



Existence and global behavior of positive solution for semilinear problems with boundary blow-up



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ABSTRACT

Using the sub-supersolution method with Karamata regular variation theory, we study the existence and asymptotic behavior of a classical solution to the following boundary blow-up semilinear Dirichlet problem

$$\begin{cases} \Delta u = a(x)f(u), & x \in \Omega, \\ u > 0 & \text{in } \Omega; \quad \lim_{\delta(x) \rightarrow 0} u(x) = \infty, \end{cases}$$

where Ω is a $C^{1,1}$ -bounded domain in \mathbb{R}^n , $n \geq 2$ and the function a belongs to $C_{loc}^\alpha(\Omega)$, $(0 < \alpha < 1)$ such that for each $x \in \Omega$,

$$c_1(\delta(x))^{-\lambda_1} \exp\left(\int_{\delta(x)}^{\eta} \frac{z_1(s)}{s} ds\right) \leq a(x) \leq c_2(\delta(x))^{-\lambda_2} \exp\left(\int_{\delta(x)}^{\eta} \frac{z_2(s)}{s} ds\right),$$

where $\eta > \text{diam}(\Omega)$, $c_1 > 0$, $c_2 > 0$, $\delta(x) = \text{dist}(x, \partial\Omega)$, $\lambda_1 \leq \lambda_2 \leq 2$ and for $i \in \{1, 2\}$, z_i is a continuous function on $[0, \eta]$ with $z_i(0) = 0$. For the function f , we assume that there exist constants k_1, k_2, p_1, p_2 with $0 < k_1 \leq k_2$, $1 < p_1 \leq p_2$ such that

$$f(t) \leq k_2 t^{p_2} \quad \text{for } t > 0 \quad \text{and} \quad f(t) \geq k_1 t^{p_1} \quad \text{for } t \geq 1.$$

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1. Introduction

Let Ω be a C^2 -bounded domain of \mathbb{R}^n , $(n \geq 2)$. In this paper, we study the existence and the boundary behavior of solutions for the following semilinear elliptic problem

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$$\begin{cases} \Delta u = a(x)f(u), & x \in \Omega, \\ u > 0 & \text{in } \Omega \\ \lim_{\delta(x) \rightarrow 0} u(x) = \infty, \end{cases} \quad (1.1)$$

where $f : [0, \infty) \rightarrow [0, \infty)$ is a C^1 function, a is a positive function, locally-Hölder continuous in Ω and satisfying some conditions related to a regular variation theory and $\delta(x)$ denotes the Euclidean distance from x to the boundary $\partial\Omega$.

A solution of (1.1) is called a large solution (or boundary blow-up or explosive solution).

The subject of large solutions has received much attention starting with the pioneering works of Bieberbach in 1916 with $a(x) = 1$, $f(u) = e^u$, $n = 2$ and with $a(x) = 1$, $f(u) = e^u$ and $n = 3$ in Rademacher's work in 1943 (see [3] and [20]). Later, many authors have considered questions such as existence, uniqueness and boundary behavior of the solution and its normal derivative. Lazer and McKenna in [14] considered (1.1) with a in $C(\overline{\Omega})$ and $f(u) = e^u$. They gave estimates near the boundary on the unique solution of problem (1.1).

In a significant development, Cîrstea and Rădulescu [6] use Karamat's regular variation theory to study the blow-up rate and uniqueness in the case where $a(x)$ decays to zero on $\partial\Omega$ at a fixed rate along the entire boundary $\partial\Omega$ and f' varies regularly at infinity. Using this approach, they were able to obtain detailed information about the qualitative behavior of large solutions under a general framework.

More recently, some results of existence and nonexistence of solutions to problem (1.1) are established when the weight $a(x)$ is unbounded near the boundary $\partial\Omega$ (see [1,3,5–15,17,19–22,25–27] and the references therein). For instance, in [17] Măagli et al. considered (1.1) in the case where $f(u) = u^p$, $p > 1$ and the function a satisfies the following conditions

(A₁) $a \in C_{loc}^\alpha(\Omega)$, $0 < \alpha < 1$ and there exists $C > 1$ such that for each $x \in \Omega$,

$$\frac{1}{C}(\delta(x))^{-\lambda}L(\delta(x)) \leq a(x) \leq C(\delta(x))^{-\lambda}L(\delta(x)),$$

where $\lambda \leq 2$, L is defined on $(0, \eta]$ for some $\eta > \text{diam}(\Omega)$ with $\int_0^\eta t^{1-\lambda}L(t) dt < \infty$ and L belongs to the set of Karamata functions \mathcal{K} defined on $(0, \eta]$ by

$$L(t) := c \exp\left(\int_t^\eta \frac{z(s)}{s} ds\right)$$

with $c > 0$ and $z \in C([0, \eta])$ such that $z(0) = 0$.

Then they proved the following theorem

Theorem 1.1. Assume (A₁) holds with $z \in C([0, \eta]) \cap C^1((0, \eta])$ such that $z(0) = \lim_{t \rightarrow 0} tz'(t) = 0$ and let $p > 1$. Then the following problem

$$\begin{cases} \Delta u = a(x)u^p, & x \in \Omega, \\ u > 0 & \text{in } \Omega \\ \lim_{\delta(x) \rightarrow 0} u(x) = \infty \end{cases} \quad (1.2)$$

has a positive solution $u \in C^{2,\alpha}(\Omega)$ satisfying for each $x \in \Omega$,

$$\frac{1}{C}(\delta(x))^{\frac{\lambda-2}{p-1}}\theta_{L,\lambda,p}(\delta(x)) \leq u(x) \leq C(\delta(x))^{\frac{\lambda-2}{p-1}}\theta_{L,\lambda,p}(\delta(x)), \quad (1.3)$$

where $C > 1$ is a constant and θ is the function defined on $(0, \eta]$ by

$$\theta_{L,\lambda,p}(t) := \begin{cases} (L(t))^{\frac{1}{1-p}}, & \text{if } \lambda < 2, \\ (\int_0^t \frac{L(s)}{s} ds)^{\frac{1}{1-p}}, & \text{if } \lambda = 2. \end{cases} \quad (1.4)$$

Remark 1.2. We note that [Theorem 1.1](#) remains true without the condition $\lim_{t \rightarrow 0} tz'(t) = 0$. Indeed, in the proof of this result in page 6 of [\[17\]](#), it is enough to consider the function $v(x) = (\int_0^{\varepsilon\varphi_1(x)} t^{1-\lambda} L(t) dt)^{\frac{1}{1-p}}$ instead of the function $\varphi_1^{\frac{2-\lambda}{1-p}}\theta_{L,\lambda,p}(\varepsilon\varphi_1)$.

The main goal of this paper is to establish existence and asymptotic behavior of positive solutions of [\(1.1\)](#) with more general nonlinearity $f(u)$ and weight $a(x)$. More precisely, we consider functions f and a satisfying the following conditions

(H₁) The function a is positive, belongs to $C_{loc}^\alpha(\Omega)$, $0 < \alpha < 1$ and there exist two α -Holder continuous functions a_1 and a_2 such that for each $x \in \Omega$,

$$a_1(\delta(x)) \leq a(x) \leq a_2(\delta(x)),$$

where $a_i(t) = t^{-\lambda_i} L_i(t)$, $\lambda_i \leq 2$ and $L_i \in \mathcal{K}$ defined on $(0, \eta]$ ($\eta > \text{diam}(\Omega)$) such that $\int_0^\eta s^{1-\lambda_i} L_i(s) ds < \infty$.

(H₂) There exist constants k_1, k_2, p_1, p_2 with $0 < k_1 \leq k_2$, $1 < p_1 \leq p_2$ such that

$$f(t) \leq k_2 t^{p_2} \quad \text{for } t > 0 \quad \text{and} \quad f(t) \geq k_1 t^{p_1} \quad \text{for } t \geq 1.$$

Using, the sub- and supersolution method, we prove the following main result

Theorem 1.3. Under hypotheses **(H₁)**–**(H₂)**, problem [\(1.1\)](#) has a classical solution $u \in C^{2,\alpha}(\Omega)$ satisfying

$$\frac{1}{C}(\delta(x))^{\frac{\lambda_2-2}{p_2-1}}\theta_{L_2,\lambda_2,p_2}(\delta(x)) \leq u(x) \leq C(\delta(x))^{\frac{\lambda_1-2}{p_1-1}}\theta_{L_1,\lambda_1,p_1}(\delta(x)), \quad (1.5)$$

where

$$\theta_{L_i,\lambda_i,p_i}(t) := \begin{cases} (L_i(t))^{\frac{1}{1-p_i}}, & \text{if } \lambda_i < 2, \\ (\int_0^t \frac{L_i(s)}{s} ds)^{\frac{1}{1-p_i}}, & \text{if } \lambda_i = 2 \end{cases} \quad (1.6)$$

and $C > 1$.

Our paper is organized as follows. In [Section 2](#), we collect some useful properties of Karamata functions. [Section 3](#) deals with the proof of our main result. The last section is reserved to some applications.

2. The Karamata class \mathcal{K}

To let the paper self-contained, we begin this section by recapitulating some properties of Karamata regular variation theory established by Karamata in 1930. This theory has been applied to study the asymptotic behavior of solutions to differential equations. We refer to [\[6,16,21,24,27\]](#) for more details.

Lemma 2.1. *The following hold.*

(i) *Let $L \in \mathcal{K}$ and $\varepsilon > 0$, then we have*

$$\lim_{t \rightarrow 0^+} t^\varepsilon L(t) = 0.$$

(ii) *Let $L_1, L_2 \in \mathcal{K}$ and $p \in \mathbb{R}$. Then we have $L_1 + L_2 \in \mathcal{K}$, $L_1 L_2 \in \mathcal{K}$ and $L_1^p \in \mathcal{K}$.*

Example 2.1. Let m be a positive integer. Let $c > 0$, $(\mu_1, \mu_2, \dots, \mu_m) \in \mathbb{R}^m$ and d be a sufficiently large positive real number such that the function

$$L(t) = c \prod_{k=1}^m \left(\log_k \left(\frac{d}{t} \right) \right)^{\mu_k}$$

is defined and positive on $(0, \eta]$, for some $\eta > 1$, where $\log_k x = \log \circ \log \circ \dots \circ \log x$ (k times). Then $L \in \mathcal{K}$.

Lemma 2.2. *A function L is in \mathcal{K} if and only if L is a positive function in $C^1((0, \eta])$ satisfying*

$$\lim_{t \rightarrow 0^+} \frac{tL'(t)}{L(t)} = 0. \quad (2.1)$$

Proof. Let $L \in \mathcal{K}$. Since $L(t) := c \exp(\int_t^\eta \frac{z(s)}{s} ds)$, then for $t \in (0, \eta]$, we have

$$\frac{tL'(t)}{L(t)} = -z(t).$$

So, using the fact that $z(0) = 0$, we deduce (2.1).

Conversely, let L be a positive function in $C^1((0, \eta])$ satisfying (2.1). For $t \in (0, \eta]$, put

$$z(t) = -\frac{tL'(t)}{L(t)}, \quad (2.2)$$

then $z \in C([0, \eta])$ and $\lim_{t \rightarrow 0^+} z(t) = 0$. This shows that $L \in \mathcal{K}$. \square

Applying Karamata's theorem (see [18,23]), we get the following.

Lemma 2.3. *Let $\mu \in \mathbb{R}$ and L be a function in \mathcal{K} defined on $(0, \eta]$. We have*

- (i) *If $\mu < -1$, then $\int_0^\eta s^\mu L(s) ds$ diverges and $\int_t^\eta s^\mu L(s) ds \sim_{t \rightarrow 0^+} -\frac{t^{1+\mu} L(t)}{\mu+1}$.*
- (ii) *If $\mu > -1$, then $\int_0^\eta s^\mu L(s) ds$ converges and $\int_0^t s^\mu L(s) ds \sim_{t \rightarrow 0^+} \frac{t^{1+\mu} L(t)}{\mu+1}$.*

Lemma 2.4. (See [4].) *Let $L \in \mathcal{K}$ be defined on $(0, \eta]$. Then we have*

$$\lim_{t \rightarrow 0^+} \frac{L(t)}{\int_t^\eta \frac{L(s)}{s} ds} = 0. \quad (2.3)$$

If further $\int_0^\eta \frac{L(s)}{s} ds$ converges, then we have

$$\lim_{t \rightarrow 0^+} \frac{L(t)}{\int_0^t \frac{L(s)}{s} ds} = 0. \quad (2.4)$$

Remark 2.5. Let $L \in \mathcal{K}$ defined on $(0, \eta]$, then using (2.1) and (2.3), we deduce that

$$t \rightarrow \int_t^\eta \frac{L(s)}{s} ds \in \mathcal{K}.$$

If further $\int_0^\eta \frac{L(s)}{s} ds$ converges, we have by (2.3) that

$$t \rightarrow \int_0^t \frac{L(s)}{s} ds \in \mathcal{K}.$$

3. Proof of Theorem 1.3

First, we recall the sub- and supersolution method developed in [7] and [26].

Definition 3.1. A function $v \in C^2(\Omega)$ is called a sub-solution of (1.1) if

$$\begin{cases} \Delta v \geq a(x)f(v), & x \in \Omega, \\ v > 0 & \text{in } \Omega \\ \lim_{\delta(x) \rightarrow 0} v(x) = \infty. \end{cases}$$

If the above inequality is reversed, v is called a super-solution of (1.1).

Lemma 3.2. (See [7] or [26].) Let $a(x)$ be a locally Hölder continuous function in Ω and f be continuously differentiable on $[0, \infty)$. Assume that there exist a sub-solution \underline{u} and a supersolution \bar{u} to the problem (1.1) such that $\underline{u} \leq \bar{u}$. Then there exists at least a classical solution u such that $\underline{u} \leq u \leq \bar{u}$.

The following proposition plays a key role in the proof of Theorem 1.3.

Proposition 3.3. Let a_1, a_2 be the functions defined in hypothesis (\mathbf{H}_1) and let p_1, p_2 be such that $1 < p_1 \leq p_2$. Let u_i be the solution of the following problem

$$\begin{cases} \Delta u_i = a_i(\delta(x))u_i^{p_i}, & x \in \Omega, \\ u_i > 0 & \text{in } \Omega \\ \lim_{\delta(x) \rightarrow 0} u_i(x) = \infty. \end{cases} \quad (3.1)$$

Then there exists a constant $c_0 > 0$ such that

$$u_2 \leq c_0 u_1 \quad \text{in } \Omega. \quad (3.2)$$

Proof. Let u_i be the unique solution of (3.1). Then, by Theorem 1.1 together with Remark 1.2, there exist two constants $c_1 > 0$, $c_2 > 0$ such that for each $x \in \Omega$ we have,

$$\frac{1}{c_i} \psi_{L_i, \lambda_i, p_i}(\delta(x)) \leq u_i(x) \leq c_i \psi_{L_i, \lambda_i, p_i}(\delta(x)), \quad (3.3)$$

where for $i \in \{1, 2\}$, $\psi_{L_i, \lambda_i, p_i}$ is the function defined on $(0, \eta]$ by

$$\psi_{L_i, \lambda_i, p_i}(t) = t^{\frac{\lambda_i-2}{p_i-1}} \theta_{L_i, \lambda_i, p_i}(t) \quad (3.4)$$

and $\theta_{L_i, \lambda_i, p_i}$ is given by (1.6). To prove Proposition 3.3, it is enough to show that $\frac{\psi_{L_2, \lambda_2, p_2}}{\psi_{L_1, \lambda_1, p_1}}$ is bounded in $(0, \eta]$. Now, using Lemma 2.1 and hypothesis (\mathbf{H}_1) , we deduce that $\lambda_1 \leq \lambda_2 \leq 2$, and since $1 < p_1 \leq p_2$, we obtain

$$0 \leq \frac{2 - \lambda_2}{p_2 - 1} \leq \frac{2 - \lambda_1}{p_1 - 1}.$$

Put $\sigma = \frac{(p_2 - p_1)(2 - \lambda_1) + (\lambda_2 - \lambda_1)(p_1 - 1)}{(p_1 - 1)(p_2 - 1)}$. Then $\sigma \geq 0$ and for each $t \in (0, \eta]$ we have

$$\frac{\psi_{L_2, \lambda_2, p_2}(t)}{\psi_{L_1, \lambda_1, p_1}(t)} = t^\sigma \frac{\theta_{L_2, \lambda_2, p_2}(t)}{\theta_{L_1, \lambda_1, p_1}(t)}.$$

Now, using Lemma 2.1 and the definition of $\theta_{L_i, \lambda_i, p_i}$, we deduce that $\frac{\theta_{L_2, \lambda_2, p_2}}{\theta_{L_1, \lambda_1, p_1}} \in \mathcal{K}$. So, we will discuss two cases.

Case 1. $\sigma > 0$. In this case, we conclude by Lemma 2.1 that $\lim_{t \rightarrow 0} \frac{\psi_{L_2, \lambda_2, p_2}(t)}{\psi_{L_1, \lambda_1, p_1}(t)} = 0$. Hence $\frac{\psi_{L_2, \lambda_2, p_2}}{\psi_{L_1, \lambda_1, p_1}}$ is bounded in $(0, \eta]$.

Case 2. $\sigma = 0$. This is equivalent to $\lambda_1 = \lambda_2 = 2$ or $\lambda_1 = \lambda_2 < 2$ and $p_1 = p_2$. In this case we have $L_1 \leq L_2$ in $(0, \eta]$. So we will discuss two subcases:

- If $\lambda_1 = \lambda_2 = 2$, then for each $t \in (0, \eta]$ we have

$$\begin{aligned} \frac{\psi_{L_2, \lambda_2, p_2}(t)}{\psi_{L_1, \lambda_1, p_1}(t)} &= \frac{\theta_{L_2, \lambda_2, p_2}(t)}{\theta_{L_1, \lambda_1, p_1}(t)} = \frac{\left(\int_0^t \frac{L_2(s)}{s} ds\right)^{\frac{1}{1-p_2}}}{\left(\int_0^t \frac{L_1(s)}{s} ds\right)^{\frac{1}{1-p_1}}} = \left(\int_0^t \frac{L_1(s)}{s} ds\right)^{\frac{1}{p_1-1}} \left(\int_0^t \frac{L_2(s)}{s} ds\right)^{\frac{-1}{p_2-1}} \\ &\leq \left(\int_0^t \frac{L_2(s)}{s} ds\right)^{\frac{1}{p_1-1}} \left(\int_0^t \frac{L_2(s)}{s} ds\right)^{\frac{-1}{p_2-1}} \\ &\leq \left(\int_0^t \frac{L_2(s)}{s} ds\right)^{\frac{p_2 - p_1}{(p_1-1)(p_2-1)}}. \end{aligned}$$

Since $1 < p_1 \leq p_2$ and $0 < \int_0^\eta \frac{L_2(s)}{s} ds < \infty$, then we deduce that $\frac{\psi_{L_2, \lambda_2, p_2}}{\psi_{L_1, \lambda_1, p_1}}$ is bounded in $(0, \eta]$.

- If $\lambda_1 = \lambda_2 < 2$ and $p_1 = p_2$, then for each $t \in (0, \eta]$ we have

$$\frac{\psi_{L_2, \lambda_2, p_2}(t)}{\psi_{L_1, \lambda_1, p_1}(t)} = \frac{\theta_{L_2, \lambda_2, p_2}(t)}{\theta_{L_1, \lambda_1, p_1}(t)} = \frac{(L_2(t))^{\frac{1}{1-p_2}}}{(L_1(t))^{\frac{1}{1-p_1}}} = \left(\frac{L_1(t)}{L_2(t)}\right)^{\frac{1}{p_1-1}} \leq 1.$$

This completes the proof of the proposition. \square

Proof of Theorem 1.3. Let u_i be the solution of problem (3.1) for $i \in \{1, 2\}$ and let $c_0 > 0$ be such that $u_2 \leq c_0 u_1$. Since $\lim_{x \rightarrow \infty} u_1(x) = \infty$, then $\inf_{x \in \Omega} u_1(x) > 0$. Let μ_1, μ_2 be two positive constants chosen so that $\mu_1 \geq \max(k_1^{\frac{1}{1-p_1}}, \frac{1}{\inf_{x \in \Omega} u_1(x)})$ and $\mu_2 \leq \min(\frac{\mu_1}{c_0}, k_2^{\frac{1}{1-p_2}})$, where k_1, k_2 are given in hypothesis (\mathbf{H}_2) . Put

$$\bar{u} = \mu_1 u_1 \quad \text{and} \quad \underline{u} = \mu_2 u_2. \quad (3.5)$$

Then using hypotheses (\mathbf{H}_1) and (\mathbf{H}_2) , we obtain

$$\begin{cases} \Delta \bar{u} = \mu_1^{1-p_1} a_1(\delta(x)) \bar{u}^{p_1} \leq k_1 a_1(\delta(x)) \bar{u}^{p_1} \leq a(x) f(\bar{u}), & x \in \Omega, \\ \bar{u} > 0 & \text{in } \Omega; \quad \lim_{\delta(x) \rightarrow 0} \bar{u}(x) = \infty \end{cases}$$

and

$$\begin{cases} \Delta \underline{u} = \frac{\mu_2^{1-p_2}}{k_2} a_2(\delta(x)) k_2 \underline{u}^{p_2} \geq a(x) f(\underline{u}), & x \in \Omega, \\ \underline{u} > 0 & \text{in } \Omega; \quad \lim_{\delta(x) \rightarrow 0} \underline{u}(x) = \infty. \end{cases}$$

So \underline{u} and \bar{u} are respectively a sub-solution and a super-solution of problem (1.1). Moreover, for each $x \in \Omega$, we have

$$\underline{u}(x) = \mu_2 u_2(x) \leq \mu_2 c_0 u_1(x) \leq \mu_1 u_1(x) = \bar{u}(x).$$

Since $a \in C_{loc}^\alpha(\Omega)$ and $f \in C^1([0, \infty))$, we deduce from Lemma 3.2 that (1.1) has a solution $u \in C^{2+\alpha}(\Omega)$ satisfying

$$\underline{u} \leq u \leq \bar{u}.$$

This together with (3.3) and (3.5) implies that u satisfies (1.5). \square

4. Applications

Let a be a function satisfying (\mathbf{H}_1) and let f be a function satisfying (\mathbf{H}_2) and $\beta \in \mathbb{R}$ with $\beta \neq 1$. In this paragraph, we are interested in the following problem

$$\begin{cases} \Delta u - \frac{\beta}{u} |\nabla u|^2 = a(x) f(u), & x \in \Omega, \\ u > 0 & \text{in } \Omega \\ \lim_{\delta(x) \rightarrow 0} u(x) = \infty. \end{cases} \quad (4.1)$$

By putting $v = u^{1-\beta}$, we obtain by a simple calculus the following.

- If $\beta < 1$, then v satisfies

$$\begin{cases} \Delta v = (1-\beta) a(x) v^{-\frac{\beta}{1-\beta}} f(v^{\frac{1}{1-\beta}}), & x \in \Omega, \\ v > 0 & \text{in } \Omega \\ \lim_{\delta(x) \rightarrow 0} v(x) = \infty. \end{cases} \quad (4.2)$$

Let g be the function defined on $(0, \infty)$ by $g(v) = (1-\beta) v^{-\frac{\beta}{1-\beta}} f(v^{\frac{1}{1-\beta}})$ and put $q_1 = \frac{p_1-\beta}{1-\beta}$ and $q_2 = \frac{p_2-\beta}{1-\beta}$. Clearly $1 < q_1 \leq q_2$ and the function g satisfies

$$k_1 r^{q_1} \leq g(r) \quad \text{for } r \geq 1 \quad \text{and} \quad g(r) \leq k_2 r^{q_2} \quad \text{for } r > 0.$$

Therefore it follows from Theorem 1.3 that problem (4.2) has a solution $v \in C^{2,\alpha}(\Omega)$ such that

$$\frac{1}{C}(\delta(x))^{\frac{(\lambda_2-2)(1-\beta)}{p_2-1}}\theta_{L_2,\lambda_2,q_2}(\delta(x)) \leq v(x) \leq C(\delta(x))^{\frac{(\lambda_1-2)(1-\beta)}{p_1-1}}\theta_{L_1,\lambda_1,q_1}(\delta(x))$$

for some constant $C > 1$. Consequently, we deduce that problem (4.1) has a solution $u \in C^{2,\alpha}(\Omega)$ satisfying

$$\frac{1}{C}(\delta(x))^{\frac{\lambda_2-2}{p_2-1}}\theta_{L_2,\lambda_2,p_2}(\delta(x)) \leq u(x) \leq C(\delta(x))^{\frac{\lambda_1-2}{p_1-1}}\theta_{L_1,\lambda_1,p_1}(\delta(x))$$

for some constant $C > 1$.

- If $\beta > 1$, then v satisfies

$$\begin{cases} -\Delta v = (\beta - 1)a(x)v^{\frac{\beta}{\beta-1}}f(v^{\frac{1}{1-\beta}}), & x \in \Omega, \\ v > 0 & \text{in } \Omega \\ \lim_{\delta(x) \rightarrow 0} v(x) = 0. \end{cases} \quad (4.3)$$

Similarly, we put $q_1 = \frac{\beta-p_1}{\beta-1}$ and $q_2 = \frac{\beta-p_2}{\beta-1}$ and we remark that $q_2 \leq q_1 < 1$. So by Theorem 2 in [2], problem (4.3) has a solution $v \in C(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$ satisfying

$$\frac{1}{C}(\delta(x))^{\min(1, \frac{(\lambda_1-2)(1-\beta)}{p_1-1})}(\Phi_{L_1,\lambda_1,q_1}(\delta(x)))^{1-\beta} \leq v(x) \leq C(\delta(x))^{\min(1, \frac{(\lambda_2-2)(1-\beta)}{p_2-1})}(\Phi_{L_2,\lambda_2,q_2}(\delta(x)))^{1-\beta}$$

where $C > 1$ and the function Φ_{L_i,λ_i,q_i} is defined in $(0, \eta]$ ($\eta > \text{diam}(\Omega)$) by

$$\Phi_{L_i,\lambda_i,q_i}(t) := \begin{cases} 1, & \text{if } \lambda_i < 1 + \frac{\beta-p_i}{\beta-1}, \\ (\int_t^\eta \frac{L_i(s)}{s} ds)^{\frac{1}{1-p_i}}, & \text{if } \lambda_i = 1 + \frac{\beta-p_i}{\beta-1}, \\ (L_i(t))^{\frac{1}{1-p_i}}, & \text{if } 1 + \frac{\beta-p_i}{\beta-1} < \lambda_i < 2, \\ (\int_0^t \frac{L_i(s)}{s} ds)^{\frac{1}{1-p_i}}, & \text{if } \lambda_i = 2. \end{cases}$$

Hence problem (4.1) has a solution $u \in C^{2,\alpha}(\Omega)$ satisfying

$$\frac{1}{C}(\delta(x))^{\max(\frac{1}{1-\beta}, \frac{\lambda_2-2}{p_2-1})}\Phi_{L_2,\lambda_2,q_2}(\delta(x)) \leq u(x) \leq C(\delta(x))^{\max(\frac{1}{1-\beta}, \frac{\lambda_1-2}{p_1-1})}\Phi_{L_1,\lambda_1,q_1}(\delta(x)).$$

Authors' contribution

All authors contributed equally and significantly in the elaboration of this article and their names are written in alphabetical order. All authors read and approved the final document.

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