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On a Kirchhoff type problem in $\mathbb{R}^N$

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ABSTRACT

In this paper, we investigate the following Kirchhoff type problem:

$$\begin{cases} \left(\alpha \displaystyle\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)dx + \beta\right)(-\Delta u + u) = |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$\mathcal{P}_{\alpha,\beta}$}$$

where $N \geq 1$, $2 < p < 2^*$ ($2^* = 2N/(N-2)$ if $N \geq 3$, $2^* = \infty$ if $N = 1, 2$) and α, β are two positive parameters. By studying the decomposition of the Nehari manifold to ($\mathcal{P}_{\alpha,\beta}$) and using the scaling technique, we give a total description on the positive solutions to ($\mathcal{P}_{\alpha,\beta}$). We also make an observation on the sign-changing solutions to ($\mathcal{P}_{\alpha,\beta}$) in the current paper.

1. Introduction

In this paper, we consider the following Kirchhoff type problem:

$$\begin{cases} \left(\alpha \displaystyle\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)dx + \beta\right)(-\Delta u + u) = |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$\mathcal{P}_{\alpha,\beta}$}$$

where $N \geq 1$, $2 < p < 2^*$ ($2^* = 2N/(N-2)$ if $N \geq 3$, $2^* = \infty$ if $N = 1, 2$) and $\alpha, \beta > 0$ are two parameters.

It is well-known that ($\mathcal{P}_{0,1}$) is the basic Schrödinger equation, which has exactly one positive solution up to a translation and infinitely many sign-changing solutions. The unique positive solution is radial symmetric and is also the unique least energy solution to ($\mathcal{P}_{0,1}$), while the energy values of the sign-changing solutions

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go to infinity. On the other hand, to the best of our knowledge, for the Kirchhoff type problem $(\mathcal{P}_{\alpha,\beta})$, only the existence result of one positive solution with $N = 1, 2, 3$ and $p \in (4, 2^*)$ was established in [2] by Alves and Figueiredo. Simultaneously, in recent years, the Kirchhoff type problems in the whole space $\mathbb{R}^N$ with $N = 1, 2, 3$ have been studied widely by the variational methods since then the nice work [8], and various existence results of the solutions to such problems were established, see for example [2,3,6,7,9–13] and the references therein. Inspired by the above facts, the purpose of the current paper is to make a detailed description on the solutions of $(\mathcal{P}_{\alpha,\beta})$.

Let us give some notations before we state our results. We respectively denote the unique positive solution and infinitely many sign-changing solutions of $(\mathcal{P}_{0,1})$ by ψ and $\{\varphi_n\}$. Then ψ is the unique least energy solution of $(\mathcal{P}_{0,1})$ and $J(\varphi_n) \to +\infty$ as $n \to +\infty$, where $J(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\|u\|_p^p$ is the corresponding functional to $(\mathcal{P}_{0,1})$, $\|\cdot\|$ and $\|\cdot\|_p$ are the usual norms in $H^1(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$, respectively. Without loss of generality, we assume $J(\varphi_n) < J(\varphi_{n+1})$ for all $n \in \mathbb{N}$. Then it is well-known that $0 < \sqrt{2}\|\psi\| < \|\varphi_1\| < \cdots < \|\varphi_n\| < \cdots$ with $\|\varphi_n\| \to +\infty$ as $n \to +\infty$. For each $\beta > 0$, let us denote

$$\alpha_1 = \frac{1}{\|\psi\|^2}, \qquad \alpha_1(\beta) = \frac{(p-2)\beta}{(4-p)\|\psi\|^2}\left(\frac{4-p}{2\beta}\right)^{\frac{2}{p-2}}, \tag{1.1}$$

$$\widetilde{\alpha}_n = \frac{1}{\|\varphi_n\|^2}, \qquad \widetilde{\alpha}_n(\beta) = \frac{(p-2)\beta}{(4-p)\|\varphi_n\|^2}\left(\frac{4-p}{2\beta}\right)^{\frac{2}{p-2}}. \tag{1.2}$$

Then it is easily see that $\frac{1}{2}\alpha_1 > \widetilde{\alpha}_1 > \cdots > \widetilde{\alpha}_n > \cdots$ and $\frac{1}{2}\alpha_1(\beta) > \widetilde{\alpha}_1(\beta) > \cdots > \widetilde{\alpha}_n(\beta) > \cdots$ with both $\widetilde{\alpha}_n \to 0$ and $\widetilde{\alpha}_n(\beta) \to 0$ as $n \to +\infty$.

Now, our first result can be stated as follows.

Theorem 1.1. *Assume $\alpha, \beta > 0$ and $n \in \mathbb{N}$. Then we have the following.*

- (a_1) *If $N = 1, 2, 3$ and $p \in (4, 2^*)$, then $(\mathcal{P}_{\alpha,\beta})$ has exactly one positive solution $u_{\alpha,\beta}$ up to a translation. Furthermore, $(\mathcal{P}_{\alpha,\beta})$ has infinitely many sign-changing solutions $\{v^i_{\alpha,\beta}\}$.*
- (a_2) *If $N = 1, 2, 3$ and $p = 4$, then $(\mathcal{P}_{\alpha,\beta})$ has exactly one positive solution $\widetilde{u}_{\alpha,\beta}$ for $\alpha \in (0, \alpha_1)$ and no positive solution for $\alpha \in [\alpha_1, +\infty)$ up to a translation. Furthermore, $(\mathcal{P}_{\alpha,\beta})$ has at least n sign-changing solutions $\{\widetilde{v}^i_{\alpha,\beta}\}$ for $\alpha \in (0, \widetilde{\alpha}_n)$.*
- (a_3) *If $p \in (2, 4) \cap (2, 2^*)$, then $(\mathcal{P}_{\alpha,\beta})$ has exactly two positive solutions $\bar{u}^1_{\alpha,\beta}$ and $\bar{u}^2_{\alpha,\beta}$ for $\alpha \in (0, \alpha_1(\beta))$, exactly one positive solution $\bar{u}^0_{\alpha,\beta}$ for $\alpha = \alpha_1(\beta)$ and no positive solution for $\alpha \in (\alpha_1(\beta), +\infty)$ up to a translation. Furthermore, $(\mathcal{P}_{\alpha,\beta})$ has at least $2n$ sign-changing solutions $\{\bar{v}^{1,i}_{\alpha,\beta}\}$ and $\{\bar{v}^{2,i}_{\alpha,\beta}\}$ for $\alpha \in (0, \widetilde{\alpha}_n(\beta))$ and at least $2n - 1$ sign-changing solutions $\{\bar{v}^{1,i}_{\alpha,\beta}\}$, $\{\bar{v}^{2,i}_{\alpha,\beta}\}$ and $\bar{v}^{0,n}_{\alpha,\beta}$ for $\alpha = \widetilde{\alpha}_n(\beta)$.*
- (a_4) *If $N = 1, 2, 3$, $p = 4$ and $\alpha \geq \frac{1}{2}\alpha_1$ or $p \in (2, 4) \cap (2, 2^*)$ and $\alpha \geq \frac{1}{2}\alpha_1(\beta)$, then $(\mathcal{P}_{\alpha,\beta})$ has no sign-changing solution.*

Remark 1.1. By checking the proof of Theorem 1.1, we can see that the positive solutions of $(\mathcal{P}_{\alpha,\beta})$ obtained by this theorem are all radial symmetric up to a translation. Furthermore, this theorem gives a total description on the existence and nonexistence of the positive solutions to $(\mathcal{P}_{\alpha,\beta})$. It is also worth to point out that Theorem 1.1 seems to be the first result for the Kirchhoff type problems in the high dimensions $(N \geq 5)$.

Since the solutions obtained by Theorem 1.1 are dependent on the parameters α and β, it is natural to discuss the concentration behaviors for α and β vary. Note that α is the parameter of the Kirchhoff type non-local term, so we mainly study the concentration behaviors of the solutions to $(\mathcal{P}_{\alpha,\beta})$ for α. Our result on this topic can be stated as follows.

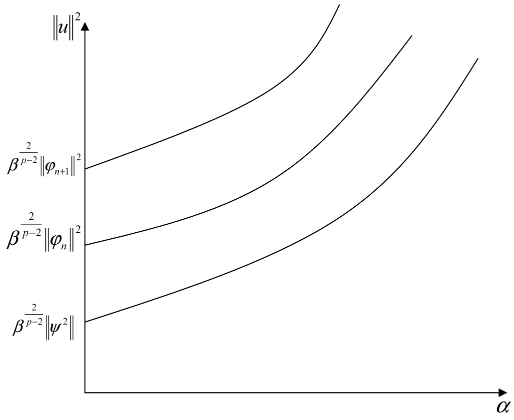

Fig. 1. Case of $p \in (4, 2^*)$.

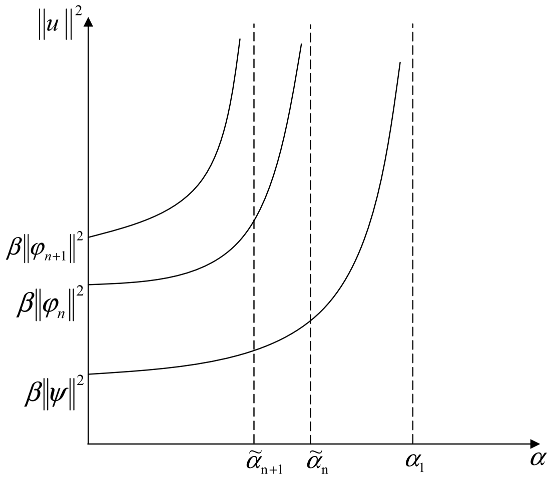

Fig. 2. Case of $p = 4$.

Theorem 1.2. *Assume $\alpha, \beta > 0$. Then we have the following.*

(b_1) *If $N = 1, 2, 3$ and $p \in (4, 2^*)$, then $\lim_{\alpha\to 0^+} u_{\alpha,\beta} = \beta^{\frac{1}{p-2}}\psi$ and $\lim_{\alpha\to 0^+} v^i_{\alpha,\beta} = \beta^{\frac{1}{p-2}}\varphi_i$ in $H^1(\mathbb{R}^N)$, while $\lim_{\alpha\to+\infty} \|u_{\alpha,\beta}\| = \lim_{\alpha\to+\infty} \|v^i_{\alpha,\beta}\| = +\infty$ for all $i \in \mathbb{N}$. Furthermore, $\|u_{\alpha,\beta}\|$ and $\|v^i_{\alpha,\beta}\|$ are all strictly increasing functions for α and $\sqrt{2}\|u_{\alpha,\beta}\| < \|v^1_{\alpha,\beta}\| < \cdots < \|v^i_{\alpha,\beta}\| < \cdots$ with $\lim_{i\to+\infty} \|v^i_{\alpha,\beta}\| = +\infty$.*

(b_2) *If $N = 1, 2, 3$ and $p = 4$, then $\lim_{\alpha\to 0^+} \widetilde{u}_{\alpha,\beta} = \beta^{\frac{1}{2}}\psi$ and $\lim_{\alpha\to 0^+} \widetilde{v}^i_{\alpha,\beta} = \beta^{\frac{1}{2}}\varphi_i$ in $H^1(\mathbb{R}^N)$, while $\lim_{\alpha\to\alpha_1^-} \|\widetilde{u}_{\alpha,\beta}\| = \lim_{\alpha\to\widetilde{\alpha}_i^-} \|\widetilde{v}^i_{\alpha,\beta}\| = +\infty$ for all $i \in \mathbb{N}$. Furthermore, $\|\widetilde{u}_{\alpha,\beta}\|$ and $\|\widetilde{v}^i_{\alpha,\beta}\|$ are all strictly increasing functions for α and $\sqrt{2}\|\widetilde{u}_{\alpha,\beta}\| < \|\widetilde{v}^1_{\alpha,\beta}\| < \cdots < \|\widetilde{v}^i_{\alpha,\beta}\|$ for $\alpha \in (0, \widetilde{\alpha}_i)$ and all $i \in \mathbb{N}$.*

(b_3) *If $p \in (2, 4) \cap (2, 2^*)$, then $\lim_{\alpha\to 0^+} \bar{u}^1_{\alpha,\beta} = \beta^{\frac{1}{p-2}}\psi$ and $\lim_{\alpha\to 0^+} \bar{v}^{1,i}_{\alpha,\beta} = \beta^{\frac{1}{p-2}}\varphi_i$ in $H^1(\mathbb{R}^N)$ and $\lim_{\alpha\to 0^+} \|\bar{u}^2_{\alpha,\beta}\| = +\infty$ and $\lim_{\alpha\to 0^+} \|\bar{v}^{2,i}_{\alpha,\beta}\| = +\infty$, while $\lim_{\alpha\to\alpha_1(\beta)^-} \bar{u}^1_{\alpha,\beta} = \lim_{\alpha\to\alpha_1(\beta)^-} \bar{u}^2_{\alpha,\beta} = \bar{u}^0_{\alpha,\beta}$ and $\lim_{\alpha\to\widetilde{\alpha}_i(\beta)^-} \bar{v}^{1,i}_{\alpha,\beta} = \lim_{\alpha\to\widetilde{\alpha}_i(\beta)^-} \bar{v}^{2,i}_{\alpha,\beta} = \bar{v}^{0,i}_{\alpha,\beta}$ for all $i \in \mathbb{N}$ in $H^1(\mathbb{R}^N)$. Furthermore, $\|\bar{u}^1_{\alpha,\beta}\|$ and $\|\bar{v}^{1,i}_{\alpha,\beta}\|$ are strictly increasing functions for α, $\|\bar{u}^2_{\alpha,\beta}\|$ and $\|\bar{v}^{2,i}_{\alpha,\beta}\|$ are strictly decreasing functions for α and $\sqrt{2}\|\bar{u}^1_{\alpha,\beta}\| < \|\bar{v}^{1,1}_{\alpha,\beta}\| < \cdots < \|\bar{v}^{1,i}_{\alpha,\beta}\| < \|\bar{v}^{2,i}_{\alpha,\beta}\| < \cdots < \|\bar{v}^{2,1}_{\alpha,\beta}\| < \|\bar{u}^2_{\alpha,\beta}\|$ for $\alpha \in (0, \alpha_i(\beta))$ and all $i \in \mathbb{N}$.*

According to Theorem 1.2, we can illustrate the concentration behaviors of the solutions to $(\mathcal{P}_{\alpha,\beta})$ obtained by Theorem 1.1 with Figs. 1–3.

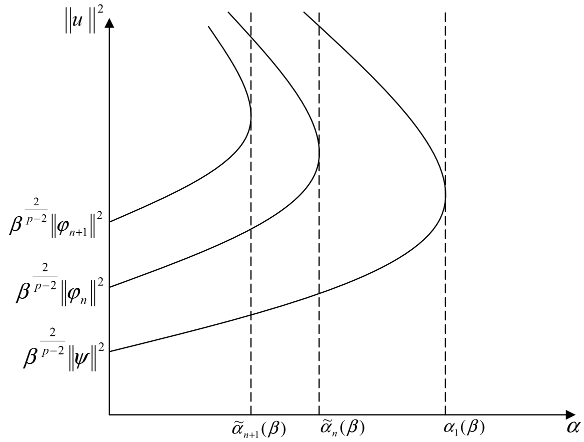

Fig. 3. Case of $p \in (2, 4)$.

In this paper, we also make some estimates on the energy values of the solutions to $(\mathcal{P}_{\alpha,\beta})$ obtained by Theorem 1.1. Let $I_{\alpha,\beta}(u) \in C^2(H^1(\mathbb{R}^N), \mathbb{R})$ be the corresponding functional to $(\mathcal{P}_{\alpha,\beta})$, which is given by

$$I_{\alpha,\beta}(u) = \frac{\alpha}{4}\|u\|^4 + \frac{\beta}{2}\|u\|^2 - \frac{1}{p}\|u\|_p^p. \tag{1.3}$$

Now, our result for the energy values of the solutions to $(\mathcal{P}_{\alpha,\beta})$ can be stated as follows.

Theorem 1.3. *Assume $\alpha, \beta > 0$. Then we have the following.*

(c_1) *If $N = 1, 2, 3$ and $p \in (4, 2^*)$, then $u_{\alpha,\beta}$ is the unique ground state solution of $(\mathcal{P}_{\alpha,\beta})$. Furthermore, $2I_{\alpha,\beta}(u_{\alpha,\beta}) + \frac{(p-4)\alpha}{2p}\|u_{\alpha,\beta}\|^4 < I_{\alpha,\beta}(v^1_{\alpha,\beta}) < \cdots < I_{\alpha,\beta}(v^i_{\alpha,\beta}) < \cdots$ with $I_{\alpha,\beta}(v^i_{\alpha,\beta}) \to +\infty$ as $i \to +\infty$.*

(c_2) *If $N = 1, 2, 3$ and $p = 4$, then $\widetilde{u}_{\alpha,\beta}$ is the unique ground state solution of $(\mathcal{P}_{\alpha,\beta})$ for $\alpha \in (0, \alpha_1)$. Furthermore, $2I_{\alpha,\beta}(u_{\alpha,\beta}) < I_{\alpha,\beta}(v^1_{\alpha,\beta}) < \cdots < I_{\alpha,\beta}(v^i_{\alpha,\beta})$ for $\alpha \in (0, \widetilde{\alpha}_i)$ and all $i \in \mathbb{N}$.*

(c_3) *If $p \in (2, 4) \cap (2, 2^*)$, then $\bar{u}^2_{\alpha,\beta}$ is the unique ground state solution of $(\mathcal{P}_{\alpha,\beta})$ with $I_{\alpha,\beta}(\bar{u}^2_{\alpha,\beta}) < 0$ for $\alpha \in (0, 2^{\frac{p}{p-2}}p^{-\frac{2}{p-2}}\alpha_1(\beta))$, $I_{\alpha,\beta}(\bar{u}^2_{\alpha,\beta}) = 0$ for $\alpha = 2^{\frac{p}{p-2}}p^{-\frac{2}{p-2}}\alpha_1(\beta)$ and $I_{\alpha,\beta}(\bar{u}^2_{\alpha,\beta}) > 0$ for $\alpha \in (2^{\frac{p}{p-2}}p^{-\frac{2}{p-2}}\alpha_1(\beta), \alpha_1(\beta))$ and $\bar{u}^1_{\alpha,\beta}$ is the unique solution of $(\mathcal{P}_{\alpha,\beta})$ satisfying $I_{\alpha,\beta}(\bar{u}^1_{\alpha,\beta}) = \inf_{u\in\mathcal{S}_*}\sup_{t\geq 0} I_{\alpha,\beta}(tu)$ with $I_{\alpha,\beta}(\bar{u}^1_{\alpha,\beta}) \in (0, \frac{(p-2)^2\beta^2}{4p(4-p)\alpha})$, where*

$$\mathcal{S}_* = \left\{u \in H^1\left(\mathbb{R}^N\right)\backslash\{0\} \mid \mathcal{S}(u) < \mathcal{D}(\alpha, \beta, p)\right\} \tag{1.4}$$

and

$$\mathcal{S}(u) = \left(\frac{\|u\|}{\|u\|_p}\right)^{\frac{2p}{4-p}} \quad \text{and} \quad \mathcal{D}(\alpha, \beta, p) = \frac{4-p}{2\beta}\left(\frac{p-2}{2\alpha}\right)^{\frac{p-2}{4-p}}. \tag{1.5}$$

Furthermore, for $\alpha \in [2^{\frac{p}{p-2}}p^{-\frac{2}{p-2}}\alpha_1(\beta), \alpha_1(\beta))$, $\bar{u}^2_{\alpha,\beta}$ is a local minimum point of $I_{\alpha,\beta}(u)$ and for $\alpha \in (0, 2^{\frac{p}{p-2}}p^{-\frac{2}{p-2}}\alpha_1(\beta))$, $\bar{u}^2_{\alpha,\beta}$ is a global minimum point of $I_{\alpha,\beta}(u)$ with $I_{\alpha,\beta}(\bar{u}^2_{\alpha,\beta}) \to -\infty$ as $\alpha \to 0^+$ and $2I_{\alpha,\beta}(\bar{u}^1_{\alpha,\beta}) - \frac{(4-p)\alpha}{2p}\|\bar{u}^1_{\alpha,\beta}\|^4 < I_{\alpha,\beta}(\bar{v}^{1,1}_{\alpha,\beta}) < \cdots < I_{\alpha,\beta}(\bar{v}^{1,i}_{\alpha,\beta}) < \frac{(p-2)^2\beta^2}{4p(4-p)\alpha}$ and $I_{\alpha,\beta}(\bar{u}^2_{\alpha,\beta}) < I_{\alpha,\beta}(\bar{v}^{2,1}_{\alpha,\beta}) < \cdots < I_{\alpha,\beta}(\bar{v}^{2,i}_{\alpha,\beta}) < \frac{(p-2)^2\beta^2}{4p(4-p)\alpha}$ for $\alpha \in (0, \alpha_i(\beta))$ and all $i \in \mathbb{N}$, while $I_{\alpha,\beta}(\bar{u}^0_{\alpha,\beta}) = I_{\alpha,\beta}(\bar{v}^{0,i}_{\alpha,\beta}) = \frac{(p-2)^2\beta^2}{4p(4-p)\alpha}$ for all $i \in \mathbb{N}$.

Remark 1.2.

(1) We point out that Theorem 1.3 gives the total description on the energy values of the positive solutions to $(\mathcal{P}_{\alpha,\beta})$. Our result on energy estimates for the sign-changing solutions to $(\mathcal{P}_{\alpha,\beta})$ with $N = 1, 2, 3$ and $p \in (4, 2^*)$ is much more precise than the corresponding result in [9]. Moreover, Theorem 1.3 seems to be the first result on the estimates of energy values for the sign-changing solutions to $(\mathcal{P}_{\alpha,\beta})$ with $p \in (2, 4) \cap (2, 2^*)$.

(2) For $\alpha, \beta > 0$, let $\mathbb{Q}_{\alpha,\beta} = \{u \in H^1(\mathbb{R}^N) \mid u \text{ is a solution of } (\mathcal{P}_{\alpha,\beta})\}$, then by checking the proof of Theorem 1.3 (see Lemma 4.10 for details), we can see that $\partial\mathbb{Q}_{\alpha,\beta} = \{\bar{u}^0_{\alpha,\beta}, \bar{u}^1_{\alpha,\beta}, \bar{u}^2_{\alpha,\beta}\}$.

(3) By Theorem 1.3, we see that $\bar{u}^2_{\alpha,\beta}$ is a local minimum point of $I_{\alpha,\beta}(u)$ in $H^1(\mathbb{R}^N)$ for $\alpha \in (0, \alpha_1(\beta))$, and 0 is clearly a local minimum point of $I_{\alpha,\beta}(u)$ in $H^1(\mathbb{R}^N)$ for all $\alpha > 0$ due to $2 < p < 2^*$. Therefore, $\bar{u}^2_{\alpha,\beta}$ and 0 are also local minimum points of the functional $I^+_{\alpha,\beta}(u)$ in $H^1_r(\mathbb{R}^N)$, where

$$I^+_{\alpha,\beta}(u) = \frac{\alpha}{4}\|u\|^4 + \frac{\beta}{2}\|u\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} (u^+)^p dx$$

and $H^1_r(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) \mid u \text{ is radial symmetric}\}$. Note that $H^1_r(\mathbb{R}^N)$ is compactly embedded into $L^p(\mathbb{R}^N)$ for $p \in (2, 2^*)$, so by the symmetric criticality principle of Palais and the maximum principle, we have $I^+_{\alpha,\beta}(\bar{u}^1_{\alpha,\beta}) = \inf_{\bar{h}\in\bar{\Gamma}} \sup_{t\in[0,1]} I^+_{\alpha,\beta}(\bar{h}(t))$ and $(I^+_{\alpha,\beta})'(\bar{u}^1_{\alpha,\beta}) = 0$ in $H^{-1}_r(\mathbb{R}^N)$, where

$$\bar{\Gamma} = \left\{\bar{h} \in C\big([0,1], H^1_r\big(\mathbb{R}^N\big)\big) \mid \bar{h}(0) = 0, \bar{h}(1) = \bar{u}^2_{\alpha,\beta}\right\}$$

and $H^{-1}_r(\mathbb{R}^N)$ is the dual space of $H^1_r(\mathbb{R}^N)$. It follows that for all $\alpha \in (0, \alpha_1(\beta))$, $\bar{u}^1_{\alpha,\beta}$ is a mountain pass solution of the following equation

$$\begin{cases} \left(\alpha \displaystyle\int_{\mathbb{R}^3} \big(|\nabla u|^2 + u^2\big)dx + \beta\right)(-\Delta u + u) = u^{p-1} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u \in H^1_r\big(\mathbb{R}^N\big). \end{cases} \tag{1.6}$$

Since $(\mathcal{P}_{\alpha,\beta})$ has no sign-changing solution for $\alpha \geq \frac{1}{2}\alpha_1(\beta)$ and $p \in (2, 4) \cap (2, 2^*)$ by Theorem 1.1, we can similarly conclude that for $\alpha \in [\frac{1}{2}\alpha_1(\beta), \alpha_1(\beta))$, $\bar{u}^1_{\alpha,\beta}$ is a mountain pass solution of the following equation

$$\begin{cases} \left(\alpha \displaystyle\int_{\mathbb{R}^3} \big(|\nabla u|^2 + u^2\big)dx + \beta\right)(-\Delta u + u) = |u|^{p-2}u & \text{in } \mathbb{R}^N, \\ u \in H^1_r\big(\mathbb{R}^N\big). \end{cases} \tag{1.7}$$

However, $\mathcal{S}_*$ is not homeomorphic to the unit sphere of $H^1(\mathbb{R}^N)$ and even not weakly low semi-continuous on $H^1(\mathbb{R}^N)$, hence we are not sure whether $\bar{u}^1_{\alpha,\beta}$ is a mountain pass solution of $(\mathcal{P}_{\alpha,\beta})$ for $p \in (2, 4) \cap (2, 2^*)$ and $\alpha \in (0, \alpha_1(\beta))$ in general due to Theorem 1.3.

The remaining part of this paper will be devoted to the proofs of Theorems 1.1 to 1.3 and will be organized as follows. In Section 2, we follow the idea of [1] to study the existence and nonexistence of the solutions to $(\mathcal{P}_{\alpha,\beta})$ and give the proof of Theorem 1.1. In Section 3, we will discuss the concentration behaviors of the solutions obtained in Theorem 1.1 by applying the implicit function theorem. Simultaneously, we will also show Theorem 1.2 in this section. In Section 4, we turn to give some estimates on the energy values of the solutions obtained in Theorem 1.1 by studying the decomposition of the Nehari manifold to $I_{\alpha,\beta}(u)$, while Theorem 1.3 will also be proved in this section.

2. The existence of solutions

The main task in this section is to investigate the existence of solutions to $(\mathcal{P}_{\alpha,\beta})$. This task will be finished essentially by the following two observations on the relation between $(\mathcal{P}_{\alpha,\beta})$ and $(\mathcal{P}_{0,1})$, which are inspired by [1].

Lemma 2.1. *Suppose u is a solution of $(\mathcal{P}_{\alpha,\beta})$ for $\alpha, \beta > 0$ and $2 < p < 2^*$. Then there exists a unique $s_{\alpha,\beta}(u) > 0$ such that $s_{\alpha,\beta}(u)u$ is a solution of $(\mathcal{P}_{0,1})$.*

Proof. Let $\varphi_s = su$ for $s > 0$. Since u is a solution of $(\mathcal{P}_{\alpha,\beta})$, we have

$$-\Delta\varphi_s + \varphi_s = \frac{s}{\alpha\|u\|^2 + \beta}|u|^{p-2}u = \frac{s^{2-p}}{\alpha\|u\|^2 + \beta}|\varphi_s|^{p-2}\varphi_s.$$

Note that $\alpha, \beta > 0$ and $2 < p < 2^*$, there exists a unique $s_{\alpha,\beta}(u) > 0$ such that $[s_{\alpha,\beta}(u)]^{2-p} = \alpha\|u\|^2 + \beta$. It follows that $s_{\alpha,\beta}(u)u$ is a solution of $(\mathcal{P}_{0,1})$. $\square$

Lemma 2.2. *Suppose $\alpha, \beta > 0$, $p \in (2, 2^*)$ and ϕ is a solution of $(\mathcal{P}_{0,1})$. Then we have the following.*

(1) *If $N = 1, 2, 3$ and $p \in (4, 2^*)$, then there exists exactly one constant $r_{\alpha,\beta,p} > 0$ such that $r_{\alpha,\beta,p}\phi \in \mathcal{D}_\phi$, where $\mathcal{D}_\phi = \{r\phi \mid r > 0, r\phi \text{ is a solution of } (\mathcal{P}_{\alpha,\beta})\}$.*
(2) *If $N = 1, 2, 3$ and $p = 4$, then $(\frac{\beta}{1-\alpha\|\phi\|^2})^{\frac{1}{2}}\phi$ is the unique one in the set $\mathcal{D}_\phi$ for $\alpha \in (0, \|\phi\|^{-2})$ and $\mathcal{D}_\phi = \emptyset$ for $\alpha \in [\|\phi\|^{-2}, +\infty)$.*
(3) *If $p \in (2, 4) \cap (2, 2^*)$, then for $\alpha \in (0, \alpha_\phi)$, there exist exactly two constants $r^1_{\alpha,\beta,p}, r^2_{\alpha,\beta,p} > 0$ such that $\{r^i_{\alpha,\beta,p}\phi\} \subset \mathcal{D}_\phi$, for $\alpha = \alpha_\phi$, $(\frac{2\beta}{4-p})^{\frac{1}{p-2}}\phi$ is the unique one in the set $\mathcal{D}_\phi$ and for $\alpha \in (\alpha_\phi, +\infty)$, $\mathcal{D}_\phi = \emptyset$, where*

$$\alpha_\phi = \frac{(p-2)\beta}{(4-p)\|\phi\|^2}\left(\frac{4-p}{2\beta}\right)^{\frac{2}{p-2}}. \tag{2.1}$$

Proof. Since ϕ is a solution of $(\mathcal{P}_{0,1})$, by a direct calculation, we can see that

$$\big(\alpha\|r\phi\|^2 + \beta\big)(-\Delta r\phi + r\phi) = f_{\alpha,\beta,p}(r)|r\phi|^{p-2}r\phi,$$

where $f_{\alpha,\beta,p}(r) = \alpha\|\phi\|^2 r^{4-p} + \beta r^{2-p}$. It follows that $r\phi$ is a solution of $(\mathcal{P}_{\alpha,\beta})$ if and only if $f_{\alpha,\beta,p}(r) = 1$.

(1) If $p > 4$, then by a direct calculation we can see that $\frac{df_{\alpha,\beta,p}(r)}{dr} < 0$ on $(0, +\infty)$. On the other hand, it is easily see that $\lim_{r\to 0^+} f_{\alpha,\beta,p}(r) = +\infty$ and $\lim_{r\to+\infty} f_{\alpha,\beta,p}(r) = 0$. Hence, there exists a unique $r_{\alpha,\beta,p} > 0$ such that $f_{\alpha,\beta,p}(r_{\alpha,\beta,p}) = 1$, which implies $r_{\alpha,\beta,p}\phi$ is the unique one in $\mathcal{D}_\phi$.

(2) If $p = 4$, then by a direct calculation we can show that $f_{\alpha,\beta,4}(r) = 1$ if and only if $\alpha \in (0, \|\phi\|^{-2})$ and $r = (\frac{\beta}{1-\alpha\|\phi\|^2})^{\frac{1}{2}}$. It follows that $(\frac{\beta}{1-\alpha\|\phi\|^2})^{\frac{1}{2}}\phi$ is the unique one in the set $\mathcal{D}_\phi$ for $\alpha \in (0, \|\phi\|^{-2})$ and $\mathcal{D}_\phi = \emptyset$ for $\alpha \in [\|\phi\|^{-2}, +\infty)$.

(3) For the case of $p \in (2, 4)$, by a direct calculation, we can see that $\frac{df_{\alpha,\beta,p}(r)}{dr} < 0$ on $(0, r^0_{\alpha,\beta,p})$, $\frac{df_{\alpha,\beta,p}(r)}{dr} > 0$ on $(r^0_{\alpha,\beta,p}, +\infty)$ and $\frac{df_{\alpha,\beta,p}(r)}{dr} = 0$ for $r = r^0_{\alpha,\beta,p}$, where $r^0_{\alpha,\beta,p} = (\frac{(p-2)\beta}{(4-p)\alpha\|\phi\|^2})^{\frac{1}{2}}$. It follows that

$$\min_{r>0} f_{\alpha,\beta,p}(r) = f_{\alpha,\beta,p}\big(r^0_{\alpha,\beta,p}\big) = \frac{\alpha}{\alpha_\phi},$$

where α_ϕ is given by (2.1). Since $\lim_{r\to 0^+} f_{\alpha,\beta,p}(r) = \lim_{r\to+\infty} f_{\alpha,\beta,p}(r) = +\infty$, there exist unique $0 < r^1_{\alpha,\beta,p} < r^0_{\alpha,\beta,p} < r^2_{\alpha,\beta,p}$ such that $f_{\alpha,\beta,p}(r^i_{\alpha,\beta,p}) = 1$ when $\alpha \in (0, \alpha_\phi)$, $f_{\alpha,\beta,p}(r) > 1$ for all $r > 0$ when

$\alpha \in (\alpha_\phi, +\infty)$ and $f_{\alpha,\beta,p}(r) = 1$ if and only if $r = r^0_{\alpha,\beta,p}$ when $\alpha = \alpha_\phi$. It follows that $r^1_{\alpha,\beta,p}\phi$ and $r^2_{\alpha,\beta,p}\phi$ are the exactly two points in $\mathcal{D}_\phi$ for $\alpha \in (0, \alpha_\phi)$, $(\frac{(p-2)\beta}{(4-p)\alpha\|\phi\|^2})^{\frac{1}{2}}\phi$ is the unique one in the set $\mathcal{D}_\phi$ for $\alpha = \alpha_\phi$ and $\mathcal{D}_\phi = \emptyset$ for $\alpha \in (\alpha_\phi, +\infty)$. Note that $(\frac{(p-2)\beta}{(4-p)\alpha\|\phi\|^2})^{\frac{1}{2}} = (\frac{2\beta}{4-p})^{\frac{1}{p-2}}$ if $f_{\alpha,\beta,p}(r^0_{\alpha,\beta,p}) = 1$, so $(\frac{2\beta}{4-p})^{\frac{1}{p-2}}\phi$ is the unique one in the set $\mathcal{D}_\phi$ for $\alpha = \alpha_\phi$. □

Note that ψ is the unique positive solution of $(\mathcal{P}_{0,1})$ up to a translation, by Lemmas 2.1 and 2.2, we can obtain the following existence and nonexistence results for the positive solutions of $(\mathcal{P}_{\alpha,\beta})$.

Proposition 2.1. *Suppose $\alpha, \beta > 0$ and $p \in (2, 2^*)$. Then we have the following.*

(1) *If $N = 1, 2, 3$ and $p \in (4, 2^*)$, then $(\mathcal{P}_{\alpha,\beta})$ has exactly one positive solution up to a translation.*
(2) *If $N = 1, 2, 3$ and $p = 4$, then $(\mathcal{P}_{\alpha,\beta})$ has exactly one positive solution up to a translation for $\alpha \in (0, \alpha_1)$ and no positive solution for $\alpha \in [\alpha_1, +\infty)$, where α_1 is given by (1.1).*
(3) *If $p \in (2, 4) \cap (2, 2^*)$, then $(\mathcal{P}_{\alpha,\beta})$ has exactly two positive solutions for $\alpha \in (0, \alpha_1(\beta))$ and exactly one positive solution for $\alpha = \alpha_1(\beta)$ up to a translation. Furthermore, $(\mathcal{P}_{\alpha,\beta})$ has no positive solution for $\alpha \in (\alpha_1(\beta), +\infty)$, where $\alpha_1(\beta)$ is given by (1.1).*

Recall that $\{\varphi_n\}$ is a sequence of sign-changing solutions for $(\mathcal{P}_{0,1})$ satisfying $\|\varphi_n\| < \|\varphi_{n+1}\|$ for all $n \in \mathbb{N}$, we can obtain the following existence results for the sign-changing solution of $(\mathcal{P}_{\alpha,\beta})$ by applying Lemmas 2.1 and 2.2.

Proposition 2.2. *Suppose $\alpha, \beta > 0$, $n \in \mathbb{N}$ and $p \in (2, 2^*)$. Then we have the following.*

(1) *If $N = 1, 2, 3$ and $p \in (4, 2^*)$, then $(\mathcal{P}_{\alpha,\beta})$ has infinitely many sign-changing solutions.*
(2) *If $N = 1, 2, 3$ and $p = 4$, then $(\mathcal{P}_{\alpha,\beta})$ has at least n sign-changing solutions for $\alpha \in (0, \widetilde{\alpha}_n)$, where $\widetilde{\alpha}_n$ is given by (1.2).*
(3) *If $p \in (2, 4) \cap (2, 2^*)$, then $(\mathcal{P}_{\alpha,\beta})$ has at least $2n$ sign-changing solutions for $\alpha \in (0, \widetilde{\alpha}_n(\beta))$ and $2n - 1$ sign-changing solutions for $\alpha = \widetilde{\alpha}_n(\beta)$, where $\widetilde{\alpha}_n(\beta)$ is given by (1.2).*

Note that for every sign-changing solutions of $(\mathcal{P}_{0,1})$, denoted by u, we have $\|u\|^2 > 2\|\psi\|^2$. By using the similar arguments as used in (3) of Proposition 2.2, we can obtain the following nonexistence result of the sign-changing solutions to $(\mathcal{P}_{\alpha,\beta})$.

Proposition 2.3. *Assume $\alpha, \beta > 0$. Then we have the following.*

(1) *If $N = 1, 2, 3$, $p = 4$ and $\alpha \geq \frac{1}{2}\alpha_1$, then $(\mathcal{P}_{\alpha,\beta})$ has no sign-changing solution.*
(2) *If $p \in (2, 4) \cap (2, 2^*)$ and $\alpha \geq \frac{1}{2}\alpha_1(\beta)$, then $(\mathcal{P}_{\alpha,\beta})$ has no sign-changing solution.*

Now, we can give a proof of Theorem 1.1.

Proof of Theorem 1.1. It follows immediately from Propositions 2.1–2.3. □

3. The concentration behaviors of solutions

In this section, we will discuss the concentration behaviors of solutions to $(\mathcal{P}_{\alpha,\beta})$ obtained by Theorem 1.1. Since $(\mathcal{P}_{\alpha,\beta})$ can be seen as $(\mathcal{P}_{0,1})$ coupled with a Kirchhoff type non-local term, we mainly concern with the concentration behaviors for α, which is the parameter on the non-local term. When $N = 1, 2, 3$ and $p = 4$, the solutions of $(\mathcal{P}_{\alpha,\beta})$ obtained by Theorem 1.1 are much more simple than other cases due to Lemma 2.2.

On the other hand, since $0 < 2\|\psi\|^2 < \|\varphi_1\|^2 < \cdots < \|\varphi_n\|^2 < \cdots$ with $\|\varphi_n\|^2 \to +\infty$ as $n \to +\infty$, the concentration behaviors of the solutions in this case is very clear and can be stated as follows.

Proposition 3.1. *Assume $\alpha, \beta > 0$, $N = 1, 2, 3$ and $p = 4$. Then $\lim_{\alpha\to 0^+} \widetilde{u}_{\alpha,\beta} = \beta^{\frac{1}{2}}\psi$ and $\lim_{\alpha\to 0^+} \widetilde{v}^i_{\alpha,\beta} = \beta^{\frac{1}{2}}\varphi_i$ in $H^1(\mathbb{R}^N)$, while $\lim_{\alpha\to\alpha_1^-} \|\widetilde{u}_{\alpha,\beta}\| = \lim_{\alpha\to\widetilde{\alpha}_i^-} \|\widetilde{v}^i_{\alpha,\beta}\| = +\infty$ for all $i \in \mathbb{N}$. Furthermore, $\|\widetilde{u}_{\alpha,\beta}\|$ and $\|\widetilde{v}^i_{\alpha,\beta}\|$ are all strictly increasing functions for α and $\sqrt{2}\|\widetilde{u}_{\alpha,\beta}\| < \|\widetilde{v}^1_{\alpha,\beta}\| < \cdots < \|\widetilde{v}^i_{\alpha,\beta}\|$ for $\alpha \in (0, \widetilde{\alpha}_i)$ and all $i \in \mathbb{N}$, where $\widetilde{u}_{\alpha,\beta} = (\frac{\beta}{1-\alpha\|\psi\|^2})^{\frac{1}{2}}\psi$ and $\widetilde{v}^i_{\alpha,\beta} = (\frac{\beta}{1-\alpha\|\varphi_i\|^2})^{\frac{1}{2}}\varphi_i$.*

In the following of this section, we will study the concentration behaviors of the solutions to $(\mathcal{P}_{\alpha,\beta})$ obtained by Theorem 1.1 in the other two cases:

(a) $N = 1, 2, 3$ and $p \in (4, 2^*)$;
(b) $p \in (2, 4) \cap (2, 2^*)$.

We first consider the case (a). By Lemma 2.2, the unique positive solution of $(\mathcal{P}_{\alpha,\beta})$ obtained by Theorem 1.1 in this case can be described as $r_{\alpha,\beta,p}\psi$, where $r_{\alpha,\beta,p} > 0$ is given in Lemma 2.2. On the other hand, due to Lemmas 2.1 and 2.2, sign-changing solutions of $(\mathcal{P}_{\alpha,\beta})$ obtained by Theorem 1.1 can also be denoted by $\{\widetilde{r}^n_{\alpha,\beta,p}\varphi_n\}$, where $\widetilde{r}^n_{\alpha,\beta,p}$ satisfies $\alpha\|\varphi_n\|^2[\widetilde{r}^n_{\alpha,\beta,p}]^{4-p} + \beta[\widetilde{r}^n_{\alpha,\beta,p}]^{2-p} = 1$ for all $n \in \mathbb{N}$. In order to get a better understanding of the concentration behaviors in this cases, we respectively re-denote $r_{\alpha,\beta,p}$ and $\widetilde{r}^n_{\alpha,\beta,p}$ by $r_{\beta,p}(\alpha)$ and $\widetilde{r}^n_{\beta,p}(\alpha)$ and consider them as functions for α. Then we have the following.

Lemma 3.1. *Suppose $\alpha, \beta > 0$, $N = 1, 2, 3$ and $p \in (4, 2^*)$. Then $r_{\beta,p}(\alpha)$ and $\{\widetilde{r}^n_{\beta,p}(\alpha)\}$ are all strictly increasing functions on $[0, +\infty)$ with $r_{\beta,p}(0) = \widetilde{r}^n_{\beta,p}(0) = \beta^{\frac{1}{p-2}}$ and $\lim_{\alpha\to+\infty} r_{\beta,p}(\alpha) = \lim_{\alpha\to+\infty} \widetilde{r}^n_{\beta,p}(\alpha) = +\infty$. Moreover, $r_{\beta,p}(\alpha) < \widetilde{r}^1_{\beta,p}(\alpha) < \cdots < \widetilde{r}^n_{\beta,p}(\alpha) < \cdots$ for all $\alpha > 0$.*

Proof. Clearly, $r_{\beta,p}(0) = \widetilde{r}^n_{\beta,p}(0) = \beta^{\frac{1}{p-2}}$. In what follows, let us consider the following function

$$g_{\beta,p}(\alpha) = \alpha\|\psi\|^2\big[r_{\beta,p}(\alpha)\big]^{4-p} + \beta\big[r_{\beta,p}(\alpha)\big]^{2-p}.$$

By Lemma 2.2, $g_{\beta,p}(\alpha) \equiv 1$ for $\alpha \geq 0$. It follows from the implicit function theorem that

$$\frac{d[r_{\beta,p}(\alpha)]}{d\alpha} = \frac{\|\psi\|^2[r_{\beta,p}(\alpha)]^{3-p}}{(p-4)\alpha\|\psi\|^2[r_{\beta,p}(\alpha)]^2 + (p-2)\beta} \geq \frac{\beta^{\frac{5-p}{p-2}}\|\psi\|^2}{(p-2)} > 0,$$

which then implies that $r_{\beta,p}(\alpha)$ is a strictly increasing function on $[0, +\infty)$. Thus, we must have $\lim_{\alpha\to+\infty} r_{\beta,p}(\alpha) = +\infty$. The same properties of $\widetilde{r}^n_{\beta,p}(\alpha)$ can be obtained in a similar way by considering functions

$$\widetilde{g}^n_{\beta,p}(\alpha) = \alpha\|\varphi_n\|^2\big[\widetilde{r}^n_{\beta,p}(\alpha)\big]^{4-p} + \beta\big[\widetilde{r}^n_{\beta,p}(\alpha)\big]^{2-p}.$$

It remains to show that $r_{\beta,p}(\alpha) < \widetilde{r}^1_{\beta,p}(\alpha)$ and $\widetilde{r}^n_{\beta,p}(\alpha) < \widetilde{r}^{n+1}_{\beta,p}(\alpha)$ for all $\alpha > 0$ and $n \in \mathbb{N}$. We only give the proof of $r_{\beta,p}(\alpha) < \widetilde{r}^1_{\beta,p}(\alpha)$ for all $\alpha > 0$, since $\widetilde{r}^n_{\beta,p}(\alpha) < \widetilde{r}^{n+1}_{\beta,p}(\alpha)$ for all $\alpha > 0$ and $n \in \mathbb{N}$ can be obtained in a similar way. For every $\phi \in H^1(\mathbb{R}^N)$, we consider the following function

$$h_{\beta,p,\phi}(\alpha, r) = \alpha\|\phi\|^2 r^{4-p} + \beta r^{2-p}. \tag{3.1}$$

Clearly $h_{\beta,p,\psi}(\alpha, r_{\beta,p}(\alpha)) = h_{\beta,p,\varphi_1}(\alpha, \widetilde{r}^1_{\beta,p}(\alpha)) = 1$. On the other hand, since $2\|\psi\|^2 < \|\varphi_1\|^2$, we can see that $h_{\beta,p,\varphi_1}(\alpha, r_{\beta,p}(\alpha)) > 1$ for all $\alpha > 0$. Note that $\frac{\partial h_{\beta,p,\varphi_1}(\alpha,r)}{\partial r} < 0$ on $[0, +\infty) \times (0, +\infty)$, we must have $\widetilde{r}^1_{\beta,p}(\alpha) > r_{\beta,p}(\alpha)$ for all $\alpha > 0$. $\square$

By Lemma 3.1, the concentration behaviors of the solutions to $(\mathcal{P}_{\alpha,\beta})$ obtained by Theorem 1.1 in the case (a) can be stated as follows.

Proposition 3.2. *Assume $\alpha, \beta > 0$, $N = 1, 2, 3$ and $p \in (4, 2^*)$. Then $\lim_{\alpha\to 0^+} u_{\alpha,\beta} = \beta^{\frac{1}{p-2}}\psi$ and $\lim_{\alpha\to 0^+} v^i_{\alpha,\beta} = \beta^{\frac{1}{p-2}}\varphi_i$ in $H^1(\mathbb{R}^N)$, while $\lim_{\alpha} \to +\infty \|u_{\alpha,\beta}\| = \lim_{\alpha\to+\infty} \|v^i_{\alpha,\beta}\| = +\infty$ for all $i \in \mathbb{N}$. Furthermore, $\|u_{\alpha,\beta}\|$ and $\|v^i_{\alpha,\beta}\|$ are all strictly increasing functions for α and $\sqrt{2}\|u_{\alpha,\beta}\| < \|v^1_{\alpha,\beta}\| < \cdots < \|v^i_{\alpha,\beta}\| < \cdots$ with $\lim_{i\to+\infty} \|v^i_{\alpha,\beta}\| = +\infty$, where $u_{\alpha,\beta} = r_{\alpha,\beta,p}\psi$ and $v^i_{\alpha,\beta} = \widetilde{r}^i_{\alpha,\beta,p}\varphi_i$.*

It remains to study the concentration behaviors of the solutions obtained by Theorem 1.1 in the case (b). Also by Lemmas 2.1 and 2.2, the solutions of $(\mathcal{P}_{\alpha,\beta})$ obtained by Theorem 1.1 in this case can be denoted by $\{\widetilde{r}^{i,k}_{\alpha,\beta,p}\varphi_k\}_{k=1,\cdots,2n}$ $(i = 1, 2)$ when $\alpha \in (\widetilde{\alpha}_{n+1}(\beta), \widetilde{\alpha}_n(\beta))$ and by $\{\widetilde{r}^{i,k}_{\alpha,\beta,p}\varphi_k\}_{k=1,\cdots,2n-2}$ $(i = 1, 2)$ and $(\frac{2\beta}{4-p})^{\frac{1}{p-2}}\varphi_n$ when $\alpha = \widetilde{\alpha}_n(\beta)$ for all $n \in \mathbb{N}$. Furthermore, $\widetilde{r}^{i,k}_{\alpha,\beta,p}$ satisfies $\alpha\|\varphi_k\|^2[\widetilde{r}^{i,k}_{\alpha,\beta,p}]^{4-p} + \beta[\widetilde{r}^{i,k}_{\alpha,\beta,p}]^{2-p} = 1$ for all $i = 1, 2$ and $k \in \mathbb{N}$. Positive solutions of $(\mathcal{P}_{\alpha,\beta})$ in this case are $r^i_{\alpha,\beta,p}\psi$ $(i = 1, 2)$ for $\alpha \in (0, \alpha_1(\beta))$ and $(\frac{2\beta}{4-p})^{\frac{1}{p-2}}\psi$ for $\alpha = \alpha_1(\beta)$. As in the case (a), we respectively re-denote $r^i_{\alpha,\beta,p}$ and $\widetilde{r}^{i,k}_{\alpha,\beta,p}$ by $r^i_{\beta,p}(\alpha)$ and $\widetilde{r}^{i,k}_{\beta,p}(\alpha)$ and consider them as functions of α. Then we have the following.

Lemma 3.2. *Suppose $\alpha, \beta > 0$ and $p \in (2, 4) \cap (2, 2^*)$. Then we have the following.*

(1) *On $(0, \alpha_1(\beta))$, $r^1_{\beta,p}(\alpha)$ is a strictly increasing function and $r^2_{\beta,p}(\alpha)$ is a strictly decreasing function. Moreover, $\lim_{\alpha\to 0^+} r^1_{\beta,p}(\alpha) = \beta^{\frac{1}{p-2}}$ and $\lim_{\alpha\to 0^+} r^2_{\beta,p}(\alpha) = +\infty$, while $\lim_{\alpha\to\alpha_1(\beta)^-} r^1_{\beta,p}(\alpha) = \lim_{\alpha\to\alpha_1(\beta)^-} r^2_{\beta,p}(\alpha) = (\frac{2\beta}{4-p})^{\frac{1}{p-2}}$.*
(2) *For every $k \in \mathbb{N}$, $\widetilde{r}^{1,k}_{\beta,p}(\alpha)$ is a strictly increasing function and $\widetilde{r}^{2,k}_{\beta,p}(\alpha)$ is a strictly decreasing function on $(0, \widetilde{\alpha}_k(\beta))$. Moreover, it holds that $\lim_{\alpha\to\widetilde{\alpha}_k(\beta)^-} \widetilde{r}^{1,k}_{\beta,p}(\alpha) = \lim_{\alpha\to\widetilde{\alpha}_k(\beta)^-} \widetilde{r}^{2,k}_{\beta,p}(\alpha) = (\frac{2\beta}{4-p})^{\frac{1}{p-2}}$, $\lim_{\alpha\to 0^+} \widetilde{r}^{1,k}_{\beta,p}(\alpha) = \beta^{\frac{1}{p-2}}$ and $\lim_{\alpha\to 0^+} \widetilde{r}^{2,k}_{\beta,p}(\alpha) = +\infty$.*
(3) *For every $k \in \mathbb{N}$, $r^1_{\beta,p}(\alpha) < \widetilde{r}^{1,1}_{\beta,p}(\alpha) < \cdots < \widetilde{r}^{1,k}_{\beta,p}(\alpha) < \widetilde{r}^{2,k}_{\beta,p}(\alpha) < \cdots < \widetilde{r}^{2,1}_{\beta,p}(\alpha) < r^2_{\beta,p}(\alpha)$ on $(0, \widetilde{\alpha}_k(\beta))$.*

Proof. (1) Consider functions

$$g^i_{\beta,p}(\alpha) = \alpha\|\psi\|^2\left[r^i_{\beta,p}(\alpha)\right]^{4-p} + \beta\left[r^i_{\beta,p}(\alpha)\right]^{2-p}, \quad i = 1, 2.$$

By Lemma 2.2, $g^i_{\beta,p}(\alpha) \equiv 1$ on $(0, \alpha_1(\beta))$. Then by the implicit function theorem, we have

$$\frac{d[r^i_{\beta,p}(\alpha)]}{d\alpha} = \frac{\|\psi\|^2[r^i_{\beta,p}(\alpha)]^{3-p}}{(p-4)\alpha\|\psi\|^2[r^i_{\beta,p}(\alpha)]^2 + (p-2)\beta}.$$

Since $r^2_{\beta,p}(\alpha) > r^0_{\beta,p}(\alpha) = (\frac{(p-2)\beta}{(4-p)\alpha\|\psi\|^2})^{1/2} > r^1_{\beta,p}(\alpha)$, we can see that $\frac{d[r^1_{\beta,p}(\alpha)]}{d\alpha} > 0$ and $\frac{d[r^2_{\beta,p}(\alpha)]}{d\alpha} < 0$ on $(0, \alpha_1(\beta))$. It follows that $r^1_{\beta,p}(\alpha)$ is strictly increasing and $r^2_{\beta,p}(\alpha)$ is strictly decreasing on $(0, \alpha_1(\beta))$. By the fact that $r^2_{\beta,p}(\alpha) > r^0_{\beta,p}(\alpha) = (\frac{(p-2)\beta}{(4-p)\alpha\|\psi\|^2})^{1/2}$ on $(0, \alpha_1(\beta))$ once more, it is easily see that $\lim_{\alpha\to 0^+} r^2_{\beta,p}(\alpha) = +\infty$. On the other hand, note that $\alpha_1(\beta)\|\psi\|^2 r^{4-p} + \beta r^{2-p} = 1$ if and only if $r = (\frac{(p-2)\beta}{(4-p)\alpha\|\psi\|^2})^{1/2}$, where $\alpha_1(\beta)$ is given by (1.1), we must have that $\lim_{\alpha\to\alpha_1(\beta)^-} r^1_{\beta,p}(\alpha) = \lim_{\alpha\to\alpha_1(\beta)^-} r^2_{\beta,p}(\alpha) = (\frac{(p-2)\beta}{(4-p)\alpha\|\psi\|^2})^{1/2} = (\frac{2\beta}{4-p})^{\frac{1}{p-2}}$. Since $r^1_{\beta,p}(\alpha)$ is a bounded function on $(0, \alpha_1(\beta))$, we can easily see that $\lim_{\alpha\to 0^+} r^1_{\beta,p}(\alpha) = \beta^{\frac{1}{p-2}}$.

(2) Due to the definition of $\widetilde{\alpha}_k(\beta)$ and $\alpha_1(\beta)$, the proof is quite similar to (1) by considering functions

$$\widetilde{g}^{i,k}_{\beta,p}(\alpha) = \alpha\|\varphi_k\|^2\left[\widetilde{r}^{i,k}_{\beta,p}(\alpha)\right]^{4-p} + \beta\left[\widetilde{r}^{i,k}_{\beta,p}(\alpha)\right]^{2-p}, \quad i = 1, 2.$$

(3) Let $h_{\beta,p,\psi}(r)$ be the function given by (3.1). Then it is easily see that for $i = 1, 2$, $h_{\beta,p,\varphi_1}(\alpha, \widetilde{r}^{i,1}_{\beta,p}(\alpha)) = 1$. Since $\|\varphi_1\| > \sqrt{2}\|\psi\|$, we can see that $h_{\beta,p,\psi}(\alpha, \widetilde{r}^{i,1}_{\beta,p}(\alpha)) < 1$ on $(0, \widetilde{\alpha}_1(\beta))$. Note that $h_{\beta,p,\psi}(\alpha, r^i_{\beta,p}(\alpha)) =$

$f_{\alpha,\beta,p}(r^i_{\beta,p}(\alpha)) = 1$ for $i = 1, 2$ and $\frac{df_{\alpha,\beta,p}(r)}{dr} < 0$ on $(0, r^0_{\alpha,\beta,p})$ and $\frac{df_{\alpha,\beta,p}(r)}{dr} > 0$ on $(r^0_{\alpha,\beta,p}, +\infty)$, by the conclusions of (1) and (2), we must have $r^1_{\beta,p}(\alpha) < \widetilde{r}^{1,1}_{\beta,p}(\alpha) < \widetilde{r}^{2,1}_{\beta,p}(\alpha) < r^2_{\beta,p}(\alpha)$ on $(0, \widetilde{\alpha}_1(\beta))$. A similar argument implies that $\widetilde{r}^{1,k}_{\beta,p}(\alpha) < \widetilde{r}^{1,k+1}_{\beta,p}(\alpha) < \widetilde{r}^{2,k+1}_{\beta,p}(\alpha) < \widetilde{r}^{2,k}_{\beta,p}(\alpha)$ on $(0, \widetilde{\alpha}_k(\beta))$ for all $k \in \mathbb{N}$, which completes the proof. □

By Lemma 3.2, for every $k \in \mathbb{N}$, we must have $2\|r^1_{\beta,p}(\alpha)\psi\|^2 < \|\widetilde{r}^{1,1}_{\beta,p}(\alpha)\varphi_1\|^2 < \cdots < \|\widetilde{r}^{1,k}_{\beta,p}(\alpha)\varphi_k\|^2$ on $(0, \widetilde{\alpha}_k(\beta))$. On the other hand, $(\alpha\|\widetilde{r}^{2,k}_{\beta,p}(\alpha)\varphi_k\|^2 + \beta)[\widetilde{r}^{2,k}_{\beta,p}(\alpha)]^{2-p} = 1$ and $\widetilde{r}^{2,k+1}_{\beta,p}(\alpha) < \widetilde{r}^{2,k}_{\beta,p}(\alpha)$ on $(0, \widetilde{\alpha}_k(\beta))$, so we must have $\|\widetilde{r}^{2,k+1}_{\beta,p}(\alpha)\varphi_{k+1}\|^2 < \|\widetilde{r}^{2,k}_{\beta,p}(\alpha)\varphi_k\|^2$ on $(0, \widetilde{\alpha}_k(\beta))$ since $p > 2$. A similar argument implies that we also have $\|\widetilde{r}^{2,1}_{\beta,p}(\alpha)\varphi_1\|^2 < \|r^2_{\beta,p}(\alpha)\psi\|^2$ on $(0, \widetilde{\alpha}_k(\beta))$. Thus, we have $\sqrt{2}\|r^1_{\beta,p}(\alpha)\psi\| < \|\widetilde{r}^{1,1}_{\beta,p}(\alpha)\varphi_1\| < \cdots < \|\widetilde{r}^{1,k}_{\beta,p}(\alpha)\varphi_k\| < \|\widetilde{r}^{2,k}_{\beta,p}(\alpha)\varphi_k\| < \cdots < \|\widetilde{r}^{2,1}_{\beta,p}(\alpha)\varphi_1\| < \|r^2_{\beta,p}(\alpha)\psi\|$ on $(0, \widetilde{\alpha}_k(\beta))$. Now, the concentration behaviors of the solutions obtained by Theorem 1.1 in the case (b) can be stated as follows.

Proposition 3.3. *Assume $\alpha, \beta > 0$ and $p \in (2, 4) \cap (2, 2^*)$. Then $\lim_{\alpha\to 0^+} \bar{u}^1_{\alpha,\beta} = \beta^{\frac{1}{p-2}}\psi$ and $\lim_{\alpha\to 0^+} \bar{v}^{1,i}_{\alpha,\beta} = \beta^{\frac{1}{p-2}}\varphi_i$ in $H^1(\mathbb{R}^N)$ and $\lim_{\alpha\to 0^+} \|\bar{u}^2_{\alpha,\beta}\| = +\infty$ and $\lim_{\alpha\to 0^+} \|\bar{v}^{2,i}_{\alpha,\beta}\| = +\infty$, while $\lim_{\alpha\to\alpha_1(\beta)^-} \bar{u}^1_{\alpha,\beta} = \lim_{\alpha\to\alpha_1(\beta)^-} \bar{u}^2_{\alpha,\beta} = \bar{u}^0_{\alpha,\beta}$ and $\lim_{\alpha\to\widetilde{\alpha}_i(\beta)^-} \bar{v}^{1,i}_{\alpha,\beta} = \lim_{\alpha\to\widetilde{\alpha}_i(\beta)^-} \bar{v}^{2,i}_{\alpha,\beta} = \bar{v}^{0,i}_{\alpha,\beta}$ in $H^1(\mathbb{R}^N)$ for all $i \in \mathbb{N}$. Furthermore, $\|\bar{u}^1_{\alpha,\beta}\|$ and $\|\bar{v}^{1,i}_{\alpha,\beta}\|$ are strictly increasing functions for α, $\|\bar{u}^2_{\alpha,\beta}\|$ and $\|\bar{v}^{2,i}_{\alpha,\beta}\|$ are strictly decreasing functions for α and $\sqrt{2}\|\bar{u}^1_{\alpha,\beta}\| < \|\bar{v}^{1,1}_{\alpha,\beta}\| < \cdots < \|\bar{v}^{1,i}_{\alpha,\beta}\| < \|\bar{v}^{2,i}_{\alpha,\beta}\| < \cdots < \|\bar{v}^{2,1}_{\alpha,\beta}\| < \|\bar{u}^2_{\alpha,\beta}\|$ for $\alpha \in (0, \alpha_i(\beta))$ and all $i \in \mathbb{N}$, where $\bar{u}^k_{\alpha,\beta} = r^k_{\beta,p}(\alpha)\psi$, $\bar{v}^{k,i}_{\alpha,\beta} = \widetilde{r}^{k,i}_{\beta,p}(\alpha)\varphi_i$, $\bar{u}^0_{\alpha,\beta} = (\frac{2\beta}{4-p})^{\frac{1}{p-2}}\psi$ and $\bar{v}^{0,i}_{\alpha,\beta} = (\frac{2\beta}{4-p})^{\frac{1}{p-2}}\varphi_i$, $k = 1, 2$ and $i \in \mathbb{N}$.*

We close this section by

Proof of Theorem 1.2. It follows immediately from Propositions 3.1–3.3. □

4. The energy of solutions

In this section, we will make some observations on the energy values of the solutions to $(\mathcal{P}_{\alpha,\beta})$ obtained by Theorem 1.1. It is well-known that the solution of $(\mathcal{P}_{\alpha,\beta})$ is the critical point of the C^2 functional $I_{\alpha,\beta}(u)$, which is given by (1.3). Let $\mathcal{N}_{\alpha,\beta} = \{u \in H^1(\mathbb{R}^N)\backslash\{0\} \mid I'_{\alpha,\beta}(u)u = 0\}$ be the Nehari manifold to $I_{\alpha,\beta}(u)$. Then every nonzero critical point of $I_{\alpha,\beta}(u)$ lies in $\mathcal{N}_{\alpha,\beta}$.

In [2], Alves and Figueiredo proved that $\inf_{\mathcal{N}_{\alpha,\beta}} I_{\alpha,\beta}(u) = m_{\alpha,\beta} > 0$ and there exists a solution of $(\mathcal{P}_{\alpha,\beta})$ with $\alpha, \beta > 0$ and $p \in (4, 2^*)$ for $N = 1, 2, 3$ such that its energy $I_{\alpha,\beta}(u)$ equals to $m_{\alpha,\beta}$. In what follows, let us give an estimate on the energy values of sign-changing solutions to $(\mathcal{P}_{\alpha,\beta})$ in this case.

Lemma 4.1. *Suppose $N = 1, 2, 3$. If u is a sign-changing solution of $(\mathcal{P}_{\alpha,\beta})$ with $\alpha, \beta > 0$ and $4 < p < 2^*$, then $I_{\alpha,\beta}(u) > 2m_{\alpha,\beta} + \frac{(p-4)\alpha}{2p}\|r_{\beta,p}(\alpha)\psi\|^4$. In particular, $I_{\alpha,\beta}(\widetilde{r}^k_{\beta,p}(\alpha)\varphi_k) > 2m_{\alpha,\beta} + \frac{(p-4)\alpha}{2p}\|r_{\beta,p}(\alpha)\psi\|^4$ for all $k \in \mathbb{N}$. Moreover, $I_{\alpha,\beta}(\widetilde{r}^k_{\beta,p}(\alpha)\varphi_k)$ increases to $+\infty$ as $k \to +\infty$.*

Proof. Since u is a sign-changing solution of $(\mathcal{P}_{\alpha,\beta})$, by Lemma 2.1, $s_{\alpha,\beta}(u)u$ is a sign-changing solution of $(\mathcal{P}_{0,1})$ for some $s_{\alpha,\beta}(u) > 0$. It follows that $2\|\psi\|^2 < \|s_{\alpha,\beta}(u)u\|^2$. Now, by Lemma 2.2 and a similar argument as used in Lemma 3.1, we can conclude that $2\|r_{\beta,p}(\alpha)\psi\|^2 < \|u\|^2$, which then implies

$$\begin{aligned} I_{\alpha,\beta}(u) &= 2m_{\alpha,\beta} + \frac{(p-4)\alpha}{4p}\left(\|u\|^4 - 2\left\|r_{\beta,p}(\alpha)\psi\right\|^4\right) \\ &\quad + \frac{(p-2)\beta}{2p}\left(\|u\|^2 - 2\left\|r_{\beta,p}(\alpha)\psi\right\|^2\right) \\ &= \frac{(\|u\|^2 - 2\|r_{\beta,p}(\alpha)\psi\|^2)}{4p}\left((p-4)\alpha\left(\|u\|^2 + 2\left\|r_{\beta,p}(\alpha)\psi\right\|^2\right) + 2\beta(p-2)\right) \end{aligned}$$

$$+ 2m_{\alpha,\beta} + \frac{(p-4)\alpha}{2p}\left\|r_{\beta,p}(\alpha)\psi\right\|^4$$

$$> 2m_{\alpha,\beta} + \frac{(p-4)\alpha}{2p}\left\|r_{\beta,p}(\alpha)\psi\right\|^4.$$

On the other hand, since $p > 4$ and $\|\widetilde{r}^k_{\beta,p}(\alpha)\varphi_k\|$ increases to $+\infty$ as $k \to +\infty$, $I_{\alpha,\beta}(\widetilde{r}^k_{\beta,p}(\alpha)\varphi_k)$ increases to $+\infty$ as $k \to +\infty$, which completes the proof of this lemma. □

Now, by Proposition 2.1 and Lemma 4.1, we can conclude that $r_{\beta,p}(\alpha)\psi$ is the unique ground state solution of $(\mathcal{P}_{\alpha,\beta})$ with $\alpha, \beta > 0$ and $p \in (4, 2^*)$ for $N = 1, 2, 3$ up to a translation. We then estimate the energy values of the solutions to $(\mathcal{P}_{\alpha,\beta})$ with $\alpha, \beta > 0$ and $p = 4$ for $N = 1, 2, 3$. By a similar argument of Lemma 4.1, we can obtain the following estimate on the energy values of the sign-changing solutions to $(\mathcal{P}_{\alpha,\beta})$ in this case.

Lemma 4.2. *Suppose $N = 1, 2, 3$. If u is a sign-changing solution of $(\mathcal{P}_{\alpha,\beta})$ with $p = 4$ and $\alpha, \beta > 0$, then $I_{\alpha,\beta}(u) > 2m_{\alpha,\beta}$. In particular, if $\alpha \in (0, \widetilde{\alpha}_n)$ for some $n \in \mathbb{N}$, then $I_{\alpha,\beta}(\widetilde{r}^k_{\beta,p}(\alpha)\varphi_k) > 2m_{\alpha,\beta}$ for $k = 1, 2, \cdots, n$. Moreover, $I_{\alpha,\beta}(\widetilde{r}^k_{\beta,p}(\alpha)\varphi_k)$ is increasing for $k = 1, 2, \cdots, n$.*

In what follows, we will estimate the energy of the positive solution to $(\mathcal{P}_{\alpha,\beta})$ in this case. Let us first study the decomposition of the Nehari manifold $\mathcal{N}_{\alpha,\beta}$ for $(\mathcal{P}_{\alpha,\beta})$ in this case. It is well-known that the decomposition of the Nehari manifold $\mathcal{N}_{\alpha,\beta}$ is tightly linked to the behavior of the fibering maps $T_{\alpha,\beta,u}(t) = I_{\alpha,\beta}(tu)$, $t > 0$. $T'_{\alpha,\beta,u}(t) = 0$ is equivalent to $tu \in \mathcal{N}_{\alpha,\beta}$. In particular, $T'_{\alpha,\beta,u}(1) = 0$ if and only if $u \in \mathcal{N}_{\alpha,\beta}$. By applying a similar argument as [2, Lemma 2.3] on $T_{\alpha,\beta,u}(t) = I_{\alpha,\beta}(tu)$, we can obtain the following.

Lemma 4.3. *Suppose $N = 1, 2, 3$, $p = 4$, $\alpha, \beta > 0$ and $u \in H^1(\mathbb{R}^N)\backslash\{0\}$.*

(1) *If $\alpha\|u\|^4 - \|u\|_4^4 \geq 0$, then $T_{\alpha,\beta,u}(t)$ is a strictly increasing function on $(0, +\infty)$. Furthermore, $\lim_{t\to+\infty} T_{\alpha,\beta,u}(t) = +\infty$.*
(2) *If $\alpha\|u\|^4 - \|u\|_4^4 < 0$, then there exists a unique $t_{\alpha,\beta}(u) > 0$ such that $t_{\alpha,\beta}(u)u \in \mathcal{N}_{\alpha,\beta}$ and $I_{\alpha,\beta}(t_{\alpha,\beta}(u)u) = \max_{t\geq 0} I_{\alpha,\beta}(tu)$. Furthermore, it holds that $m_{\alpha,\beta} > 0$.*

With Lemma 4.3 in hands, we can obtain the following.

Lemma 4.4. *Suppose $N = 1, 2, 3$, $p = 4$ and $\beta > 0$. If $0 < \alpha < S_4^{-2}$, then $(\frac{\beta}{1-\alpha\|\psi\|^2})^{\frac{1}{2}}\psi$ is the unique ground state solution of $(\mathcal{P}_{\alpha,\beta})$ up to a translation, where S_4 is given by*

$$S_4 = \inf_{u\in H^1(\mathbb{R}^N)\backslash\{0\}} \frac{\|u\|^2}{\|u\|_4^2}.$$

Proof. Clearly, $m_{\alpha,\beta} > 0$ and $\|u\|^2 \geq \beta S_4^2$ for $u \in \mathcal{N}_{\alpha,\beta}$. It follows that $\|u\|_4^4 - \alpha\|u\|^4 \geq \beta^2 S_4^2$ for $u \in \mathcal{N}_{\alpha,\beta}$, which then implies that $\mathcal{N}_{\alpha,\beta}$ is a natural constraint due to $I_{\alpha,\beta}(u) \in C^2$. Now, applying the Ekeland principle and the implicit function theorem in a standard way (cf. [5]), we can conclude that there exists $\{u_n\} \subset \mathcal{N}_{\alpha,\beta}$ such that $I_{\alpha,\beta}(u_n) = m_{\alpha,\beta} + o_n(1)$ and $I'_{\alpha,\beta}(u_n) = o_n(1)$ strongly in $H^{-1}(\mathbb{R}^N)$. Clearly, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Without loss of generality, we may assume $u_n \rightharpoonup u_*$ weakly in $H^1(\mathbb{R}^N)$ as $n \to +\infty$. Let $v_n = u_n - u_*$. Then $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$ as $n \to +\infty$, which implies $I'_{\alpha,\beta}(v_n) = o_n(1)$ strongly in $H^{-1}(\mathbb{R}^N)$. Note that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, we must have $I'_{\alpha,\beta}(v_n)v_n = o_n(1)$. For the sake of clarity, we divide the following proof into two cases.

Case. 1 $\alpha\|v_n\|^4 - \|v_n\|_4^4 \geq o_n(1)$.

Since $I'_{\alpha,\beta}(v_n)v_n = o_n(1)$, $v_n = o_n(1)$ strongly in $H^1(\mathbb{R}^N)$ due to $\alpha\|v_n\|^4 - \|v_n\|_4^4 \geq o_n(1)$. It follows that $I_{\alpha,\beta}(u_*) = m_{\alpha,\beta}$. Note that $I_{\alpha,\beta}(|u_*|) = I_{\alpha,\beta}(u_*) = m_{\alpha,\beta}$ and $\mathcal{N}_{\alpha,\beta}$ is a natural constraint, by a similar argument as [4, Theorem 2.3], $I'_{\alpha,\beta}(|u_*|) = 0$ in $H^{-1}(\mathbb{R}^N)$. Thanks to the maximum principle, u_* can be chosen positive, which implies that $(\mathcal{P}_{\alpha,\beta})$ has a positive ground state solution. Thanks to Proposition 2.1 and Lemma 4.2, $(\frac{\beta}{1-\alpha\|\psi\|^2})^{\frac{1}{2}}\psi$ is the unique ground state solution of $(\mathcal{P}_{\alpha,\beta})$ up to a translation.

Case. 2 There exists a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, satisfying $\alpha\|v_n\|^4 - \|v_n\|_4^4 < -C$.

Since $\alpha\|v_n\|^4 - \|v_n\|_4^4 < -C$, by Lemma 4.3, there exists $\{t_n\} \subset \mathbb{N}$ such that $t_n v_n \in \mathcal{N}_{\alpha,\beta}$. It follows from $I'_{\alpha,\beta}(v_n)v_n = o_n(1)$ that $t_n \to 1$ as $n \to +\infty$. This together with $u_n \rightharpoonup u_*$ weakly in $H^1(\mathbb{R}^N)$ as $n \to +\infty$, implies

$$\begin{aligned} m_{\alpha,\beta} &\leq I_{\alpha,\beta}(t_n v_n) - \frac{1}{4} I'_{\alpha,\beta}(t_n v_n) t_n v_n \\ &= \frac{\beta}{4}\|v_n\|^2 + o_n(1) \\ &= \frac{\beta}{4}\|u_n\|^2 - \frac{\beta}{4}\|u_*\|^2 + o_n(1) \\ &= I_{\alpha,\beta}(u_n) - \frac{\beta}{4}\|u_*\|^2 + o_n(1). \end{aligned} \tag{4.1}$$

Note that $I_{\alpha,\beta}(u_n) = m_{\alpha,\beta} + o_n(1)$, (4.1) implies $u_* = 0$. On the other hand, since $\{u_n\} \subset \mathcal{N}_{\alpha,\beta}$, $\|u_n\| \geq \beta^{1/2} S_4$ for all $n \in \mathbb{N}$. Thanks to the Lions lemma [14, Lemma 1.21], there exist $R > 0$ and $\{x_n\} \subset \mathbb{R}^3$ satisfying $|x_n| \to +\infty$ such that

$$\delta = \limsup_{n \to +\infty} \int_{B_R(x_n)} |u_n|^4 > 0.$$

Let $w_n(x) = u_n(x + x_n)$. Then $w_n \rightharpoonup w_* \neq 0$ weakly in $H^1(\mathbb{R}^N)$ as $n \to +\infty$. Denote $\widetilde{w}_n = w_n - w_*$. Then $I'_{\alpha,\beta}(\widetilde{w}_n) = o_n(1)$ strongly in $H^{-1}(\mathbb{R}^N)$ and $\{\widetilde{w}_n\}$ is bounded in $H^1(\mathbb{R}^N)$, which implies $I'_{\alpha,\beta}(\widetilde{w}_n)\widetilde{w}_n = o_n(1)$. If $\alpha\|\widetilde{w}_n\|^4 - \|\widetilde{w}_n\|_4^4 < -C$ for some subsequence, then by a similar argument as (4.1), we can conclude that $w_* = 0$, which is impossible. It follows that $\alpha\|\widetilde{w}_n\|^4 - \|\widetilde{w}_n\|_4^4 \geq o_n(1)$. This together with $I'_{\alpha,\beta}(\widetilde{w}_n)\widetilde{w}_n = o_n(1)$, implies $w_n = w_* + o_n(1)$ strongly in $H^1(\mathbb{R}^N)$. Thus, we must have

$$\begin{aligned} I_{\alpha,\beta}(w_*) &= I_{\alpha,\beta}(w_n) - I_{\alpha,\beta}(\widetilde{w}_n) - \frac{\alpha}{2}\|\widetilde{w}_n\|^2\|w_*\|^2 + o_n(1) \\ &= I_{\alpha,\beta}(u_n) + o_n(1) \\ &= m_{\alpha,\beta}. \end{aligned}$$

Since $I_{\alpha,\beta}(|w_*|) = I_{\alpha,\beta}(w_*)$ and $\mathcal{N}_{\alpha,\beta}$ is a natural constraint, by a similar argument as [4, Theorem 2.3], we have $I'_{\alpha,\beta}(|w_*|) = 0$. Thanks to the maximum principle, w_* can be chosen to be positive, which implies that $(\mathcal{P}_{\alpha,\beta})$ has a positive ground state solution. Thanks to Proposition 2.1 and Lemma 4.2, $(\frac{\beta}{1-\alpha\|\psi\|^2})^{\frac{1}{2}}\psi$ is the unique ground state solution of $(\mathcal{P}_{\alpha,\beta})$ up to a translation, which completes the proof. □

Remark 4.1. It is well known that S_4 can be achieved by ψ and $\|\psi\|^2 = S_4^2$. Then by Proposition 2.1 and Lemma 4.4, for $N = 1, 2, 3$, $p = 4$, $\beta > 0$, $(\mathcal{P}_{\alpha,\beta})$ has a unique ground state solution if and only if $\alpha \in (0, \alpha_1)$.

In the final of this section, we will estimate the energy values of the solutions for $(\mathcal{P}_{\alpha,\beta})$ with $\alpha, \beta > 0$ and $p \in (2, 4) \cap (2, 2^*)$. Since $p < 4$ and $\alpha > 0$, it is easily observe the following.

Lemma 4.5. *Suppose $\alpha, \beta > 0$ and $p \in (2,4) \cap (2, 2^*)$. Then $I_{\alpha,\beta}(u)$ is conceive on $H^1(\mathbb{R}^N)$.*

Similar to the case of $p = 4$, we will make an observation on the fibering maps $T_{\alpha,\beta,u}(t)$ in this case for a better understanding of the energy values of the solutions to $(\mathcal{P}_{\alpha,\beta})$. Note that $T_{\alpha,\beta,u}(t)$ is C^2, so by Lemma 4.5, we can divide the Nehari manifold into the following three parts:

$$\mathcal{N}^+_{\alpha,\beta} = \{u \in \mathcal{N}_{\alpha,\beta} \mid T''_{\alpha,\beta,u}(1) > 0\};$$
$$\mathcal{N}^-_{\alpha,\beta} = \{u \in \mathcal{N}_{\alpha,\beta} \mid T''_{\alpha,\beta,u}(1) < 0\};$$
$$\mathcal{N}^0_{\alpha,\beta} = \{u \in \mathcal{N}_{\alpha,\beta} \mid T''_{\alpha,\beta,u}(1) = 0\}.$$

Now, let us begin with

Lemma 4.6. *Suppose $\alpha, \beta > 0$, $p \in (2,4) \cap (2, 2^*)$ and $u \in H^1(\mathbb{R}^N)\backslash\{0\}$. Then we have the following.*

(1) *If $\mathcal{S}(u) > \mathcal{D}(\alpha,\beta,p)$, then $T'_{\alpha,\beta,u}(t) > 0$ for all $t > 0$, where $\mathcal{S}(u)$ and $\mathcal{D}(\alpha,\beta,p)$ are given by (1.5).*
(2) *If $\mathcal{S}(u) = \mathcal{D}(\alpha,\beta,p)$, then there exists a unique $t^0_{\alpha,\beta}(u) > 0$ such that $t^0_{\alpha,\beta}(u)u \in \mathcal{N}_{\alpha,\beta}$. Moreover, $t^0_{\alpha,\beta}(u)u \in \mathcal{N}^0_{\alpha,\beta}$.*
(3) *If $\mathcal{S}(u) < \mathcal{D}(\alpha,\beta,p)$, then there exist unique $0 < t^-_{\alpha,\beta}(u) < t^0_{\alpha,\beta}(u) < t^+_{\alpha,\beta}(u)$ such that $t^\pm_{\alpha,\beta}(u)u \in \mathcal{N}_{\alpha,\beta}$. Moreover, $t^\pm_{\alpha,\beta}(u)u \in \mathcal{N}^\pm_{\alpha,\beta}$.*

Proof. Clearly, $T'_{\alpha,\beta,u}(t) = \alpha\|u\|^4t^3 + \beta\|u\|^2t - \|u\|^p_pt^{p-1}$. Let $T^*_{\alpha,\beta,u}(t) = \alpha\|u\|^4t^2 - \|u\|^p_pt^{p-2}$. Then $T'_{\alpha,\beta,u}(t) = t(\beta\|u\|^2 + T^*_{\alpha,\beta,u}(t))$. By a direct calculation, we can see that $T^*_{\alpha,\beta,u}(t)$ is strictly decreasing for $0 < t < t^0_{\alpha,\beta}(u)$ and strictly increasing for $t > t^0_{\alpha,\beta}(u)$, where

$$t^0_{\alpha,\beta}(u) = \left(\frac{(p-2)\|u\|^p_p}{2\alpha\|u\|^4}\right)^{\frac{1}{4-p}}.$$

It follows that

$$\beta\|u\|^2 + T^*_{\alpha,\beta,u}(t) \geq \beta\|u\|^2 + T^*_{\alpha,\beta,u}\big(t^0_{\alpha,\beta}(u)\big) = \frac{\beta\|u\|^2}{\mathcal{S}(u)}\big(\mathcal{S}(u) - \mathcal{D}(\alpha,\beta,p)\big).$$

Now, if $\mathcal{S}(u) > \mathcal{D}(\alpha,\beta,p)$, then $T'_{\alpha,\beta,u}(t) > 0$ for all $t > 0$. If $\mathcal{S}(u) = \mathcal{D}(\alpha,\beta,p)$, then $T'_{\alpha,\beta,u}(t) = 0$ if and only if $t = t^0_{\alpha,\beta}(u)$. Note that

$$T''_{\alpha,\beta,u}\big(t^0_{\alpha,\beta}(u)\big) = \beta\|u\|^2 + T^*_{\alpha,\beta,u}\big(t^0_{\alpha,\beta}(u)\big) + t\big[T^*_{\alpha,\beta,u}\big]'\big(t^0_{\alpha,\beta}(u)\big) = 0,$$

so we also have $t^0_{\alpha,\beta}(u)u \in \mathcal{N}^0_{\alpha,\beta}$. If $\mathcal{S}(u) < \mathcal{D}(\alpha,\beta,p)$, then there exist unique $0 < t^-_{\alpha,\beta}(u) < t^0_{\alpha,\beta}(u) < t^+_{\alpha,\beta}(u)$ such that $T'_{\alpha,\beta,u}(t^\pm_{\alpha,\beta}(u)u) = 0$. Since $0 < t^-_{\alpha,\beta}(u) < t^0_{\alpha,\beta}(u) < t^+_{\alpha,\beta}(u)$, we can see that

$$T''_{\alpha,\beta,u}\big(t^-_{\alpha,\beta}(u)\big) = \beta\|u\|^2 + T^*_{\alpha,\beta,u}\big(t^-_{\alpha,\beta}(u)\big) + t\big(T^*_{\alpha,\beta,u}\big)'\big(t^-_{\alpha,\beta}(u)\big) < 0,$$

and

$$T''_{\alpha,\beta,u}\big(t^+_{\alpha,\beta}(u)\big) = \beta\|u\|^2 + T^*_{\alpha,\beta,u}\big(t^+_{\alpha,\beta}(u)\big) + t\big(T^*_{\alpha,\beta,u}\big)'\big(t^+_{\alpha,\beta}(u)\big) > 0.$$

It follows that $t^\pm_{\alpha,\beta}(u)u \in \mathcal{N}^\pm_{\alpha,\beta}$. $\square$

By Lemma 4.6, we can see that $\mathcal{N}^\pm_{\alpha,\beta} \subset \{u \in H^1(\mathbb{R}^N) \mid \mathcal{S}(u) < \mathcal{D}(\alpha,\beta,p)\}$ and $\mathcal{N}^0_{\alpha,\beta} \subset \{u \in H^1(\mathbb{R}^N) \mid \mathcal{S}(u) = \mathcal{D}(\alpha,\beta,p)\}$. The following lemma gives an energy estimate on $\mathcal{N}_{\alpha,\beta}$.

Lemma 4.7. *Suppose $\alpha, \beta > 0$ and $p \in (2,4) \cap (2, 2^*)$. If $\{u \in H^1(\mathbb{R}^N) \mid \mathcal{S}(u) < \mathcal{D}(\alpha,\beta,p)\} \neq \emptyset$ and $\{u \in H^1(\mathbb{R}^N) \mid \mathcal{S}(u) = \mathcal{D}(\alpha,\beta,p)\} \neq \emptyset$, then we have the following.*

(1) *For $u \in \mathcal{N}^0_{\alpha,\beta}$, we have $\|u\|^2 = \frac{(p-2)\beta}{(4-p)\alpha}$ and $I_{\alpha,\beta}(u) = \frac{(p-2)^2\beta^2}{4p(4-p)\alpha}$.*

(2) *For $u \in \mathcal{N}^-_{\alpha,\beta}$, we have $\|u\|^2 < \min\{(\frac{p-2}{2\alpha} S_p^{-\frac{p}{2}})^{\frac{1}{4-p}}, \frac{(p-2)\beta}{(4-p)\alpha}\}$ and $I_{\alpha,\beta}(u) \in (0, \frac{(p-2)^2\beta^2}{4p(4-p)\alpha})$, where $S_p = \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^2}{\|u\|_p^2}$.*

(3) *For $u \in \mathcal{N}^+_{\alpha,\beta}$, we have $\|u\|^2 > \max\{(\frac{2\beta}{4-p} S_p^{\frac{p}{2}})^{\frac{1}{p-2}}, \frac{(p-2)\beta}{(4-p)\alpha}\}$ and $I_{\alpha,\beta}(u) < \frac{(p-2)^2\beta^2}{4p(4-p)\alpha}$.*

Proof. (1) Let $u \in \{u \in H^1(\mathbb{R}^N) \mid \mathcal{S}(u) = \mathcal{D}(\alpha,\beta,p)\}$. Then by Lemma 4.6, there exists a unique $t^0_{\alpha,\beta}(u) > 0$ such that $t^0_{\alpha,\beta}(u)u \in \mathcal{N}^0_{\alpha,\beta}$. Without loss of generality, we assume $t^0_{\alpha,\beta}(u) = 1$. Now, we have $\alpha\|u\|^4 + \beta\|u\|^2 - \|u\|_p^p = 0$ and $3\alpha\|u\|^4 + \beta\|u\|^2 - (p-1)\|u\|_p^p = 0$. It follows that $(4-p)\alpha\|u\|^2(\|u\|^2 - \frac{(p-2)\beta}{4-p}\alpha) = 0$. Since $u \neq 0$, we must have $\|u\|^2 = \frac{(p-2)\beta}{4-p}\alpha$. Note that $I_{\alpha,\beta}(u) = (\frac{1}{4} - \frac{1}{p})\|u\|^4 + (\frac{1}{2} - \frac{1}{p})\|u\|^2$ for $u \in \mathcal{N}_{\alpha,\beta}$, so $I_{\alpha,\beta}(u) = \frac{(p-2)^2\beta^2}{4p(4-p)\alpha}$.

(2) Let $u \in \{u \in H^1(\mathbb{R}^N) \mid \mathcal{S}(u) < \mathcal{D}(\alpha,\beta,p)\}$. Then by Lemma 4.6, there exists a unique $t^-_{\alpha,\beta}(u) > 0$ such that $t^-_{\alpha,\beta}(u)u \in \mathcal{N}^-_{\alpha,\beta}$. Without loss of generality, we assume $t^-_{\alpha,\beta}(u) = 1$. Now, we have $\alpha\|u\|^4 + \beta\|u\|^2 - \|u\|_p^p = 0$ and $3\alpha\|u\|^4 + \beta\|u\|^2 - (p-1)\|u\|_p^p < 0$. It follows that $\|u\|^2 < \min\{(\frac{p-2}{2\alpha} S_p^{-\frac{p}{2}})^{\frac{1}{4-p}}, \frac{(p-2)\beta}{(4-p)\alpha}\}$. On the other hand, by a direct calculation, it is easily see that $\max_{t\geq 0} f(t) = f(\frac{(p-2)\beta}{4-p}\alpha) = \frac{(p-2)^2\beta^2}{4p(4-p)\alpha}$, where $f(t) = (\frac{1}{4} - \frac{1}{p})t^2 + (\frac{1}{2} - \frac{1}{p})t$. Note that $I_{\alpha,\beta}(u) = (\frac{1}{4} - \frac{1}{p})\|u\|^4 + (\frac{1}{2} - \frac{1}{p})\|u\|^2$ and $\|u\|^2 < \frac{(p-2)\beta}{(4-p)\alpha}$ for $u \in \mathcal{N}^-_{\alpha,\beta}$, we must have $I_{\alpha,\beta}(u) < \frac{(p-2)^2\beta^2}{4p(4-p)\alpha}$. By Lemma 4.6, we can see that $t^-_{\alpha,\beta}(u)$ is the unique maximum point of $T_{\alpha,\beta}(t)$ on $[0, t^+_{\alpha,\beta}(u)]$. Hence, we also have $I_{\alpha,\beta}(u) > 0$ for $u \in \mathcal{N}^-_{\alpha,\beta}$.

(3) Similar to the proof of (2). □

By Propositions 2.1 and 2.2, we can see that the solutions of $(\mathcal{P}_{\alpha,\beta})$ appear in pairs for $\beta > 0$, $\alpha \in (0, \alpha_1(\beta)]$ and $p \in (2,4) \cap (2, 2^*)$ except $\alpha \in \mathcal{A}$, where $\mathcal{A} = \{\alpha_1(\beta), \widetilde{\alpha}_1(\beta), \cdots, \widetilde{\alpha}_n(\beta), \cdots\}$. The following result gives more information for the case of $\alpha \in \mathcal{A}$.

Lemma 4.8. *Suppose $\beta > 0$ and $p \in (2,4) \cap (2, 2^*)$. Then $(\frac{2\beta}{4-p})^{\frac{1}{p-2}}\varphi_n \in \mathcal{N}^0_{\alpha,\beta}$ for $\alpha = \widetilde{\alpha}_n(\beta)$ and all $n \in \mathbb{N}$. In particular, $\mathcal{N}_{\alpha,\beta} = \mathcal{N}^0_{\alpha,\beta} = \{(\frac{2\beta}{4-p})^{\frac{1}{p-2}}\psi\}$ if $\alpha = \alpha_1(\beta)$.*

Proof. If $\alpha = \alpha_1(\beta)$, then by the definition of S_p, it is easily see that $\mathcal{S}(u) \geq S_p^{\frac{p}{4-p}}$. Furthermore, it is well-known that $\mathcal{S}(u) = S_p^{\frac{p}{4-p}}$ if and only if $u = \psi$, where ψ is the unique positive solution of $(\mathcal{P}_{0,1})$. It follows that $\|\psi\|^2 = S_p^{\frac{p}{p-2}}$. On the other hand, since $\alpha = \alpha_1(\beta)$, where $\alpha_1(\beta)$ is given by (1.1), by a direct calculation, we can see that $\mathcal{D}(\alpha_1(\beta), \beta, p) = \|\psi\|^{\frac{2(p-2)}{4-p}} = S_p^{\frac{p}{4-p}} = \mathcal{S}(\psi) < \mathcal{S}(u)$ for $u \neq \psi$. By Lemma 4.6, we can see that $\mathcal{N}_{\alpha,\beta} = \mathcal{N}^0_{\alpha,\beta} = \{t^0_{\alpha,\beta}(\psi)\psi\}$, where $t^0_{\alpha,\beta}(\psi) = (\frac{(p-2)\|\psi\|_p^p}{2\alpha\|\psi\|^4})^{\frac{1}{4-p}} = (\frac{p-2}{2\alpha\|\psi\|^2})^{\frac{1}{4-p}}$. Thanks to Lemma 4.7, we must have $(\frac{p-2}{2\alpha\|\psi\|^2})^{\frac{1}{4-p}} = (\frac{2\beta}{4-p})^{\frac{1}{p-2}}$. On the other hand, by Lemmas 2.2 and 4.6, we can see that $t^0_{\alpha,\beta}(\varphi_n)\varphi_n \in \mathcal{N}^0_{\alpha,\beta}$ if $\alpha = \widetilde{\alpha}_n(\beta)$ for all $n \in \mathbb{N}$. Since φ_n is a solution of $(\mathcal{P}_{0,1})$, by Lemma 4.7 again, we can see that $(\frac{2\beta}{4-p})^{\frac{1}{p-2}}\varphi_n \in \mathcal{N}^0_{\alpha,\beta}$, which completes the proof. □

In what follows, let us give some estimates on the solutions of $(\mathcal{P}_{\alpha,\beta})$ with $\beta > 0$, $\alpha \in (0, \alpha_1(\beta))$ and $p \in (2,4) \cap (2, 2^*)$. We start by

Lemma 4.9. *Suppose $\beta > 0$, $\alpha \in (0, \alpha_1(\beta))$ and $p \in (2,4) \cap (2, 2^*)$. If u is a sign-changing solution of $(\mathcal{P}_{\alpha,\beta})$, then $2\|r^1_{\beta,p}(\alpha)\psi\|^2 < \|u\|^2 < \|r^2_{\beta,p}(\alpha)\psi\|^2$.*

Proof. Since u is a sign-changing solution of $(\mathcal{P}_{\alpha,\beta})$, by Lemma 2.1, there exists $S_{\alpha,\beta}(u) > 0$ such that $S_{\alpha,\beta}(u)u$ is a sign-changing solution of $(\mathcal{P}_{0,1})$, which implies $\|S_{\alpha,\beta}(u)u\|^2 > 2\|\psi\|^2$. By Lemma 2.2, there exist unique $\bar{r}^2_{\beta,p}(\alpha) > r^0_{\beta,p}(\alpha) > \bar{r}^1_{\beta,p}(\alpha)$ such that $\bar{r}^2_{\beta,p}(\alpha)S_{\alpha,\beta}(u)u$ and $\bar{r}^1_{\beta,p}(\alpha)S_{\alpha,\beta}(u)u$ are two sign-changing solutions of $(\mathcal{P}_{\alpha,\beta})$. By Lemma 2.2 again, we known that either $\bar{r}^1_{\beta,p}(\alpha)S_{\alpha,\beta}(u) = 1$ or $\bar{r}^2_{\beta,p}(\alpha)S_{\alpha,\beta}(u) = 1$. On the other hand, by a similar argument as (3) of Lemma 3.2, we have $r^1_{\beta,p}(\alpha) < \bar{r}^1_{\beta,p}(\alpha) < \bar{r}^2_{\beta,p}(\alpha) < r^2_{\beta,p}(\alpha)$. It follows that $2\|r^1_{\beta,p}(\alpha)\psi\|^2 < \|u\|^2$. Note that $(\alpha\|\bar{r}^2_{\beta,p}(\alpha)S_{\alpha,\beta}(u)u\|^2 + \beta)[\bar{r}^2_{\beta,p}(\alpha)]^{2-p} = 1$ and $(\alpha\|r^2_{\beta,p}(\alpha)\psi\|^2 + \beta)[r^2_{\beta,p}(\alpha)]^{2-p} = 1$, we also have $\|u\|^2 < \|r^2_{\beta,p}(\alpha)\psi\|^2$. $\square$

With Lemma 4.9, we can obtain the following.

Lemma 4.10. *Suppose $\beta > 0$, $\alpha \in (0, \alpha_1(\beta))$ and $p \in (2, 4) \cap (2, 2^*)$. Then we have the following.*

(1) *$r^1_{\beta,p}(\alpha)\psi$ is the unique one achieved $m^-_{\alpha,\beta}$. Moreover, $\|r^1_{\beta,p}(\alpha)\psi\| = \min\{\|u\| \mid u \in \mathcal{N}_{\alpha,\beta}\}$, where $m^-_{\alpha,\beta} = \inf_{\mathcal{N}^-_{\alpha,\beta}} I_{\alpha,\beta}(u)$.*
(2) *$r^2_{\beta,p}(\alpha)\psi$ is the unique one achieved $m^+_{\alpha,\beta}$. Moreover, $\|r^2_{\beta,p}(\alpha)\psi\| = \max\{\|u\| \mid u \in \mathcal{N}_{\alpha,\beta}\}$, where $m^+_{\alpha,\beta} = \inf_{\mathcal{N}^+_{\alpha,\beta}} I_{\alpha,\beta}(u)$.*

Proof. (1) By Lemma 2.2, we have $r^1_{\beta,p}(\alpha) < r^0_{\beta,p}(\alpha)$, where $r^0_{\beta,p}(\alpha) = (\frac{(p-2)\beta}{(4-p)\alpha\|\psi\|^2})^{\frac{1}{2}}$. It follows from Proposition 2.1 that $r^1_{\beta,p}(\alpha)\psi \in \mathcal{N}^-_{\alpha,\beta}$. Let

$$m^{-,R}_{\alpha,\beta} = \inf_{\mathcal{N}^-_{\alpha,\beta}\cap H^1_0(B_R(0))} I_{\alpha,\beta}(u),$$

where $B_R(0) = \{x \in \mathbb{R}^N \mid |x| \le R\}$. Then it is well-known that there exists $u^{-,R}_{\alpha,\beta} \in \mathcal{N}^-_{\alpha,\beta} \cap H^1_0(B_R(0))$ with $u^{-,R}_{\alpha,\beta} > 0$ such that $I_{\alpha,\beta}(u^{-,R}_{\alpha,\beta}) = m^{-,R}_{\alpha,\beta}$ and $I'_{\alpha,\beta}(u^{-,R}_{\alpha,\beta}) = 0$ in $H^{-1}(B_R(0))$ for all $R > 0$. By a similar argument as [15, Lemma 2.4], we can see that $\lim_{R\to+\infty} m^{-,R}_{\alpha,\beta} = m^-_{\alpha,\beta}$. On the other hand, by a similar argument as used in Lemma 2.1, we can see that there exists $S_{\alpha,\beta}(u^{-,R}_{\alpha,\beta}) > 0$ such that $S_{\alpha,\beta}(u^{-,R}_{\alpha,\beta})u^{-,R}_{\alpha,\beta} = \psi_R$, where ψ_R is the positive solution of the following equation

$$-\Delta\psi_R + \psi_R = \psi_R^{p-1}, \quad \psi_R \in H^1_0\big(B_R(0)\big).$$

It is well-known that ψ_R is unique and $\psi_R \to \psi$ strongly in $H^1(\mathbb{R}^N)$ as $R \to +\infty$. Note that $[S_{\alpha,\beta}(u^{-,R}_{\alpha,\beta})]^{2-p} = \alpha\|u^{-,R}_{\alpha,\beta}\|^2 + \beta$, we can see that $\{S_{\alpha,\beta}(u^{-,R}_{\alpha,\beta})\}$ is bounded away both from 0 and $+\infty$. Without loss of generality, we may assume that $u^{-,R}_{\alpha,\beta} = [S_{\alpha,\beta}(u^{-,R}_{\alpha,\beta})]^{-1}\psi_R \to [S^{-,0}_{\alpha,\beta}]^{-1}\psi$ strongly in $H^1(\mathbb{R}^N)$ as $R \to +\infty$ for some $[S^{-,0}_{\alpha,\beta}]^{-1} > 0$, which together with $I'_{\alpha,\beta}(u^{-,R}_{\alpha,\beta}) = 0$ in $H^{-1}(B_R(0))$, implies $I'_{\alpha,\beta}([S^{-,0}_{\alpha,\beta}]^{-1}\psi) = 0$ in $H^{-1}(\mathbb{R}^3)$. By Lemma 2.2, either $[S^{-,0}_{\alpha,\beta}]^{-1} = r^1_{\beta,p}(\alpha)$ or $[S^{-,0}_{\alpha,\beta}]^{-1} = r^2_{\beta,p}(\alpha)$. Since $u^{-,R}_{\alpha,\beta} \in \mathcal{N}^-_{\alpha,\beta}$, we have $\|u^{-,R}_{\alpha,\beta}\|^2 < \frac{(p-2)\beta}{(4-p)\alpha}$ by Lemma 4.7. Therefore, $\|[S^{-,0}_{\alpha,\beta}]^{-1}\psi\|^2 \le \frac{(p-2)\beta}{(4-p)\alpha}$. Hence, we must have $[S^{-,0}_{\alpha,\beta}]^{-1} = r^1_{\beta,p}(\alpha)$ by Lemma 2.2 again. It follows that $I_{\alpha,\beta}(r^1_{\beta,p}(\alpha)\psi) = m^-_{\alpha,\beta}$. Now, by Proposition 2.1 and Lemma 4.9, $r^1_{\beta,p}(\alpha)\psi$ is the unique one achieved $m^-_{\alpha,\beta}$ and $\|r^1_{\beta,p}(\alpha)\psi\| = \min\{\|u\| \mid u \in \mathcal{N}_{\alpha,\beta}\}$.

(2) Similar to the proof of (1). $\square$

For $r^1_{\beta,p}(\alpha)\psi$ and $r^2_{\beta,p}(\alpha)\psi$, we also have the following.

Lemma 4.11. *Suppose $\beta > 0$, $\alpha \in (0, \alpha_1(\beta))$ and $p \in (2, 4) \cap (2, 2^*)$. Then*

(1) *$I_{\alpha,\beta}(r^1_{\beta,p}(\alpha)\psi) = \inf_{u\in\mathcal{S}_*} \sup_{t\ge0} I_{\alpha,\beta}(tu)$, where $\mathcal{S}_*$ is given by (1.4).*
(2) *$r^2_{\beta,p}(\alpha)\psi$ is a local minimum point of $I_{\alpha,\beta}(u)$ for $\alpha \in [2^{\frac{p}{p-2}}p^{-\frac{2}{p-2}}\alpha_1(\beta), \alpha_1(\beta))$ and is a global minimum point of $I_{\alpha,\beta}(u)$ for $\alpha \in (0, 2^{\frac{p}{p-2}}p^{-\frac{2}{p-2}}\alpha_1(\beta))$. Moreover, it holds that $m^+_{\alpha,\beta} > 0$ for*

$\alpha \in (2^{\frac{p}{p-2}} p^{-\frac{2}{p-2}} \alpha_1(\beta), \alpha_1(\beta))$, $m^+_{\alpha,\beta} = 0$ for $\alpha = 2^{\frac{p}{p-2}} p^{-\frac{2}{p-2}} \alpha_1(\beta)$ and $m^+_{\alpha,\beta} < 0$ for $\alpha \in (0, 2^{\frac{p}{p-2}} p^{-\frac{2}{p-2}} \alpha_1(\beta))$ with $m^+_{\alpha,\beta} \to -\infty$ as $\alpha \to 0^+$.

Proof. (1) It follows immediately from Lemmas 4.6 and 4.10.

(2) Let $g(r) = \frac{\alpha}{4}\|\psi\|^2 r^2 - \frac{1}{p} r^{p-2} + \frac{\beta}{2}$. Then by a direct calculation, we can see that $g(r_0) = \min_{r\geq 0} g(r)$, where $r_0 = (\frac{2(p-2)}{\alpha\|\psi\|^2 p})^{\frac{1}{4-p}}$. Since $I_{\alpha,\beta}(r\psi) = \|\psi\|^2 r^2 g(r)$, by a direct calculation, we can see that $\min_{r\geq 0} I_{\alpha,\beta}(r\psi) < 0$ for $\alpha \in (0, 2^{\frac{p}{p-2}} p^{-\frac{2}{p-2}} \alpha_1(\beta))$ and $\min_{r\geq 0} I_{\alpha,\beta}(r\psi) = 0$ for $\alpha \in [2^{\frac{p}{p-2}} p^{-\frac{2}{p-2}} \alpha_1(\beta), \alpha_1(\beta))$. Furthermore, $I_{\alpha,\beta}(r) > 0$ for $r > 0$ if $\alpha \in (2^{\frac{p}{p-2}} p^{-\frac{2}{p-2}} \alpha_1(\beta), \alpha_1(\beta))$ and $I_{\alpha,\beta}(r) > 0$ for $r > 0$ and $r \neq r_0$ if $\alpha = 2^{\frac{p}{p-2}} p^{-\frac{2}{p-2}} \alpha_1(\beta)$. By Lemma 2.2 and (2) of Lemma 4.10, we must have $m^+_{\alpha,\beta} > 0$ for $\alpha \in (2^{\frac{p}{p-2}} p^{-\frac{2}{p-2}} \alpha_1(\beta), \alpha_1(\beta))$, $m^+_{\alpha,\beta} = 0$ for $\alpha = 2^{\frac{p}{p-2}} p^{-\frac{2}{p-2}} \alpha_1(\beta)$ and $m^+_{\alpha,\beta} < 0$ for $\alpha \in (0, 2^{\frac{p}{p-2}} p^{-\frac{2}{p-2}} \alpha_1(\beta))$. Since

$$I_{\alpha,\beta}(r_0\psi) = \|\psi\|^2 \left(\frac{2(p-2)}{\alpha\|\psi\|^2 p}\right)^{\frac{2}{4-p}} \left(\frac{\beta}{2} - \frac{4-p}{2p}\left(\frac{2(p-2)}{\alpha\|\psi\|^2 p}\right)^{\frac{p-2}{4-p}}\right) \to -\infty \quad \text{as } \alpha \to 0^+,$$

we also have $m^+_{\alpha,\beta} \to -\infty$ as $\alpha \to 0^+$ by Lemma 2.2 and (2) of Lemma 4.10 once more. It remains to show that $r^2_{\beta,p}(\alpha)\psi$ is a local minimum point of $I_{\alpha,\beta}(u)$ for $\alpha \in [2^{\frac{p}{p-2}} p^{-\frac{2}{p-2}} \alpha_1(\beta), \alpha_1(\beta))$ and is a global minimum point of $I_{\alpha,\beta}(u)$ for $\alpha \in (0, 2^{\frac{p}{p-2}} p^{-\frac{2}{p-2}} \alpha_1(\beta))$. Without loss of generality, we assume $u \in H^1(\mathbb{R}^N)$ with $\|u\|^2 > \frac{(p-2)\beta}{(4-p)\alpha}$ and $\mathcal{S}(u) < \mathcal{D}(\alpha, \beta, p)$. Then by Lemmas 4.6 and 4.10, there exists $t^+_{\alpha,\beta}(u) > 0$ such that $t^+_{\alpha,\beta}(u)u \in \mathcal{N}^+_{\alpha,\beta}$. Moreover,

$$I_{\alpha,\beta}\big(t^+_{\alpha,\beta}(u)u\big) = \min_{t > \sqrt{\frac{(p-2)\beta}{(4-p)\alpha\|u\|^2}}} I_{\alpha,\beta}(tu).$$

In particular, $I_{\alpha,\beta}(t^+_{\alpha,\beta}(u)u) \leq I_{\alpha,\beta}(u)$. Note that $I_{\alpha,\beta}(r^2_{\beta,p}(\alpha)\psi) = m^+_{\alpha,\beta}$, we have $I_{\alpha,\beta}(r^2_{\beta,p}(\alpha)\psi) \leq I_{\alpha,\beta}(u)$. It follows that $r^2_{\beta,p}(\alpha)\psi$ is a local minimum point of $I_{\alpha,\beta}(u)$ for $\alpha \in [2^{\frac{p}{p-2}} p^{-\frac{2}{p-2}} \alpha_1(\beta), \alpha_1(\beta))$ and is a global minimum point of $I_{\alpha,\beta}(u)$ for $\alpha \in (0, 2^{\frac{p}{p-2}} p^{-\frac{2}{p-2}} \alpha_1(\beta))$. □

Finally, we give some estimates on the energy values of the sign-changing solutions to $(\mathcal{P}_{\alpha,\beta})$ for $\beta > 0$, $\alpha \in (0, \widetilde{\alpha}_n(\beta))$ with some $n \in \mathbb{N}$ and $p \in (2,4) \cap (2, 2^*)$.

Lemma 4.12. *Suppose $\beta > 0$, $\alpha \in (0, \widetilde{\alpha}_n(\beta))$ for some $n \in \mathbb{N}$ and $p \in (2,4) \cap (2, 2^*)$. If u is a sign-changing solution of $(\mathcal{P}_{\alpha,\beta})$, which lies in $\mathcal{N}^-_{\alpha,\beta}$, then $I_{\alpha,\beta}(u) > 2m^-_{\alpha,\beta} - \frac{(4-p)\alpha}{2p}\|r^1_{\beta,p}(\alpha)\psi\|^4$. In particular, $I_{\alpha,\beta}(\widetilde{r}^{1,k}_{\beta,p}(\alpha)\varphi) > 2m^-_{\alpha,\beta} - \frac{(4-p)\alpha}{2p}\|r^1_{\beta,p}(\alpha)\psi\|^4$ for $k = 1, 2, \cdots, n$. Moreover, we also have $I_{\alpha,\beta}(\widetilde{r}^{1,1}_{\beta,p}(\alpha)\varphi) < \cdots < I_{\alpha,\beta}(\widetilde{r}^{1,k}_{\beta,p}(\alpha)\varphi)$ and $I_{\alpha,\beta}(\widetilde{r}^{2,k}_{\beta,p}(\alpha)\varphi) > \cdots > I_{\alpha,\beta}(\widetilde{r}^{2,1}_{\beta,p}(\alpha)\varphi)$.*

Proof. Since u is a sign-changing solution of $(\mathcal{P}_{\alpha,\beta})$, which lies in $\mathcal{N}^-_{\alpha,\beta}$, by Lemmas 4.7, 4.9 and 4.10, we have

$$\begin{aligned}
I_{\alpha,\beta}(u) &= 2m^-_{\alpha,\beta} + \frac{(p-4)\alpha}{4p}\big(\|u\|^4 - 2\big\|r^1_{\beta,p}(\alpha)\psi\big\|^4\big) \\
&\quad + \frac{(p-2)\beta}{2p}\big(\|u\|^2 - 2\big\|r^1_{\beta,p}(\alpha)\psi\big\|^2\big) \\
&= \frac{(\|u\|^2 - 2\|r^1_{\beta,p}(\alpha)\psi\|^2)}{4p}\big(-(4-p)\alpha\big(\|u\|^2 + 2\big\|r^1_{\beta,p}(\alpha)\psi\big\|^2\big) + 2\beta(p-2)\big) \\
&\quad + 2m^-_{\alpha,\beta} - \frac{(4-p)\alpha}{2p}\big\|r^1_{\beta,p}(\alpha)\psi\big\|^4 \\
&> 2m^-_{\alpha,\beta} - \frac{(4-p)\alpha}{2p}\big\|r^1_{\beta,p}(\alpha)\psi\big\|^4.
\end{aligned}$$

Note that $\widetilde{r}^{1,k}_{\beta,p}(\alpha)\varphi$ are sign-changing solutions of $(\mathcal{P}_{\alpha,\beta})$, which lies in $\mathcal{N}^-_{\alpha,\beta}$ for all $k \in \mathbb{N}$ due to Proposition 2.2 and Lemma 4.7, so we have $I_{\alpha,\beta}(\widetilde{r}^{1,k}_{\beta,p}(\alpha)\varphi) > 2m^-_{\alpha,\beta} - \frac{(4-p)\alpha}{2p}\|r^1_{\beta,p}(\alpha)\psi\|^4$ for all $k \in \mathbb{N}$. On the other hand, it is easily see that $I_{\alpha,\beta}(u) = (\frac{1}{4} - \frac{1}{p})\|u\|^4 + (\frac{1}{2} - \frac{1}{p})\|u\|^2$ if u is a solution of $(\mathcal{P}_{\alpha,\beta})$. Clearly, $f(t) = (\frac{1}{4} - \frac{1}{p})t^2 + (\frac{1}{2} - \frac{1}{p})t$ is increasing on $(0, \frac{(p-2)\beta}{4-p}\alpha)$ and decreasing on $(\frac{(p-2)\beta}{4-p}\alpha, +\infty)$. Now, by Proposition 2.2, Lemma 4.7 and the concentration behaviors of $\{\widetilde{r}^{i,k}_{\beta,p}(\alpha)\varphi\}$, we can see that $I_{\alpha,\beta}(\widetilde{r}^{1,1}_{\beta,p}(\alpha)\varphi) < \cdots < I_{\alpha,\beta}(\widetilde{r}^{1,k}_{\beta,p}(\alpha)\varphi)$ and $I_{\alpha,\beta}(\widetilde{r}^{2,k}_{\beta,p}(\alpha)\varphi) > \cdots > I_{\alpha,\beta}(\widetilde{r}^{2,1}_{\beta,p}(\alpha)\varphi)$ for all $k \in \mathbb{N}$, which completes the proof. □

We close this section by

Proof of Theorem 1.3. The conclusion of (1) follows immediately from the result of [2] and Lemma 4.1. The conclusion of (2) follows immediately from Lemmas 4.2 and 4.4. The conclusion of (3) follows from Lemmas 4.8 and 4.11–4.12. □

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