bar{\gamma}_i W$. Obviously, condition (3.11) holds if and only if (5.3) holds. □

Remark 5. In Theorem 8, the description of a self-adjoint domain $D(T)$ is given in terms of a direct sum decomposition $\tilde{S} = \tilde{S}_- \oplus \tilde{S}_+$.

Note that $[\gamma_i : \gamma_j]$ here is the symplectic product in $\mathbb{C}^{2d}$, since $\tilde{S}$ is symplectically isomorphic to $\mathbb{C}^{2d}$, we do not distinguish it from its corresponding element in $\tilde{S}$.

Since there exists a natural one-to-one correspondence between the set $\{T\}$ of all self-adjoint extensions T of $T_{\min}$ and the set $\{\tilde{L}\}$ of all complete Lagrangian subspaces $\tilde{L}$, let us consider the specific form of the complete Lagrangian subspace associated with self-adjoint domains $D(T)$ which are described in Theorem 8 in terms of symplectic geometry.

Theorem 9. *Let M be a symmetric differential expression on J, let the notation and hypotheses of Theorem 2 hold, and assume that a linear submanifold $D(T)$ of $D_{\max}$ is the domain of a self-adjoint extension T of $T_{\min}$; then:*

(1) *$\tilde{L} = D(T)/D_{\min}$ is a complete Lagrangian subspace of $\tilde{S} = D_{\max}/D_{\min}$.*

(2) *$\tilde{L} = \text{span}\{\tilde{w}_1, \ldots, \tilde{w}_d\},\ w_1, \ldots, w_d \in D_{\max}$, where*

$$\tilde{w}_i = \bar{\gamma}_i\tilde{W}, \quad i = 1, \ldots, d. \tag{5.4}$$

Here $\tilde{W} = (\tilde{u}_1, \ldots, \tilde{u}_{m_1}, \tilde{v}_1, \ldots, \tilde{v}_{m_2})^T$, and $\gamma_i,\ i = 1, \ldots, d$, satisfy conditions (5.1) and (5.2).

Proof. We obtain (1) directly from the GKN–EM Theorem. Next we prove (2).

Firstly, we prove that $w_1, \ldots, w_d$ are linearly independent modulo $D_{\min}$. Assume that this does not hold; then there exist constants $c_1, \ldots, c_d$, not all zero, such that

$$\zeta = \sum_{i=1}^{d} c_i w_i \in D_{\min},$$

since $D_{\min} = \{f \in D_{\max} : [f : D_{\max}] = 0\}$; so

$$(0, \ldots, 0) = ([\zeta : u_1], \ldots, [\zeta : u_{m_1}]) = (c_1, \ldots, c_d)(-\bar{a}_{ij})_{d\times m_1}U_{m_1\times m_1}.$$

Notice that U is nonsingular; so

$$(c_1, \ldots, c_d)(-\bar{a}_{ij})_{d\times m_1} = 0.$$

Similarly, we have

$$(c_1, \ldots, c_d)(\bar{b}_{ij})_{d\times m_2} = 0.$$

So

$$(c_1, \ldots, c_d)(\bar{\gamma}_1, \ldots, \bar{\gamma}_d)^T = 0.$$

Note that $c_1, \ldots, c_d$, are not all zero; so $\gamma_i,\ i = 1, \ldots, d$, are linearly dependent, and this contradicts the condition (5.1) in Theorem 8. So $w_1, \ldots, w_d$ are linearly independent modulo $D_{\min}$.

Secondly, we prove that $[w_i : w_j] = 0$, i.e. $w_i \in D(T),\ i, j = 1, \ldots, d$. We have

$$[w_i : w_j] = [\bar{\gamma}_i\tilde{W} : \bar{\gamma}_j\tilde{W}] = \bar{\gamma}_i H \bar{\gamma}_j = [\bar{\gamma}_i : \bar{\gamma}_j],$$

and from (5.2)

$$[\bar{\gamma}_i : \bar{\gamma}_j] = \overline{[\gamma_i : \gamma_j]} = 0 = [\gamma_i : \gamma_j],$$

so $[w_i : w_j] = 0,\ i, j = 1, \ldots, d$, and from (5.3), $w_1, \ldots, w_d \in D(T)$; then from the GKN–EM Theorem, $\tilde{w}_1, \ldots, \tilde{w}_d \in \tilde{L}$. And from Theorem 2 in [20], we know that a complete Lagrangian subspace $\tilde{L}$, $\dim \tilde{L} = \Delta = d$, has the form $\tilde{L} = \operatorname{span}\{\tilde{w}_1, \ldots, \tilde{w}_d\}$. □

From Theorem 9, we get other descriptions of the self-adjoint domains $D(T)$ as follows:

Corollary 3. *Let the notation and hypotheses of Theorem 2 hold, and assume T is a self-adjoint extension of $T_{\min}$; then*

$$D(T) = c_1 w_1 + \cdots + c_d w_d \oplus D_{\min}$$

where $w_i,\ i = 1, \ldots, d$, are given in (5.3), and $c_i \in \mathbb{C}$.

5.3. The classification of complete Lagrangian subspaces

From [6,20] we know that the complete Lagrangian subspaces can be divided into three mutually exclusive classes: strictly separated, totally coupled and another type which we define as mixed. In this subsection, we will give the specific descriptions of all three forms of the corresponding complete Lagrangian subspaces. Note that in this subsection the Balanced Intersection Principle plays an important role: it provides an algebraic criterion for describing and classifying the three different kinds of self-adjoint boundary conditions.

Definition 9. Consider the complex symplectic space

$$S = S_- \oplus S_+ \quad \text{with } [S_- : S_+] = 0,$$

with finite dimension $D \geq 1$. Let $p_\pm, q_\pm, \Delta_\pm, Ex_\pm$ denote the corresponding symplectic invariants for $S_\pm$. Assume $D = 2\Delta$, $Ex = 0$ (so $Ex_- = -Ex_+$).

For each Lagrangian Δ-space $L \subset S$ define the coupling grade of L:

$$\text{grade } L = \Delta_- - \dim L \cap S_- = \Delta_+ - \dim L \cap S_+.$$

Also define the necessary coupling of L:

$$\text{Nec-coupling } L = \Delta - \dim L \cap S_- - \dim L \cap S_+ = 2\text{grade } L + |Ex_\pm|,$$

and note that $|Ex_-| = |Ex_+| = |Ex_\pm|$ since $Ex = 0$.

Definition 10. Consider the complex symplectic space

$$S = S_- \oplus S_+ \quad \text{with } [S_- : S_+] = 0,$$

with finite dimension $D = 2\Delta$ and excess $Ex = 0$ (so $Ex_- = Ex_+$).

A non-zero vector $v \in S$ is separated at the left in the case where $v \in S_-$; $v \in S$ is separated at the right in the case where $v \in S_+$; and v is coupled otherwise. If $S_- = 0$ (or $S_+ = 0$), then no such v is coupled.

For any Lagrangian Δ-space $L \in S$, a basis for L is minimally coupled if it contains exactly Nec-coupling L vectors, each of which is coupled.

A Lagrangian Δ-space $L \in S$ is

strictly separated if Nec-coupling $L = 0$,

or

totally coupled if Nec-coupling $L = \Delta$.

Theorem 10. *Let $\tilde{S} = D_{\max}/D_{\min}$ be the complex vector space as defined in Section 4, having a prescribed direct sum decomposition*

$$\tilde{S} = \tilde{S}_- \oplus \tilde{S}_+ \quad \text{with } [\tilde{S}_- : \tilde{S}_+] = 0.$$

Then for each complete Lagrangian subspace $\tilde{L} \subset \tilde{S}$, the Balanced Intersection Principle holds:

$$0 \le \frac{m_1}{2} - \dim \tilde{L} \bigcap \tilde{S}_- = \frac{m_2}{2} - \dim \tilde{L} \bigcap \tilde{S}_+ \le \min\left\{\frac{m_1}{2}, \frac{m_2}{2}\right\}.$$

Proof. Note that

$$D = 2\Delta, \qquad Ex = 0 \quad \text{and} \quad \Delta_- = \frac{m_1}{2}, \qquad \Delta_+ = \frac{m_2}{2}.$$

Then this theorem is a specific case of Theorem 4 (the Balanced Intersection Principle) in [20]. □

Theorem 11. *Consider the complex symplectic space $\tilde{S} = D_{\max}/D_{\min}$, having a prescribed direct sum decomposition*

$$\tilde{S} = \tilde{S}_- \oplus \tilde{S}_+ \quad \text{with } [\tilde{S}_- : \tilde{S}_+] = 0.$$

Then for each integer $l = 0, 1, 2, \dots, \min\{\frac{m_1}{2}, \frac{m_2}{2}\}$, there exists a complete Lagrangian subspace $\tilde{L}_l$ with

$$\text{grade } \tilde{L}_l = l.$$

Proof. From Lemma 7 and Corollary 2, the symplectic invariants of the complex symplectic spaces $\tilde{S}$ and $\tilde{S}_\pm$ satisfy

$$D = 2\Delta, \qquad Ex = 0 \quad \text{and} \quad Ex_- = Ex_+ = 0, \qquad \Delta = \Delta_- + \Delta_+.$$

Obviously this is a special case of Theorem 4 (the Balanced Intersection Principle) in [20]. □

Note that the Balanced Intersection Principle is one of the major results of the Everitt–Markus monograph [6], and it provides an algebraic criterion for describing and classifying the different kinds of self-adjoint boundary conditions.

Theorem 12. *Consider the complex symplectic space $\tilde{S} = \tilde{S}_- \oplus \tilde{S}_+$; then the complete Lagrangian subspace $\tilde{L} \subset \tilde{S}$ is strictly separated if and only if any one of the following conditions holds:*

(1) grade $\tilde{L} = 0$.

(2) *There exists a basis $\tilde{w}_i = \bar{\gamma}_i \tilde{W},\ i = 1, \dots, d$, for $\tilde{L}$ such that*

$$\gamma_i = \begin{cases} (-a_{i1}, \dots, -a_{i,m_1}, 0, \dots, 0), & 1 \le i \le \dfrac{m_1}{2}, \\ (0, \dots, 0, b_{i1}, \dots, b_{i,m_2}), & \dfrac{m_1}{2} + 1 \le i \le d, \end{cases}$$

where $\gamma_i,\ i = 1, \dots, d$, are linearly independent and satisfy (5.1) and (5.2).

Proof. Assume a complete Lagrangian subspace $\tilde{L}$ is strictly separated, and note that $Ex_\pm = 0$; so

$$0 = \text{Nec-coupling } \tilde{L} = 2\text{grade } \tilde{L},$$

which is necessary and sufficient for condition (1). Hence we need only show that conditions (1), (2) are logically equivalent. If the γ_i satisfy (2), then it is clear that

$$\tilde{w}_i \in \tilde{S_-}, \quad i = 1, \dots, \frac{m_1}{2}, \qquad \tilde{w}_i \in \tilde{S_+}, \quad i = \frac{m_1}{2} + 1, \dots, d,$$

and hence

$$\tilde{L} \cap \tilde{S}_- = \frac{m_1}{2}, \qquad \tilde{L} \cap \tilde{S}_+ = \frac{m_2}{2},$$

and so grade $\tilde{L} = 0$.

Conversely, if $\tilde{L}$ is 0-grade then

$$\dim \tilde{L} = d, \qquad \dim \tilde{L} \cap \tilde{S}_- = \frac{m_1}{2}, \qquad \dim \tilde{L} \cap \tilde{S}_+ = \frac{m_2}{2}.$$

So there exist $a_{i,j} \in \mathbb{C},\ i = 1, \dots, \frac{m_1}{2},\ j = 1, \dots, m_1$, and $b_{k,l} \in \mathbb{C},\ k = 1, \dots, \frac{m_2}{2},\ l = 1, \dots, m_2$, and

$$\tilde{L} = \text{span}\{-\bar{a}_{11}\tilde{u}_1 - \dots - \bar{a}_{1,m_1}\tilde{u}_{m_1}, \dots, -\bar{a}_{\frac{m_1}{2},1}\tilde{u}_1 - \dots - \bar{a}_{\frac{m_1}{2},m_1}\tilde{u}_{m_1}$$
$$\bar{b}_{\frac{m_1}{2}+1,1}\tilde{v}_1 + \dots + \bar{b}_{\frac{m_1}{2}+1,m_2}\tilde{v}_{m_2}, \dots, \bar{b}_{d,1}\tilde{v}_1 + \dots + \bar{b}_{d,m_2}\tilde{v}_{m_2}\}.$$

Define

$$\gamma_i = (-a_{i1}, \dots, -a_{im_1}, b_{i1}, \dots, b_{im_1}) \in \mathbb{C}^{2d}, \quad i = 1, \dots, d$$

and let $\tilde{w}_i = \bar{\gamma}_i \tilde{W},\ i = 1, \dots, d$, be the basis which spans $\tilde{L}$ as above. Then $\gamma_i = \begin{cases} (-a_{i1}, \dots, -a_{i,m_1}, 0, \dots, 0), & 1 \le i \le \frac{m_1}{2}, \\ (0, \dots, 0, b_{i1}, \dots, b_{i,m_2}), & \frac{m_1}{2} + 1 \le i \le d. \end{cases}$

□

Theorem 13. *Consider the complex symplectic space*

$$\tilde{S} = \tilde{S}_- \oplus \tilde{S}_+ \quad \text{with } [\tilde{S}_- : \tilde{S}_+] = 0.$$

Then a complete Lagrangian subspace $\tilde{L} \subset \tilde{S}$ is totally coupled if and only if either one of the following conditions holds:

(1) $m_1 = m_2 = m$ *and* grade $\tilde{L} = \frac{m}{2}$.

(2) *Every basis $\tilde{w}_i = \bar{\gamma}_i \tilde{W},\ i = 1, \dots, d$, of $\tilde{L}$ is of the form*

$$\gamma_i = (-a_{i1}, \dots, -a_{i,d}, b_{i1}, \dots, b_{i,d}) \quad 1 \le i \le d,$$

where the $\gamma_i,\ i = 1, \dots, d$, are linearly independent and satisfy (5.1) and (5.2).

Proof. This is similar to the proof of Theorem 12 and hence omitted. □

Remark 6. Note that only when $m_1 = m_2$, i.e. $d_1 = d_2$, is it possible that there exists a totally coupled complete Lagrangian subspace.

Definition 11. We say that a complete Lagrangian subspace $\tilde{L}$ is mixed if it is neither strictly separated nor totally coupled.

Theorem 14. *Consider the complex symplectic space*

$$\tilde{S} = \tilde{S}_- \oplus \tilde{S}_+ \quad \text{with } [\tilde{S}_- : \tilde{S}_+] = 0.$$

If $m_1 \neq m_2$, then a complete Lagrangian subspace $\tilde{L} \subset \tilde{S}$ is mixed if and only if either one of the following two conditions holds:

(1) grade $\tilde{L} = k,\ 0 < k \le \min\{\frac{m_1}{2}, \frac{m_2}{2}\}$.

(2) *There exists a basis $\tilde{w}_i = \bar{\gamma}_i \tilde{W},\ i = 1, \dots, d$, for $\tilde{L}$ such that*

$$\gamma_i = \begin{cases} (-a_{i1}, \dots, -a_{i,m_1}, 0, \dots, 0), & 1 \le i \le \dfrac{m_1}{2} - k, \\ (-a_{i1}, \dots, -a_{i,m_1}, b_{i1}, \dots, b_{i,m_2}), & \dfrac{m_1}{2} - k + 1 \le i \le \dfrac{m_1}{2} + k, \\ (0, \dots, 0, b_{i1}, \dots, b_{i,m_2}), & \dfrac{m_1}{2} + k + 1 \le i \le d. \end{cases}$$

And if $m_1 = m_2 = m$, we should only change condition (1) *to* $0 < k < \frac{m}{2}$.
Here the $\gamma_i,\ i = 1, \dots, d$, satisfy (5.1), (5.2). And for $\frac{m_1}{2} - k + 1 \le i \le \frac{m_1}{2} + k$,

$$(-a_{i1}, \dots, -a_{i,m_1}) \neq (0, \dots, 0), \qquad (b_{i1}, \dots, b_{i,m_2}) \neq (0, \dots, 0).$$

Proof. This is similar to the proof of Theorem 12. □

Remark 7. Assume $\tilde{L}$ is mixed, and grade$\tilde{L} = k$; then there exists a basis for $\tilde{L}$ containing:

exactly $2k$ vectors, each of which is coupled;

exactly $\dfrac{m_1}{2} - k$ vectors, each of which is separated at the left;

exactly $\dfrac{m_2}{2} - k$ vectors, each of which is separated at the right.

Note that because of the Balanced Intersection Principle, the number of vectors which are coupled in the basis is even; specifically it is $2k$.

Remark 8. From the GKN–EM Theorem and the equivalence of Theorems 4 and 8, we may conclude that when a complete Lagrangian subspace $\tilde{L}$ is strictly separated, totally coupled or mixed, the self-adjoint boundary condition is correspondingly separated, coupled or mixed as defined in [21]. Thus the Everitt–Markus definitions of strictly separated, totally coupled, and mixed self-adjoint boundary conditions, defined in terms of subspaces of a symplectic space, correspond exactly to the Hao–Sun–Wang–Zettl definitions of separated, coupled, and mixed self-adjoint boundary conditions defined in terms of LC solutions and matrices A, B as in Theorem 2.

6. The EM characterization and eigenvalues

Let M be a symmetric even order differential expression with real-valued coefficients on J and assume that T is a self-adjoint extension of $T_{\min}$. Then a necessary symplectic geometry condition for a real number λ to be an eigenvalue of a self-adjoint realization of M is given by the next theorem.

Theorem 15. *Let the notation and hypotheses of Theorem 2 hold. If $\lambda_0 \in \mathbb{R}$ is an eigenvalue of a self-adjoint extension T of $T_{\min}$, and y_0 is an eigenfunction of λ_0, then y_0 satisfies the differential equation $My = \lambda_0 wy$ on $J = (a, b)$, and further satisfies*

$$[y_0 : w_i] = 0, \quad i = 1, \dots, d,$$

where

$$w_i = \bar{\gamma}_i W, \quad i = 1, \dots, d.$$

Here the $\bar{\gamma}_i = (-\bar{a}_{i1}, \dots, -\bar{a}_{im_1}, \bar{b}_{i1}, \dots, \bar{b}_{im_2})$ satisfy the conditions (5.1) and (5.2); $W = (u_1, \dots, u_{m_1}, v_1, \dots, v_{m_2})^T$.

Proof. If y_0 is an eigenfunction of T, then y_0 is a solution of the differential equation $My = \lambda_0 wy$ and $y_0 \in D(T)$. Combining this with Theorem 8, we get

$$[y_0 : w_i] = 0 \quad i = 1, \dots, d. \quad \square$$

Acknowledgments

The authors thank the referee for his/her careful reading of the manuscript and for the constructive comments which have led to improvement of the presentation of the paper.

The work of the first and second authors was supported by the National Natural Science Foundation of China (Grant No. 10861008 and Grant No. 11161030). The first author was also supported by the "211 project" innovative talents training program of Inner Mongolia University. The third author was supported by the Ky and Yu-fen Fan US–China Exchange Fund through the American Mathematical Society. This made his visit to the University of Inner Mongolia possible, where some of this work was done. The third author thanks the School of Mathematical Sciences of Inner Mongolia University for its extraordinary hospitality.

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