



Smoothness dependent stability in corrosion detection



Eva Sincich

Laboratory for Multiphase Processes, University of Nova Gorica, Slovenia

ARTICLE INFO

Article history:

Received 23 June 2014

Available online 18 October 2014

Submitted by H. Kang

Keywords:

Corrosion detection

Global stability

Boundary impedance

Inverse problems

ABSTRACT

We consider the stability issue for the determination of a linear corrosion in a conductor by a single electrostatic measurement. We established a global *log-log* type stability when the corroded boundary is simply Lipschitz. We also improve such a result obtaining a global *log* stability by assuming that the damaged boundary is $C^{1,1}$ -smooth.

© 2014 Elsevier Inc. All rights reserved.

1. Introduction

In this paper we study the stable determination of a corrosion coefficient on an inaccessible boundary by means of electrostatic measurements. Indeed, we adopt the model proposed by Inglese and Santosa, see [16] and the references therein (see also [10]), where the corrosion is represented by a non-negative exchange coefficient γ in a third-kind boundary condition. Such an adopted model can be regarded as a first linear approximation of a more accurate and nonlinear exchange condition discussed by Vogelius and others [12,21,24].

More precisely, we consider

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = g, & \text{on } \Gamma_A, \\ \frac{\partial u}{\partial \nu} + \gamma u = 0, & \text{on } \Gamma_I, \end{cases} \quad (1.1)$$

where Γ_A and Γ_I are two open, disjoint portions of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma_A} \cup \overline{\Gamma_I}$ and $\Omega \subset \mathbb{R}^n$, $n \geq 2$. The portion Γ_A corresponds to the part of boundary which is accessible to measurements while Γ_I is the portion which is out of reach and where the corrosion damage occurs. The function $\gamma(x)$ is known as

E-mail address: eva.sincich@ung.si.

corrosion coefficient and its amplitude is related to the corrosion rate at the point x . The inverse problem we address here consists in the determination of such γ by means of the current density g prescribed on Γ_A and the corresponding measured potential $u|_{\Gamma_A}$. In particular, we are interested in providing *global* stability estimates for γ , or namely avoiding the a priori hypothesis that the unknown corrosion coefficient is a small perturbation of a given and known one.

Our first aim is to investigate the continuous dependence of γ upon the data when the corroded boundary Γ_I is merely *Lipschitz*. To this purpose, we notice that by the impedance condition in (1.1) we can formally compute γ as

$$\gamma(x) = -\frac{1}{u(x)} \frac{\partial u(x)}{\partial \nu}. \quad (1.2)$$

Since the potential u may vanish in some points on Γ_I , it follows that the above quotient may be highly unstable. In this respect it is necessary to compute the local vanishing rate of u on Γ_I . Indeed, we proved that such a rate can be controlled in an exponential manner as follows

$$\int_{\Delta_r(x_0)} u^2 \geq \exp(-Kr^{-K}) \quad (1.3)$$

where $K > 0$ and $\Delta_r(x_0) = B_r(x_0) \cap \Gamma_I$ with $x_0 \in \Gamma_I^{2r} \subset \Gamma_I$ (see Section 2.1 for a precise definition) for sufficiently small radius r (see Section 4.1). By combining such a control with a logarithmic stability estimate for the underlying Cauchy problem we are able to prove a *global* stability estimate for γ with a *log-log* type modulus of continuity.

The second purpose of this paper is to strengthen the hypothesis on the corroded boundary assuming that Γ_I is $C^{1,1}$ -smooth in order to obtain a better rate of stability. Indeed, under such additional a priori hypothesis, we derive a surface doubling inequality of this sort for sufficiently small radius r (see Section 4.2).

$$\int_{\Delta_{2r}(x_0)} u^2 \leq \text{const.} \int_{\Delta_r(x_0)} u^2, \quad (1.4)$$

which allows us to deduce that the vanishing rate of u at the boundary is at most polynomial, that is

$$\int_{\Delta_r(x_0)} u^2 \geq \frac{1}{K} r^K, \quad (1.5)$$

for sufficiently small radius r (see Section 4.2). Again, by gathering a logarithmic stability estimate for the Cauchy problem and the above vanishing rate we provide a *global* stability estimate for γ with a *single log*.

In addition we also give an alternative proof of the above mentioned *global logarithmic* stability estimate. Such an alternative argument mostly relies on the application of the theory of the Muckenhoupt weights which justifies the computation in (1.2) in the $L^{\frac{2}{p-1}}$ sense for some $p > 1$.

Indeed, such a dependence of the modulus of continuity upon the smoothness of the boundary has been already observed in other contexts. In [3], inverse problems for the determination of unknown defects with Dirichlet and Neumann condition have been studied. The authors proved that when the unknown boundary is smooth enough and hence a doubling inequality at the boundary is available then stability turns out to be of *logarithmic* type. On the contrary relaxing the regularity assumptions on the unknown domain the rate of stability degenerates into a *log-log* type one.

Let us mention that global stability estimates for unknown boundary impedance coefficients have been previously discussed under analogous boundary smoothness assumptions in [8] and [22] for an inverse acoustic scattering problem.

The present inverse problem has been studied in [4] and in [13] in a two-dimensional setting where the authors provided a *global logarithmic* stability estimate for the corrosion coefficient for $C^{1,\alpha}$ corroded boundary.

Similar inverse problems have been studied for the heat equation [9] and for the Stokes equations [11], where *logarithmic* stability estimates for the Robin coefficient γ have been provided. However in such papers the analysis on the local vanishing control of the solution has not been carried over and as a consequence the stability results are stated *only* on a compact set where the solution does not vanish.

The paper is organized as follows. In Section 2 we introduce notation and definition, the main assumptions and we state our main results in Theorem 2.1 and in Theorem 2.2. In Section 3 we preliminary analyze the direct problem recalling some regularity properties of the solution in Lemma 3.1 and Lemma 3.2. Moreover, in Theorem 3.4 we provide an a priori bound of the boundary trace of the solution in the H^1 norm. The proof of such a bound relies on the well-known Rellich's identity. In Section 4.1 we discuss the inverse problem under the a priori hypothesis of a merely Lipschitz boundary. In Theorem 4.2 we recall a known stability result for the underlying Cauchy problem based on unique continuation tools, while in Corollary 4.3 we use the latter result in order to deduce the stability for negative norms of the normal derivative of u . In Theorem 4.4 we provide a lower bound on the local vanishing rate of the solution u . The main ingredient of the proof is the so-called Lipschitz Propagation of Smallness, see also [3,19]. Finally in Proposition 4.5 we state a weighted interpolation inequality which was previously introduced in [8] and we conclude by giving the proof of Theorem 2.1. In Section 4.2 we treat the inverse problem under the further $C^{1,1}$ a priori smoothness assumption on Γ_I . In Theorem 4.6 we recall a stability result for the Dirichlet trace of the solution in C^1 norm. The increased smoothness regularity hypothesis on Γ_1 allows us to refine the analysis on the local vanishing control of the solution, indeed in Proposition 4.7 a surface doubling inequality is provided. We use such an inequality as a tool to state in Theorem 4.8 the polynomial rate of decay of the solution at the boundary. The main argument of this proof again relies on Lipschitz Propagation of Smallness estimates, see also [6]. In Proposition 4.9 we state a weighted interpolation inequality for a weight satisfying a polynomial vanishing rate. We conclude by giving a proof of Theorem 2.2. As already mentioned, we also provide another way to obtain the logarithmic stability results which involves in Proposition 4.10 the notion of Muckenhoupt weights [14]. We complete Section 4 with an alternative proof of Theorem 2.2 relying on the result achieved in Proposition 4.10.

2. Main results

2.1. Notation and definitions

We introduce some notation that we shall use in the sequel.

For any $x_0 \in \partial\Omega$ and for any $\rho > 0$ we shall denote

$$\Gamma_A^\rho = \{x \in \Gamma_A : \text{dist}(x, \Gamma_I) > \rho\}, \quad (2.1)$$

$$\Gamma_I^\rho = \{x \in \Gamma_I : \text{dist}(x, \Gamma_A) > \rho\}, \quad (2.2)$$

$$B_\rho^\Omega(x_0) = B_\rho(x_0) \cap \overline{\Omega}, \quad (2.3)$$

$$\Gamma_\rho(x_0) = B_\rho(x_0) \cap \partial\Omega. \quad (2.4)$$

Definition 2.1. We shall say that a domain Ω is of *Lipschitz class with constants* $r_0, M > 0$ if for any $P \in \partial\Omega$, there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap B_{r_0} = \{(x', x_n) : x_n > \varphi(x')\} \quad (2.5)$$

where

$$\varphi : B'_{r_0} \subset \mathbb{R}^{n-1} \rightarrow \mathbb{R} \quad (2.6)$$

is a Lipschitz function satisfying

$$|\varphi(0)| = 0 \quad \text{and} \quad \|\varphi\|_{C^{0,1}(B'_{r_0})} \leq Mr_0, \quad (2.7)$$

where we denote

$$\|\varphi\|_{C^{0,1}(B'_{r_0}(x_0))} = \|\varphi\|_{L^\infty(B'_{r_0}(x_0))} + r_0 \sup_{\substack{x,y \in B'_{r_0}(z_0) \\ x \neq y}} \frac{|\varphi(x) - \varphi(y)|}{|x - y|}$$

and $B'_{r_0}(x_0)$ denotes a ball in \mathbb{R}^{n-1} .

Definition 2.2. Given α , $0 < \alpha \leq 1$, we shall say that a domain Ω is of class $C^{1,\alpha}$ with constants $r_0, M > 0$ if for any $P \in \partial\Omega$, there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap B_{r_0} = \{(x', x_n) : x_n > \varphi(x')\} \quad (2.8)$$

where

$$\varphi : B'_{r_0} \subset \mathbb{R}^{n-1} \rightarrow \mathbb{R} \quad (2.9)$$

is a $C^{1,\alpha}$ function satisfying

$$|\varphi(0)| = |\nabla\varphi(0)| = 0 \quad \text{and} \quad \|\varphi\|_{C^{1,\alpha}(B'_{r_0})} \leq Mr_0, \quad (2.10)$$

where we denote

$$\begin{aligned} \|\varphi\|_{C^{1,\alpha}(B'_{r_0})} &= \|\varphi\|_{L^\infty(B'_{r_0})} + r_0 \|\nabla\varphi\|_{L^\infty(B'_{r_0})} \\ &\quad + r_0^{1+\alpha} \sup_{\substack{x,y \in B'_{r_0} \\ x \neq y}} \frac{|\nabla\varphi(x) - \nabla\varphi(y)|}{|x - y|^\alpha}. \end{aligned} \quad (2.11)$$

2.2. Assumptions and a priori information

Assumption on the domain. Given $r_0, M > 0$ constants, we assume that $\Omega \subset \mathbb{R}^n$ and

$$\Omega \text{ is of Lipschitz class with constants } r_0, M. \quad (2.12)$$

Moreover, we assume that

$$\text{the diameter of } \Omega \text{ is bounded by } d_0. \quad (2.13)$$

Assumption on γ . Given $\gamma_0 > 0$ constant we assume that the Robin coefficient $\gamma \geq 0$ is such that $\text{supp } \gamma \subset \Gamma_I$ and

$$\|\gamma\|_{C^{0,1}(\Gamma_I)} \leq \gamma_0. \quad (2.14)$$

Assumption on g . Given E, \hat{r} positive constants we assume that the current flux g is such that $\text{supp } g \subset \Gamma_A^{\hat{r}}$ and

$$\|g\|_{C^{0,\alpha}(\Gamma_A)} \leq E. \quad (2.15)$$

From now on we shall refer to the a priori data as the following set of quantities $r_0, M, d_0, \gamma_0, E, \hat{r}$.

In the sequel we shall denote with $\eta(t)$ a positive increasing concave function defined on $(0, +\infty)$, that satisfies

$$\eta(t) \leq C |\log(t)|^{-\vartheta}, \quad \text{for every } 0 < t < 1, \quad (2.16)$$

where $C > 0, \vartheta > 0$ are constants depending on the *a priori* data only.

Let us fix an open connected portion Γ of the boundary of Ω . We introduce the trace space $H_{00}^{\frac{1}{2}}(\Gamma)$ as the interpolation space $[H_0^1(\Gamma), L^2(\Gamma)]_{\frac{1}{2}}$, we refer to [18, Chap. 1] for further details. The functions in $H_{00}^{\frac{1}{2}}(\Gamma)$ might be also characterized as the elements in $H^{\frac{1}{2}}(\partial\Omega)$ which are identically zero outside Γ , this identification shall be understood throughout. We denote by $H_{00}^{-\frac{1}{2}}(\Gamma)$ its dual space, which also can be interpreted as a subspace of $H^{-\frac{1}{2}}(\partial\Omega)$.

2.3. The main results

Theorem 2.1. Let Ω be a Lipschitz domain and let γ_1, γ_2 satisfy (2.14). Let $u_i, i = 1, 2$ be the weak solution to the problem (1.1) with $\gamma = \gamma_i$ respectively. If for some ε , we have

$$\|u_1 - u_2\|_{L^2(\Gamma_A)} \leq \varepsilon \quad (2.17)$$

then

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Gamma_I^{r_0})} \leq \eta \circ \eta(\varepsilon). \quad (2.18)$$

Theorem 2.2. Let Ω be a $C^{1,\alpha}$ domain with $0 < \alpha \leq 1$ and let γ_1, γ_2 satisfy (2.14). Furthermore, we assume that Γ_I is of class $C^{1,1}$ with constants r_0, M . Let $u_i, i = 1, 2$ be the weak solution to the problem (1.1) with $\gamma = \gamma_i$ respectively. If for some ε , we have

$$\|u_1 - u_2\|_{L^2(\Gamma_A)} \leq \varepsilon \quad (2.19)$$

then

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Gamma_I^{r_0})} \leq \eta(\varepsilon). \quad (2.20)$$

3. The direct problem

Lemma 3.1. Let Ω be a Lipschitz domain. Let $u \in H^1(\Omega)$ be a solution to (1.1) with γ and g satisfying the a priori assumptions stated above. Then there exists a constant $0 < \alpha < 1$ and a constant $C > 0$ depending on the a priori data only, such that $u \in C^\alpha(\bar{\Omega})$ and such that

$$\|u\|_{C^\alpha(\bar{\Omega})} \leq C. \quad (3.1)$$

Proof. This is a standard regularity estimate up to the boundary. The Moser iteration technique [15, Theorem 8.18] fits to this task. More details can be found in [23]. Such arguments only require the Lipschitz regularity of $\partial\Omega$. \square

Lemma 3.2. Let Ω be a $C^{1,\alpha}$ domain with $0 < \alpha \leq 1$. Let $u \in H^1(\Omega)$ be a solution to (1.1) with γ and g satisfying the a priori assumptions stated above. Then there exists a constant $0 < \alpha' < 1$, $\alpha' \leq \alpha$ and a constant $C > 0$ depending on the a priori data only, such that $u \in C^{1,\alpha'}(\bar{\Omega})$, such that

$$\|u\|_{C^{1,\alpha'}(\bar{\Omega})} \leq C. \quad (3.2)$$

Proof. Again the proof relies in a slight adaptation of the arguments developed in [23] based on the Moser iteration technique and by well-known regularity bounds for the Neumann problem [2, p. 667]. \square

Theorem 3.3. Let Ω be a Lipschitz domain and let $v \in H^1(\Omega)$ be a solution to

$$\Delta v = 0 \quad \text{in } \Omega. \quad (3.3)$$

If its trace $v|_{\partial\Omega} \in H^1(\partial\Omega)$ then $\frac{\partial v}{\partial \nu}|_{\partial\Omega} \in L^2(\partial\Omega)$ and we have

$$\left\| \frac{\partial v}{\partial \nu} \right\|_{L^2(\partial\Omega)}^2 \leq C(\|\nabla_T v\|_{L^2(\partial\Omega)}^2 + \|v\|_{H^1(\Omega)}^2). \quad (3.4)$$

Conversely, if $\frac{\partial v}{\partial \nu}|_{\partial\Omega} \in L^2(\partial\Omega)$ then $v|_{\partial\Omega} \in H^1(\partial\Omega)$ and

$$\|\nabla_T v\|_{L^2(\partial\Omega)}^2 \leq C\left(\left\| \frac{\partial v}{\partial \nu} \right\|_{L^2(\partial\Omega)}^2 + \|v\|_{H^1(\Omega)}^2\right). \quad (3.5)$$

Here $\nabla_T v$ denotes the tangential gradient of v on $\partial\Omega$ and C depends on M , r_0 and d_0 only.

Proof. These inequalities follow from well-known Rellich's identity [20]. Related estimates were first proven by Jerison and Kenig [17]. A detailed proof in the present form can be found in [5, Proposition 5.1]. \square

Theorem 3.4. Let u be as in Lemma 3.1, then

$$\|u\|_{H^1(\partial\Omega)} \leq C, \quad (3.6)$$

where $C > 0$ depends on the a priori data only.

Proof. The proof is a consequence of (3.5) in combination with the impedance condition in (1.1), the regularity assumption (2.15) on g and standard estimates for solution to boundary value problem for the Laplace equation. \square

4. The inverse problem

In this section we shall discuss the desired stability estimates. For a sake of exposition we first discuss in Section 4.1 the case when the boundary Γ_I is of Lipschitz class only. While the treatment of the case when Γ_I is $C^{1,1}$ -smooth will follow in Section 4.2.

4.1. The Lipschitz corroded boundary case

Lemma 4.1. Let $u \in H^1(\Omega) \cap C^0(\bar{\Omega})$ be a solution to

$$\Delta u = 0 \quad \text{in } \Omega. \quad (4.1)$$

We have

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{H^{-1}(\Gamma_I)} \leq C \|u\|_{L^\infty(\Omega)} \quad (4.2)$$

where C depends on M , r_0 and d_0 only.

Proof. By standard result on the elliptic boundary value problem, for any $\zeta \in H_0^1(\Gamma_I)$ we can consider the unique solution $\varphi \in H^1(\Omega)$ to the Dirichlet problem

$$\begin{cases} \Delta \varphi = 0 & \text{in } \Omega, \\ \varphi = \zeta & \text{on } \Gamma_I, \\ \varphi = 0 & \text{on } \Gamma_A. \end{cases} \quad (4.3)$$

Moreover we have

$$\|\varphi\|_{H^1(\Omega)} \leq C \|\zeta\|_{H_{00}^{1/2}(\Gamma_I)} \quad (4.4)$$

with $C > 0$ only depending on the a priori data. By the Green's identity we have that

$$\int_{\Gamma_I} \varphi \frac{\partial u}{\partial \nu} = \int_{\partial \Omega} u \frac{\partial \varphi}{\partial \nu} \quad (4.5)$$

hence

$$\left| \int_{\Gamma_I} \zeta \frac{\partial u}{\partial \nu} \right| \leq \int_{\partial \Omega} u \frac{\partial \varphi}{\partial \nu}; \quad (4.6)$$

applying (3.4) to φ and taking into account (3.1) and (4.4) we get

$$\left| \int_{\Gamma_I} \zeta \frac{\partial u}{\partial \nu} \right| \leq C \|u\|_{L^\infty(\Omega)} (\|\zeta\|_{H_0^1(\Gamma_I)} + \|\varphi\|_{H^1(\Omega)}) \quad (4.7)$$

$$\leq C \|u\|_{L^\infty(\Omega)} \|\zeta\|_{H_0^1(\Gamma_I)} \quad (4.8)$$

and the thesis follows by duality. \square

Theorem 4.2. Let u_i , $i = 1, 2$ be as in Theorem 2.1. If for some ε (2.17) holds, then

$$\|u_1 - u_2\|_{L^\infty(\Gamma_I)} \leq \eta(\varepsilon) \quad (4.9)$$

where η is the modulus of continuity introduced in (2.16).

Proof. The proof follows by a slight adaptation of the argument developed in Proposition 4.4 in [23]. \square

Corollary 4.3. Let u_i , $i = 1, 2$ be as in Theorem 2.1, then we have that

$$\left\| \frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} \right\|_{H_{00}^{-\frac{1}{2}}(\Gamma_I)} \leq \eta(\varepsilon). \quad (4.10)$$

Proof. By interpolation and the impedance condition we have that

$$\left\| \frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} \right\|_{H_{00}^{-\frac{1}{2}}(\Gamma_I)} \leq C \left\| \frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} \right\|_{H^{-1}(\Gamma_I)}^\theta \|\gamma_1 u_1 - \gamma_2 u_2\|_{L^2(\Gamma_I)}^{1-\theta} \quad (4.11)$$

where $C > 0$, $0 < \theta < 1$ are constants depending on the a priori data only. Finally by [Lemma 4.1](#) and [Theorem 3.1](#) we get the thesis. \square

Theorem 4.4. Let u be a weak solution to (1.1). For every r , $0 < r < r_1$ and for every $x_0 \in \Gamma_I^{r_0}$ we have that

$$\int_{\Gamma_r(x_0)} u^2 \geq \exp(-Kr^{-K}) \quad (4.12)$$

where $r_1 = \min\{\frac{\rho}{2}, r_0, \frac{1}{4}, k_1^{\frac{1}{k_2+1}}\}$ and $k_1, k_2, K > 0$ only depend on the a priori data.

Proof. By the local stability estimates for the Cauchy problem discussed in [\[7, Theorem 1.7\]](#) and the bounds established earlier in [Theorem 3.1](#) and [Theorem 3.4](#), we get that for any $x_0 \in \Gamma_I^{r_0}$ and any $0 < r < r_1$ we have

$$\|u\|_{L^2(B_{\frac{\rho}{2}}^\Omega(x_0))} \leq C \left(\|u\|_{H^{\frac{1}{2}}(\Gamma_r(x_0))} + \|\partial_\nu u\|_{H^{-\frac{1}{2}}(\Gamma_r(x_0))} \right)^\delta \left(\|u\|_{L^2(B_r^\Omega(x_0))} \right)^{1-\delta} \quad (4.13)$$

where $C > 0$, $0 < \delta < 1$ are constants depending on the a priori data only. Moreover, by the following interpolation inequality

$$\|u\|_{H^{\frac{1}{2}}(\Gamma_r(x_0))} \leq C \|u\|_{L^2(\Gamma_r(x_0))}^{\frac{1}{2}} \|u\|_{H^1(\Gamma_r(x_0))}^{\frac{1}{2}} \quad (4.14)$$

where $C > 0$ depends on the a priori data only, by the a priori bound in [Theorem 3.4](#) and the impedance boundary condition we have that

$$\left(\int_{\Gamma_r(x_0)} u^2 \right)^{\frac{\delta}{2}} \geq C \int_{B_{\frac{\rho}{2}}^\Omega(x_0)} u^2. \quad (4.15)$$

Let us consider $\bar{x} \in B_r^\Omega(x_0)$ such that $B_{\frac{r}{8}}(\bar{x}) \subset B_{\frac{\rho}{2}}^\Omega(x_0)$. We now recall that using the arguments of Lipschitz propagation of smallness developed in [\[19, Proposition 3.1\]](#) we have that

$$\int_{B_{\frac{r}{16}}(\bar{x})} |\nabla u|^2 \geq C \exp(-k_1 r^{-k_2}) \int_{\Omega} |\nabla u|^2 \quad (4.16)$$

where k_1 and k_2 are positive constants depending on the a priori data only.

Combining the standard inequality

$$\int_{\Omega} |\nabla u|^2 \geq C_1 \|g\|_{H^{-\frac{1}{2}}(\Gamma_A)} \quad (4.17)$$

and the Caccioppoli inequality

$$\int_{B_{\frac{r}{16}}(\bar{x})} |\nabla u|^2 \leq C_2 r^{-2} \int_{B_{\frac{r}{8}}(\bar{x})} u^2 \quad (4.18)$$

where $C_1, C_2 > 0$ are constants depending on the a priori data only we have that

$$\int_{B_{\frac{r}{8}}(\bar{x})} u^2 \geq C r^2 \exp(-k_1 r^{-k_2}) \quad (4.19)$$

where C is a constant depending on the a priori data only.

We observe that if $r < \min\{\frac{1}{4}, k_1^{\frac{1}{k_2+1}}\}$, we have that

$$\int_{B_{\frac{r}{8}}(\bar{x})} u^2 \geq C \exp(-2k_1 r^{-k_2}). \quad (4.20)$$

Moreover, combining the trivial inequality $\int_{B_{\frac{r}{2}}(x_0)} u^2 \geq \int_{B_{\frac{r}{8}}(\bar{x})} u^2$ with (4.15) we have that

$$\int_{\Gamma_r(x_0)} u^2 \geq C \exp\left(-\frac{4k_1}{\delta} r^{-k_2}\right). \quad (4.21)$$

Finally, we observe that it is possible to find a number $K > 0$ depending on C, k_1, k_2, δ only such that the thesis follows. \square

Proposition 4.5. *Given $M, K > 0$, let $w \geq 0$ be a measurable function on $\Gamma_I^{r_0}$ satisfying the conditions*

$$\|w\|_{L^\infty(\Gamma_I^{r_0})} \leq M \quad (4.22)$$

and

$$\|w\|_{L^2(\Delta_r(x_0))} \geq \exp(-Kr^{-K}) \quad \text{for every } x \in \Gamma_I^{r_0} \text{ and } r \in (0, r_1) \quad (4.23)$$

where r_1 is as in Theorem 4.4 with $\rho = r_0$. Let $f \in C^\alpha(\Gamma_I^{r_0})$ such that

$$|f(x) - f(y)| \leq E|x - y|^\alpha \quad \text{for every } x, y \in \Gamma_I^{r_0}. \quad (4.24)$$

If

$$\int_{\Gamma_I^{r_0}} |f|w \leq \varepsilon \quad (4.25)$$

then

$$\|f\|_{L^\infty(\Gamma_I^{r_0})} \leq E\eta\left(\frac{\varepsilon}{E}\right) \quad (4.26)$$

where η satisfies (2.16) with constants only depending on $M, K, r_0, \alpha, k_1, k_2$.

Proof. The proof of such weighted interpolation inequality relies on slight adaptation of the arguments in [8, Proposition 1]. \square

Proof of Theorem 2.1. By a standard interpolation result we have that

$$\|u_1(\gamma_1 - \gamma_2)\|_{L^2(\Gamma_I)} \leq C \|u_1(\gamma_1 - \gamma_2)\|_{H^1(\Gamma_I)}^{\frac{1}{3}} \|u_1(\gamma_1 - \gamma_2)\|_{H_{00}^{-\frac{1}{2}}(\Gamma_I)}^{\frac{2}{3}} \quad (4.27)$$

where $C > 0$ is a constant depending on the a priori data only.

We observe that

$$\|u_1(\gamma_1 - \gamma_2)\|_{H^1(\Gamma_I)} \leq \|\gamma_1 - \gamma_2\|_{C^{0,1}(\Gamma_I)} \|u_1\|_{H^1(\Gamma_I)} \leq C \quad (4.28)$$

where $C > 0$ is a constant depending on the a priori data only.

Moreover, by the impedance condition on Γ_I it follows that

$$\|u_1(\gamma_1 - \gamma_2)\|_{H_{00}^{-\frac{1}{2}}(\Gamma_I)} \leq \left\| \frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} \right\|_{H_{00}^{-\frac{1}{2}}(\Gamma_I)} + C \|u_1 - u_2\|_{H_{00}^{-\frac{1}{2}}(\Gamma_I)} \quad (4.29)$$

where $C > 0$ is a constant depending on the a priori data only.

By combining the estimate in Theorem 4.2 and in Corollary 4.3 we obtain

$$\|u_1(\gamma_1 - \gamma_2)\|_{H_{00}^{-\frac{1}{2}}(\Gamma_I)} \leq \eta(\varepsilon). \quad (4.30)$$

Hence by (4.27) we have that

$$\|u_1(\gamma_1 - \gamma_2)\|_{L^2(\Gamma_I^{r_0})} \leq \eta(\varepsilon). \quad (4.31)$$

The conclusion follows by applying Proposition 4.5 with $w = u_1^2$ and $f = (\gamma_1 - \gamma_2)^2$. \square

4.2. The $C^{1,1}$ -smooth corroded boundary case

Theorem 4.6. Let u_i , $i = 1, 2$ be as in Theorem 2.2. If for some ε , (2.17) holds, we have that

$$\|u_1 - u_2\|_{C^1(\Gamma_I)} \leq \eta(\varepsilon) \quad (4.32)$$

where η is given by (2.16).

Proof. The proof can be achieved along the lines of Proposition 4.4 in [23] and Theorem 4.2 in [22]. \square

Proposition 4.7. Let Γ_I be of class $C^{1,1}$ with constants r_0 , M . Let u be the solution to the problem (1.1), then there exist constants $K_1 > 0$, $\bar{r} > 0$ depending on the a priori data only, such that for every $x_0 \in \Gamma_I^{r_0}$ and every $r \in (0, \bar{r})$ the following holds

$$\int_{\Gamma_{2r}(x_0)} u^2 \leq K_1 \int_{\Gamma_r(x_0)} u^2. \quad (4.33)$$

Proof. We provide here a sketch of the proof. Let $v \in H^1(\Omega)$ be the weak solution to the problem

$$\begin{cases} \Delta v = 0, & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = 1, & \text{on } \Gamma_A, \\ \frac{\partial v}{\partial \nu} + \gamma u = 0, & \text{on } \Gamma_I. \end{cases} \quad (4.34)$$

Dealing as in the proof of Lemma 3.3 of [23] and relying on an iterated use of the Harnack inequality as well as the Giraud's maximum principle, we may infer that there exists a constant $C > 0$ depending on the a priori data only such that $v(x) \geq C$ in $\bar{\Omega}$.

It is trivial to check that the function $z = \frac{u}{v} \in H^1(\Omega)$ satisfies

$$\begin{cases} \operatorname{div}(v^2 \nabla z) = 0, & \text{in } \Omega, \\ v^2 \frac{\partial z}{\partial \nu} = gv - u, & \text{on } \Gamma_A, \\ v^2 \frac{\partial z}{\partial \nu} = 0, & \text{on } \Gamma_I. \end{cases} \quad (4.35)$$

Let us observe that such change of variable allows us to treat a new boundary problem with a homogeneous Neumann condition on Γ_I instead of the Robin one. By the arguments due to Adolfsson and Escauriaz in [1] (see also [3, Proposition 3.5]) we have that $u \in H^1(\Omega)$ satisfies the so-called doubling inequality at the boundary which can be stated as follows. There exists a radius \bar{r} depending on the a priori data only such that for any $x_0 \in \Gamma_I^{r_0}$ the following holds

$$\int_{B_{\beta r}^\Omega(x_0)} z^2 \leq C \beta^K \int_{B_r^\Omega(x_0)} z^2 \quad (4.36)$$

for every r, β such that $\beta > 1$ and $0 < \beta r < 4\bar{r}$.

Now, we observe that repeating the arguments in Theorem 4.5 and in Theorem 4.6 in [22] and mainly based on well-known stability estimate for the Cauchy problem we can reformulate the above volume doubling inequality at the boundary for the solution z into a surface doubling inequality for the solution u . Indeed, we have that there exists a constant $K_1 > 0$ depending on the a priori data only, such that for any $x_0 \in \Gamma_I^{r_0}$ and for every $r \in (0, \bar{r})$ the following holds

$$\int_{\Gamma_{2r}(x_0)} u^2 \leq K_1 \int_{\Gamma_r(x_0)} u^2, \quad (4.37)$$

and the thesis follows. \square

Theorem 4.8. Let Γ_I be of class $C^{1,1}$ with constants r_0, M . Let u be a weak solution to (1.1). For every $r, 0 < r < r_2$ and for every $x_0 \in \Gamma_I^{r_0}$ we have that

$$\int_{\Gamma_r(x_0)} u^2 \geq \frac{1}{K} r^K \quad (4.38)$$

where $r_2 = \min\{\bar{r}, r_1\}$ and $K > 0$ only depends on the a priori data.

Proof. Let $x_0 \in \Gamma_I^{r_0}$. Dealing as in [6, Remark 4.11], we have that

$$\int_{\Gamma_{2^{1-j}r_2}(x_0)} u^2 \leq K_1 \int_{\Gamma_{2^{-j}r_2}(x_0)} u^2, \quad \text{for every } j = 2, 3, \dots \quad (4.39)$$

By iteration over j we get

$$\int_{\Gamma_{\frac{r_2}{2}}(x_0)} u^2 \leq K_1^{j-1} \int_{\Gamma_{2^{-j}r_2}(x_0)} u^2, \quad \text{for every } j = 2, 3, \dots \quad (4.40)$$

Hence for any $r < \frac{r_2}{2}$ and choosing j such that

$$2^{-j}r_2 \leq r \leq 2^{1-j}r_2 \quad (4.41)$$

and

$$q = \frac{\log(K_1)}{\log(2)} \quad (4.42)$$

we have that

$$\int_{\Gamma_r(x_0)} u^2 \geq \left(\frac{r}{r_2}\right)^q \int_{\Gamma_{\frac{r_2}{2}}(x_0)} u^2. \quad (4.43)$$

By (4.43) and (4.15) we find that

$$\int_{\Gamma_r(x_0)} u^2 \geq \left(\frac{r}{r_2}\right)^q C \left(\int_{B_{\frac{r_2}{2}}^\Omega(x_0)} u^2 \right)^{\frac{1}{\delta}} \quad (4.44)$$

where $C > 0$ is a constant depending on the a priori data only.

Let $\bar{x} \subset B_{\frac{r_2}{4}}^\Omega(x_0)$ be such that $B_{\frac{r_2}{32}}(\bar{x}) \subset B_{\frac{r_2}{4}}^\Omega(x_0)$. By (4.20) with $r = \frac{r_2}{4}$ we have that

$$\left(\int_{B_{\frac{r_2}{4}}^\Omega(x_0)} u^2 \right)^{\frac{1}{\delta}} \geq C \quad (4.45)$$

where C is a constant depending on the a priori data only. Combining (4.44) and (4.45) we have that

$$\int_{\Gamma_r(x_0)} u^2 \geq C \left(\frac{r}{r_2}\right)^q \quad (4.46)$$

where $C > 0$ is a constant depending on the a priori data only.

We conclude by observing that we may find a constant $K > 0$ depending on the a priori data only such that the thesis follows. \square

Proposition 4.9. Given $M, K > 0$, let $w \geq 0$ be a measurable function on $\Gamma_I^{r_0}$ satisfying the conditions

$$\|w\|_{L^\infty(\Gamma_I^{r_0})} \leq M \quad (4.47)$$

and

$$\|w\|_{L^2(\Gamma_r(x_0))} \geq \frac{1}{K} r^K \quad \text{for every } x \in \Gamma_I^{r_0} \text{ and } r \in (0, r_2) \quad (4.48)$$

where r_2 is as in [Theorem 4.8](#). Let $f \in C^\alpha(\Gamma_I^{r_0})$ such that

$$|f(x) - f(y)| \leq E|x - y|^\alpha \quad \text{for every } x, y \in \Gamma_I^{r_0}. \quad (4.49)$$

If

$$\int_{\Gamma_I^{r_0}} |f|w \leq \varepsilon \quad (4.50)$$

then

$$\|f\|_{L^\infty(\Gamma_I^{r_0})} \leq C \left(\frac{\varepsilon}{E} \right)^{\delta'} \quad (4.51)$$

where $C > 0$, $0 < \delta' < 1$ are constants only depending on M , K , r_0 , α , r_2 .

Proof. By the bound in [\(4.47\)](#) we have that

$$\int_{\Gamma_r(x)} w^2 \geq M^{-1} \frac{r^{2K}}{K^2}, \quad \text{for every } x \in \Gamma_I^{r_0} \text{ and } r \in (0, r_2). \quad (4.52)$$

Let now \bar{x} be such that $|f(\bar{x})| = \|f\|_{L^\infty(\Gamma_I^{r_0})}$. By the Hölder regularity of f we have that for every $r > 0$ and $x \in \Gamma_r(\bar{x})$ the following holds

$$|f(\bar{x})| \leq |f(x)| + Er^\alpha. \quad (4.53)$$

Multiplying the above inequality by the weight w and integrating both sides over $\Gamma_r(\bar{x})$ we obtain that

$$|f(\bar{x})| \int_{\Gamma_r(\bar{x})} w \leq \int_{\Gamma_r(\bar{x})} w|f| + Er^\alpha \int_{\Gamma_r(\bar{x})} w, \quad (4.54)$$

from which we deduce that

$$\|f\|_{L^\infty(\Gamma_I^{r_0})} \leq \frac{\varepsilon}{\int_{\Gamma_r(\bar{x})} w} + Er^\alpha \quad (4.55)$$

$$\leq \varepsilon M K^2 r^{-2K} + Er^\alpha. \quad (4.56)$$

Now minimizing over $r \in (0, r_2)$, the thesis follows with $\delta' = \frac{\alpha}{2K+\alpha}$. \square

Proof of Theorem 2.2. By the impedance condition we have that

$$\|u_1(\gamma_1 - \gamma_2)\|_{L^2(\Gamma_I^{r_0})} \leq \left\| \frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} \right\|_{L^2(\Gamma_I^{r_0})} + C \|u_1 - u_2\|_{L^2(\Gamma_I^{r_0})} \quad (4.57)$$

where $C > 0$ is a constant depending on the a priori data only.

By [Theorem 4.6](#) we obtain that

$$\|u_1(\gamma_1 - \gamma_2)\|_{L^2(\Gamma_I^{r_0})} \leq \eta(\varepsilon). \quad (4.58)$$

By applying [Proposition 4.9](#) the thesis follows with $w = u_1^2$ and $\lambda = (\gamma_1 - \gamma_2)^2$ up to a possible replacement of the constants C and ϑ in [\(2.16\)](#). \square

We now follow a slightly different strategy in order to prove [Theorem 2.2](#). The main difference is based on the introduction of the notion of Muckenhoupt weights in [Proposition 4.10](#).

Proposition 4.10. *Let Γ_I be of class $C^{1,1}$ with constants r_0, M . Let u be the solution to the problem [\(1.1\)](#), then there exist constant $p > 1$, $A > 0$ depending on the a priori data only, such that for every $\Gamma_I^{r_0}$ and every $r \in (0, \bar{r})$ the following holds*

$$\left(\frac{1}{|\Gamma_r(x_0)|} \int_{\Delta_r(x_0)} u^2 \right) \left(\frac{1}{|\Gamma_r(x_0)|} \int_{\Gamma_r(x_0)} u^{-\frac{2}{p-1}} \right)^{p-1} \leq A. \quad (4.59)$$

Proof. For a detailed proof we refer to Corollary 4.7 in [\[22\]](#). The main tools of the proof rely on the above mentioned surface doubling inequality [\(4.33\)](#) and the theory of Muckenhoupt weights [\[14\]](#) as well. \square

Alternative proof of Theorem 2.2. Let $x_0 \in \Gamma_I^{r_0}$. Let us choose $r = \frac{\bar{r}}{2}$, where \bar{r} is the radius in [Proposition 4.10](#). By the lower bound in [\(4.38\)](#) with $r = \frac{\bar{r}}{2}$ and with $u = u_2$ we have that

$$\int_{\Gamma_{\frac{\bar{r}}{2}}(x_0)} u_2^2 \geq C \quad (4.60)$$

where $C > 0$ is a constant depending on the a priori data only.

Combining [\(4.59\)](#) and [\(4.60\)](#), we have that for every $x_0 \in \Gamma_I^{r_0}$ the following holds

$$\left(\int_{\Gamma_{\frac{\bar{r}}{2}}(x_0)} |u_2|^{-\frac{2}{p-1}} \right)^{p-1} \leq C, \quad (4.61)$$

where $C > 0$ is a constant depending on the a priori data only.

Let us now consider $x \in \Gamma_{\frac{\bar{r}}{2}}(x_0)$, then by [Theorem 4.6](#) and by [\(2.14\)](#) we have that

$$|\gamma_1(x) - \gamma_2(x)| \leq (\gamma_0 + 1)\eta(\varepsilon) \frac{1}{|u_2(x)|}. \quad (4.62)$$

Denoting $\beta = \frac{2}{p-1}$ and combining [\(4.61\)](#) and [\(4.62\)](#) we find that

$$\left(\int_{\Gamma_{\frac{\bar{r}}{2}}(x_0)} |\gamma_1(x) - \gamma_2(x)|^\beta \right)^{\frac{1}{\beta}} \leq \eta(\varepsilon). \quad (4.63)$$

By the a priori bound [\(2.14\)](#), we get that

$$\|\gamma_1 - \gamma_2\|_{L^2(\Gamma_{\frac{\bar{r}}{2}}(x_0))} \leq (2\gamma_0)^{1-\frac{\beta}{2}} \left(\int_{\Gamma_{\frac{\bar{r}}{2}}(x_0)} |\gamma_1(x) - \gamma_2(x)|^\beta \right)^{\frac{1}{2}} \quad (4.64)$$

which in turn combined with (4.63) implies that by a possible further replacement of the constants C, θ in (2.16) we have

$$\|\gamma_1 - \gamma_2\|_{L^2(\Gamma_{\frac{\bar{r}}{2}}(x_0))} \leq \eta(\varepsilon). \quad (4.65)$$

By interpolation we have that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Gamma_{\frac{\bar{r}}{2}}(x_0))} \leq C \|\gamma_1 - \gamma_2\|_{L^2(\Gamma_{\frac{\bar{r}}{2}}(x_0))}^{\frac{1}{2}} \|\gamma_1 - \gamma_2\|_{\dot{C}^{0,1}(\Gamma_{\frac{\bar{r}}{2}}(x_0))}^{\frac{1}{2}} \quad (4.66)$$

where $C > 0$ is a constant depending on the a priori data only.

Hence by the a priori bound (2.14) and (4.65) we have that by a possible further replacement of the constants C, θ in (2.16) we have

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Gamma_{\frac{\bar{r}}{2}}(x_0))} \leq \eta(\varepsilon). \quad (4.67)$$

By a covering argument we finally deduce the thesis. \square

References

- [1] V. Adolfsson, L. Escauriaza, $C^{1,\alpha}$ domains and unique continuation at the boundary, *Comm. Pure Appl. Math.* 50 (1997) 935–969.
- [2] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solution of elliptic partial differential equations satisfying general boundary conditions I, *Comm. Pure Appl. Math.* 12 (1959) 623–727.
- [3] G. Alessandrini, E. Beretta, E. Rosset, S. Vessella, Optimal stability for inverse elliptic boundary value problems with unknown boundaries, *Ann. Sc. Norm. Super. Pisa Sci. Fis. Mat. Ser. IV XXXIX* (4) (2000).
- [4] G. Alessandrini, L. Del Piero, L. Rondi, Stable determination of corrosion by a single electrostatic boundary measurement, *Inverse Problems* 19 (4) (2003) 973–984.
- [5] G. Alessandrini, A. Morassi, E. Rosset, Detecting cavities by electrostatic boundary measurements, *Inverse Problems* 18 (2002) 1333–1353.
- [6] G. Alessandrini, A. Morassi, E. Rosset, Size estimates, in: G. Alessandrini, G. Uhlmann (Eds.), *Inverse Problems: Theory and Applications*, in: *Contemp. Math.*, vol. 333, American Mathematical Society, Providence, RI, 2003, pp. 1–33.
- [7] G. Alessandrini, L. Rondi, E. Rosset, S. Vessella, The stability for the Cauchy problem for elliptic equations, *Inverse Problems* 25 (2009) 123004, 47 pp.
- [8] G. Alessandrini, E. Sincich, S. Vessella, Stable determination of surface impedance on a rough obstacle by far field data, *Inverse Probl. Imaging* 7 (2013) 341–351.
- [9] M. Bellassoued, J. Cheng, M. Choulli, Stability estimate for an inverse boundary coefficient problem in thermal imaging, *J. Math. Anal. Appl.* 343 (2008) 328–336.
- [10] Y.S. Bhat, S. Moskow, Linearization of a nonlinear periodic boundary condition related to corrosion modeling, *J. Comput. Math.* 25 (6) (2007) 645–660.
- [11] M. Boulakia, A.C. Egloff, C. Grandmont, Stability estimates for a Robin coefficient in the two-dimensional Stokes problem, *Math. Control Related Fields* 3 (2012) 21–49.
- [12] K. Bryan, M. Vogelius, Singular solutions to a nonlinear elliptic boundary value problem originating from corrosion modeling, *Quart. Appl. Math.* 60 (2002) 675–694.
- [13] S. Chaabane, I. Fellah, M. Jaoua, J. Leblond, Logarithmic stability estimates for a Robin coefficient in two-dimensional Laplace inverse problems, *Inverse Problems* 20 (1) (2004) 47–59.
- [14] R.R. Coifman, C.L. Fefferman, Weighted norm inequalities for maximal function and singular integrals, *Studia Math.* 51 (1976) 241–250.
- [15] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, second ed., Springer-Verlag, Berlin, Heidelberg, New York, 1977.
- [16] G. Inglese, F. Santosa, An Inverse Problem in Corrosion Detection, *Pubblicazioni IAGA*, vol. 75, CNR, Firenze, 1995.
- [17] D.S. Jerison, C.E. Kenig, The Neumann problem on Lipschitz domains, *Bull. Amer. Math. Soc. (N.S.)* 4 (2) (1981) 203–207 (in English, Russian original: 1980, pp. 181–189).
- [18] J.L. Lions, E. Magenes, *Non-homogeneous Boundary Value Problems and Applications*, vol. 1, Springer-Verlag, 1972.
- [19] A. Morassi, E. Rosset, Stable determination of cavities in elastic bodies, *Inverse Problems* 20 (2004) 453–480.
- [20] F. Rellich, Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral, *Math. Z.* 46 (1940) 635–636.
- [21] F. Santosa, M. Vogelius, J. Xu, An effective nonlinear boundary condition for a corroding surface. Identification of the damage based on steady state electric data, *Z. Angew. Math. Phys.* 49 (4) (1998) 656–679.

- [22] E. Sincich, Stable determination of the surface impedance of an obstacle by far field measurements, *SIAM J. Math. Anal.* 38 (2006) 434–451.
- [23] E. Sincich, Stability for the determination of unknown boundary and impedance with a Robin boundary condition, *SIAM J. Math. Anal.* 6 (2010) 2922–2943.
- [24] M. Vogelius, J. Xu, A nonlinear elliptic boundary value problem related to corrosion modeling, *Quart. Appl. Math.* 56 (1998) 479–505.