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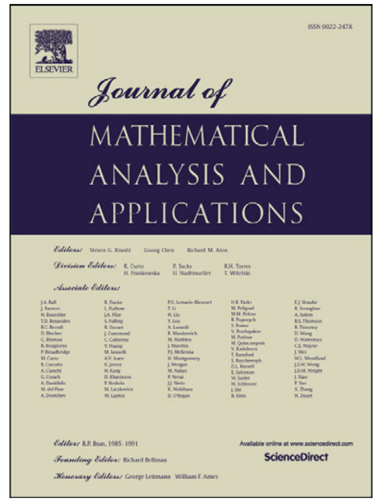

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Turnpike theorem for terminal functionals in infinite horizon optimal control problems

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Abstract

An optimal control problem for continuous time systems described by a special class of multi-valued mappings and quasi-concave utility functions is considered. The objective is defined as an analogue of the terminal functional defined over an infinite time horizon. An upper bound of this functional over all solutions to the system is established. The turnpike property is proved which states that all optimal solutions converge to some unique optimal stationary point.

Keywords: turnpike property, differential inclusions, optimal control, terminal functional, asymptotic stability
2000 MSC: 49J24,, 37C70.

1. Introduction

In this paper the turnpike property is investigated for a special class of non-convex optimal control problems in continuous time. Simple put this property states that, regardless of initial conditions, all *optimal* trajectories spend most of the time within a small neighborhood of some optimal stationary point when the planning period is long enough. For a classification of different definitions of the turnpike property, we refer the reader to [1, 5, 13, 16, 24], and also [2] for the so called *exponential* turnpike property. Possible applications in Markov Games can be found in a recent study [11].

Many approaches have been developed when considering continuous time and discrete time systems. The type of functional involved turns out to be very crucial in the proof of the turnpike property. Discounted and undiscounted integrals are the most commonly studied functionals. Among the most successful approaches developed for these types of functionals, we mention the approaches developed by Rockafellar [21, 22] and by Scheinkman, Brock and collaborators (see, for example, [12]). Several other approaches in this area have been developed including those considering special classes of problems (e.g. [10, 17, 23, 25]). An interesting class of control problems considered in [8, 9] involves long run average cost functions where the asymptotic behaviour of optimal solutions is defined in terms of a probability measure.

This paper considers a special class of terminal functionals defined as a lower limit at infinity of utility functions. This approach is introduced in [14] where stability results are established for some classes of non-convex problems with applications to environment pollution models. This class of terminal functionals is also used to establish the turnpike theory in terms of statistical convergence ([15, 18]) and A-statistical convergence ([4]), where the convergence of optimal trajectories to some stationary point is proved in the sense of "weak" convergence while ordinary convergence may not be true.

In this paper the turnpike property is established for optimal control problems involving continuous time systems described by differential inclusions. It generalizes some results from [14] obtained for a particular macroeconomic model of air pollution and establishes the turnpike property for a much broader class of optimal control problems by relaxing the assumptions imposed on the set of stationary points as well as on the utility function.

In this study the set of stationary points is not assumed to be bounded as required in the proof of the turnpike property in [14]. Moreover, the utility function is assumed to be quasi concave (instead of concavity in [14]). Obviously, a concave function is also quasi concave but not vice versa; for example, any monotonically increasing or decreasing function is quasi concave. Note that utility functions are often used to describe preferences that are usually assumed to be convex. If a preference relation is given by a continuous utility function, then this preference is convex if and only if the utility function is quasi concave. In this sense, the class of quasi concave utility functions is in some meaningful sense the largest class of functions representing convex preferences.

The assumptions and techniques used in this paper are essentially different from those developed for discrete systems in [4, 15, 18] where the main assumptions involve both the multi-valued mapping and the utility function. The main assumptions of this paper are imposed on the multi-valued mapping trying to keep (as much as possible) the utility function arbitrary. In this way we establish the class of multi-valued mappings (called Class $\mathcal{A}$) for which the turnpike property is true for any quasi concave utility function.

The reminder of the article is organized as follows. In the next section we formulate the problem and provide the notations and assumptions used throughout the paper. Section 3 presents the main results of the paper demonstrated with examples. Some preliminary results are provided in Section 4. The main theorems are proved in Section 5.

2. Problem formulation and assumptions

Consider the system

$$\dot{x}(t) \in a(x(t)), \qquad \text{a.e. } t \geq 0; \tag{1}$$

where x is an element of the Euclidean space R^n. The multi-valued mapping a is defined on a convex closed set $\mathcal{D}_a$ with non-empty interior, has compact images and is upper semi-continuous (u.s.c.) in the Hausdorff metric. The assumption that $\mathcal{D}_a$ has a non-empty interior is not restrictive; otherwise one could consider system (1) in a subspace of R^n (the affine hull of $\mathcal{D}_a$) by reducing the dimensionality of the space where the corresponding multi-valued mapping has a non-empty interior.

We will use the notation $a(A) = \cup_{x \in A} a(x)$ and, given a point x, we do not distinguish between $a(x)$ and $a(\{x\})$. Throughout the paper, " $\cdot$ ", "co" and "int" stand for the scalar product, convex hull and interior, respectively.

2.1. Solutions to (1)

An absolutely continuous function $\mathbf{x} = x(t), t \geq 0$, satisfying (1) is called a solution. We assume that system (1) has a bounded solution defined on an infinite horizon $[0, \infty)$. This is not a restrictive assumption and is satisfied for many practical models. Denote by $\mathbf{X}$ the set of all bounded solutions to (1); that is, for every $\mathbf{x} \in \mathbf{X}$ there exists $K_{\mathbf{x}} < \infty$ such that

$$\|x(t)\| \leq K_{\mathbf{x}}, \quad \forall\ t \in [0, \infty). \tag{2}$$

The existence of solutions to (1) is not a focus of this paper; we refer to [7] (Chapter 2, Section 7) for the related results. Here we just note that the existence of solutions defined on some small interval $[0, \varepsilon]$ can be guaranteed, for example, by assuming that (a) mapping a has convex compact images and is upper semi-continuous; or alternatively, (b) mapping a has compact images and is continuous.

Note that condition (2) means each particular solution is bounded, while the number $K_{\mathbf{x}}$ is not assumed to be uniformly bounded on the set of solutions $\mathbf{X}$. For example, in the Ramsey growth model ([20]) where $a(x) = \{\lambda\, f(x) - \delta\, x : \ \lambda \in [0,1]\}$, x is capital, $\delta > 0$ and $f(x) : [0,\infty) \to [0,\infty)$ is a concave (production) function satisfying $f'(x) \to 0$ as $x \to \infty$, the set $\mathcal{D}_a = [0,\infty)$ is unbounded and there exists a bounded solution from any initial point $x(0) \in \mathcal{D}_a$.

2.2. Class of multi-valued mappings $\mathcal{A}$

We define a class of mappings a, denoted by $\mathcal{A}$, satisfying the following condition: given any set $A \subset \mathcal{D}_a$

$$\text{if } 0 \in \operatorname{co} a(A) \quad \text{then} \quad \exists x \in \operatorname{co} A : \quad 0 \in \operatorname{co} a(x). \tag{A}$$

Clearly if a has convex images then the condition (A) can be represented as:

$$0 \in \operatorname{co} a(A) \quad \Rightarrow \quad 0 \in a(\operatorname{co} A). \tag{A^c}$$

This class of mapping was introduced in [14] when considering a particular model of air pollution control. The class $\mathcal{A}$ is quite broad; we provide here two examples that are commonly studied in the literature.

1: Convex mappings. Denote the graph of mapping a by

$$\operatorname{graph} a \triangleq \{(x,y) : \quad x \in \mathcal{D}_a,\ y \in a(x)\}.$$

It is easy to verify that if $\operatorname{graph} a$ is a convex set then condition (A) holds.

Note that the mappings with convex graphs are very important in many applications. Macroeconomic models are usually convex ([13, 16]); for example, the Ramsey growth model is one of the most studied convex models.

2: Linear mappings. Consider linear systems where mapping a is given by

$$a(x) = \{Bx + Cu; \quad u \in U\}.$$

Here B and C are $n \times n$ and $n \times r$ matrices and $U \subset R^r$ is any given set (not necessarily convex). Again, it is not difficult to verify that condition (A) holds without imposing any assumptions on matrices B, C and set U.

Indeed, let $0 \in \mathrm{co}\, a(A)$ for some set $A \subset R^n$. Then, there are points $x_i \in A$, $u_i \in U$ and numbers $\lambda_i \geq 0$, $i = 1, \cdots, k$, satisfying

$$0 = \sum_{i=1}^{k} \lambda_i \,(Bx_i + Cu_i) \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i = 1.$$

Denoting $x = \sum_{i=1}^{k} \lambda_i \, x_i \in \mathrm{co}\, A$ and $y_i = Bx + Cu_i \in a(x)$, we have

$$0 = Bx + \sum_{i=1}^{k} \lambda_i \, Cu_i = \sum_{i=1}^{k} \lambda_i \,(Bx + Cu_i) = \sum_{i=1}^{k} \lambda_i \, y_i.$$

This means that $0 \in \mathrm{co}\, a(x)$; that is, $a \in \mathcal{A}$.

2.3. Stationary points

Note that a stationary point is usually defined by the relation $0 \in a(x)$. In this paper we will use a different definition given by

Definition 2.1. *Point x is called a stationary point if* $0 \in \mathrm{co}\, a(x)$.

This definition in some sense defines a "generalized stationary point"; however, for the sake of simplicity we use the term "stationary point". The meaning of such a stationary point is discussed below (see also Example 3.5 in Section 3).

Clearly, if $0 \in \mathrm{co}\, a(\tilde{x})$ but $0 \notin a(\tilde{x})$, then the constant function $x(t) \equiv \tilde{x}$ is not a (stationary) trajectory starting from $\tilde{x}$. However, if in addition mapping a is Lipschitz continuous then (see [6]) given any $T > 0$ and any small ε there is a trajectory $\tilde{x}(t)$ such that $\tilde{x}(0) = \tilde{x}$ and $\|\tilde{x}(t) - \tilde{x}\| < \varepsilon$ for all $t \in [0, T]$. In other words, in this case $\tilde{x}$ is "almost" a stationary point. Although this result requires Lipschitz continuity of a (otherwise it may not be true; for a counterexample when a is only u.s.c. see [19]) we consider the points defined by Definition 2.1 as stationary points.

We denote the set of stationary points by

$$M \triangleq \{x \in \mathcal{D}_a : \quad 0 \in \mathrm{co}\, a(x)\}.$$

We will show that if $a \in \mathcal{A}$ and $\mathbf{X} \neq \emptyset$ then the set M is not empty. Moreover, from upper-semi continuity of a it will follow that M is a closed set.

2.4. Objective Functional

Given a continuous function $u : \mathcal{D}_u \to R^1$, consider the following objective

$$\text{Maximize :} \quad J(\mathbf{x}) = \liminf_{t\to\infty} u(x(t)). \tag{3}$$

For the sake of simplicity we assume $\mathcal{D}_a \subset \operatorname{int} \mathcal{D}_u$. The main results of the paper are obtained under the assumption that function u is quasi-concave or strictly quasi-concave on $\mathcal{D}_u$ as defined below (see, for example, [3]).

Definition 2.2. *The function u is called quasi-concave if for every $x_1 \neq x_2$*

$$u(\lambda x_1 + (1-\lambda)x_2) \geq \min\{u(x_1), u(x_2)\}, \quad \forall \lambda \in (0,1).$$

If the above inequality is strong, $u(x)$ is called strictly quasi-concave.

As mentioned above, the class of quasi concave utility functions is the largest class of functions representing convex preferences. A concave function is also quasi concave but not vice versa; for example, any monotonically increasing or decreasing function is quasi concave. The following is two examples for quasi concave functions in practical applications that are generated from simple concave utility functions:

- if $g(x)$ is concave and $h(g)$ is strictly increasing then $u(x) = h(g(x))$ is quasi concave;
- if $g_i(x), i = 1, \cdots, m$, are concave then $u(x) = \max_{i=1,\cdots,m} g_i(x)$ is quasi concave.

For example, a utility function in the form $u(x) = \max_{i=1,\cdots,m} a_i \ln(k_i x + b_i)$, $a_i, k_i, b_i > 0$, is considered in [26] to describe the fairness in a general telecommunications network.

Summarizing the above, we consider problem (1),(3) under the following assumptions:

(i) multi-valued mapping a is defined on convex closed set $\mathcal{D}_a$ with non-empty interior, has compact images and is upper semi-continuous in the Hausdorff metric;

(ii) there exists a bounded solution defined on $[0, \infty)$; that is, $\mathbf{X} \neq \emptyset$;

(iii) function u is continuous on $\mathcal{D}_u$, where $\mathcal{D}_a \subset \operatorname{int} \mathcal{D}_u$.

Note that $\mathcal{D}_a$ is not necessarily the whole space R^n. For the sake of simplicity, the assumptions **(i)-(iii)** are not mentioned in the theorems and lemmas below. Rather we emphasize the major assumptions, such as $a \in \mathcal{A}$ and the quasi-concavity of function u.

3. Main results

Under the assumptions of the theorems below, we will show the set of stationary points M is non-empty. We denote

$$u^* = \sup_{x \in M} u(x). \tag{4}$$

$x^* \in M$ is called an optimal stationary point (o.s.p.) if $u(x^*) = u^*$.

Given solution $\mathbf{x} = x(t)$ we denote by $P(\mathbf{x})$ the set of ω-limit points defined by

$$P(\mathbf{x}) \triangleq \{\xi : \; x(t_k) \to \xi \text{ for some } t_k \to \infty\}. \tag{5}$$

Since solutions are bounded, the set $P(\mathbf{x})$ is bounded and closed for every $\mathbf{x} \in \mathbf{X}$.

Theorem 3.1. (Upper bound of the functional) *Assume that $a \in \mathcal{A}$ and the function u is quasi-concave. Then*

$$J(\mathbf{x}) \leq u^* \quad \text{for all solutions } \mathbf{x} \in \mathbf{X}. \tag{6}$$

Note that in this theorem, M is not necessarily bounded. Moreover, it might be that $u^* = \infty$.

The proof of Theorem 3.1 is based on Lemma 4.2 in Section 4 which states that if $a \in \mathcal{A}$ then for every bounded solution $\mathbf{x}$ the following relation holds:

$$\operatorname{co} P(\mathbf{x}) \cap M \neq \emptyset. \tag{7}$$

This relation in particular means $M \neq \emptyset$. The following question related to the class $\mathcal{A}$ and relation (7) is of interest. Given a bounded solution $\mathbf{x}$, can (7) be satisfied if $a \notin \mathcal{A}$?

It appears quite reasonable to expect that on the plane the convex hull of the ω-limit set of any bounded solution should contain some stationary point; that is, relation (7) should hold in R^2 (we do not have proof for this statement). However the following example shows that this is not the case for R^n, $n \geq 3$, where relation (7) may not be satisfied for some bounded solution $\mathbf{x}$ if $a \notin \mathcal{A}$.

Example 3.2. *Consider system (1) where*

$$a(x, y, z) = \{(-y, x, 1 - x^2 - y^2) : \ (x, y, z) \in R^3\}.$$

The solution from the initial point $(1, 0, 0)$ can be obtained as follows:

$$x(t) = \cos(t), \ y(t) = \sin(t), \ z(t) = 0, \quad t \in [0, \infty).$$

This solution is bounded and its ω-limit set is given by

$$P = \{(x, y, z) : \ z = 0, \ x^2 + y^2 = 1\}.$$

It easy to verify that the set $\operatorname{co} P = \{(x, y, z) : \ z = 0, \ x^2 + y^2 \leq 1\}$ does not contain any stationary point; that is (7) does not hold. Indeed, first we note that for any (x, y, z) the set $a(x, y, z)$ is a singleton which means that the relation $0 \in \operatorname{co} a(x, y, z)$ is equivalent to $0 = a(x, y, z)$. Now let $(\dot{x}, \dot{y}, \dot{z}) = a(x, y, z)$. Clearly, if $x^2 + y^2 = 1$ either $\dot{x} \neq 0$ or $\dot{y} \neq 0$; on the other hand, if $x^2 + y^2 < 1$ then $\dot{z} \neq 0$. Thus, $0 \notin a(x, y, z)$ for all $(x, y, z) \in \operatorname{co} P$.

Now we show that the mapping a does not belong to class $\mathcal{A}$. Since the images of a are convex (i.e. singleton) we verify condition (A^c).

Consider the set of two points $A = \{(1, 0, 0), \ (-1, 0, 0)\} \subset P$. We have $a(1, 0, 0) = (0, 1, 0)$, $a(-1, 0, 0) = (0, -1, 0)$, and therefore

$$(0, 0, 0) = \frac{1}{2} a(1, 0, 0) + \frac{1}{2} a(-1, 0, 0) \in \operatorname{co} a(A).$$

However, $(0, 0, 0) \notin a(\operatorname{co} A)$. Indeed, for any $\lambda \in [0, 1]$ for the points

$$(x_\lambda, y_\lambda, z_\lambda) = \lambda\,(1, 0, 0) + (1 - \lambda)(-1, 0, 0) = (2\lambda - 1, 0, 0) \in \operatorname{co} A$$

we have

$$a(x_\lambda, y_\lambda, z_\lambda) = (0, 2\lambda - 1, 1 - (2\lambda - 1)^2) \neq (0, 0, 0), \quad \forall \lambda \in [0, 1]$$

which means that $a \notin \mathcal{A}$.

This example shows that the assumption $a \in \mathcal{A}$ is important for (7) and therefore for Theorem 3.1. Now we formulate the turnpike theorem.

Theorem 3.3. (Turnpike property) *Assume that* $a \in \mathcal{A}$, *function* u *is strictly quasi-concave and there exists a unique o.s.p.* x^*. *Then any solution* $\mathbf{x} \in \mathbf{X}$ *satisfying* $J(\mathbf{x}) = u(x^*)$ *(i.e. optimal by Theorem 3.1) converges to* x^*; *that is,*

$$\lim_{t\to\infty} x(t) = x^*, \quad \forall \mathbf{x} \in \mathbf{X},\ J(\mathbf{x}) = u(x^*). \tag{8}$$

Note that any continuous and strictly quasi-concave function achieves its maximum on a convex compact set at a unique point. This means that if the set of stationary points M is bounded and convex the assumption about the existence and uniqueness of x^* can be removed. Thus from Theorem 3.3 we have

Corollary 3.4. *Assume that* $a \in \mathcal{A}$ *and* M *is nonempty, convex and bounded. Then given any strictly quasi-concave function* u, *there exists a unique o.s.p.* x^* *and (8) holds.*

This corollary generalizes Theorem 1 from [14] where utility function u is assumed to be strictly concave on R^n.

Note that o.s.p. x^* does not necessarily satisfy $0 \in a(x^*)$ even if it attracts all optimal solutions. We demonstrate this fact on the next example where the equality $J(\mathbf{x}) = u(x^*)$ is satisfied (thus $\mathbf{x}$ is optimal and converges to x^* by Theorem 3.3); however $0 \notin a(x^*)$. In this example, there is no stationary point x satisfying $0 \in a(x)$; that is, $M = \{x^*\}$ and $0 \in \operatorname{co} a(x^*)$.

Example 3.5. *Let*

$$a(x, y) = \{(1-x, -1), (1-x, 1)\} \quad \text{and} \quad u(x, y) = x - y^2, \qquad \forall (x, y) \in R^2.$$

It can easily be verified that mapping a is in Class $\mathcal{A}$. Indeed, if $(0, 0) \in \operatorname{co} a(A)$ then A contains at least two points (x_1, y_1) and (x_2, y_2) with $x_1 \le 1 \le x_2$; that is, $1 = \lambda x_1 + (1-\lambda) x_2$ for some $\lambda \in [0, 1]$. In this case, $(0, 0) \in \operatorname{co} a(1, y_\lambda)$, where $y_\lambda = \lambda y_1 + (1-\lambda) y_2$ and $(1, y_\lambda) \in \operatorname{co} A$; that is, (A) holds.

Moreover, $(0, 0) \in \operatorname{co} a(1, y)$ for all y; that is, $M = \{(1, y) : y \in (-\infty, \infty)\}$. Clearly, function u is strictly quasi-concave (but not strictly concave) and $(1, 0)$ is a unique optimal stationary point:

$$(1, 0) = \arg\max\{u(x, y) :\ (x, y) \in M\}, \quad u^* = u(1, 0) = 1.$$

Consider an arbitrary solution $(x(t), y(t))$ starting from an initial point (x^0, y^0). We have $x(t) = 1 + (x^0 - 1)\exp(-t)$ and therefore $x(t) \to 1$ as $t \to \infty$.

It is not difficult to construct a function $y(t)$ satisfying $y(0) = y^0$, $\dot{y}(t) \in \{-1, 1\}$ for almost all $t \geq 0$, such that $y(t) \to 0$ as $t \to \infty$. In this case, $(x(t), y(t)) \to (1, 0)$ as $t \to \infty$, and therefore $(x(t), y(t))$ is an optimal solution with the maximum possible objective function value $u^* = 1$. Then, Theorem 3.3 ensures that all optimal solutions, not depending on the initial state, converge to the unique o.s.p. $(1, 0)$.

On the other hand, $(0, 0) \notin a(1, 0)$; that is, $(x(t), y(t)) \equiv (1, 0)$ is not a solution to the differential inclusion given by mapping $a(x, y)$.

4. Preliminary lemmas

Throughout the paper, given closed set B the notations $V(B, \varepsilon)$ and $\overline{V}(B, \varepsilon)$ stand for the open and closed ε-neighborhood of $B \subset R^n$:

$$V(B, \varepsilon) = \{x \in R^n : \ \rho(x, B) < \varepsilon\}, \ \ \overline{V}(B, \varepsilon) = \{x \in R^n : \ \rho(x, B) \leq \varepsilon\};$$

where $\rho(x, B) = \min_{y \in B} \|x - y\|$ is the distance from x to the set B.

The following lemma directly follows from the definition of class $\mathcal{A}$; that is, from condition (A) (see also Lemma 2, [14]).

Lemma 4.1. *Assume that mapping $a \in \mathcal{A}$ and that $B \subset R^n$ is a convex set. Then*

$$0 \notin \operatorname{co} a(x), \ \forall x \in B \quad \Rightarrow \quad 0 \notin \operatorname{co} a(B). \tag{9}$$

The next lemma is proved in [14] (Lemma 3), by assuming that the set M is bounded. In fact it is also true if M is unbounded. We provide this lemma with its proof for the completeness of the presentation.

Lemma 4.2. *Assume that $a \in \mathcal{A}$. Then for every solution $\mathbf{x} \in \mathbf{X}$ the following relation holds:*

$$\operatorname{co} P(\mathbf{x}) \cap M \neq \emptyset. \tag{10}$$

Proof. On the contrary assume that the intersection in (10) is empty for some solution $\mathbf{x} \in \mathbf{X}$. Since both sets $\operatorname{co} P(\mathbf{x})$ and M are closed and in addition $P(\mathbf{x})$ is bounded, there is $\varepsilon > 0$ such that $\overline{V}(\operatorname{co} P(\mathbf{x}), \varepsilon) \cap M = \emptyset$; that is,

$$0 \notin \operatorname{co} a(x), \ \forall x \in \overline{V}(\operatorname{co} P(\mathbf{x}), \varepsilon).$$

From Lemma 4.1 we have

$$0 \notin \operatorname{co} a(\overline{V}(\operatorname{co} P(\mathbf{x}), \varepsilon)).$$

Now, since a is upper-semi continuous, the set $a(\overline{V}(\operatorname{co} P(\mathbf{x}), \varepsilon))$ is compact. Then, there are $q \in R^n$ and $\delta > 0$ such that

$$q \cdot y \geq \delta > 0, \quad \forall y \in \operatorname{co} a(\overline{V}(\operatorname{co} P(\mathbf{x}), \varepsilon)).$$

Since $P(\mathbf{x})$ is the set of ω-limit points of $x(t)$, there is $t_\varepsilon > 0$ such that the inclusion $x(t) \in \overline{V}(P(\mathbf{x}), \varepsilon)$ holds for all $t \geq t_\varepsilon$. Then

$$q \cdot \dot{x}(t) \geq \delta, \qquad \text{a.e. } t \geq t_\varepsilon$$

or

$$q \cdot x(t) = q \cdot x(t_\varepsilon) + \int_{t_\varepsilon}^{t} q \cdot \dot{x}(s)\, ds \geq q \cdot x(t_\varepsilon) + \delta\, (t - t_\varepsilon) \to \infty, \quad \text{as } t \to \infty,$$

which means that $x(t)$ is not bounded. This is a contradiction.

□

For the rest of paper, we will assume that $a \in \mathcal{A}$, solution $\mathbf{x} \in \mathbf{X}$ is fixed and u is a given strictly quasi-concave function. Lemma 4.2 in particular implies $M \neq \emptyset$. We will also assume that o.s.p. x^* exists and is unique:

$$u(x^*) = \max_{x \in M} u(x) = u^*.$$

Denote

$$\Omega \ \triangleq \ \{x \in \mathcal{D}_a : \quad \|x\| \leq \overline{K}_{\mathbf{x}}\},$$

where $\overline{K}_{\mathbf{x}} = \max\{\|x^*\|, K_{\mathbf{x}}\}$ and $K_{\mathbf{x}}$ satisfies (2). We note that Ω is convex with non-empty interior and

$$\{x(t), \ t \in [0, \infty)\} \cup P(\mathbf{x}) \subset \Omega.$$

Consider the set

$$L \ \triangleq \ \{x \in \Omega : \quad u(x) \geq u^*\}. \tag{11}$$

Clearly, L is a non-empty closed bounded set and $x^* \in L$.

Lemma 4.3. *Assume that $a \in \mathcal{A}$, function u is strictly quasi-concave and there exists a unique o.s.p. x^*. Then there is a non-zero vector $p \in R^n$ such that*

$$p \cdot (x - x^*) > 0, \quad \forall x \in L, x \neq x^*. \tag{12}$$

Proof. Consider

$$\mathcal{L} \triangleq \{x \in \mathcal{D}_u : \quad u(x) \geq u^*\}. \tag{13}$$

Since u is strictly quasi-concave the set $\mathcal{L}$ is convex and $L \subset \mathcal{L}$. In particular, $x^* \in \mathcal{L}$. Now if $x^* \in \operatorname{int} \mathcal{L}$, then $x^* = (x_1 + x_2)/2$ for some $x_1, x_2 \in \mathcal{L}$. In this case we have

$$u(x^*) > \min\{u(x_1), u(x_2)\} \geq u^*$$

which is a contradiction. Thus, x^* belongs to the boundary of the convex set $\mathcal{L}$; that is, there exists a non-zero vector $p \in R^n$ such that

$$p \cdot (x - x^*) \geq 0, \quad \forall x \in \mathcal{L}. \tag{14}$$

Now, we show (12). On the contrary, assume that it is not true; that is, for some $\tilde{x} \in L, \tilde{x} \neq x^*$ the relation $p \cdot (\tilde{x} - x^*) = 0$ holds. Then for the point $x' = (\tilde{x} + x^*)/2 \in L$ we have

$$p \cdot (x' - x^*) = 0, \quad \text{and} \quad u(x') > \min\{u(\tilde{x}), u(x^*)\} \geq u^*.$$

Since $x^*, \tilde{x} \in \Omega \subset \mathcal{D}_a \subset \operatorname{int} \mathcal{D}_u$ we have $x' \in \operatorname{int} \mathcal{D}_u$. On the other hand, u is continuous. Then, there is a sufficiently small $\lambda > 0$ such that

$$x' - \lambda p \in \mathcal{D}_u, \quad \text{and} \quad u(x' - \lambda p) > u^*.$$

This in particular means $x' - \lambda p \in \mathcal{L}$. Thus, from the relation $p \cdot (x' - x^*) = 0$ we have $p \cdot (x' - \lambda p - x^*) < 0$. This contradicts (14).

□

For given positive number δ we define

$$B_\delta = \{x \in L : \quad p \cdot (x - x^*) \geq \delta\}. \tag{15}$$

Here, p is defined in Lemma 4.3. Clearly, if not empty, B_δ is a convex set and $x^* \notin B_\delta$ for $\delta > 0$. According to the assumption that o.s.p. x^* is unique, the set B_δ does not contain any stationary points: $0 \notin \operatorname{co} a(x), \ \forall x \in B_\delta$.

Then by Lemma 4.1 we have $0 \notin \operatorname{co} a(B_\delta)$. Since $\operatorname{co} a(B_\delta)$ is convex and compact, there is a non-zero vector $q_\delta \in R^n$, for the sake of simplicity we assume $\|q_\delta\| = 1$, and $\varepsilon_\delta > 0$, such that

$$q_\delta \cdot y \geq \varepsilon_\delta, \quad \forall y \in \operatorname{co} a(B_\delta). \tag{16}$$

We also denote

$$P_\delta \triangleq P(\mathbf{x}) \cap B_\delta. \tag{17}$$

Lemma 4.4. *Assume that all assumptions in Lemma 4.3 hold. In addition let $P(\mathbf{x}) \subset L$ and $P(\mathbf{x}) \cap B_{\delta'} \neq \emptyset$ for some $\delta' > 0$. Then for all $\delta \in (0, \delta'/2]$ the following relations hold*

$$\max_{x \in P_\delta} (q_\delta \cdot x) > \min_{x \in P_\delta} (q_\delta \cdot x); \tag{18}$$

$$p \cdot (x - x^*) = \delta, \quad \text{for all } x \in (\arg\max_{x \in P_\delta} (q_\delta \cdot x)) \ \cup \ (\arg\min_{x \in P_\delta} (q_\delta \cdot x)). \tag{19}$$

Proof. Take any $z \in P_{\delta'}$ and let $\delta \in (0, \delta'/2]$.

(i) Since z is a limit point of $x(t)$, there is $t_k \to \infty$ such that $x(t_k) \to z$. From (16) we have $q_\delta \cdot y \geq \varepsilon_\delta > 0$ for all $y \in a(z)$. Since a is u.s.c., there is a sufficiently small neighborhood $\overline{V}(z, \gamma)$ of z such that

$$q_\delta \cdot y \geq \varepsilon_\delta/2 > 0 \quad \text{for all } y \in a(\overline{V}(z, \gamma)). \tag{20}$$

Noticing that $z \in B_{\delta'}$ implies $p \cdot (z - x^*) \geq \delta' \geq 2\,\delta$, the number $\gamma > 0$ can be chosen so small that

$$p \cdot (x - x^*) \geq \delta, \quad \forall x \in \overline{V}(z, \gamma). \tag{21}$$

In addition, there are $\tilde{t} > 0$ and $\eta > 0$ such that

$$x(t) \in \overline{V}(z, \gamma) \quad \text{for all} \quad t \in [t_k - \eta, t_k + \eta], \ t_k \geq \tilde{t}. \tag{22}$$

Then from (20) we have

$$q_\delta \cdot x(t_k + \eta) = q_\delta \cdot x(t_k) + \int_{t_k}^{t_k+\eta} q_\delta \cdot \dot{x}(s)\, ds \geq q_\delta \cdot x(t_k) + \eta\, \varepsilon_\delta/2.$$

For any convergent subsequence $x(t_{k_m} + \eta) \to \tilde{x}^1$ by taking limit we obtain

$$q_\delta \cdot \tilde{x}^1 \geq q_\delta \cdot z + \eta\, \varepsilon_\delta/2 > q_\delta \cdot z.$$

Clearly, $\tilde{x}^1 \in P(\mathbf{x}) \subset L$. Moreover, from (21), (22) we have $p \cdot (\tilde{x}^1 - x^*) \geq \delta$ and therefore $\tilde{x}^1 \in P_\delta$. On the other hand $z \in B_{\delta'} \subset B_\delta$ which implies $z \in P_\delta$. Thus, the inequality $q_\delta \cdot \tilde{x}^1 > q_\delta \cdot z$ holds at two different points $\tilde{x}^1, z \in P_\delta$; that is, (18) is true.

(ii) Now we prove (19) for the set $\arg\max\{q_\delta \cdot x : \ x \in P_\delta\}$. On the contrary assume that

$$p \cdot (z - x^*) > \delta \quad \text{for some} \quad z \in \arg\max\{q_\delta \cdot x : \ x \in P_\delta\}.$$

Similar to the case **(i),** since, $z_\delta \in P_\delta \subset B_\delta$ from (16) the relation $q_\delta \cdot y \geq \varepsilon_\delta > 0$ holds for all $y \in a(z)$. And then there is a sufficiently small neighborhood $\overline{V}(z, \gamma)$ of z such that (20) holds.

Now, since $p \cdot (z - x^*) > \delta$ the number $\gamma > 0$ can be chosen so small that (21) holds. Then, following the steps in the case **(i),** we can construct a point $\tilde{x}^1 \in P_\delta$ for which the inequality $q_\delta \cdot \tilde{x}^1 > q_\delta \cdot z$ holds. This contradicts $z \in \arg\max\{q_\delta \cdot x : \ x \in P_\delta\}$.

(iii) The proof of relation (19) for any point $z \in \arg\min\{q_\delta \cdot x : \ x \in P_\delta\}$ is similar. In this case, we repeat the above procedure by considering any convergent subsequence $x(t_{k_m} - \eta) \to \tilde{x}^2$. Then from

$$q_\delta \cdot x(t_k - \eta) = q_\delta \cdot x(t_k) + \int_{t_k}^{t_k - \eta} q_\delta \cdot \dot{x}(s)\, ds \leq q_\delta \cdot x(t_k) - \eta\, \varepsilon_\delta/2$$

we have

$$q_\delta \cdot \tilde{x}^2 \leq q_\delta \cdot z - \eta\, \varepsilon_\delta/2 < q_\delta \cdot z.$$

It can be verified that $\tilde{x}^2 \in P_\delta$. Then the above inequality contradicts the assumption that $z \in \arg\min\{q_\delta \cdot x : \ x \in P_\delta\}$.

□

Finally, we formulate the following simple property that is provided with the proof for the completeness of the presentation.

Lemma 4.5. *Assume that the sequence q_k converges to q and the sequence of compact sets P_k converges to a compact set P in the Hausdorff metric. Then*

$$\lim_{k\to\infty} \max_{x\in P_k}(q_k \cdot x) \;=\; \max_{x\in P}(q\cdot x) \quad \text{and} \quad \lim_{k\to\infty} \min_{x\in P_k}(q_k \cdot x) \;=\; \min_{x\in P}(q\cdot x).$$

Proof. Denote $\xi_k = \max_{x\in P_k}(q_k \cdot x)$ and $\xi = \max_{x\in P}(q\cdot x)$. Let $\tilde{x} \in P$ such that $\xi = q{\cdot}\tilde{x}$. Consider any limit point of ξ_k; say $\xi_{k_m} \to \tilde{\xi}$. We show that $\tilde{\xi} = \xi$.

Let $x_{k_m} \in P_{k_m}$ such that $\xi_{k_m} = q_{k_m} \cdot x_{k_m}$ and for sake of simplicity let $x_{k_m} \to x^0$. Clearly $x^0 \in P$. Then by taking limit we have

$$\tilde{\xi} = q\cdot x^0 \le \max_{x\in P}(q\cdot x) = \xi. \tag{23}$$

On the other hand, since $\tilde{x} \in P$ and P_k converges to P in the Hausdorff metric, there is a sequence $x_{k_i} \in P_{k_i}$ such that $x_{k_i} \to \tilde{x}$. Then,

$$\xi_{k_i} = \max_{x\in P_{k_i}}(q_{k_i}\cdot x) \ge q_{k_i}\cdot x_{k_i}$$

and by taking limit we have $\tilde{\xi} \ge q\cdot\tilde{x} = \xi$. This, together with (23) implies $\tilde{\xi} = \xi$.

The second assertion with "min" follows from the first one by replacing q_k with $(-q_k)$.

□

Below we consider small numbers δ satisfying $\delta \le \delta'/2$ as in Lemma 4.4.

Lemma 4.6. *Assume that all assumptions in Lemma 4.4 hold. If $q_{\delta_k} \to q$ as $\delta_k \downarrow 0$, then*

$$q\cdot(x - x^*) = 0, \quad \forall x \in P(\mathbf{x}). \tag{24}$$

Proof. Consider the sets P_{δ_k} defined by (17) and for the sake of simplicity assume that $\delta_1 > \delta_2 > \dots$. Clearly $B_{\delta_1} \subset B_{\delta_2} \subset \dots$, and therefore $P_{\delta_1} \subset P_{\delta_2} \subset \dots \subset P(\mathbf{x})$.

Take any sequence $z_{\delta_k} \in \arg\max\{q_{\delta_k} \cdot x : \ x \in P_{\delta_k}\}$. From Lemma 4.4 it follows that the relation $p \cdot (z_{\delta_k} - x^*) = \delta_k$ holds. Thus any limit point z' of z_{δ_k} satisfies $z' \in P(\mathbf{x})$ and $p \cdot (z' - x^*) = 0$. Then from (12) we obtain $z' = x^*$. Therefore any limit point z' of the sequence z_{δ_k} coincides with x^*; that is, $z_{\delta_k} \to x^*$.

We show that $P_{\delta_k} \to P(\mathbf{x})$ in the Hausdorff metric. On the contrary assume that there exist $\tilde{x} \in P(\mathbf{x})$ and a small number $\varepsilon > 0$, for which $V(\tilde{x}, \varepsilon) \cap P_{\delta_k} = \emptyset$ for all k. This implies $z_{\delta_k} \notin V(\tilde{x}, \varepsilon)$; that is, $\tilde{x} \neq x^*$. On the other hand the relation $p \cdot (\tilde{x} - x^*) = 0$ holds. This contradicts (12) since $\tilde{x} \in P(\mathbf{x}) \subset L$.

Therefore $P_{\delta_k} \to P(\mathbf{x})$ in the Hausdorff metric. Then from Lemma 4.5 we obtain

$$\lim_{k\to\infty} \max_{x \in P_k} (q_{\delta_k} \cdot x) \ = \max_{x \in P} (q \cdot x). \tag{25}$$

On the other hand

$$\lim_{k\to\infty} \max_{x \in P_k} (q_{\delta_k} \cdot x) \ = \lim_{k\to\infty} (q_{\delta_k} \cdot z_{\delta_k}) = q \cdot x^*.$$

This relation together with (25) implies $\max_{x\in P}(q \cdot x) = q \cdot x^*$. In a similar way it can be proved that $\min_{x \in P}(q \cdot x) = q \cdot x^*$. Thus (24) is true.

□

5. Proofs of main theorems

5.1. Proof of Theorem 3.1

Since function u is continuous it is not difficult to show that for any solution $\mathbf{x} \in \mathbf{X}$ the representation

$$J(\mathbf{x}) = \liminf_{t\to\infty} u(x(t)) = \min_{x \in P(\mathbf{x})} u(x). \tag{26}$$

is true (see similar results in terms of the statistical limit [18] and A-statistical limit [4]). In addition, from Lemma 4.2 we know that $\operatorname{co} P(\mathbf{x}) \cap M \neq \emptyset$. Then since u is quasi-concave, we obtain

$$J(\mathbf{x}) = \min_{x \in P(\mathbf{x})} u(x) = \min_{x \in \operatorname{co} P(\mathbf{x})} u(x) \leq \min_{x \in \operatorname{co} P(\mathbf{x}) \cap M} u(x) \leq \sup_{x \in M} u(x) = u^*.$$

The theorem is proved.

5.2. Proof of Theorem 3.3

Consider any optimal solution $\mathbf{x} = x(t)$ satisfying $J(\mathbf{x}) = u^*$. From Lemma 4.3 it follows that $M \neq \emptyset$. By the assumption of theorem, o.s.p. $x^* \in M$ is unique. Note that $u^* = u(x^*)$ and (26) implies

$$u(x) \geq u^*, \quad \forall x \in P(\mathbf{x}).$$

This in particular means that $P(\mathbf{x}) \subset L$. Since u is strictly quasi-concave, the set L is convex and therefore $\operatorname{co} P(\mathbf{x}) \subset L$. Moreover, $L \cap M = \{x^*\}$ and $\operatorname{co} P(\mathbf{x}) \cap M \neq \emptyset$ (Lemma 4.2). Thus $x^* \in \operatorname{co} P(\mathbf{x})$.

We claim that $x^* \in P(\mathbf{x})$. Indeed, otherwise, $x^* = \sum_i \lambda_i z_i$, where $\lambda_i > 0$, $\sum_i \lambda_i = 1$ and $z_i \in P(\mathbf{x})$. Since u is strictly quasi-concave

$$u^* = u(x^*) > \min_i u(z_i) \geq u^*.$$

This is a contradiction.

Thus, $x^* \in P(\mathbf{x})$. To prove the theorem, we need to show that $P(\mathbf{x}) \setminus \{x^*\} = \emptyset$.

On the contrary, assume that $z' \in P(\mathbf{x})$ and $z' \neq x^*$. Denote $\delta' = p \cdot (z' - x^*)$. The relation (12) implies $\delta' > 0$. From Lemma 4.4, for all $\delta \in (0, \delta'/2]$ the inequality (18) holds; that is,

$$\max_{x \in P_\delta}(q_\delta \cdot x) > \min_{x \in P_\delta}(q_\delta \cdot x), \quad \forall \delta \in (0, \delta'/2]. \tag{27}$$

Recall that $\|q_\delta\| = 1$ and (16) holds.

Consider a sequence $\delta_k \to 0$ such that $q_{\delta_k} \to q^1$. Clearly, $\|q^1\| = 1$ and according to Lemma 4.6

$$q^1 \cdot (x - x^*) = 0, \quad \forall x \in P(\mathbf{x}). \tag{28}$$

(i). We observe that, $q_\delta \neq \xi q^1$ for all numbers ξ and $\delta \in (0, \delta'/2]$; otherwise, since $P_\delta \subset P(\mathbf{x})$ the relation (27) contradicts (28). Denote

$$q_\delta^2 = \frac{1}{\|q_\delta - \xi_1 q^1\|}(q_\delta - \xi_1 q^1), \quad \text{where } \xi_1 = q^1 \cdot q_\delta.$$

It is not difficult to verify that, for all $\delta \in (0, \delta'/2]$ the following two conditions hold:

- $q_\delta^2 \cdot q^1 = 0$;
- $\|q_\delta^2\| = 1$.

Consider any point point $z_\delta^1 \in \arg\max_{x\in P_\delta}(q_\delta \cdot x)$. From (28) we know that $q^1 \cdot z_\delta^1 = q^1 \cdot x^* = q^1 \cdot x$ for all $x \in P_\delta \subset P(\mathbf{x})$. Then, we have

$$q_\delta^2 \cdot z_\delta^1 = \frac{1}{\|q_\delta - \xi_1\, q^1\|}\ \left[(q_\delta \cdot z_\delta^1) - \xi_1\,(q^1 \cdot z_\delta^1)\right]$$
$$= \frac{1}{\|q_\delta - \xi_1\, q^1\|}\ \left[\max_{x\in P_\delta}(q_\delta \cdot x) - \xi_1\,(q^1 \cdot x^*)\right]$$
$$= \frac{1}{\|q_\delta - \xi_1\, q^1\|}\ \max_{x\in P_\delta}\left[(q_\delta - \xi_1\, q^1)\cdot x\right] = \max_{x\in P_\delta}(q_\delta^2 \cdot x).$$

A similar relation can be obtained for any point $z_\delta^2 \in \arg\min_{x\in P_\delta}(q_\delta \cdot x)$ to show that $q_\delta^2 \cdot z_\delta^2 = \min_{x\in P_\delta}(q_\delta^2 \cdot x)$. Therefore,

$$z_\delta^1 \in \arg\max_{x\in P_\delta}(q_\delta^2 \cdot x) \quad \text{and} \quad z_\delta^2 \in \arg\min_{x\in P_\delta}(q_\delta^2 \cdot x).$$

Now we note that for the points z_δ^1 and z_δ^2 the relation (19) holds. This means that Lemma 4.6 can be applied to the sequence $q_{\delta_k}^2$.

Consider any convergent subsequence $q_{\delta_k}^2 \to q^2$. Clearly $q^2 \cdot q^1 = 0$ and $\|q^2\| = 1$. Lemma 4.6 states that

$$q^2 \cdot (x - x^*) = 0, \quad \forall x \in P(\mathbf{x}). \tag{29}$$

(ii). Now, if the relation $q_\delta = \lambda_1 q^1 + \lambda_2 q^2$ holds for some λ_1, λ_2, then from (28) and (29) it follows that $q_\delta \cdot x = \lambda_1 q^1 \cdot x^* + \lambda_2 q^2 \cdot x^* = \mathit{const}$ for all $x \in P(\mathbf{x})$. This contradicts (27). Then, similar to q_δ^2, we define

$$q_\delta^3 = \frac{1}{\|q_\delta - \xi_1\, q^1 - \xi_2\, q^2\|}\,(q_\delta - \xi_1\, q^1 - \xi_2\, q^2), \quad \text{where } \xi_1 = q^1 \cdot q_\delta,\ \xi_2 = q^2 \cdot q_\delta.$$

Then it can be shown that, for all $\delta \in (0, \delta'/2]$ the following two conditions hold:

- $q_\delta^3 \cdot q^1 = q_\delta^3 \cdot q^2 = 0$;
- $\|q_\delta^3\| = 1$.

Moreover,

$$z_\delta^1 \in \arg\max_{x\in P_\delta}(q_\delta^3 \cdot x) \quad \text{and} \quad z_\delta^2 \in \arg\min_{x\in P_\delta}(q_\delta^3 \cdot x).$$

Then, again we can apply Lemma 4.6 to the sequence $q^3_{\delta_k}$. Taking a convergent subsequence $q^3_{\delta_k} \to q^3$, we ensure that $\|q^3\| = 1$, $q^3 \cdot q^i = 0$, for $i = 1, 2$ and

$$q^3 \cdot (x - x^*) = 0, \quad \forall x \in P(\mathbf{x}).$$

(iii). We can continue this procedure by constructing n pairwise orthogonal unit vectors q^i, $i = 1, 2, \ldots, n$, such that the following relation holds

$$q^i \cdot (x - x^*) = 0, \quad \forall x \in P(\mathbf{x}) \subset R^n, \quad i = 1, 2, \ldots, n. \tag{30}$$

This contradicts the assumption that $z' \in P(\mathbf{x})$ and $z' \neq x^*$.

The theorem is proved.

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