J. Math. Anal. Appl. 400 (2013) 452–459

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

On the linear combinations of harmonic univalent mappings

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ARTICLE INFO

Article history:
Received 7 May 2012
Available online 7 November 2012
Submitted by S. Ruscheweyh

Keywords:
Harmonic univalent mappings
Linear combination
Shear construction

ABSTRACT

In this paper, we derive several sufficient conditions of the linear combinations of harmonic univalent mappings to be univalent and convex in the direction of the real axis. Furthermore, some illustrative examples and imagine domains of the linear combinations satisfying the desired conditions are enumerated.

1. Introduction and preliminaries

A complex-valued harmonic mapping f in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ has a canonical decomposition $f = h + \overline{g}$, where h and g are analytic functions in $\mathbb{D}$. We call h the analytic part and g the co-analytic part of f, respectively. Throughout this paper, we will discuss harmonic mappings that are univalent and sense-preserving in $\mathbb{D}$. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathbb{D}$ is that $|g'| < |h'|$, or equivalently if $h'(z) \neq 0$ and the dilatation $\omega = g'/h'$ has the property $|\omega| < 1$ in $\mathbb{D}$ (see [1,2]). For some recent investigations on planar harmonic mappings, see (for example) the earlier works [3–34] and the references cited therein.

Denote by $\mathcal{S}_{\mathcal{H}}$ the class of functions f of the form $f = h + \overline{g}$ which are harmonic univalent and sense-preserving in $\mathbb{D}$, normalized so that $f(0) = f_z(0) - 1 = 0$. Such functions can be written as

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}.$$

The classical family $\mathcal{S}$ of analytic univalent and normalized functions in $\mathbb{D}$ is a subclass of $\mathcal{S}_{\mathcal{H}}$ with $g(z) \equiv 0$. For convenience, we denote

$$\mathcal{S}_{\mathcal{H}}^0 = \{f \in \mathcal{S}_{\mathcal{H}} : b_1 = f_{\overline{z}}(0) = 0\}.$$

Clearly, we have the relationship $\mathcal{S} \subset \mathcal{S}_{\mathcal{H}}^0 \subset \mathcal{S}_{\mathcal{H}}$.

The shear construction is also essential to the present work as it allows one to study harmonic mappings through their related analytic functions (see [35–38]). A domain $\Omega \in \mathbb{C}$ is said to be convex in the direction of $e^{i\beta}$, if for all $a \in \mathbb{C}$, the set $\Omega \cap \{a + te^{i\beta} : t \in \mathbb{R}\}$ is either connected or empty. Specifically, a domain is convex in the direction of the imaginary (or real) axis if all lines parallel to the imaginary (or real) axis have a connected intersection with the domain. The shear

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doi:10.1016/j.jmaa.2012.09.011

construction produces a harmonic univalent function which maps $\mathbb{D}$ onto a region convex in the direction of the real axis, it relies on the following result due to Clunie and Sheil-Small [1].

Theorem A. *A harmonic function $f = h + \overline{g}$ locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of the real axis if and only if $h - g$ is a conformal univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of the real axis.*

The following result determines whether a function f maps $\mathbb{D}$ onto a domain convex in the direction of the imaginary axis.

Theorem B (*See [37]*)**.** *Suppose f is analytic and non-constant in $\mathbb{D}$. Then*

$$\Re\left(\left(1-z^2\right)f'(z)\right) \geqq 0 \quad (z \in \mathbb{D})$$

if and only if

(1) *f is univalent in $\mathbb{D}$;*
(2) *f is convex in the direction of the imaginary axis;*
(3) *there exists two points z_n' and z_n'' converging to $z = 1$ and $z = -1$, respectively, such that*

$$\begin{aligned} \lim_{n\to\infty} \Re\left(f(z_n')\right) &= \sup_{|z|<1} \Re\left(f(z)\right), \\ \lim_{n\to\infty} \Re\left(f(z_n'')\right) &= \inf_{|z|<1} \Re(f(z)). \end{aligned} \tag{1}$$

A common way to try to construct new functions with a given property is to take the linear combination of two functions with that property. Let $f_1 = h_1 + \overline{g_1}$ and $f_2 = h_2 + \overline{g_2}$ be two harmonic univalent mappings in $\mathbb{D}$, we construct a new harmonic mapping

$$f_3 = tf_1 + (1-t)f_2 = [th_1 + (1-t)h_2] + [t\overline{g_1} + (1-t)\overline{g_2}] = h_3 + \overline{g_3} \tag{2}$$

with the dilatation $\omega_3 = g_3'/h_3'$.

In a recent monograph, by using Theorem B, Dorff [39, p. 242] proved the following sufficient condition for the linear combination $f_3 = tf_1 + (1-t)f_2$ to be univalent and convex in the direction of the imaginary axis. Moreover, one can find in [40] regarding the applications of this result.

Theorem C. *Let $f_1 = h_1 + \overline{g_1}$ and $f_2 = h_2 + \overline{g_2}$ be univalent harmonic mappings convex in the direction of the imaginary axis and $\omega_1 = \omega_2$. If f_1 and f_2 satisfy the conditions given by Eq. (1), then $f_3 = tf_1 + (1-t)f_2$ $(0 \leqq t \leqq 1)$ is univalent and convex in the direction of the imaginary axis.*

The following two lemmas are required in the proof of our main results, which are due to Pommenrenke [41] and Clunie and Sheil-Small [1], respectively.

Lemma 1. *Let f be an analytic function in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) \neq 0$. Suppose also that*

$$\varphi(z) = \frac{z}{(1+ze^{i\theta})(1+ze^{-i\theta})} \quad (\theta \in \mathbb{R}). \tag{3}$$

If

$$\Re\left(\frac{zf'(z)}{\varphi(z)}\right) > 0 \quad (z \in \mathbb{D}),$$

then f is convex in the direction of the real axis.

Lemma 2. *Let $\Omega \subset \mathbb{C}$ be a domain convex in the direction of the real axis. Also let p be a real-valued continuous function in Ω. Then the mapping $\omega \mapsto \omega + p(\omega)$ is univalent in Ω if and only if it is locally univalent. If it is univalent, then its range is convex in the direction of the real axis.*

In the present paper, we aim at deriving several sufficient conditions on f_1 and f_2 for the linear combination $f_3 = tf_1 + (1-t)f_2$ to be univalent and convex in the direction of the real axis. Some examples are also presented to demonstrate our main results.

2. Main results

We begin by stating the following result.

Theorem 1. *Let $f_j = h_j + \overline{g_j} \in \mathcal{S}_{\mathcal{H}}$ $(j = 1, 2)$ with $\omega_1 = \omega_2$. Suppose also that $F_j = h_j - g_j$ $(j = 1, 2)$ satisfy the conditions $\Re\left(\frac{zF_j'(z)}{\varphi(z)}\right) > 0$ for all $z \in \mathbb{D}$, where φ is given by Eq. (3). Then $f_3 = tf_1 + (1-t)f_2$ $(0 \leqq t \leqq 1)$ is univalent and convex in the direction of the real axis.*

Proof. By noting that $g_1' = \omega_1 h_1'$ and $g_2' = \omega_2 h_2' = \omega_1 h_2'$, we have

$$\omega_3 = \frac{tg_1' + (1-t)g_2'}{th_1' + (1-t)h_2'} = \frac{t\omega_1 h_1' + (1-t)\omega_1 h_2'}{th_1' + (1-t)h_2'} = \omega_1,$$

which implies that f_3 is locally univalent.

Next, we show that f_3 is convex in the direction of the real axis. Since $f_1, f_2 \in \mathcal{S}_{\mathcal{H}}$, then $F_j = h_j - g_j$ $(j = 1, 2)$ are analytic functions in $\mathbb{D}$. By virtue of

$$\Re\left(\frac{z(h_j' - g_j')}{\varphi(z)}\right) > 0 \quad (z \in \mathbb{D};\ j = 1, 2),$$

we know that

$$\begin{aligned}\Re\left(\frac{z(h_3' - g_3')}{\varphi(z)}\right) &= \Re\left(\frac{z}{\varphi(z)}\left[t\left(h_1' - g_1'\right) + (1-t)\left(h_2' - g_2'\right)\right]\right)\\ &= t\Re\left(\frac{z}{\varphi(z)}\left(h_1' - g_1'\right)\right) + (1-t)\Re\left(\frac{z}{\varphi(z)}\left(h_2' - g_2'\right)\right) > 0.\end{aligned}$$

Hence, by Lemma 1, we know that $h_3 - g_3$ is convex in the direction of the real axis. Moreover, by Theorem A, we deduce that f_3 is univalent and convex in the direction of the real axis. □

Next, we generalize Theorem 1 as follows.

Corollary 1. *Let $f_j = h_j + \overline{g_j}$ $(j = 1, 2, \ldots, n)$ be harmonic univalent mappings in $\mathbb{D}$ with $\omega_1 = \omega_2 = \cdots = \omega_n$. Suppose also that $F_j(z) = h_j - g_j$ $(j = 1, 2, \ldots, n)$ satisfy the conditions $\Re\left(\frac{zF_j'(z)}{\varphi(z)}\right) > 0$ for all $z \in \mathbb{D}$, where φ is given by Eq. (3). Then $F = t_1 f_1 + \cdots + t_n f_n$ is univalent and convex in the direction of the real axis, where $0 \leqq t_j \leqq 1$ $(j = 1, 2, \ldots, n)$ and $t_1 + t_2 + \cdots + t_n = 1$.*

Theorem 2. *Let $f_j = h_j + \overline{g_j} \in \mathcal{S}_{\mathcal{H}}$ $(j = 1, 2)$ be harmonic univalent mappings convex in the direction of the real axis. Suppose also that $\Re\left((1 - \omega_1\overline{\omega_2})h_1'\overline{h_2'}\right) \geqq 0$. Then $f_3 = tf_1 + (1-t)f_2 \in \mathcal{S}_{\mathcal{H}}$ $(0 \leqq t \leqq 1)$ is convex in the direction of the real axis.*

Proof. For $g_1' = \omega_1 h_1'$ and $g_2' = \omega_2 h_2'$ satisfy the conditions $|\omega_j| < 1$ $(j = 1, 2)$, we have

$$|\omega_3| = \left|\frac{tg_1' + (1-t)g_2'}{th_1' + (1-t)h_2'}\right| = \frac{|t\omega_1 h_1' + (1-t)\omega_2 h_2'|}{|th_1' + (1-t)h_2'|}. \tag{4}$$

By assumption, we know that

$$\begin{aligned}|th_1' + (1-t)h_2'|^2 - |t\omega_1 h_1' + (1-t)\omega_2 h_2'|^2 &= (th_1' + (1-t)h_2')\overline{(th_1' + (1-t)h_2')}\\ &\quad - (t\omega_1 h_1'(z) + (1-t)\omega_2 h_2')\overline{(t\omega_1 h_1' + (1-t)\omega_2 h_2')}\\ &= t^2(1 - |\omega_1|^2)|h_1'|^2 + (1-t)^2(1 - |\omega_2|^2)|h_2'|^2\\ &\quad + 2t(1-t)\Re((1 - \omega_1\overline{\omega_2})h_1'\overline{h_2'}) > 0,\end{aligned}$$

hence $|\omega_3| < 1$, which implies that f_3 is locally univalent.

Now, we show that $f_3 \in \mathcal{S}_{\mathcal{H}}$. For

$$f_1(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n \in \mathcal{S}_{\mathcal{H}}$$

and

$$f_2(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n \overline{z}^n \in \mathcal{S}_{\mathcal{H}},$$

we get

$$\begin{aligned} f_3(z) &= tf_1(z) + (1-t)f_2(z) \\ &= t\left(z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n\right) + (1-t)\left(z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n \overline{z}^n\right) \\ &= z + \sum_{n=2}^{\infty} [ta_n + (1-t)A_n] z^n + \sum_{n=1}^{\infty} [tb_n + (1-t)B_n] \overline{z}^n \in \mathcal{S}_{\mathcal{H}}. \end{aligned}$$

By Theorem A, we know that $F_j = h_j - g_j$ $(j = 1, 2)$ is univalent in $\mathbb{D}$ and that $\Omega_j = F_j(\mathbb{D})$ are domains convex in the direction of the real axis. Then $f_j = F_j + 2\Re(g_j)$ and

$$f_j\left[F_j^{-1}(w)\right] = w + 2\Re\left(g_j(F_j^{-1}(w))\right) = w + q_j(w) \quad (j = 1, 2),$$

where $q_j(w)$ $(j = 1, 2)$ are real-valued continuous functions. Thus, we know that

$$\begin{aligned} f_3\left[F_3^{-1}(w)\right] &= tf_1\left[F_1^{-1}(w)\right] + (1-t)f_2\left[F_2^{-1}(w)\right] \\ &= t[w + q_1(w)] + (1-t)[w + q_2(w)] \\ &= w + [tq_1(w) + (1-t)q_2(w)] \\ &= w + q_3(w) \end{aligned}$$

is univalent in Ω. Moreover, by Lemma 2, we conclude that f_3 is univalent in Ω and its range is a domain convex in the direction of the real axis. □

Theorem 3. *Let $f_j = h_j + \overline{g_j} \in \mathcal{S}_{\mathcal{H}}$ with $h_j + g_j = \frac{z}{1-z}$ for $j = 1, 2$. Then $f_3 = tf_1 + (1-t)f_2$ $(0 \leqq t \leqq 1)$ is univalent and convex in the direction of the real axis.*

Proof. Since $h_j + g_j = \frac{z}{1-z}$ and $g_j' = \omega_j h_j'$ for $j = 1, 2$, we get

$$h_j' = \frac{1}{(1+\omega_j)(1-z)^2} \quad (j = 1, 2).$$

It follows that

$$|\omega_3| = \left|\frac{tg_1' + (1-t)g_2'}{th_1' + (1-t)h_2'}\right| = \left|\frac{t\omega_1 h_1' + (1-t)\omega_2 h_2'}{th_1' + (1-t)h_2'}\right| = \frac{|t\omega_1 + (1-t)\omega_2 + \omega_1\omega_2|}{|1 + (1-t)\omega_1 + t\omega_2|}. \tag{5}$$

Now, we show that $|\omega_3| < 1$. Let

$$\omega_j := \rho_j(\cos\theta_j + i\sin\theta_j) \quad (0 \leqq \rho_j < 1;\ j = 1, 2).$$

Suppose also that

$$\begin{aligned} \phi(t) &= |1 + (1-t)\omega_1 + t\omega_2|^2 - |t\omega_1 + (1-t)\omega_2 + \omega_1\omega_2|^2 \\ &= \left|[1 + (1-t)\rho_1\cos\theta_1 + t\rho_2\cos\theta_2] + i[(1-t)\rho_1\sin\theta_1 + t\rho_2\sin\theta_2]\right|^2 - \big|[t\rho_1\cos\theta_1 + (1-t)\rho_2\cos\theta_2 \\ &\quad + \rho_1\rho_2\cos(\theta_1+\theta_2)] + i[t\rho_1\sin\theta_1 + (1-t)\rho_2\sin\theta_2 + \rho_1\rho_2\sin(\theta_1+\theta_2)]\big|^2 \\ &= [1 + (1-t)\rho_1\cos\theta_1 + t\rho_2\cos\theta_2]^2 + [(1-t)\rho_1\sin\theta_1 + t\rho_2\sin\theta_2]^2 \\ &\quad - \{[t\rho_1\cos\theta_1 + (1-t)\rho_2\cos\theta_2 + \rho_1\rho_2\cos(\theta_1+\theta_2)]^2 \\ &\quad + [t\rho_1\sin\theta_1 + (1-t)\rho_2\sin\theta_2 + \rho_1\rho_2\sin(\theta_1+\theta_2)]^2\} \\ &= [1 + (1-t)^2\rho_1^2 + t^2\rho_2^2 + 2t(1-t)\rho_1\rho_2\cos(\theta_1-\theta_2) + 2(1-t)\rho_1\cos\theta_1 + 2t\rho_2\cos\theta_2] \\ &\quad - [t^2\rho_1^2 + (1-t)^2\rho_2^2 + \rho_1^2\rho_2^2 + 2t(1-t)\rho_1\rho_2\cos(\theta_1-\theta_2) + 2(1-t)\rho_1\rho_2^2\cos\theta_1 + 2t\rho_1^2\rho_2\cos\theta_2] \\ &= [2\rho_2\cos\theta_2(1-\rho_1^2) - 2\rho_1\cos\theta_1(1-\rho_2^2) + 2(\rho_2^2-\rho_1^2)]t + (1-\rho_2^2)(\rho_1^2 + 2\rho_1\cos\theta_1 + 1). \end{aligned}$$

Then, we know that $\phi(t)$ is a continuous and monotone function of t in the interval $[0, 1]$.

Moreover, we observe that

$$\phi(0) = (1-\rho_2^2)(\rho_1^2 + 2\rho_1\cos\theta_1 + 1) = (1-\rho_2^2)[(\rho_1 + \cos\theta_1)^2 + \sin^2\theta_1] > 0,$$

and

$$\phi(1) = (1-\rho_1^2)[(\rho_2 + \cos\theta_2)^2 + \sin^2\theta_2] > 0,$$

which implies that $\phi(t) > 0$ for all $t \in [0, 1]$. It follows that $|\omega_3| < 1$, and f_3 is locally univalent.

In what follows, we prove that f_3 is convex in the direction of the real axis. Note that

$$h_j' - g_j' = (h_j' + g_j')\left(\frac{h_j' - g_j'}{h_j' + g_j'}\right) = (h_j' + g_j')\left(\frac{1-\omega_j}{1+\omega_j}\right) = \frac{p_j}{(1-z)^2} \quad (j = 1, 2),$$

where $p_j = \frac{1-\omega_j}{1+\omega_j}$ $(j = 1, 2)$ satisfy the conditions $\Re(p_j) > 0$. Thus, by setting

$$\varphi(z) := \frac{z}{(1-z)^2},$$

we have

$$\begin{aligned} \Re\left(\frac{z(h_3' - g_3')}{\varphi(z)}\right) &= \Re\left(\frac{z}{\varphi(z)}[t(h_1' - g_1') + (1-t)(h_2' - g_2')]\right) \\ &= t\Re\left((1-z)^2(h_1' - g_1')\right) + (1-t)\Re\left((1-z)^2(h_2' - g_2')\right) \\ &= t\Re(p_1) + (1-t)\Re(p_2) > 0. \end{aligned}$$

Therefore, by Lemma 1, we know that $h_3 - g_3$ is convex in the direction of the real axis. Furthermore, by Theorem A, we deduce that f_3 is univalent and convex in the direction of the real axis. The proof of Theorem 3 is thus completed. □

We observe that $f = h + \overline{g}$ is an asymmetric vertical strip mapping if

$$h + g = \frac{1}{2i\sin\theta}\log\left(\frac{1+ze^{i\theta}}{1+ze^{-i\theta}}\right) \quad (0 < \theta < \pi).$$

Thus, Theorem 3 can be stated in terms of the asymmetric vertical strip mappings instead of the right half-plane mappings.

Corollary 2. *Let $f_j = h_j + \overline{g_j} \in \mathcal{S}_{\mathcal{H}}$ $(j = 1, 2)$ with*

$$h_j + g_j = \frac{1}{2i\sin\theta}\log\left(\frac{1+ze^{i\theta}}{1+ze^{-i\theta}}\right) \quad (j = 1, 2;\ 0 < \theta < \pi).$$

Then $f_3 = tf_1 + (1-t)f_2$ $(0 \leqq t \leqq 1)$ is univalent and convex in the direction of the real axis.

3. Two examples

In this section, we give two examples to illuminate our main results.

Example 1. *Consider the functions*

$$f_1 = z - \frac{1}{2}\overline{z}^2, \qquad f_2 = z + \frac{1}{3}\overline{z}^3,$$

and

$$f_3 = tf_1 + (1-t)f_2.$$

Obviously, $f_1, f_2 \in \mathcal{S}_{\mathcal{H}},\ \omega_1 = -z,\ \omega_2 = z^2$ and $\omega_3 = -tz + (1-t)z^2$. Then we have

$$|\omega_3| \leqq t|z| + (1-t)|z|^2 < t|z| + (1-t)|z| < |z| < 1,$$

and

$$\Re\left((1 - \omega_1\overline{\omega_2})h_1'\overline{h_2'}\right) = \Re\left(1 + |z|^2\overline{z}\right) \geqq 0,$$

which are satisfied by the conditions of Theorem 2. So f_3 is univalent and convex in the direction of the real axis. The images of $\mathbb{D}$ under f_1 and f_2 are shown in Figs. 1 and *2, respectively, the image of $\mathbb{D}$ under f_3 with $t = 1/3$ is presented in Fig. 3.*

Example 2. *Let $f_1 = h_1 + \overline{g_1}$, where $h_1 + g_1 = \frac{z}{1-z}$ and $\omega_1 = z$. Then we have*

$$h_1 = \frac{1}{4}\log\frac{1+z}{1-z} + \frac{1}{2}\frac{z}{1-z},$$

and

$$g_1 = -\frac{1}{4}\log\frac{1+z}{1-z} + \frac{1}{2}\frac{z}{1-z}.$$

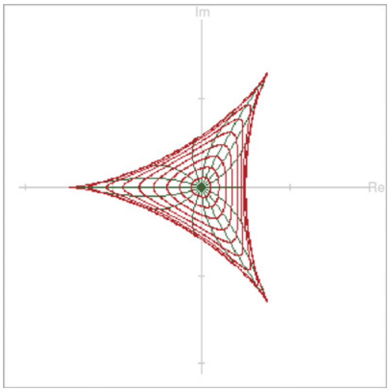

Fig. 1. Image of $\mathbb{D}$ under $f_1 = z - \frac{1}{2}\overline{z}^2$.

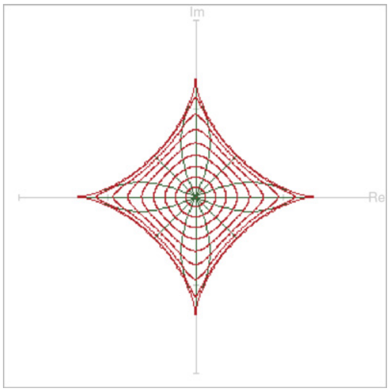

Fig. 2. Image of $\mathbb{D}$ under $f_2 = z + \frac{1}{3}\overline{z}^3$.

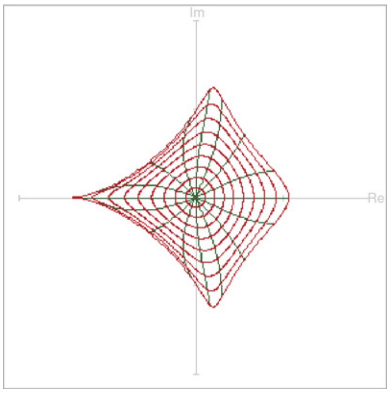

Fig. 3. Image of $\mathbb{D}$ under $f_3 = \frac{1}{3}f_1 + \frac{2}{3}f_2$.

Suppose also that $f_2 = h_2 + \overline{g_2}$, *where* $h_2 + g_2 = \frac{z}{1-z}$ *and* $\omega_2 = -z^2$. *Then we get*

$$h_2 = \frac{1}{8}\log\frac{1+z}{1-z} + \frac{\frac{3}{4}z - \frac{1}{2}z^2}{(1-z)^2},$$

and

$$g_2 = -\frac{1}{8}\log\frac{1+z}{1-z} + \frac{\frac{1}{4}z - \frac{1}{2}z^2}{(1-z)^2}.$$

Since f_1 *and* f_2 *satisfy the conditions of Theorem 3 (*Figs. 4 and 5*), we know that* $f_3 = tf_1 + (1-t)f_2$ $(0 \leqq t \leqq 1)$ *is convex in the direction of the real axis, the image of* $\mathbb{D}$ *under* f_3 *with* $t = 1/2$ *is shown in Fig. 6.*

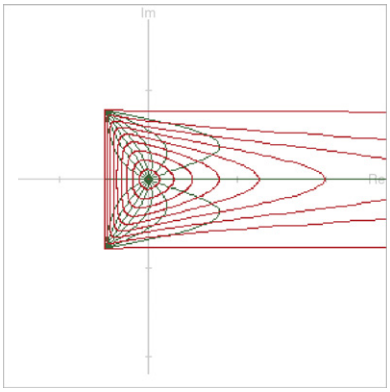

Fig. 4. Image of $\mathbb{D}$ under f_1.

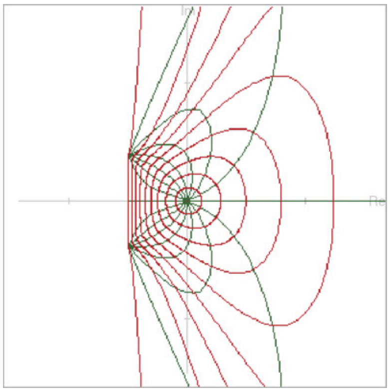

Fig. 5. Image of $\mathbb{D}$ under f_2.

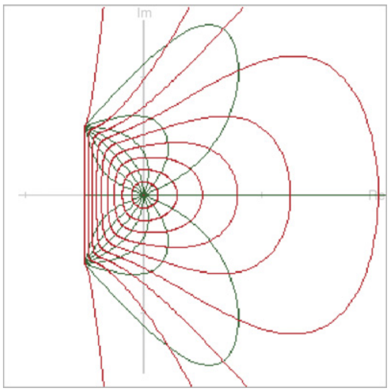

Fig. 6. Image of $\mathbb{D}$ under $f_3 = \frac{1}{2}f_1 + \frac{1}{2}f_2$.

Acknowledgments

The present investigation was supported by the *National Natural Science Foundation* under Grant 11101053, the *Key Project of Chinese Ministry of Education* under Grant 211118, the *Excellent Youth Foundation of Educational Committee of Hunan Province* under Grant 10B002, the *Key Project of Natural Science Foundation of Educational Committee of Henan Province* under Grant 12A110002, and the *Key Courses Construction of Honghe University* under Grant ZDKC1003 of the People's Republic of China. The authors would like to thank Mr. Yu-Dong Wu for his stimulating discussions on the proof of Theorem 3 of this paper; they are also grateful to the referees for their valuable comments and suggestions which essentially improved the quality of the paper.

References

[1] J. Clunie, T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. AI 9 (1984) 3–25.
[2] H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, Bull. Amer. Math. Soc. 42 (1936) 689–692.
[3] Z. Abdulhadi, Y. Abu-Muhanna, Landau's theorem for biharmonic mappings, J. Math. Anal. Appl. 338 (2008) 705–709.
[4] O.P. Ahuja, Use of theory of conformal mappings in harmonic univalent mappings with directional convexity, Bull. Malays. Math. Sci. Soc. 35 (2) (2012) 775–784.
[5] D. Bshouty, S.S. Joshi, S.B. Joshi, On close-to-convex harmonic mappings, Complex Var. Elliptic Equ. (in press).
[6] D. Bshouty, A. Lyzzaik, Close-to-convexity criteria for planar harmonic mappings, Complex Anal. Oper. Theory 5 (2011) 767–774.
[7] D. Bshouty, A. Lyzzaik, A. Weitsman, On the boundary behaviour of univalent harmonic mappings, Ann. Acad. Sci. Fenn. Math. 37 (2012) 135–147.
[8] H.-H. Chen, Some new results on planar harmonic mappings, Sci. China Math. 53 (2010) 597–604.
[9] H.-H. Chen, The Schwarz-Pick lemma for planar harmonic mappings, Sci. China Math. 54 (2011) 1101–1118.
[10] H.-H. Chen, P.M. Gauthier, The Landau theorem and Bloch theorem for planar harmonic and pluriharmonic mappings, Proc. Amer. Math. Soc. 139 (2011) 583–595.
[11] X. Chen, A. Fang, A note on harmonic quasiconformal mappings, J. Math. Anal. Appl. 348 (2008) 607–613.
[12] X. Chen, A. Fang, A Schwarz-Pick inequality for harmonic quasiconformal mappings and its applications, J. Math. Anal. Appl. 369 (2010) 22–28.
[13] S. Chen, S. Ponnusamy, X. Wang, Bloch constant and Landau's theorem for planar p-harmonic mappings, J. Math. Anal. Appl. 373 (2011) 102–110.
[14] S. Chen, S. Ponnusamy, X. Wang, Properties of some classes of planar harmonic and planar biharmonic mappings, Complex Anal. Oper. Theory 5 (2011) 901–916.
[15] S. Chen, S. Ponnusamy, X. Wang, Integral means and coefficient estimates on planar harmonic mappings, Ann. Acad. Sci. Fenn. Math. 37 (2012) 1–11.
[16] M. Chuaqui, R. Hernandez, Harmonic univalent mappings and linearly connected domains, J. Math. Anal. Appl. 332 (2007) 1189–1194.
[17] M. Dorff, Convolutions of planar harmonic convex mappings, Complex Var. Theory Appl. 45 (2001) 263–271.
[18] M. Dorff, M. Nowak, M. Wołoszkiewicz, Convolutions of harmonic convex mappings, Complex Var. Elliptic Equ. 57 (2012) 489–503.
[19] P. Duren, Harmonic Mappings in the Plane, Cambridge University Press, Cambridge, 2004.
[20] T. Hayami, Coefficient conditions for harmonic close-to-convex functions, Abstr. Appl. Anal. (2012) 12. http://dx.doi.org/10.1155/2012/413965. Article ID 413965.
[21] X.-Z. Huang, Estimates on Bloch constants for planar harmonic mappings, J. Math. Anal. Appl. 337 (2008) 880–887.
[22] X.-Z. Huang, Locally univalent harmonic mappings with linearly connected imagine domains, Chin. Ann. Math. Ser. A 31 (2010) 625–630.
[23] D. Kalaj, Quasiconformal harmonic mappings and close-to-convex domains, Filomat 24 (2010) 63–68.
[24] M.-S. Liu, Landau's theorem for biharmonic mappings, Complex Var. Elliptic Equ. 53 (2008) 843–855.
[25] M.-S. Liu, Estimates on Bloch constants for planar harmonic mappings, Sci. China Math. 52 (2009) 87–93.
[26] M.-S. Liu, Landau's theorem for planar harmonic mappings, Comput. Math. Appl. 57 (2009) 1142–1146.
[27] W. Majchrzak, Harmonic univalent mappings into a half-plane with nonreal vertical slits, J. Math. Anal. Appl. 255 (2001) 519–534.
[28] P.T. Mocanu, Three-cornered hat harmonic functions, Complex Var. Elliptic Equ. 54 (2009) 1079–1084.
[29] P.T. Mocanu, Injectivity conditions in the complex plane, Complex Anal. Oper. Theory 5 (2011) 759–766.
[30] S. Muir, Weak subordination for convex univalent harmonic functions, J. Math. Anal. Appl. 348 (2008) 862–871.
[31] S. Muir, Harmonic mappings convex in one or every direction, Comput. Methods Funct. Theory 12 (2012) 221–239.
[32] J. Qiao, X. Wang, Extreme points of subordination and weak subordination families of harmonic mappings, Czechoslovak Math. J. 61 (2011) 145–155.
[33] X.-T. Wang, X.-Q. Liang, Y.-L. Zhang, Precise coefficient estimates for close-to-convex harmonic univalent functions, J. Math. Anal. Appl. 263 (2001) 501–509.
[34] Z.-G. Wang, Z.-H. Liu, Y. Sun, On harmonic starlike functions with respect to symmetric, conjugate and symmetric conjugate points, Quaest. Math. (in press).
[35] M. Dorff, J. Szynal, Harmonic shears of elliptic integrals, Rocky Mountain J. Math. 35 (2005) 485–499.
[36] K. Driver, P. Duren, Harmonic shears of regular polygons by hypergeometric functions, J. Math. Anal. Appl. 239 (1999) 72–84.
[37] W. Hengartner, G. Schober, On schlicht mappings to domains convex in one direction, Comment. Math. Helv. 45 (1970) 303–314.
[38] L. Schaubroeck, Growth, distortion and coefficient bounds for plane harmonic mappings convex in one direction, Rocky Mountain J. Math. 31 (2001) 625–639.
[39] M. Dorff, Anamorphosis, mapping problems, and harmonic univalent functions, in: Explorations in Complex Analysis, Math. Assoc. of America, Inc., Washington, DC, 2012, pp. 197–269.
[40] M. Dorff, R. Viertel, M. Woloszkiewicz, Convex combinations of minimal graphs, Int. J. Math. Math. Sci. (2012) 9. http://dx.doi.org/10.1155/2012/724268. Article ID 724268.
[41] C. Pommenrenke, On starlike and close-to-convex functions, Proc. Lond. Math. Soc. (3) 13 (1963) 290–304.