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Option pricing and hedging in incomplete market driven by Normal Tempered Stable process with stochastic volatility ☆

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ABSTRACT

This paper develops a distribution class, termed Normal Tempered Stable, by subordinating a drifted Brownian motion through a strictly increasing Tempered Stable process that generalizes the Variance Gamma and the Normal Inverse Gaussian and is used to model the logarithm asset returns. The newly added parameter is to create subclasses for all the distributions discovered in financial market. The empirical test suggests that time series of Technology stock returns in US market reject both the Variance Gamma distribution and the Normal Inverse Gaussian distribution and admit instead another subclass of the Normal Tempered Stable distribution. Furthermore, we introduce stochastic volatilities into the Normal Tempered Stable process and derive explicit formulae for option pricing and hedging by means of the characteristic function based methods. To answer the question of how well different models work in practice, we investigate four models adopting data on daily equity option prices and obtain several findings from the numerical results. To sum up, the Normal Tempered Stable process with stochastic volatility is able to adequately capture implied volatility dynamics and seen as a superior model relative to the jump-diffusion stochastic volatility model, based on the construction methodology that incorporates more sophisticated and flexible jump structure and the systematic and realistic treatment of volatility dynamics. The Normal Tempered Stable model turns out to have the competitive performance in an efficient manner given that it only requires three parameters.

1. Introduction

Ever since the ground-breaking work of Black and Scholes [6], both academics and practitioners alike have devoted significant attention to the option-pricing theory, one of the most fascinating and prolific areas within financial mathematics. During the same period, overwhelming statistical evidence shows that the Black Scholes model is inconsistent with the market prices which gives rise to the volatility "smirk", and the

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limitations of exponential Brownian process describing the empirical behavior of asset price have also been recognized and corrected for with varying degrees of success. In particular, Cont [10] made an exhaustive list of stylized features of financial time series, including non-Gaussian character and volatility clustering, and argued that these features should be viewed as constraints that a stochastic process of the asset return will satisfy in financial modeling. In light of the mentioned characteristics, the main stream focuses the study on the replacement of Brownian dynamics by a Lévy process, which is a natural extension of the normal random variable conserving the statistical tractability of its i.i.d. increments and allowing jumps to occur for non-normal innovations. Albeit simple, the Lévy process can be quite capable of accommodating the common features of financial asset returns, such as asymmetry, high kurtosis and heavy tails. Another essential aspect of financial asset modeling is to live with the volatility smile and reflect the volatility persistence. Pioneered by the influential work of Heston [14], the time-integral of a mean-reverting square-root process is chosen to describe the variance and induce the time-varying and clustering property, which is found very useful in fitting implied volatility surface across both maturity and moneyness.

There is a large literature considering the possibility of discontinuous information flows accompanied by stochastic volatility in price process, with three streams that draw the most attention. The first trend finds strong evidence of regime shifts in the underlying variables and incorporates the continuous model with the Markov Chain process to govern the shifts and reflect the market movements in a finite number of states of the underlying economy, which results in jumps over the boundaries or the shifts of regime. See Yao et al. [24] for the regime-switching geometric Brownian model and Elliott et al. [11] for the regime-switching version of Heston model, and the bibliography therein. The second follows Bakshi et al. [2], Kaeck and Alexander [15] and Bao et al. [3], to name but a few, with the exclusive use of compound Poisson process representing the large rare jumps in underlying asset returns, known as the jump-diffusion models and constructed by incorporating the stochastic volatility into the diffusion part. These models can be seen as applications of affine jump-diffusion framework in Yang and Lee [23], where the uncertainties all have affine dependency of the state variable. The third stream centered around the more general jump structure by building pure-jump Lévy processes with infinite activity mainly in two ways: 1) subordinating the Brownian motion by a strictly increasing subordinator, see Variance Gamma (VG) process in Madan et al. [17] and Normal Inverse Gaussian (NIG) process in Barndorff-Nielsen [4]; 2) specifying a new Lévy measure directly, see CGMY process in Carr et al. [7], and Madan and Yor [18] has described the CGMY as a time-changed Brownian motion. Furthermore, Geman et al. [13] has suggested that price process viewed as the semimartingale are time-changed Brownian motions. In the infinite activity Lévy model, one need not introduce a Brownian motion, due to its infinitely small jumps near the origin at any time interval which account for the high activity events and act as a counterpart to the continuous part in jump-diffusion model. This is deemed to give a more realistic description of the financial price processes, for the reason that the price can never be “continuous” in the exchange. In order to allow for the volatility fluctuation in the context of pure-jump Lévy models, Carr et al. [8] has developed a general setting that is extended and empirically applied by Figueroa-López [12] and Yamazaki [22] for very recent work. In summary, while the second stream assumes that volatility varies only in the continuous part, the third offers a systematic treatment which allows both “continuous” and larger jump part to enjoy the same volatility dynamics, which is more reliable in the real world.

The purpose of this paper is threefold. Firstly, we synthesize and extend the subordinated Brownian class, abbreviated as the Normal Tempered Stable (NTS) process, by employing a Tempered Stable process as the subordinator. In this framework, the NIG process and the VG process are only parametric examples, hence one has a greater modeling freedom to control the distribution shape of asset returns time series. Since the three-parameter stochastic processes, the NIG and the VG, have empirically proven themselves quite good to provide a good control on the distribution of the logarithm stock return, the additional fourth parameter of the new process is expected to identify and categorize the asset distributions according to the economic behavior implicit in the prices. From this point of view, we model the stock price by an exponential NTS process and adopt data on 15 indices and stocks in the US market, with the conclusion that stocks in the same industry are most likely in the same subclass of the NTS distribution.

Secondly, volatility fluctuations are integrating into the exponential stochastic process when pricing options, in which we run the NTS process by integral of the instantaneous volatility and denote the new process by NTSSV. We only introduce stochastic volatility into the option pricing model by reason of the need to describe the complex and multi-dimensional information content of options across the whole spectrum of strikes and maturities at a point of time. Additionally, option pricing models should be able to capture real world behavior, such as volatility effect, in an intuitive manner. Compared with the NTS process, NTSSV maintains and enhances the flexibility in rebuilding and capturing a variety of implied volatility smile patterns observed in financial market. The closed form representation of the vanilla option price can be easily obtained, by treating the option price as the convolution of the modified terminal payoff and the characteristic function of the NTSSV process, and is computed via the efficient fast Fourier transform (FFT) using the same procedure in Carr and Madan [9] and Pillay et al. [19].

It is well-known that the popularity of Black Scholes model is attributed to its simplicity and for the most part the ease with whom a closed-form of hedging strategy is derived, which is in fact of great interest both to scholars and market participants for the purpose of making predictions and being immunized to changes in prices. In general, a perfect hedge strategy such as Delta ratio can't be achieved in incomplete market. With the explosion of studies on Lévy driven models in financial modeling, the usual practise is to minimize the expected squared hedging errors over all admissible strategies for the so-called variance-optimal hedging strategy. To quote the most popular ones, Tankov [21] has derived the explicit formula by Galtchouk–Kunita–Watanabe decomposition in terms of the exponential Lévy model, and Kallsen and Pauwels [16] has worked out the corresponding results for the time-changed Lévy model. The third contribution of this paper is that we take an alternative approach to the previous studies to make the strategies explicitly computable in the pure-jump Lévy model, which is relied on the stochastic integral representation of the martingale version of stock and option prices.

It should be stressed at the outset that the pricing and hedging method developed in this paper are for the general case and not limited to the NTS or the NTSSV models. As a matter of fact, the option price is explicitly obtained as long as the characteristic function of the underlying asset return is available; the extension of hedging method to models involving diffusions requires basically no further conceptual evolution but just a matter of details handling as done in Appendix B. We conduct empirical study to test both fitting capacity and predictive quality of four models: the NTSSV model, the Bakshi jump-diffusion model, the NTS model and the Black Scholes model. Following a wide variety of experiment designs to investigate the alternative models, we judge the pricing and hedging performance based on in-sample test, out-of-sample test and hedge test. To give additional insight into model performance, the market implied volatility is also compared with those generated by the alternative models. More specifically, the NTSSV is the best performing model in all tests; despite similar results in all tests, we'd like to draw the conclusion that the NTS is better than the Bakshi jump-diffusion model regarding its requirement of fewer parameters; in spite of its poor performance in all tests, Black Scholes model remains its competence in options pricing and hedging because of its simplicity and relatively high efficiency.

The rest of this paper is organized as follows. In Section 2 we define the NTS process and present its basic properties. Details about the Lévy measure and the integration framework are also considered and set into place. Section 3 is dedicated to the modeling of logarithm stock returns and the generalized method of pricing and hedging options in the time-changed pure-jump model. Section 4 focuses on the empirical work and the analysis of the results. Finally, Section 5 summarizes the principle findings and outlines the topics for future study.

2. The Normal Tempered Stable process

We begin by assuming a given filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ along with a filtration $(\mathcal{F}_t)_{0\le t<\infty}$ satisfying the usual hypothesis. Consider a real-valued stochastic process Y_t, termed the Normal Tempered Stable process (NTS), constructed by evaluating a drifted Brownian motion at an independent random time G_t, which follows a Tempered Stable process (TS). All the processes are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. We further denote the law of Y_1 by $\mathrm{NTS}(\mu, \sigma, \nu, \theta)$, for some $\mu \in \mathbb{R}$, $\sigma > 0$, $\nu > 0$, $\theta \in (0, 1)$. Equivalently, let W_t be a standard Brownian process, and G_1 follows the distribution $\mathrm{TS}(\theta, 2^{-\theta}\theta^{-1}(\nu/(1-\theta))^{\theta-1}, 2^{\theta}(\nu/(1-\theta))^{-\theta})$, then

$$Y_t = \mu G_t + \sigma W(G_t) \tag{1}$$

Note that $Y_t \overset{d}{=} \mu G_t + \sigma\sqrt{G_t}W(t)$ due to the independence of the Brownian process and the TS process, suggesting that (1) is a parsimonious model with stochastic volatility $\sigma\sqrt{G}$. Closed form for the TS density is not generally known, however, G_t is well characterized by its moment generating function

$$Ee^{uG_t} = \exp\left\{\frac{(1-\theta)t}{\nu\theta}\left[1 - \left(1 - \frac{u\nu}{1-\theta}\right)^{\theta}\right]\right\}$$

and by its Lévy measure Π_G with explicit formula

$$\Pi_G(dx) = \frac{1}{\Gamma(1-\theta)}\left(\frac{\nu}{1-\theta}\right)^{\theta-1} x^{-\theta-1}\exp\left\{\frac{\theta-1}{\nu}x\right\}1_{\{x>0\}}dx$$

where Γ is the Gamma function and $1_{\{x>0\}}$ is the indicator function. Note from the Lévy measure that TS process generate only positive jumps and hence strictly increasing, which ensures G_t a subordinator and in turn Y_t a Lévy process.

Characteristic function of Y_t can now be derived as

$$\begin{aligned} Ee^{iuY_t} &= \exp\left\{\frac{(1-\theta)t}{\nu\theta}\left[1 - \left(1 - \frac{(iu\mu - \sigma^2u^2/2)\nu}{1-\theta}\right)^{\theta}\right]\right\} \\ &= e^{t\phi_Y(u)} \end{aligned}$$

where ϕ_Y is the characteristic exponent of Y_1. From knowledge of the probability density function of the Generalized Inverse Gaussian distribution, explicit form of the Lévy measure Π_Y of Y_t can be obtained

$$\begin{aligned} \Pi_Y(ds) &= \int_0^\infty f_x(s)\Pi_G(dx)ds \\ &= \sqrt{\frac{2}{\pi}}\left(\frac{\nu}{1-\theta}\right)^{\theta-1}\frac{e^{\mu s/\sigma^2}|s|^{-\theta-\frac{1}{2}}}{\sigma\Gamma(1-\theta)}\left(\mu^2 - \frac{2(\theta-1)\sigma^2}{\nu}\right)^{\frac{\theta}{2}+\frac{1}{4}} \\ &\quad \times \mathbf{K}_{\theta+\frac{1}{2}}\left(\sqrt{\frac{s^2}{\sigma^2}\left(\frac{\mu^2}{\sigma^2} - \frac{2(\theta-1)}{\nu}\right)}\right)1_{\{s\neq 0\}}ds \end{aligned}$$

where f_x is the density of the process $\mu x + \sigma W(x)$ for a fixed x, and $\mathbf{K}$ is the modified Bessel function of the third kind. Then Y_t is a well-defined Lévy process with characteristic triplet given by $(\int_{-1}^{1} s\Pi_Y(ds), 0, \Pi_Y(ds))$, and $\Pi_Y(\mathbb{R}) = \Pi_G(\mathbb{R}) = \infty$, indicating that the NTS is a pure-jump process and has infinite activity. With the given characteristic functions, denote by J_Y the Poisson random measure associated with the jumps defined on $[0,t]\times\mathbb{R}$ and intensity measure $\Pi_Y(dx)dz$, then Y_t can be represented by a stochastic integral

$$Y_t=\int_0^t\int_{\mathbb{R}} xJ_Y(dz,dx)$$

Furthermore, the characteristic function can be inverted to the desired probability density function

$$f_{Y_t}(x)=\frac{1}{\pi}\int_0^\infty \Re\big[e^{-iux}Ee^{iuY_t}\big]du \tag{2}$$

where $\Re[z]$ stands for the real part of some complex number z.

Moments of the NTS process give an insight into how each parameter accounts for the asset return distribution and provide the convenience for calibration. The explicit expressions for the higher moments are as following

$$\begin{aligned}
E(Y_t)&=\mu t\\
E\big(Y_t-E(Y_t)\big)^2&=\big(\sigma^2+\mu^2\nu\big)t\\
E\big(Y_t-E(Y_t)\big)^3&=\left(3\sigma^2+\frac{\mu^2\nu(2-\theta)}{1-\theta}\right)\mu\nu t\\
E\big(Y_t-E(Y_t)\big)^4&=\left(\mu^4\nu^2\frac{(2-\theta)(3-\theta)}{(1-\theta)^2}+6\mu^2\sigma^2\nu\frac{2-\theta}{1-\theta}+3\sigma^4\right)\nu t+3\big(\sigma^2+\mu^2\nu\big)^2t^2
\end{aligned} \tag{3}$$

Obviously when $\mu=0$, the skewness is zero and the expectation of Y_t equals to zero, suggesting that μ controls the skewness degree of Y_t and represents the average return in financial application; the whole risk can be seen as composition of the self-risk σ^2 and the one from reaction to the economic environment $\mu^2\nu$; moreover, fix $\mu=0$, we obtain the kurtosis $3\nu/t$, meaning ν determines the excess kurtosis relative to the normal distribution. Note that kurtosis tends to zero when $t\to\infty$, which coincides with the law of large numbers. Last but not least, θ controls the shape of the TS distribution G to represent different economic timing or events, allowing the NTS process to better describe the asset returns and nest different Lévy processes of interest. Note from (3) that the third and fourth central moments are increasing functions with respect to θ, implying that data series with higher kurtosis would most likely enjoy the greater value of θ. Specifically, $\theta=1/2$ generates the $\mathrm{IG}(\frac{1}{\sqrt{\nu}},\frac{1}{\sqrt{\nu}})$, and hence the NTS turns into the NIG process; as $\theta\to 0$, the $\mathrm{Gamma}(\frac{1}{\nu},\frac{1}{\nu})$ can be derived, and hence the NTS is a VG process, in this case, letting $\nu\to 0$ generates the classic Black Scholes model.

3. Financial applications

This section is concerned with the stock price modeling and options pricing and hedging in exponential pure-jump Lévy models. The parameter θ helps to provide a wide range of possible distributions for the stock returns and hence fit various levels of skewness and kurtosis. As such, we price the statistical logarithm stock price process by an NTS process under the actual probability $\mathbb{P}$, and then suppose a different NTS process with stochastic volatility to be followed by the risk-neutral logarithm stock price process under the risk-neutral probability $\widetilde{\mathbb{P}}$. We derives explicit formulae for the option pricing and hedging, beginning with simple volatility structure models up to delicate and complex volatility structure models, in which the NIG, the VG and the BS models are viewed as special cases.

3.1. Statistical and risk-neutral stock prices

We describe the statistical and risk-neutral stock processes in different forms of the NTS process, because of the lack of unique or "right" equivalent probability measures in the infinite activity Lévy market which is a more realistic market and known as the incomplete financial environment. Further assume that the time 0 price of a bond with unit face value that matures at time t is given by $P_t = e^{-rt}$, where r is the constant continuously compound riskless interest rate.

The statistical stock price is assumed to follow the evolution as

$$S_t = S_0 e^{Y_t} \tag{4}$$

S_0 is the spot price, then the logarithm of the price relative follows NTS$(\mu, \sigma, \nu, \theta)$ distribution, and the arguments in the bracket are all estimated under the actual probability $\mathbb{P}$ and have their own economic explanations.

As to the risk-neutral pricing, denote the discounted stock price process by $\hat{S}_t = S_t e^{-rt}$. Assume that under the risk-neutral probability $\widetilde{\mathbb{P}}$, the NTS process is of the form $\widetilde{Y}_t = -\tilde{\sigma}^2 G_t/2 + \tilde{\sigma} W(G_t)$, which follows NTS$(-\tilde{\sigma}^2/2, \tilde{\sigma}, \tilde{\nu}, \tilde{\theta})$. By definition in Section 2, the integral $\int_{\mathbb{R}} (e^x - 1) \Pi_{\widetilde{Y}}(dx)$ equals to zero, where $\Pi_{\widetilde{Y}}$ is the Lévy measure of $\widetilde{Y}$, indicating that the stochastic exponential of the NTS process is a martingale under $\widetilde{\mathbb{P}}$. This martingale construction method aims to produce a subordinated exponential Brownian martingale, and has been discussed in Carr et al. [8] for the exclusion of dynamic arbitrage. We have also adopted the other martingale construction method by mean correcting the exponential Lévy process as done in Madan et al. [17], and found that the former is more efficient and requires less parameters. As a consequence, we present here only the former martingale version. Recall that in the celebrated Heston model, the diffusion part of the logarithm asset process is modeled by $\int_0^t \sqrt{v_t} dW_t \overset{d}{=} W(\int_0^t v_t dt)$ with the mean-reverting square-root process

$$dv_t = \iota(\xi - v_t)dt + \varsigma \sqrt{v_t} dW_t^v \tag{5}$$

Denote time integral of the instantaneous volatility by $V_t = \int_0^t v_t dt$, then the Heston model can be viewed as a time-changed Brownian process, provided that $\iota\xi > \varsigma^2/2$. Heuristically, by replacing the Brownian process with an NTS process and assuming that $\widetilde{Y}_t$ and V_t are independent, a general stochastic volatility pure-jump process $\widetilde{Y}(V_t)$, termed the NTSSV, can be obtained, in which the volatility dynamics in both the "continuous" part (infinitely small jumps in the NTS) and the "jump" part (finitely larger jumps in the NTS) are depicted simultaneously.

As proved in Tankov [20], the discounted stock process constructed by evaluating a positive martingale $\exp\{\widetilde{Y}_t\}$ at an independent time change V_t is again a martingale and has the form

$$\hat{S}_t = \hat{S}_0 e^{\widetilde{Y}(V_t)} \tag{6}$$

in which the parameters are all estimated under $\widetilde{\mathbb{P}}$; obviously $\widetilde{E} S_t = e^{rt}$. As a result, the characteristic function of the NTSSV process is given by

$$\begin{aligned}\widetilde{E} e^{iu \ln \frac{S_t}{S_0}} &= \exp\left\{ iurt + \frac{2\phi_{\widetilde{Y}}(u) V_0 (1 - e^{-\gamma t})}{2\gamma - (\gamma - \iota)(1 - e^{-\gamma t})} - \frac{\iota\xi}{\varsigma^2} \left[2 \ln\left[1 - \frac{\gamma - \iota}{2\gamma} \left(1 - e^{-\gamma t}\right) \right] + (\gamma - \iota) t \right] \right\} \\ &= \exp\left\{ iurt + \wp(t, u) \right\}\end{aligned}$$

where $\gamma = \sqrt{\iota^2 - 2\phi_{\widetilde{Y}}(u)\varsigma^2}$, and $\phi_{\widetilde{Y}}$ is the characteristic exponent of the NTS process $\widetilde{Y}$. Using results of Bentata and Cont [5], we write $\widetilde{Y}(V_t)$ in the following form

$$\widetilde{Y}(V_t) = \int_0^t \int_{\mathbb{R}} x v_z J_{\widetilde{Y}}(dz, dx)$$

where $J_{\widetilde{Y}}$ is the Poisson random measure of $\widetilde{Y}$. Moreover, apply the Itô formula to (6) and we have

$$\hat{S}_t = \hat{S}_0 + \int_0^t \int_{\mathbb{R}} \hat{S}_{z-}\big(e^x - 1\big) v_z J_{\widetilde{Y}}(dz, dx)$$

3.2. Option pricing

Let C_t be the time t value of a European call option price that matures at time T with strike K and underlying asset S_t given by (6). Under the risk-neutral measure $\widetilde{\mathbb{P}}$, for fixed S_t, T and t,

$$\begin{aligned} C_t(\kappa) &= e^{-r\tau} \widetilde{E}_t\big[(S_T - K)^+\big] \\ &= K\widetilde{E}\big[\big(e^{\widetilde{Y}(V_\tau)-\kappa} - e^{-r\tau}\big)^+\big] \end{aligned} \tag{7}$$

where $\tau = T - t$, and $\kappa = \ln(K/S_t)$ serves as a gauge of moneyness. Define a modified claim payoff function by

$$\varphi(x) = e^{(1+\alpha)x}\big(e^{-x} - e^{-r\tau}\big)^+$$

for some $\alpha > 0$, which is chosen such that the functions $\varphi(x)$ and $\psi(x) = e^{(1+\alpha)x} f_{\widetilde{Y}(V_\tau)}(x)$ are absolutely integrable, where $f_{\widetilde{Y}(V_\tau)}$ is the risk-neutral probability density function of $\widetilde{Y}(V_\tau)$. Now consider the option price

$$C_t(\kappa) = K \int_{\mathbb{R}} \varphi(\kappa - y) e^{(1+\alpha)(y-\kappa)} \widetilde{f}_{\widetilde{Y}(V_\tau)}(y) dy = S_t e^{-\alpha\kappa}(\varphi * \psi)(\kappa)$$

which is equal to

$$\begin{aligned} \mathcal{F}\bigg(\frac{e^{\alpha\kappa}}{S_t} C_t(\kappa)\bigg)(u) &= \mathcal{F}(\varphi)(u)\mathcal{F}(\psi)(u) \\ &= \frac{\exp\{(iu+\alpha)r\tau + \wp(\tau, u - i - i\alpha)\}}{(iu+\alpha+1)(iu+\alpha)} \end{aligned}$$

Then we recover analytical expression of the initial call option via the inverse Fourier transform

$$C_t(\kappa) = \frac{S_t e^{\alpha(r\tau-\kappa)}}{\pi} \int_0^\infty \Re\bigg\{e^{-iu\kappa} \frac{ecp\{iur\tau + \wp(\tau, u - i - i\alpha)\}}{(iu+\alpha+1)(iu+\alpha)}\bigg\} du \tag{8}$$

The outcome is the same as that in Carr and Madan [9], which is derived by a modified option price designed to overcome the singularity of the original one, while ours is through the modified payoff function utilizing the convenience provided by Fourier transform and convolution theorem. Now consider the appropriate α for C_t, note that (8) is well defined when the expectation $\widetilde{E}[e^{(1+\alpha)\widetilde{Y}(V_t)}]$ exists, which in our case is equivalent to

$$\alpha < \sqrt{\frac{1}{4} + \frac{2(1-\tilde{\theta})}{\tilde{\sigma}^2\tilde{\nu}}} - \frac{1}{2}$$

As Carr and Madan [9] mentioned that in practice α can be determined to be one fourth of the upper bound, we follow their choice in the empirical calibration. For the issue of European put option pricing, one may follow the same steps but with the modified claim payoff function redefined as

$$\bar{\varphi}(x) = e^{-\bar{\alpha}x}\left(e^{-r\tau} - e^{-x}\right)^{+}$$

3.3. Option hedging

As stated in the previous sections, the exponential NTSSV market is incomplete, meaning the exact replicating portfolios for option hedging impossible in contrast with the BS model. The objective of this subsection is to hedge a short position in the vanilla call option. To state the issue more clearly, imagine that an agent is holding a short position in the option contract and seeking to avoid the risk by long positions in the riskless bond and the underlying stock. Following Tankov [21] and Kallsen and Pauwels [16], we introduce a new criteria through which the variance-optimal hedging strategies can be defined by minimizing the time zero mean squared hedging error under the risk-neutral measure $\widetilde{\mathbb{P}}$

$$\epsilon_0 = \min_{\vartheta \in \mathcal{S}} \widetilde{E}\left[e^{-rT}(S_T - K)^{+} - C_0 - \int_0^T \vartheta_t d\hat{S}_t\right]^2 \tag{9}$$

where C_0 is the initial endowment or the option price received by the agent, and $\mathcal{S}$ denotes the set of admissible self-financing strategies such that $\widetilde{E}[\int_0^T \vartheta_t d\hat{S}_t]^2 < \infty$. Then the expression in the parentheses is the total loss of a self-financing hedging portfolio, and the solutions ϑ to Eq. (9) is the so-called variance-optimal hedging strategy. For fixed K and T, referred to Appendix A for the detailed computation process and the technical conditions to be satisfied, we rewrite the discounted option price $\hat{C}_t = e^{-rt}C_t$ as follows

$$e^{-rT}(S_T - K)^{+} = C_0 + \int_0^T \int_{\mathbb{R}} \left[\hat{C}\left(t, S_{t-}e^{x}\right) - \hat{C}(t-, S_{t-})\right] v_t J_{\widetilde{Y}}(dt, dx)$$

where $\widetilde{J}_{\widetilde{Y}}(dt, dx) = J_{\widetilde{Y}}(dt, dx) - \mathit{\Pi}_{\widetilde{Y}}(dx)dt$ is the compensated version of $J_{\widetilde{Y}}$. Now that the stochastic integrals of the option and stock processes are derived, we deal with the problem in (9) in the following procedure

$$\begin{aligned}
&\widetilde{E}\left\{\int_0^T \int_{\mathbb{R}} \left[\hat{C}\left(t, S_{t-}e^{x}\right) - \hat{C}(t-, S_{t-}) - \vartheta_t \hat{S}_{t-}\left(e^{x} - 1\right)\right] v_t J_{\widetilde{Y}}(dt, dx)\right\}^2 \\
&\quad = \int_0^T \int_{\mathbb{R}} \widetilde{E}\left\{\left[\hat{C}\left(t, S_{t-}e^{x}\right) - \hat{C}(t-, S_{t-}) - \vartheta_t \hat{S}_{t-}\left(e^{x} - 1\right)\right]^2 v_t^2\right\} \mathit{\Pi}_{\widetilde{Y}}(dx)dt
\end{aligned} \tag{10}$$

Differentiate (10) with respect to ϑ_t and equate the resulting formula to zero, then the self-financing variance-optimal hedging strategy at time t is given by

$$\begin{aligned}
\vartheta_t &= \frac{\int_{\mathbb{R}}[C(t, S_{t-}e^{x}) - C(t-, S_{t-})](e^{x} - 1)\mathit{\Pi}_{\widetilde{Y}}(dx)}{S_{t-}\int_{\mathbb{R}}(e^{x} - 1)^2 \mathit{\Pi}_{\widetilde{Y}}(dx)} \\
&= \frac{e^{\alpha(r\tau - \kappa)}}{\pi} \int_0^\infty \Re\left\{\left[\frac{\phi_{\widetilde{Y}}(u - 2i - i\alpha) - \phi_{\widetilde{Y}}(u - i - i\alpha) - \phi_{\widetilde{Y}}(-i)}{\phi_{\widetilde{Y}}(-2i) - 2\phi_{\widetilde{Y}}(-i)}\right] \right. \\
&\quad \left. \times e^{-iu\kappa} \frac{e^{iur\tau + \wp(\tau, u - i - i\alpha)}}{(iu + \alpha + 1)(iu + \alpha)}\right\} du
\end{aligned} \tag{11}$$

The validity to interchange the integration has been assured in Appendix A. Note that $\vartheta_t = \widetilde{Cov}(\Delta C_t, \Delta S_t)/\widetilde{Var}(\Delta S_t)$, where the expectation is computed under $\widetilde{\mathbb{P}}$.

4. Empirical performance

In this section, we conduct an extensive empirical investigation and discuss the results from two perspectives. Firstly, relying on data of 15 indices and stocks listed in US market, we calibrate statistical individual equity return assuming it follows an NTS distribution, which is considered as a distribution class that covers the NIG, the VG and the BS. The additional parameter is also tested to show if there's necessity to add it into the model. Next, we tackle the question of which option pricing model to use by investigating the NTS models with and without stochastic volatility and the jump-diffusion models, with the Black Scholes model as a benchmark. Daily option prices are used to estimate parameters of the risk neutral logarithm return. A logical next step is to investigate the hedging ability of each model, both dynamically and statically. Our analysis intends to get a complete picture of the extent to which each model has outperformed the benchmark BS and whether each generation of the BS produce a tradeoff between improvements and costs.

4.1. Calibrate the statistical stock prices

We begin the analysis by estimating the statistical parameters of the NTS distribution and using the time series return data of the underlying assets over the period from December 31, 2010 to December 30, 2013, with a total of 754 trading days (closing prices) and the return sample size of 753. 3 market indices and 12 stocks are selected to fit the NTS density, with tickers: DJI, NYA, SPX, AAPL, AMD, AMZN, BAC, BCS, DAKT, GOOG, LRTR, MARPS, MU, UBS and YHOO. All of the data is available in Yahoo Finance.

Maximal likelihood Estimation is applied to the probability density function (2) to obtain the statistical parameters, in which the probability density function can be recovered by FFT as follows

$$f_{Y_t}(x) \approx \frac{1}{\pi} \sum_{j=0}^{N-1} \Re\big[e^{-iu_j x} E e^{iu_j Y_t}\big]\eta$$

where $u_j = j\eta$, $a = N\eta$ is the upper bound, let $\lambda = \frac{2\pi}{N\eta}$, $x_k = -b + \lambda(k-1)$, $b = \frac{N\lambda}{2}$, $k = 1, \dots, N$, then $x_k \in (-b, b)$, and further

$$f_{Y_t}(x_k) \approx \frac{1}{\pi} \sum_{j=1}^{N} \Re\big[e^{-i(j-1)(k-1)\frac{2\pi}{N}} e^{ij\eta b} E e^{iu_j Y_t}\big]\eta \tag{12}$$

The FFT can be very efficient to calculate the sum (12) when N is a power of 2.

For each stock or index prices, daily time series of logarithm price relatives are used to estimate the NTS density through (12). The integration spacing η used in this study is set to be 0.26, $N = 2^{15}$ and hence the return spacing λ is 0.0007 ranging from −12 to 12, which is large enough for practice. Since the density will be evaluated at the predetermined point x_k, we fist assign each return observation to the closest predetermined point to form a binned return data series as done in Carr et al. [7], and then search the parameters that maximize the likelihood of the binned series. To obtain the standard errors, we employ the inverse of the information matrix of the log-likelihood at the maximum likelihood estimates.

Results of the estimation can be found in Table 1, with the standard errors of estimators in the parentheses. The estimated densities have a variety of shapes, with the value of θ ranging from 0 to 1. The stocks and indices in the table are displayed by the magnitude of θ for the purpose of finding the "right" distribution class for each stock and industry. To further appreciate the effect of change of θ on the model, we present some of the stock density fit in Fig. 1. We also estimate the parameters for the NIG model and

Table 1
Estimation of statistical parameters of the NTS distribution for daily returns during the period from December 31, 2010 to December 30, 2013. Numbers in the second to fifth column are maximal likelihood estimates of the binned data, with standard errors in the parentheses. The final two columns report p-values of the likelihood ratio tests under the 5% significance level, using NTS as the unrestricted model and NIG or VG as the restricted models, respectively.

TICKER	μ	σ	ν	θ	LRT-NIG	LRT-VG
MARPS	−0.0007	0.0235	1.0012	0.0000	0.0008	1.0000
	(0.0638)	(0.0023)	(0.1303)	(0.0092)		
LRTR	0.000454	0.074406	4.700259	0.030271	0.0000	0.0473
	(0.0027)	(0.0040)	(0.0243)	(0.0063)		
SPX	0.0005	0.0104	1.7124	0.4600	0.7750	0.0321
	(0.0004)	(0.0005)	(0.1203)	(0.0535)		
BAC	0.0002	0.0267	2.0603	0.5051	0.9636	0.0005
	(0.0136)	(0.0013)	(0.0196)	(0.0040)		
DJI	0.0005	0.0095	1.8706	0.5650	0.6540	0.0241
	(0.0008)	(0.0004)	(0.1013)	(0.0336)		
NYA	0.0003	0.0110	1.7145	0.5802	0.5504	0.0036
	(0.0324)	(0.0007)	(0.0821)	(0.0278)		
BCS	0.0001	0.0302	2.2151	0.6240	0.3379	0.0025
	(0.0112)	(0.0058)	(0.0369)	(0.0070)		
UBS	0.0002	0.0244	2.3292	0.6647	0.2349	0.0011
	(0.0013)	(0.0114)	(0.2172)	(0.0255)		
YHOO	0.0012	0.0192	2.5095	0.7540	0.0493	0.0008
	(0.0233)	(0.0007)	(0.0158)	(0.0112)		
MU	0.0013	0.0318	2.1641	0.7603	0.0382	0.0002
	(0.0689)	(0.0011)	(0.0125)	(0.0147)		
AMD	−0.0010	0.0343	4.0470	0.7724	0.0146	0.0000
	(0.0560)	(0.0244)	(0.7359)	(0.0630)		
DAKT	0.0000	0.0276	7.1066	0.7955	0.0009	0.0000
	(0.0008)	(0.0010)	(0.5125)	(0.0245)		
AAPL	0.0007	0.0177	1.9109	0.8175	0.0315	0.0011
	(0.0125)	(0.0007)	(0.0404)	(0.0091)		
AMZN	0.0010	0.0207	5.0810	0.8372	0.0005	0.0000
	(0.0325)	(0.0009)	(0.0828)	(0.0314)		
GOOG	0.0008	0.0157	13.8409	0.8426	0.0001	0.0000
	(0.0155)	(0.0007)	(0.7684)	(0.0503)		

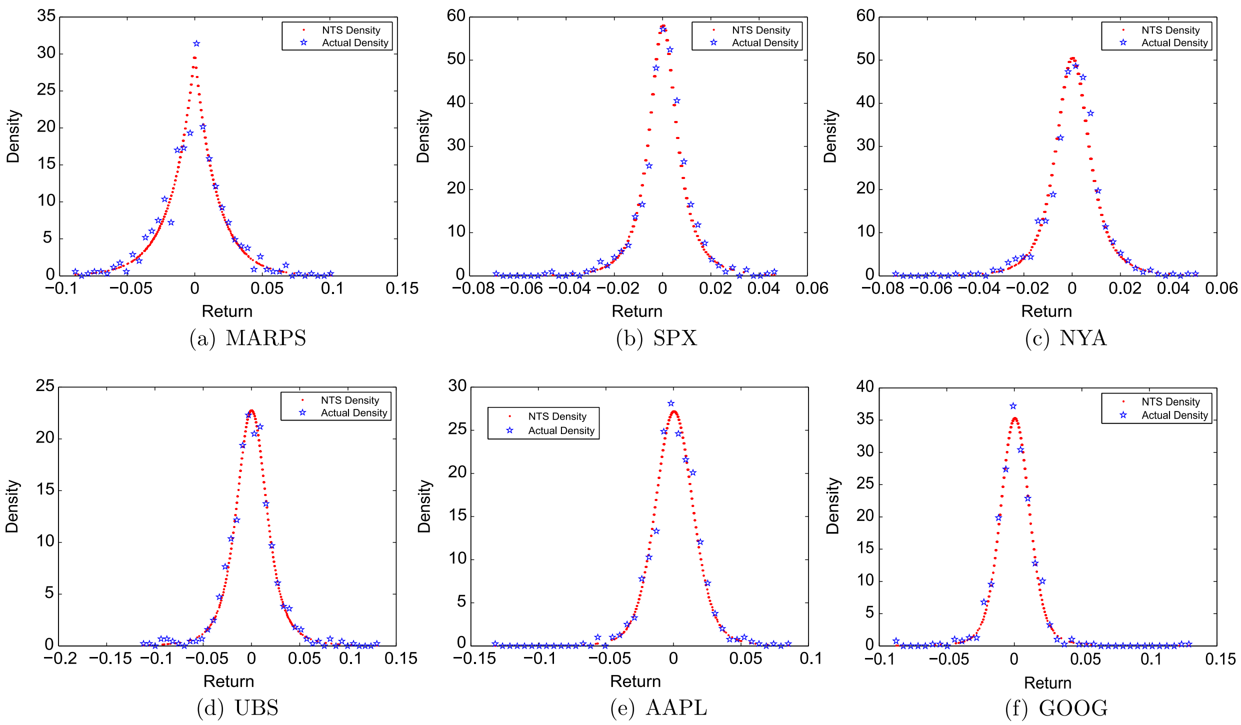


Fig. 1. Graph of NTS Density fit for the MARPS, SPX, NYA, UBS, AAPL and GOOG returns using parameters in Table 1. Actual density is also plotted for comparison, and the relative bin size is set according to the Freedman–Diaconis rule.

the VG model, and represent only p-value of the likelihood ratio tests for saving space, in which NTS is the unrestricted model and NIG and VG are the restricted model. As shown, θ of the indices are estimated around 0.5, and the p-values indicates that all indices prefer the NIG model than the other two. Based on our results, under the 5% significance level, for $\theta < 0.1$, VG is the desired model for the return data series; for $0.1 < \theta < 0.7$, NIG model can't be rejected; for $\theta > 0.7$, there is strong evidence suggesting that the other subclass of NTS distribution fits the data better than VG and NIG. Heuristically, it seems that companies in the same industry have similar value of θ and hence the same distribution shape: the NTS model fit the stocks that belong to technology and the electronic equipment industries quite well while both the NIG and VG are rejected; the NIG model seems the most appropriate for the financial stocks; the diversified investment stock observes the VG models and rejects the NIG model.

4.2. Analysis of option pricing and hedging

In this subsection, we concentrate on four alternative models to price options: the NTSSV model, the Bakshi jump-diffusion model termed BSJSV, the NTS model and the Black Scholes model termed BS. The model selection aims at covering and comparing the following features: a model exhibiting infinite jump activity and small jumps versus a model exhibiting finite jump activity and large jumps; a model with stochastic volatility versus a model without one; a model with relatively simpler jump structure and more complicated volatility dynamics versus one with more complicated jump structure and simpler volatility dynamics. The BS model is a limiting case of every other model and viewed as the benchmark in our empirical study. For more information about the characteristics of the popular BSJSV model, the reader is referred to Appendix B. We assess and investigate the model performances from three angles: 1) in-sample pricing errors, 2) out-of-sample pricing errors and 3) hedging errors. The indicators employed to measure the magnitude of the pricing errors are mean absolute errors (MAEs) and mean absolute percentage errors (MAPEs).

The option prices used in this study are for the AAPL stock options traded in the Chicago Board Options Exchange. We employ the delayed market quotes on December 30, 2013 as the in-sample data to calibrate the risk-neutral parameters, with the underlying price 554.52, and those on December 30, 2013 are used for the out-of-sample test, with the underlying price 561.02. Note that the closing hour of the options and of the stock are the same, thus there is no nonsynchronous issue here. In order to illustrate the hedge performance in the various models, we calculate the empirical hedging errors, static as well as dynamic ones, applying variance-optimal strategy and the traditional Delta hedging strategy as the comparison, with the realized AAPL stock price sample extending from December 31, 2013 to April 17, 2014. All of the data is available on the CBOE site. To ensure sufficient liquidity and alleviate the influences of price discreetness during the valuation, we preclude the option quotes that lower than \$0.3 in the sample data.The one-month US Treasury Bonds Rates 0.01% on December 30, 2013 is chosen to be the risk free interest rate.

Finally the option sample contains 413 call options on December 30, 2013 and 397 call options on Decembder 31, 2013, respectively, with available maturities: January 3, 2014; January 10, 2014; January 18, 2014; February 22, 2014; March 22, 2014; April 19, 2014. We divide the option data into 3 categories according to the moneyness S/K, where S and K denote respectively the AAPL stock price and the exercise price: Out-of-the-Money (OTM), At-the-Money (ATM), In-the-Money (ITM). As shown in Table 2, we describe the sample by exhibiting the average prices and the corresponding sample size for each moneyness category. Note that there are a total of 809 call options with ITM and ATM options taking up 47.3% and 21.6%, respectively.

The risk neutral parameters are backed out by minimizing the MAE loss function

$$Loss^{\mathrm{MAE}} = \sum_{n=1}^{N} |C_n - \hat{C}_n|$$

Table 2
Description of AAPL option data. The reported numbers are respectively the average option price and the number of observations, which is shown in the parenthesis, for the overall sample and each moneyness category on December 30, 2013 and on December 31, 2013. S denote the AAPL stock price and K is the exercise price of the option contract. OTM, ATM, ITM denote Out-of-the-Money, At-the-Money, In-the-Money options, respectively.

Date	Total	Moneyness S/K		
		OTM <0.94	ATM 0.94–1.06	ITM >1.06
December 30, 2013	\$72.60 (413)	\$4.20 (138)	\$19.73 (87)	\$147.27 (188)
December 31, 2013	\$76.41 (397)	\$4.19 (113)	\$19.68 (88)	\$143.52 (196)

where N is the number of the sample data, C_n and $\hat{C}_n$ represent the market price and the model price, respectively. Each $\hat{C}_n$ available in the option sample, regardless of moneyness and maturity, is taken as inputs and computed using the formula (8) and the same estimation procedure as in the calibration of statistical stock prices. For the NTSSV model, the volatility parameter σ, the jump-related parameters (ν, θ), and the structural parameters that induce stochastic volatility (ι, ξ, ς), are estimated, and the unobserved spot volatility is set to be 0.0003, which is the variance of the AAPL stock sample from December 31, 2010 to December 30, 2013; For the BSJSV model, we estimate the volatility parameter σ along with its structural parameters (ι, ξ, ς), and the jump-related parameters $(\mu_J, \sigma_J, \lambda_J)$, with the unobserved spot volatility being assumed to be 0.0001, for the reason that the total variance is decomposed into two components, the continuous and the pure-jump part; Both the NTS and BS models are nested in the NTSSV model and hence are processed in the same way.

For all the hedging exercises conducted in this section, only a single instrument, i.e. the underlying AAPL stock, is employed. To make the point precise, consider the situation in which an agent is to hedge a call option with initial price C_0, strike price K and T periods to expiration, then the hedging is implemented by interpreting Eq. (9) in a discrete-time and absolute-value version

$$\hat{\epsilon}_\Delta = \left| \sum_{t=0}^{[T/\Delta]-1} \left\{ C_0 + \vartheta_{t\Delta} \left(S_{t\Delta} e^{-rt\Delta} - S_{t(\Delta-1)} e^{-rt(\Delta-1)} \right) - e^{-rT} (S_T - K)^+ \right\} \right| \tag{13}$$

where we assume that the portfolio re-balancing takes place at intervals of length Δ days. For the details of hedging strategies in the BSJSV and the BS models, refer to Appendix B. The steps are as follows. First, the agent shorts an option on day 0 and constructs the hedging portfolio $\vartheta_0 S_0$, in which ϑ is derived using formula (11) and (15) in the corresponding model. Second, he liquidates the position on the next trading day Δ and calculate the hedging error. As such, repeat the above procedure on day 2Δ, and so on. Finally, compute the sum of errors for every iteration.

With a view to a fully and complete description, the results are presented in four tables and two graphs after implementing the above procedures. The statistics in Table 3 and Table 4 are respectively the estimated risk-neutral parameters and the corresponding in-sample pricing errors, being quite informative to explain the internal working of each model, with several observations followed in order. First, regarding the reported MAE value, we find that NTSSV can best fit the market prices, followed by NTS and BSJSV. According to the magnitude of MAPE, all models deliver large errors for OTM options fitting, which is in line with the estimation method that assigns more weight to the ATM/ITM options and less weight to the OTM options; NTSSV still has the fewest errors; however, BSJSV resulted in a slightly smaller error than NTS, due to the reason that BSJSV fits the ATM options better than the latter, which can be seen in Table 4, and the poor fitting in ITM options has increased the value of BSJSV's MAE. Given that BSJSV has more parameters than NTS, we'd like to draw the conclusion that NTS performed better than BSJSV for in-sample pricing. Compared to the other three models, BS gives much larger errors by looking at values of both MAE and MAPE. The generations of BS all provide large pricing improvements, for the most part, in OTM and ATM

Table 3
Estimates of risk-neutral parameters. By minimizing the sum of the absolute errors between the market price and the model-determined price for each option on December 30, 2013, the estimated parameter for a given model is reported first, followed by its standard error in parenthesis. MAE and MAPE in the given row groups display the mean absolute errors and the mean absolute percentage errors for all options. NTSSV and BSJSV stand respectively for the NTS model with stochastic volatility and the jump-diffusion model with stochastic volatility.

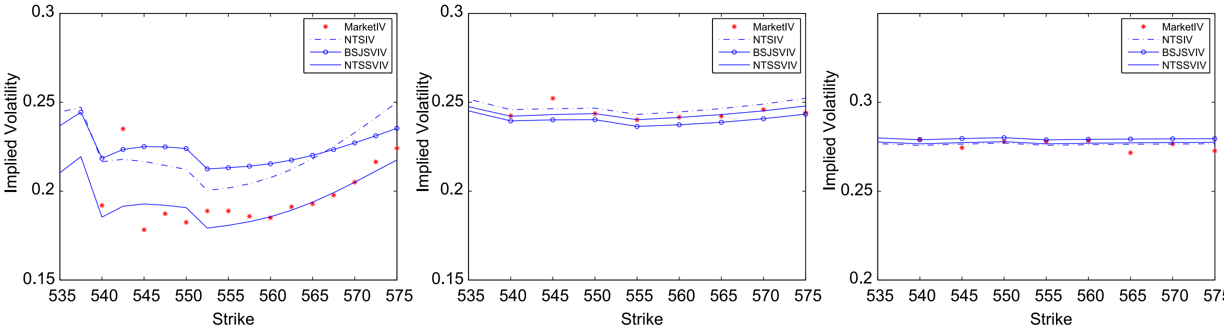
Parameters	NTSSV	BSJSV	NTS	BS
σ	0.1197		0.0157	0.0146
	(0.0217)		(0.0000)	(0.0001)
ν	5.3941		136.1473	
	(0.0135)		(1.0254)	
θ	0.8432		0.7870	
	(0.0019)		(0.1501)	
ι	1.1414	0.0682		
	(0.0385)	(0.3471)		
ξ	0.0182	0.0002		
	(0.0044)	(0.0001)		
ς	0.0683	0.0086		
	(0.0468)	(0.0201)		
μ_J		−0.0046		
		(0.0099)		
σ_J		0.0035		
		(0.0193)		
λ_J		0.7728		
		(0.1358)		
MAE	\$5.95	\$6.01	\$5.97	\$6.22
MAPE	11.20%	12.86%	12.92%	23.15%

Table 4
In-sample pricing errors. For a given model, we compute the price of each option on December 30, 2013, with a total of 413 options, using the parameters estimated in Table 3. The group under the heading MAPE reports the sample average of the absolute difference between the market price and the model price for each option in a given moneyness category; the group under the heading MAE reports the sample average of the value derived through dividing the MAE by the market price.

Models	MAPE			MAE		
	OTM	ATM	ITM	OTM	ATM	ITM
NTSSV	20.01%	4.57%	7.77%	\$0.51	\$0.79	\$12.37
BSJSV	23.21%	7.33%	7.79%	\$0.58	\$0.88	\$12.40
NTS	22.25%	9.18%	7.78%	\$0.52	\$0.87	\$12.36
BS	43.56%	23.73%	7.82%	\$0.89	\$1.26	\$12.45

options; BS, however, makes quite a good fit especially in ITM options considering it requires only one parameter.

Second, parameters of all models except BSJSV have small standard deviations and hence are stable. BSJSV, however, in our experience requires the most fitting time. Take a closer look at the estimates, the jump related parameters in BSJSV are significant, suggesting that AAPL risk-neutral stock price can't reject jump component. Moreover, the estimated structural parameters (ι, ξ, ς) in NTSSV are more notable than in BSJSV. Recall from the construction of NTSSV that the square-root structure is built into both the "continuous" and the jump part while it's not the case in BSJSV, signifying that jumps in price process can't deny stochastic volatility. In addition to inheriting characteristic of NTS, NTSSV has the added advantage of allowing time-varying variances in stock price to occur; θ in both the two models are similar and estimated around 0.8, which makes sense since this parameter determines the distribution subclass of the stock price; however, ν in the latter is a lot larger than the former. As stated previously, NTS can be seen as a BS model with stochastic volatility $\sigma\sqrt{G}$ and ν is exactly the variance of G, which indicates that without more intricate structure in volatility, NTS can only roughly account for the information implicit in the volatility skew by enlarging the fluctuation of G. To get a sense of the capability of each model capturing features of the volatility, we back out model's implied-volatility series from the Black–Scholes formula by taking the model-determined prices as input and plot those of the ATM options respectively with three maturities as



(a) 4-day options (b) 19-day options (c) 110-day options

Fig. 2. Graph of AAPL's ATM options volatility smirk. Implied volatilities on December 30, 2013 are computed using the market prices and the model-determined prices as inputs to the inverse BS formula to obtain the MarketIV and the corresponding ModelIV. Only three maturities are chosen to present.

Table 5
Out-of-sample pricing errors. For a given model, we compute the price of each option on December 31, 2013, with a total of 397 options, using the parameters estimated in Table 3. The group under the heading MAPE reports the sample average of the absolute difference between the market price and the model price for each option in a given moneyness category; the group under the heading MAE reports the sample average of the value derived through dividing the MAE by the market price.

Models	MAPE			MAE		
	OTM	ATM	ITM	OTM	ATM	ITM
NTSSV	27.42%	10.82%	8.04%	\$0.64	\$0.97	\$12.79
BSJSV	21.94%	15.80%	8.07%	\$0.62	\$1.13	\$12.83
NTS	27.12%	16.83%	8.04%	\$0.62	\$1.09	\$12.76
BS	37.28%	33.45%	7.99%	\$0.75	\$1.65	\$12.76

reflected in Fig. 2. The NTSSV's implied-volatility pattern fit the market skew quite well across different maturities, while BSJSV and NTS are not so close to it.

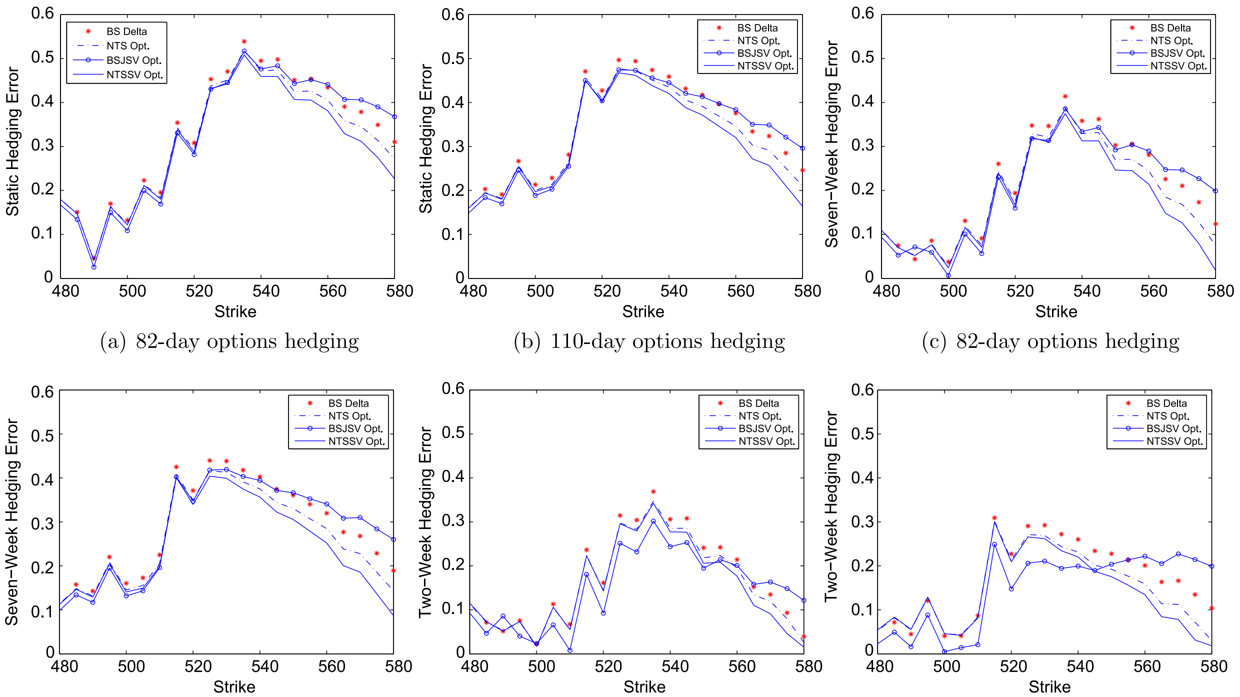
Third, there's a big difference of the estimated parameter ν of the NTS model from the options and that from the history data of the stock, mostly due to the reason that the historical time-series data only reflect the past behavior of the stock while the option prices represent expectation by the market of a stock's future volatility, which on the other hand contains a lot more uncertainty. On the whole, NTSSV shows the best in-sample performance, being capable of fitting market prices as well as producing volatility skew, and NTS on the other hand, gives competitive performance in consideration of its fewer parameters requirements and good qualifying performance.

Now that the in-sample fit is increasingly better from BS, BSJSV, NTS and NTSSV, one may argue that the outcome can be biased due to the larger number of parameters and the over-fitting to the data. Moreover, a model that performs well in fitting option prices may have poor predictive qualities. Given these concerns, we design the out-of-sample test by using the parameters estimated in Table 3 as inputs to compute the model-based option prices on December 31, 2013 and report the corresponding pricing errors in Table 5. According to the results, NTS and NTSSV still generate fewer pricing error in ATM options, with BSJSV slightly better in OTM options and BS the best performer in ITM options. The overall MAPE for NTSSV, BSJSV, NTS and BS are respectively 14.17%, 13.73%, 15.42% and 21.97%, with the overall MAE \$6.71, \$6.76, \$6.72 and \$6.88. All the models except BS generate larger percentage errors in the out-of-sample test, which shows that BS is quite competent in out-of-sample pricing with the advantage of its simplicity.

To further gauge and analyze the forecasting power of the alternative models, we conduct the hedging test, using parameters in Table 3 and realized stock price from December 31, 2013 to April 17, 2014 as inputs to Eq. (13) to obtain the absolute errors and the percentage errors. The annul volatility employed to calculate the Delta Hedge strategy is set to be 0.3401, which is backed out as the volatility of the Black

Table 6
Mean absolute errors of alternative hedging strategies using options with strike range from 480 to 580 and maturities 81 days and 110 days on December 30, 2013. Results under Static Hedge, Seven-Week and Two-Week are backed out respectively by setting Δ in Eq. (13) to be the corresponding maturities, seven weeks and two weeks. The variance in Delta Hedge is set to be 0.3401.

Strategies	Mean absolute hedging error		
	Static Hedge	Seven-Week	Two-Week
NTSSV Optimal	\$13.57	\$9.59	\$6.65
BSJSV Optimal	\$14.10	\$10.43	\$6.00
NTS Optimal	\$14.01	\$10.17	\$6.92
Delta Hedge	\$14.79	\$11.09	\$7.48
No Hedge		\$35.87	



(a) 82-day options hedging (b) 110-day options hedging (c) 82-day options hedging

(d) 110-day options hedging (e) 82-day options hedging (f) 110-day options hedging

Fig. 3. Graph of percentage hedging errors of options with strike range from 480 to 580 and maturities 81 days and 110 days on December 30, 2013, computed by dividing the absolute hedging error by the market price of the corresponding option contract. Static, Seven-Week and Two-Week hedging errors are backed out respectively by setting Δ in Eq. (13) to be the corresponding maturities, seven weeks and two weeks. The variance in Delta Hedge is set to be 0.3401.

Scholes formula using AAPL stock price data from December 31, 2010 to December 30, 2013. The results are outlined in Table 6 and Fig. 3. We see that firstly, each line in Table 6 shows discrete-time hedging errors by the same strategy but different re-balancing intervals, and the value declines as the trading frequency increase and makes sense to us. Secondly, all optimal strategies have better performance than the classical delta hedge. In the optimal hedge test, NTSSV and NTS have fewer errors except in the high-frequency hedging, probably for the reason that BSJSV is weaker in anticipating the longer-term tendency of the stock price, or in other word, NTS-like models can better describe and predict the stock market. Thirdly, by comparing the errors between NTSSV and NTS hedging, note that the inclusion of stochastic volatility does enhance the ability to hedge market risks. Moreover, we plot the errors in percentage of the corresponding market option prices and find that NTSSV does pretty good job in ATM and OTM options, while BSJSV produce the most errors in OTM options. Regarding the ITM options, BSJSV performs slightly better than the other three strategies, followed by NTSSV, NTS and Delta strategies. To sum up, NTSSV works the best in the hedging test.

5. Conclusion

This paper has developed a parsimonious model, the NTS model, that is an extension of the famous VG model and NIG model and allows for more flexible parameters than the latter two. In order to verify the greater flexibility, likelihood ratio test is executed using statistical log return data to compare the alternative model, the NTS model, and the null model, the VG or the NIG model. Results show that all technology equities in our test enjoy θ with value higher than 0.7 and both the VG and the NIG are the "wrong" model choice to fit these stock returns, in other words, another subclass of NTS process is preferred and the newly added θ can help to identify the distribution subclass that a stock belongs to.

Inspired by the statistical analysis, we introduce a new variant of NTS process, NTSSV which combines the Heston-type stochastic volatility and the NTS process, into the option pricing model and derive general explicit solutions to price and hedge vanilla options in the pure-jump market. Models of particular interest, i.e., the NTSSV, the BSJSV, the NTS and the BS, are compared on one-day AAPL option price data. In conclusion, the NTSSV model is found to have the best overall performance given its intuitively acceptable dynamics and numerical results. The NTS ranks next, followed by the BSJSV. The significant performance gap between the BS model and the remaining models suggests that all the generations of the BS are more efficient with greater improvements. On comparison of models fitting market implied volatility surface, all the three models are able to generate the volatility smirk effects, yet, only the NTSSV perfectly fit the volatilities in our test. Compared with NTS, NTSSV incorporates the stochastic time in a form of more complicated structure that reproduces time-varying and clustering volatilities; the NTSSV has more flexibility of the jump component and allows the "whole" process to be affected by the volatility dynamics, versus BSJSV which has relatively simple jump design and exposes only the continuous part to the volatility effects, with the former being more realistic and better synthesizing information represented by jumps and stochastic volatility. Summarizing all findings, the introduction of square-root time change into the NTS model results in a significant improvement in option pricing and hedging, however, the NTS model itself still have the competitive performance and a wide range of applications.

To finish this section, we point out several issues that haven't been discusses in this paper and hope the continuing research will shed further light on them. First, our model hasn't included the leverage effect, referred to as the correlation between asset returns and their volatilities, and this effect in BSJSV also disappear because of our desire to make the comparison between models as meaningful as possible. The important question is whether a model containing the leverage variable is relevant and useful from an economic point of view, and if the answer is yes, how to construct the framework to make the pricing and hedging possible and efficient. Second, the option pricing method developed in this paper can be used in any financial environment where the characteristic function of the underlying is available, and the hedging method therein can be adapted to any pure-jump Lévy model in the context of stochastic volatility, providing an even more flexible and reliable foundation for various financial models. Third, the new processes in this paper have delivered excellent numerical performance and can be applied to price and hedge other contingent claims, such as the path-dependent options and credit derivatives. All of these questions and possible extensions are left for the future research.

Appendix A

To derive the stochastic integral version of the European call option price, for fixed K and T, apply the Itô formula to (8) and rewrite the discounted call option price $\hat{C}(t, S_t) = e^{-rt} C(t, S_t)$ as a stochastic differential equation

$$d\hat{C}(t, S_t) = \int_{\mathbb{R}} \left[\hat{C}\left(t, S_{t-} e^x\right) - \hat{C}(t-, S_{t-})\right] v_t J_{\widetilde{Y}}(dt, dx)$$

$$+ e^{-rt}\Big\{-rC(t, S_{t-}) + \frac{\partial C}{\partial t}(t, S_{t-}) + rS_{t-}\frac{\partial C}{\partial S}(t, S_{t-})$$
$$+ \int_{\mathbb{R}} \big[C\big(t, S_{t-}e^x\big) - C(t-, S_{t-})\big] v_t \varPi_{\widetilde{Y}}(dx)\Big\}dt$$

in which

$$\int_{\mathbb{R}} \big[C\big(t, S_{t-}e^x\big) - C(t-, S_{t-})\big] v_t \varPi_{\widetilde{Y}}(dx)$$
$$= \frac{S_{t-}e^{\alpha(r\tau-\kappa)}}{\pi} \int_0^\infty \Re\Big\{e^{-iu\kappa}\frac{\partial \wp}{\partial \tau}(t, u-i-i\alpha)\frac{\exp\{iur\tau + \wp(\tau, u-i-i\alpha)\}}{(iu+\alpha+1)(iu+\alpha)}\Big\}du$$

The feasibility of changing the order of integration in the last equation is due to the Stochastic Fubini Theorem with respect to compensated Poisson random measures under the applicability condition

$$\widetilde{E}\bigg(\int_{\mathbb{R}}\int_{\mathbb{R}}\bigg|\big[e^{(iu+\alpha+1)x} - 1\big] \times \frac{\exp\{iur\tau + \wp(\tau, u-i-i\alpha)\}}{(iu+\alpha+1)(iu+\alpha)}\bigg|^2 \varPi_{\widetilde{Y}}(dx)du\bigg)$$
$$\le \widetilde{E}\bigg(\int_{\mathbb{R}} \big[\phi_{\widetilde{Y}}(-2i-2i\alpha) - 2\Re\big\{\phi_{\widetilde{Y}}(u-i-i\alpha)\big\}\big] \times \bigg|\frac{e^{\wp(\tau,u-i-i\alpha)}}{(iu+\alpha+1)(iu+\alpha)}\bigg|^2 du\bigg) < \infty$$

The last inequality holds by recalling the choice of α that

$$\big|e^{\wp(\tau,u-i-i\alpha)}\big| = e^{\Re\{\wp(\tau,u-i-i\alpha)\}} < \infty$$

For more details, the reader is referred to Applebaum [1]. Then dt term equals to zero, and $e^{-rt}C(t, S_t)$ is a martingale under $\widetilde{\mathbb{P}}$, which can be verified also by the definition (7).

Appendix B

Now let's assume that under the risk-neutral probability $\widetilde{\mathbb{P}}$, the stock return follows the lognormal jump-diffusion process with stochastic volatility as in Bakshi et al. [2], named BSJSV and given by

$$\frac{dS_t}{S_t} = (r - \lambda_J\mu_J)dt + \sqrt{v_t}dW_t + J_t dN_t$$
$$\ln(1 + J_t) \sim N\bigg(\ln(1 + \mu_J) - \frac{1}{2}\sigma_J^2, \sigma_J^2\bigg) \tag{14}$$

where r is the riskless interest rate, N_t is the Poisson jump with intensity λ_J and is independent of the standard Brownian motion W_t, J_t is the percentage jump size that's i.i.d. lognormally, and v_t is defined as in (5). The characteristic function of the asset return is explicitly known as

$$\widetilde{E}e^{iu\ln\frac{S_t}{S_0}} = \exp\bigg\{iu(r - \lambda_J\mu_J)t + \frac{2\varpi V_0(1 - e^{-\gamma_J t})}{2\gamma_J - (\gamma_J - \iota)(1 - e^{-\gamma_J t})}$$
$$- \frac{\iota\xi}{\varsigma^2}\bigg[2\ln\bigg[1 - \frac{\gamma_J - \iota}{2\gamma_J}\big(1 - e^{-\gamma_J t}\big)\bigg] + (\gamma_J - \iota)t\bigg] + \lambda_J t\big[(1 + \mu_J)^{iu}e^{\varpi\sigma_J^2} - 1\big]\bigg\}$$
$$= \exp\big\{iurt + \bar{\wp}(t, u)\big\}$$

where $\varpi = -(u^2 + iu)/2$, $\gamma_J = \sqrt{\iota^2 - 2\varpi\varsigma^2}$. To hedge options in a BSJSV model, solve Eq. (9) and hence

$$\begin{aligned}&\widetilde{E}\left[e^{-rT}(S_T - K)^+ - C_0 - \int_0^T \vartheta_t d\hat{S}_t\right]^2 \\ &= \widetilde{E}\left\{\int_0^T \left[\frac{\partial C}{\partial S}(t, S_{t-}) - \vartheta_t\right]\hat{S}_{t-}\sqrt{v_t}dW_t\right\}^2 \\ &\quad + \widetilde{E}\left\{\int_0^T \int_{\mathbb{R}} \left[\hat{C}\left(t, S_{t-}e^x\right) - \hat{C}(t-, S_{t-}) - \vartheta_t\hat{S}_{t-}\left(e^x - 1\right)\right]\widetilde{J}(dz, dx)\right\}^2\end{aligned}$$

where $\widetilde{J}(dz, dx) = J(dz, dx) - \lambda_J f_J(dx)dt$ is the compensated random measure of the compound Poisson process $\int_0^t \ln(1 + J_t)dN_t$, and $f_J(dx)$ is the probability distribution function of $\ln(1 + J_t)$. Following the same procedure as in Subsection 3.3, we have

$$\begin{aligned}\vartheta_t &= \frac{v_t\frac{\partial C}{\partial S}(t, S_{t-}) + \frac{1}{S_{t-}}\int_{\mathbb{R}}[C(t, S_{t-}e^x) - C(t-, S_{t-})](e^x - 1)\lambda_J f_J(dx)}{v_t + \int_{\mathbb{R}}(e^x - 1)^2\lambda_J f_J(dx)} \\ &= \frac{e^{\alpha(r\tau - \kappa)}}{\pi}\int_0^\infty \Re\left\{\left[\frac{(1 + \alpha + iu)v_t + \phi_J(u - 2i - i\alpha) - \phi_J(u - i - i\alpha) - \phi_J(-i)}{v_t + \phi_J(-2i) - 2\phi_J(-i)}\right]\right. \\ &\quad \left. \times e^{-iu\kappa}\frac{e^{iur\tau + \bar{\wp}(\tau, u - i - i\alpha)}}{(iu + \alpha + 1)(iu + \alpha)}\right\}du \qquad (15)\end{aligned}$$

where $\phi_J(u) = \lambda_J[(1 + \mu_J)^{iu}\exp\{-\frac{1}{2}(u^2 + iu)\sigma_J^2\} - 1]$.

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