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Spectral property of a class of Moran measures on $\mathbb{R}$ ☆

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ABSTRACT

Let $b \geq 2$ be a positive integer. Let $\mathcal{D}$ be a finite subset of $\mathbb{Z}$ and $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ be a sequence of strictly increasing numbers. A Moran measure $\mu_{b,\mathcal{D},\{n_k\}}$ is a Borel probability measure generated by the Moran iterated function system (Moran IFS) $\{f_{k,d}(x) = b^{n_{k-1}-n_k}(x+d) : d \in \mathcal{D}, k \in \mathbb{N}, n_0 = 0\}$. In this paper we study one of the basic problems in Fourier analysis associated with $\mu_{b,\mathcal{D},\{n_k\}}$. More precisely, we give some conditions under which the measure $\mu_{b,\mathcal{D},\{n_k\}}$ is a spectral measure, i.e., there exists a discrete subset $\Lambda \subseteq \mathbb{R}$ such that $E(\Lambda) = \{e^{2\pi i\lambda x} : \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\mu_{b,\mathcal{D},\{n_k\}})$.

1. Introduction

Let μ be a compactly supported Borel probability measure on $\mathbb{R}$. We call μ a *spectral measure* if there exists a discrete set $\Lambda \subseteq \mathbb{R}$ such that $E(\Lambda) = \{e^{2\pi i\lambda x} : \lambda \in \Lambda\}$ forms an orthonormal basis (Fourier basis) for $L^2(\mu)$. The set Λ is called a *spectrum* for μ; we also say that (μ, Λ) is a *spectral pair*. In this paper we will study the spectral property of a class of *Moran measures* $\mu_{b,\mathcal{D},\{n_k\}}$ on $\mathbb{R}$ generated by the Moran iterated function system (Moran IFS)

$$\{f_{k,d}(x) = b^{n_{k-1}-n_k}(x+d) : d \in \mathcal{D}, k \in \mathbb{N}\},$$

where $b \geq 2$ is an integer, $\mathcal{D}$ is a finite subset of $\mathbb{Z}$, $n_0 = 0$ and $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ is a strictly increasing sequence. In this paper we use $\mathbb{N}$ to denote the set of positive integers, and $\#\mathcal{D}$ to denote the cardinality of the finite set $\mathcal{D}$.

More explicitly, the Moran measure $\mu_{b,\mathcal{D},\{n_k\}}$ considered here can be expressed as the infinite convolution product measure

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$$\mu_{b,\mathcal{D},\{n_k\}} = \delta_{b^{-n_1}\mathcal{D}} * \delta_{b^{-n_2}\mathcal{D}} * \cdots * \delta_{b^{-n_k}\mathcal{D}} * \cdots \tag{1.1}$$

in the *weak convergence* and is supported on a uniquely nonempty compact set $T_{b,\mathcal{D},\{n_k\}}$, which is called a *Moran set* of the Moran IFS $\{f_{k,d} : d \in \mathcal{D}, k \in \mathbb{N}\}$ (see [1,28]). Here, the symbol $\delta_{\mathcal{D}}$ denotes the atomic measure

$$\delta_{\mathcal{D}} = \frac{1}{\#\mathcal{D}} \sum_{d\in\mathcal{D}} \delta_d, \tag{1.2}$$

where δ_d is the Dirac point mass measure at the point d and $rE = \{rx : x \in E\}$. The compact set $T_{b,\mathcal{D},\{n_k\}}$ has an explicit expression by the radix expansion

$$T_{b,\mathcal{D},\{n_k\}} := \sum_{k=1}^{\infty} b^{-n_k}\mathcal{D} = \{\sum_{k=1}^{\infty} b^{-n_k} d_k : d_k \in \mathcal{D}\}. \tag{1.3}$$

In general, the Moran sets $T(b, \mathcal{D}, \{n_k\}) := T_{b,\mathcal{D},\{n_k\}}$ behave like a fractal, highly non-linear, and they include complicated geometries, and the Moran measures $\mu_{b,\mathcal{D},\{n_k\}}$ include the classical self-similar measure $\mu_{b,\mathcal{D}} := \mu_{b,\mathcal{D},\mathbb{N}}$ and the restriction of 1-dimensional Lebesgue measure $\mathcal{L}$. These Moran sets and Moran measures have connections with a number of areas in mathematics, such as harmonic analysis, wavelet theory, multifractal analysis, algebraic number theory, dynamical system, and others (see [8,12,26] and the references cited therein).

It is well known that the study of spectral measure has a long history. It was attracted more attention from Fuglede's spectral set conjecture [13] in 1974, which said that the d-dimensional Lebesgue measure $\mathcal{L}^d$ restricted on a Borel set $\Omega \subseteq \mathbb{R}^d$ is a spectral measure if and only if Ω is a translation tile. Although the conjecture has been proved to be false by Tao and others in both directions in dimension three or higher [17,18,25,30], it is still suggestive in the study of spectral measure.

In recent years, there has been a wide range of interests in expanding the classical Fourier analysis to fractal or more general probability measures after the pioneer work of Jorgensen and Pedersen [16] in 1998 [1–7,9–11,14,15,19–24,27–29,31]. One of the central themes of this area of research involves constructing Fourier bases in $L^2(\mu)$, where μ is a measure which is generated by the iterated function systems. If so, one can develop L^2-Fourier representation of periodic functions on the real line (or $\mathbb{R}^d$) for general classes of fractal measures including $\mu_{b,\mathcal{D},\{n_k\}}$. Indeed, there is relatively few classes of fractal spectral measures that is known. The aim of this paper is to give some new singular spectral measures.

Recall that a pair $(b, \mathcal{D})$ is called *admissible* if there exists a finite subset $\mathcal{C} \subseteq \mathbb{Z}$ such that $\#\mathcal{D} = \#\mathcal{C} = q$ and the matrix

$$q^{-1/2}[e^{2\pi i b^{-1} dc}]_{d\in\mathcal{D},c\in\mathcal{C}}$$

is unitary (usually the pair $(b^{-1}\mathcal{D}, \mathcal{C})$ is called a *compatible pair*, see Definition 2.2). In $\mathbb{R}$, the well known result proved by Łaba and Wang [19] is that if $(b, \mathcal{D})$ is admissible, then $\mu_{b,\mathcal{D}}$ is a spectral measure. Recently, An and He [1] proved that $\mu_{b,\mathcal{D},\{n_k\}}$ is a spectral measure for any increasing sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ and a consecutive digit set $\mathcal{D} = \{0, 1, \cdots, r-1\}$ with $r \mid b$. In the present paper, we follow their research to consider the spectral property of $\mu_{b,\mathcal{D},\{n_k\}}$ and formulate the following conjecture.

Conjecture 1.1. *Let $b \geq 2$ be an integer. Let $\mathcal{D}$ be a finite subset of $\mathbb{Z}$ such that $(b, \mathcal{D})$ is admissible. Then the Moran measure $\mu_{b,\mathcal{D},\{n_k\}}$ is a spectral measure for any increasing sequence $\{n_k\}_{k=1}^{\infty}$ in $\mathbb{N}$.*

In the present paper we will prove that Conjecture 1.1 is true in several cases. We first show that Conjecture 1.1 is true under a condition of Strichartz [16] in Section 3; as a consequence, we obtain Conjecture 1.1

when the cardinality of the finite digit set $\mathcal{D}$ is 2 or 3. Also, we obtain that $\mu_{b,\mathcal{D},\{n_k\}}$ is a spectral measure for certain 'lacunary' sequence $\{n_k\}_{k=1}^{\infty}$ in Section 3. We then get a sufficient condition such that Conjecture 1.1 holds in Section 4, in which the condition is inspired by [2]. In the final section, we get that the measure $\mu_{b,\mathcal{D},\{n_k\}}$ is a spectral measure when the sequence $\{n_k\}_{k=1}^{\infty}$ is ultimately periodic (see Definition 5.1).

2. Basic definition and facts

The purpose of the present section is to introduce a basic definition and some facts; see especially Proposition 2.1 and Lemma 2.4. The data of compatible pair (Definition 2.2) allows us to create orthogonal set and then to check the conditions in Proposition 2.1 in the following sections.

It is known that the Fourier transform plays an important role in the study of spectral measure. From (1.1), the Fourier transform $\widehat{\mu}_{b,\mathcal{D},\{n_k\}}$ of the Moran measure $\mu_{b,\mathcal{D},\{n_k\}}$ is

$$\widehat{\mu}_{b,\mathcal{D},\{n_k\}}(\xi) = \int e^{2\pi i\xi x} d\mu_{b,\mathcal{D},\{n_k\}}(x) = \prod_{k=1}^{\infty} \widehat{\delta}_{\mathcal{D}}(b^{-n_k}\xi) \qquad (\xi \in \mathbb{R}), \tag{2.1}$$

where

$$\widehat{\delta}_{\mathcal{D}}(\xi) = \int e^{2\pi i\xi x} d\delta_{\mathcal{D}}(x) = \frac{1}{\#\mathcal{D}} \sum_{d\in\mathcal{D}} e^{2\pi i d\xi} \qquad (\xi \in \mathbb{R}). \tag{2.2}$$

Then, it is clear that a discrete set $\Lambda \subseteq \mathbb{R}$ such that the family $E(\Lambda) = \{e^{2\pi i\lambda x} : \lambda \in \Lambda\}$ is an orthogonal system in $L^2(\mu_{b,\mathcal{D},\{n_k\}})$ if and only if

$$\sum_{\lambda\in\Lambda} \left|\widehat{\mu}_{b,\mathcal{D},\{n_k\}}(\xi + \lambda)\right|^2 \le 1 \qquad (\xi \in \mathbb{R}). \tag{2.3}$$

In this case, we also say that Λ is an orthogonal set for $\mu_{b,\mathcal{D},\{n_k\}}$.

Let

$$Q_\Lambda(\xi) := \sum_{\lambda\in\Lambda} \left|\widehat{\mu}_{b,\mathcal{D},\{n_k\}}(\xi + \lambda)\right|^2 \qquad (\xi \in \mathbb{R}).$$

The well known result of Jorgensen and Pedersen [16] shows that Q_Λ has an entire analytic extension to $\mathbb{C}$ if Λ is an orthogonal set for $\mu_{b,\mathcal{D},\{n_k\}}$. The following is a universal test which allows us to decide whether an orthogonal set Λ is a spectrum for the measure $\mu_{b,\mathcal{D},\{n_k\}}$.

Proposition 2.1. *(See [16].) If Λ is an orthogonal set for $\mu_{b,\mathcal{D},\{n_k\}}$, then the following statements are equivalent:*

(i) *The orthogonal set Λ is a spectrum for $\mu_{b,\mathcal{D},\{n_k\}}$;*
(ii) $Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} \left|\widehat{\mu}_{b,\mathcal{D},\{n_k\}}(\xi + \lambda)\right|^2 = 1$, $\forall\ \xi \in \mathbb{R}$;
(iii) $Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} \left|\widehat{\mu}_{b,\mathcal{D},\{n_k\}}(\xi + \lambda)\right|^2 = 1$, $\forall\ \xi \in (-a, a)$, $a > 0$.

The equivalence of (i) and (ii) is a direct consequence of Stone–Weierstrass theorem and Parseval identity, while the equivalence of (ii) and (iii) is due to the analytic property of the function Q_Λ.

Definition 2.2. Let $b \ge 2$ be a positive integer, and let $\mathcal{D}$ and $\mathcal{C}$ be two finite subsets of $\mathbb{Z}$. We say that $(b^{-1}\mathcal{D}, \mathcal{C})$ forms a compatible pair if $\#\mathcal{D} = \#\mathcal{C} = q$ and the matrix

$$H_{b^{-1}\mathcal{D},\mathcal{C}} := q^{-1/2}[e^{2\pi i b^{-1}dc}]_{d\in\mathcal{D},c\in\mathcal{C}} \tag{2.4}$$

is unitary, i.e. $H_{b^{-1}\mathcal{D},\mathcal{C}}H^*_{b^{-1}\mathcal{D},\mathcal{C}} = I_q$.

Standing hypotheses. Throughout the paper, we will assume that $(b^{-1}\mathcal{D},\mathcal{C})$ is a compatible pair and $0 \in \mathcal{D}\cap\mathcal{C}$. Given any strictly increasing sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$, we will always assume that

$$\mu_k = \delta_{b^{-n_1}\mathcal{D}} * \cdots * \delta_{b^{-n_k}\mathcal{D}} \qquad (k \in \mathbb{N}),$$
$$\Lambda_k(b,\mathcal{C},\{n_k\}) := b^{n_1-1}\mathcal{C} + \cdots + b^{n_k-1}\mathcal{C} \qquad (k \in \mathbb{N}),$$
$$\Lambda(b,\mathcal{C},\{n_k\}) := \Big\{ \sum_{finite} b^{n_j-1}c_j : c_j \in \mathcal{C}, j \in \mathbb{N}\Big\}.$$

It is easy to check that the following proposition holds.

Proposition 2.3. *(See [16].) Let $b \geq 2$ be a positive integer, $\mathcal{D}$ and $\mathcal{C}$ be two finite subsets of $\mathbb{Z}$ with the same cardinality such that $(b^{-1}\mathcal{D},\mathcal{C})$ is a compatible pair and $0 \in \mathcal{D}\cap\mathcal{C}$. Then*

(i) $\sum_{\lambda\in\Lambda_k(b,\mathcal{C},\{n_k\})} \left|\widehat{\mu}_k(\xi+\lambda)\right|^2 = 1$ *($\xi \in \mathbb{R}$);*
(ii) *The set $\Lambda(b,\mathcal{C},\{n_k\})$ is an infinitely orthogonal set for $\mu_{b,\mathcal{D},\{n_k\}}$.*

We denote the zero set of the continuous function g by $\mathcal{Z}(g)$:

$$\mathcal{Z}(g) = \{\xi : g(\xi) = 0\}.$$

The following lemma gives a characteristic of the zero set $\mathcal{Z}(\widehat{\delta}_{\mathcal{D}})$, provided that $(b^{-1}\mathcal{D},\mathcal{C})$ is a compatible pair.

Lemma 2.4. *Let $b \geq 2$ be a positive integer, $\mathcal{D}$ and $\mathcal{C}$ be two finite subsets of $\mathbb{Z}$ with the same cardinality such that $(b^{-1}\mathcal{D},\mathcal{C})$ is a compatible pair. Then for each $\beta \in \mathcal{Z}(\widehat{\delta}_{\mathcal{D}})$ there exists $c \in \mathcal{C}$ depending on β such that $\beta + \frac{c}{b} \notin \mathcal{Z}(\widehat{\delta}_{\mathcal{D}})$.*

Proof. Let $\mathcal{D} = \{d_1, \cdots, d_q\} \subseteq \mathbb{Z}$ $(q \leq b)$. On the contrary, we suppose that there exists $\beta \in \mathcal{Z}(\widehat{\delta}_{\mathcal{D}})$ such that $\beta + \frac{c}{b} \in \mathcal{Z}(\widehat{\delta}_{\mathcal{D}})$ for all $c \in \mathcal{C}$. Then by (2.2), we have

$$e^{2\pi i d_1\beta}e^{2\pi i d_1\frac{c}{b}} + \cdots + e^{2\pi i d_q\beta}e^{2\pi i d_q\frac{c}{b}} = 0 \ \text{ for all } \ c \in \mathcal{C},$$

which means that the non-zero column vector $(e^{2\pi i d_1\beta}, \cdots, e^{2\pi i d_q\beta})^T$ is the solution of the $q\times q$ linear system $Mx = 0$, where $M = [e^{2\pi i d_j\frac{c}{b}}]_{1\leq j\leq q, c\in\mathcal{C}}$ is a $q \times q$ non-singular matrix. This leads to a contradiction. □

3. A condition of Strichartz

In this section we prove, under a condition of Strichartz [16], that $\mu_{b,\mathcal{D},\{n_k\}}$ is a spectral measure for any finite subset $\mathcal{D}$ of $\mathbb{Z}$ and any strictly increasing sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{Z}$. Furthermore, we show that $\mu_{b,\mathcal{D},\{n_k\}}$ is a spectral measure for any finite digit set $\mathcal{D}$ when the strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ keeps a certain gap.

Theorem 3.1. *Let $b \geq 2$ be a positive integer, $\mathcal{D}$ and $\mathcal{C}$ be two finite subsets of $\mathbb{Z}$ with the same cardinality such that $(b^{-1}\mathcal{D},\mathcal{C})$ is a compatible pair and $0 \in \mathcal{D}\cap\mathcal{C}$. If the zero set $\mathcal{Z}(\widehat{\delta}_{b^{-1}\mathcal{D}})$ is disjoint from the set $T(b,\mathcal{C})$, then $\big(\mu_{b,\mathcal{D},\{n_k\}}, \Lambda(b,\mathcal{C},\{n_k\})\big)$ is a spectral pair.*

Proof. The proof is identical to Theorem 2.8 of [28]. We know by Theorem 3.1 in [24], that the following two statements are equivalent:

(i) $\mathcal{Z}(\widehat{\delta}_{b^{-1}\mathcal{D}})\bigcap T(b,\mathcal{C})=\emptyset$;
(ii) $\mathcal{Z}(\widehat{\mu}_{b,\mathcal{D}})\bigcap T(b,\mathcal{C})=\emptyset$.

Note that $\mathcal{Z}(\widehat{\mu}_{b,\mathcal{D}})$ is closed and $T(b,\mathcal{C})$ is compact. It follows from (ii) that there is a $\delta>0$ such that $d(\mathcal{Z}(\widehat{\mu}_{b,\mathcal{D}}),T(b,\mathcal{C}))\geq\delta$. Here, d denotes the usual metric on the real line $\mathbb{R}$. Since the function $\widehat{\mu}_{b,\mathcal{D}}$ is continuous on $\mathbb{R}$, there is an $\epsilon>0$ such that

$$|\widehat{\mu}_{b,\mathcal{D}}(t)|^2>\epsilon, \tag{3.1}$$

if $t\in\{x\in\mathbb{R}:d(x,T(b,\mathcal{C}))\leq\frac{1}{2}\delta\}$.

Fix $\xi\in(-1,1)$ and set

$$f_k(\lambda)=\begin{cases}|\widehat{\mu}_k(\xi+\lambda)|^2, & \lambda\in\Lambda_k(b,\mathcal{C},\{n_k\});\\ 0, & \text{otherwise,}\end{cases}$$

and

$$f(\lambda)=\begin{cases}|\widehat{\mu}_{b,\mathcal{D},\{n_k\}}(\xi+\lambda)|^2, & \lambda\in\Lambda(b,\mathcal{C},\{n_k\}),\\ 0, & \text{otherwise.}\end{cases}$$

Then, $\lim\limits_{k\to\infty}f_k(\lambda)=f(\lambda)$ for all $\lambda\in\Lambda(b,\mathcal{C},\{n_k\})$. By Proposition 2.3, we have

$$\sum_{\lambda\in\Lambda_k(b,\mathcal{C},\{n_k\})}f_k(\lambda)=1.$$

Obviously, $|\widehat{\delta}_{b^{-n_k}\mathcal{D}}|\leq1$ for $k\geq1$. From (2.1), we have

$$\begin{aligned}|\widehat{\mu}_{b,\mathcal{D},\{n_k\}}(\xi+\lambda)|^2&=|\widehat{\mu}_k(\xi+\lambda)|^2\prod_{j=1}^{\infty}\left|\widehat{\delta}_{b^{-n_{k+j}}\mathcal{D}}(\xi+\lambda)\right|^2\\&\geq|\widehat{\mu}_k(\xi+\lambda)|^2\prod_{j=n_{k+1}}^{\infty}\left|\widehat{\delta}_{b^{-j}\mathcal{D}}(\xi+\lambda)\right|^2\\&=|\widehat{\mu}_k(\xi+\lambda)|^2\left|\widehat{\mu}_{b,\mathcal{D}}(b^{-(n_{k+1}-1)}(\xi+\lambda))\right|^2,\qquad\lambda\in\Lambda_k(b,\mathcal{C},\{n_k\}).\end{aligned}$$

Note that $b^{-(n_{k+1}-1)}\lambda\in T(b,\mathcal{C})$ if $\lambda\in\Lambda_k(b,\mathcal{C},\{n_k\})$. Hence we can choose k_0 large enough such that $k\geq k_0$ implies that $\left|b^{-(n_{k+1}-1)}\right|\leq\frac{1}{2}\delta$. Whence from (3.1), we have

$$f_k(\lambda)\leq\epsilon^{-1}f(\lambda),\qquad k\geq k_0.$$

By (2.3), we have

$$\sum_{\lambda\in\Lambda(b,\mathcal{C},\{n_k\})}\epsilon^{-1}f(\lambda)\leq\epsilon^{-1}<\infty.$$

Applying Lebesgue's dominated convergence theorem to $\{f_k\}_{k=1}^{\infty}$, we obtain that

$$\sum_{\lambda\in\Lambda(b,\mathcal{C},\{n_k\})} \left|\widehat{\mu}_{b,\mathcal{D},\{n_k\}}(\xi+\lambda)\right|^2 = 1, \qquad \xi\in(-1,1).$$

So by Proposition 2.1, $\Lambda(b,\mathcal{C},\{n_k\})$ is a spectrum for $\mu_{b,\mathcal{D},\{n_k\}}$. This completes the proof of Theorem 3.1. □

Inspired by the proof of Theorem 3.1, we can determine the following class of sequences $\{n_k\}_{k=1}^{\infty}$ such that the measures $\mu_{b,\mathcal{D},\{n_k\}}$ are spectral measures.

Theorem 3.2. *Let $b\ge 2$ be a positive integer, and let $\mathcal{D}$ be a finite subset of $\mathbb{Z}$ such that $(b,\mathcal{D})$ is admissible. If there is a $K_0\in\mathbb{N}$ such that $k\ge K_0$ implies that $n_{k+1}\ge n_k+N$ holds for some N (only depending on b, $\mathcal{D}$), then $\left(\mu_{b,\mathcal{D},\{n_k\}},\Lambda(b,\mathcal{C},\{n_k\})\right)$ is a spectral pair.*

Proof. We first choose $\mathcal{C}\subseteq[-b+2,b-2]$ such that $(b^{-1}\mathcal{D},\mathcal{C})$ forms a compatible pair. Next let f_k, f be as in the proof of Theorem 3.1. By doing the same procedure for f_k as in the proof of Theorem 3.1, we obtain that

$$f(\lambda)\ge f_k(\lambda)\left|\widehat{\mu}_{b,\mathcal{D}}(b^{-(n_{k+1}-1)}(\xi+\lambda))\right|^2, \qquad \lambda\in\Lambda_k(b,\mathcal{C},\{n_k\}).$$

In order to use Lebesgue's dominated convergence theorem, it is sufficient to show that there is an $\epsilon>0$ such that

$$\left|\widehat{\mu}_{b,\mathcal{D}}(b^{-(n_{k+1}-1)}(\xi+\lambda))\right|^2>\epsilon, \qquad \forall\ \lambda\in\Lambda_k(b,\mathcal{C},\{n_k\}), \tag{3.2}$$

for k large enough and ξ small.

Since $\widehat{\mu}_{b,\mathcal{D}}$ is continuous and $\widehat{\mu}_{b,\mathcal{D}}(0)=1$, there exist positive constants ϵ and δ such that

$$|\widehat{\mu}_{b,\mathcal{D}}(x)|^2>\epsilon, \quad \forall\ x\in(-\delta,\delta). \tag{3.3}$$

Let N be the integer satisfied that $b^{-(N-2)}<\delta/2$. Note that

$$b^{-(n_{k+1}-1)}\lambda\in b^{-(n_{k+1}-1)}(b^{n_1-1}\mathcal{C}+b^{n_2-1}\mathcal{C}+\cdots+b^{n_k-1}\mathcal{C})$$

for any $\lambda\in\Lambda_k(b,\mathcal{C},\{n_k\})$. Hence, it follows from the assumption of the sequence $\{n_k\}$ that

$$\left|b^{-(n_{k+1}-1)}\lambda\right|\le(b-1)^{-1}b^{-(n_{k+1}-n_k-2)}\le(b-1)^{-1}b^{-(N-2)}<\delta/2 \tag{3.4}$$

holds for $k\ge K_0$. Whence, from (3.3) and (3.4), the desired result (3.2) holds for all $\xi\in[-\delta/4,\delta/4]$. By Proposition 2.1, we finished the proof of Theorem 3.2. □

The following two examples serve to illustrate Theorem 3.2.

Example 3.3. If the sequence $\{n_k\}_{k=1}^{\infty}\subseteq\mathbb{N}$ is Hadamard *lacunary*, i.e., there exists a constant $q>1$ such that $n_{k+1}>qn_k$ for all $k\in\mathbb{N}$, then the measure $\mu_{b,\mathcal{D},\{n_k\}}$ is a spectral measure with a spectrum $\Lambda(b,\mathcal{C},\{n_k\})$.

Indeed, we know, from the proof of Theorem 3.2, that the condition about the sequence $\{n_k\}_{k=1}^{\infty}\subseteq\mathbb{N}$ in Theorem 3.2 can be replaced by $n_{k+1}\ge n_k+f(k)$ for $k\in\mathbb{N}$, where $f(k)$ is an increasing sequence. The lacunary sequence is contained in the latter case. Also, there are many other sequences, such as classical Fibonacci sequence $1,2,3,5,8,13,21,34,55,\cdots$.

Example 3.4. The purpose of this example is to show how to determine N in concrete case. Let $b = 10$, and let $\mathcal{D} = \{0, 1, 2, 3, 4\}$. Let $\mathcal{C} \subseteq \mathbb{Z}$ such that $(b^{-1}\mathcal{D}, \mathcal{C})$ is a compatible pair. So we can check that if the sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ satisfies that $n_{k+1} \geq n_k + 3$ eventually, then the measure $\mu_{b,\mathcal{D},\{n_k\}}$ admits a spectrum $\Lambda(b, \mathcal{C}, \{n_k\})$.

Proof. The proof is similar to that of Theorem 3.2. From (2.1) and (2.2), one has

$$\mathcal{Z}(\widehat{\mu}_{10,\mathcal{D}}) = \bigcup_{k=1}^{\infty} 10^k \mathcal{Z}(\widehat{\delta}_{\mathcal{D}}),$$

where $\mathcal{Z}(\widehat{\delta}_{\mathcal{D}}) = \{\frac{a}{5} : a \in \mathbb{Z}\backslash 5\mathbb{Z}\}$. Hence

$$\begin{aligned}\mathcal{Z}(\widehat{\mu}_{10,\mathcal{D}}) &= \{10^k \frac{a}{5} : k \geq 1 \text{ and } a \in \mathbb{Z}\backslash 5\mathbb{Z}\} \\ &= 2\{10^k a : k \geq 0 \text{ and } a \in \mathbb{Z}\backslash 5\mathbb{Z}\}.\end{aligned}$$

This means that 2 is the positive and smallest zero of $\widehat{\mu}_{10,\mathcal{D}}$. Throughout the proof of Theorem 3.2, any number in the interval $(0, 2)$ can be chosen as the δ in (3.3). Without loss of generality, we choose $\delta = 1$. It follows from (3.4) that the inequality $10^{-(N-2)} < 1/2$ implies that $N \geq 3$. Hence if the sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ satisfies that $n_{k+1} \geq n_k + 3$ eventually, then (3.2) holds for all $\xi \in [-1/4, 1/4]$. By Proposition 2.1, the corresponding measure $\mu_{10,\mathcal{D},\{n_k\}}$ is a spectral measure. $\square$

At the end of this section, we will deal with the spectral property of the measure $\mu_{b,\mathcal{D},\{n_k\}}$ when the cardinality of the finite set is 2 or 3.

Theorem 3.5. *Let $b \geq 2$ be a positive integer, and let $\mathcal{D}$ be a finite subset of $\mathbb{Z}$ with the cardinality 2 such that $(b, \mathcal{D})$ is admissible. Then for any strictly increasing sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$, the measure $\mu_{b,\mathcal{D},\{n_k\}}$ is a spectral measure.*

Proof. Without loss of generality, we assume that $\{n_k\}_{k=1}^{\infty}$ is a proper subset of $\mathbb{N}$, and $\mathcal{D} = \{0, 1\}$. Since $(b, \mathcal{D})$ is admissible, there exists $\mathcal{C} = \{0, c\} \subseteq \mathbb{Z}$ such that $(b^{-1}\mathcal{D}, \mathcal{C})$ is a compatible pair, which is equivalent to say that 2×2 matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & e^{2\pi i \frac{c}{b}} \end{pmatrix}$$

is unitary. This implies that $c \in \frac{b}{2}(2\mathbb{Z} + 1)$. Hence $b \in 2\mathbb{N}$.

Let $b = 2b_0$, $b_0 \in \mathbb{N}$. By taking $c = b_0 p$ with $p \in 2\mathbb{Z}+1$, we get a compatible pair $(b^{-1}\mathcal{D}, \mathcal{C})$. Furthermore, by (2.2) and (1.3), we obtain that

$$\widehat{\delta}_{b^{-1}\mathcal{D}}(\xi) = \frac{1}{2}(1 + e^{2\pi i b^{-1}\xi}) = 0 \quad \Leftrightarrow \quad \xi \in b_0(2\mathbb{Z} + 1), \tag{3.5}$$

and

$$T(b, \mathcal{C}) = \{\sum_{j=1}^{\infty} b^{-j} c_j : c_j \in \mathcal{C}\} = b_0 p \{\sum_{j=1}^{\infty} b^{-j} c_j : c_j \in \{0, 1\}\} \subseteq b_0 p [0, \frac{1}{b-1}). \tag{3.6}$$

Therefore if $p = 1$, we obtain, from (3.5) and (3.6), that $T(b, \mathcal{C}) \cap \mathcal{Z}(\widehat{\delta}_{b^{-1}\mathcal{D}}) = \emptyset$. By Theorem 3.1, $\mu_{b,\mathcal{D},\{n_k\}}$ is a spectral measure. The proof of Theorem 3.5 is complete. $\square$

Theorem 3.6. *Let $b \geq 2$ be a positive integer, and let $\mathcal{D}$ be a finite subset of $\mathbb{Z}$ with the cardinality 3 such that $(b, \mathcal{D})$ is admissible. Then for any strictly increasing sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$, the measure $\mu_{b,\mathcal{D},\{n_k\}}$ is a spectral measure.*

Proof. We may assume that $\mathcal{D} = \{0, d_1, d_2\} \subseteq \mathbb{Z}$. Since $(b, \mathcal{D})$ is admissible, there exists $\mathcal{C} = \{0, c_1, c_2\} \in \mathbb{Z}$ such that $(b^{-1}\mathcal{D}, \mathcal{C})$ is a compatible pair, which is equivalent to say that 3×3 matrix

$$H_{b^{-1}\mathcal{D},\mathcal{C}} := \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i \frac{d_1}{b} c_1} & e^{2\pi i \frac{d_1}{b} c_2} \\ 1 & e^{2\pi i \frac{d_2}{b} c_1} & e^{2\pi i \frac{d_2}{b} c_2} \end{pmatrix} \tag{3.7}$$

is unitary. Because the distinct rows of the matrix $H_{b^{-1}\mathcal{D},\mathcal{C}}$ are mutually orthogonal, we obtain that $d_j c_k \in b(\pm\frac{1}{3} + \mathbb{Z})$ for $j, k = 1, 2$. Hence $b \in 3\mathbb{N}$.

Let $b = 3b_0$, $b_0 \in \mathbb{N}$. With an easy computation, from (2.2), we have

$$\widehat{\delta}_{\mathcal{D}}(\xi) = \frac{1}{3}(1 + e^{2\pi i d_1 \xi} + e^{2\pi i d_2 \xi}) = 0 \quad \Leftrightarrow \quad \xi \in \pm\frac{1}{3} + \mathbb{Z} \text{ and } \{d_1, d_2\} \equiv \{1, 2\} \pmod 3.$$

Hence,

$$\widehat{\delta}_{b^{-1}\mathcal{D}}(\xi) = 0 \quad \Leftrightarrow \quad \xi \in b_0(\pm 1 + 3\mathbb{Z}) \text{ and } \{d_1, d_2\} \equiv \{1, 2\} \pmod 3. \tag{3.8}$$

Taking $\mathcal{C} = \{0, (3n+1)b_0, (3n-1)b_0\}$, where $n \in \mathbb{N} \cup \{0\}$, we obtain that $(b^{-1}\mathcal{D}, \mathcal{C})$ is a compatible pair. Furthermore,

$$T(b, \mathcal{C}) = \Big\{ \sum_{j=1}^{\infty} b^{-j} c_j : c_j \in \mathcal{C} \Big\} \subseteq (3n+1)[0, \frac{b_0}{b-1}]. \tag{3.9}$$

It follows from (3.8) and (3.9) that $\mathcal{Z}(\widehat{\delta}_{b^{-1}\mathcal{D}}) \bigcap T(b, \mathcal{C}) = \emptyset$ if $\mathcal{C} = \{0, b_0, -b_0\}$. Then by Theorem 3.1, the measure $\mu_{b,\mathcal{D},\{n_k\}}$ is a spectral measure. The proof of Theorem 3.6 is complete. □

4. Additional assumption on $T(b, \mathcal{C} \cup (-\mathcal{C}))$

In the present section we will prove that $\mu_{b,\mathcal{D},\{n_k\}}$ is a spectral measure for any increasing sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ under the assumption that $0 \in \mathcal{C} \subseteq \{0, 1, \cdots, b-1\}$ such that each element in the attractor $T(b, \mathcal{C} \cup (-\mathcal{C}))$ (see formula (1.3)) has a unique radix expansion in base b. Before stating our main result, we first introduce and not include the proof of the following technique lemma, since it follows readily from Lemma 3.1 and Lemma 4.3 in [2].

Lemma 4.1. *Let $b \geq 2$ be a positive integer and $\mathcal{D}$ be a finite set of $\mathbb{Z}$. Suppose that $\mathcal{C} + \mathcal{C} \subseteq \{0, 1, \cdots, b-1\}$ such that $(b^{-1}\mathcal{D}, \mathcal{C})$ forms a compatible pair. Then there is a positive integer L and an integer set $\widetilde{\mathcal{C}}$ such that $(b^{-1}\mathcal{D} + b^{-2}\mathcal{D} + \cdots + b^{-L}\mathcal{D}, \widetilde{\mathcal{C}})$ is a compatible pair and*

$$\mathcal{Z}(\widehat{\mu}_{b,\mathcal{D}}) \cap T(b^L, \widetilde{\mathcal{C}}) = \emptyset.$$

Corollary 4.2. *Let $b \geq 2$ be a positive integer, and $\mathcal{D}$ be a finite set of $\mathbb{Z}$. Suppose that $0 \in \mathcal{C} \subseteq \{0, 1, \cdots, b-1\}$ such that $(b^{-1}\mathcal{D}, \mathcal{C})$ is a compatible pair and each element in $T(b, \mathcal{C} \cup (-\mathcal{C}))$ has a unique radix expansion in base b. Then there is a positive integer L and an integer set $\widetilde{\mathcal{C}}$ such that $(b^{-1}\mathcal{D} + b^{-2}\mathcal{D} + \cdots + b^{-L}\mathcal{D}, \widetilde{\mathcal{C}})$ is a compatible pair and*

$$\mathcal{Z}(\widehat{\mu}_{b,\mathcal{D}}) \cap T(b^L, \widetilde{\mathcal{C}}) = \emptyset. \tag{4.1}$$

Proof. We will omit the proof, one can see Lemma 3.1 and Lemma 4.3 in [2]. We would like to point out that the proof of Lemma 4.3 in [2] is easier with the help of Lemma 2.4 in our setting. The reader may refer to [2] for more detail. □

As a consequence of Corollary 4.2, we obtain our main result.

Theorem 4.3. *Let $b \geq 2$ be a positive integer, $\mathcal{D}$ and $\mathcal{C}$ be two finite subsets of $\mathbb{Z}$ with the same cardinality such that $(b^{-1}\mathcal{D}, \mathcal{C})$ is a compatible pair and $0 \in \mathcal{C} \subseteq \{0, 1, \cdots, b-1\}$. Suppose that each element in $T(b, \mathcal{C} \cup (-\mathcal{C}))$ has a unique radix expansion in base b. Then for any strictly increasing sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$, $\mu_{b,\mathcal{D},\{n_k\}}$ is a spectral measure.*

Proof. Given a sequence $\{n_k\}_{k=1}^{\infty}$. We choose L as in Corollary 4.2 and set

$$\mathcal{A}_j := \big(\{n_k\}_{k=1}^{\infty} - jL\big) \cap (0, L], \qquad j \in \mathbb{N} \cup \{0\}.$$

Then, from properties of compatible pairs (see Lemma 2.2 in [19]) and Corollary 4.2, there exist $\widetilde{\mathcal{C}}_j \subseteq \widetilde{\mathcal{C}}$ such that $(\sum_{i\in\mathcal{A}_j} b^{-i}\mathcal{D}, \widetilde{\mathcal{C}}_j)$ form compatible pairs, where $\widetilde{\mathcal{C}}$ as in Corollary 4.2. Denote by

$$\widetilde{\mathcal{D}}_j = \sum_{i\in\mathcal{A}_j} b^{L-i}\mathcal{D}, \qquad j \in \mathbb{N} \cup \{0\}.$$

Then $\widetilde{\mathcal{D}}_j \subseteq \mathbb{Z}$ and $(b^{-L}\widetilde{\mathcal{D}}_j, \widetilde{\mathcal{C}}_j)$ are compatible pairs. It is easy to check that

$$\mu_{b,\mathcal{D},\{n_k\}} = \delta_{b^{-L}\widetilde{D}_0} * \delta_{b^{-2L}\widetilde{D}_1} * \cdots.$$

The proof to be left only need us to use the same standard approximating method as in Lemma 3.1, but we write it down for the sake of completeness.

Fix $k \in \mathbb{N}$. Let $\Gamma_k = \widetilde{\mathcal{C}}_0 + b^L\widetilde{\mathcal{C}}_1 + \cdots + b^{(k-1)L}\widetilde{\mathcal{C}}_{k-1}$ and $\Gamma = \bigcup_{k=1}^{\infty} \Gamma_k$. We will show that Γ is a spectrum for $\mu_{b,\mathcal{D},\{n_k\}}$. For the purpose of this, we assume that

$$\nu_k = \delta_{b^{-L}\widetilde{\mathcal{D}}_0} * \delta_{b^{-2L}\widetilde{\mathcal{D}}_1} * \cdots \delta_{b^{-kL}\widetilde{\mathcal{D}}_k}.$$

Then, (ν_k, Γ_k) are spectral pairs, and Γ is an orthogonal set for $\mu_{b,\mathcal{D},\{n_k\}}$ as a direct result of compatible pairs. According to Proposition 2.3, we have

$$\sum_{\gamma\in\Gamma_k} \left|\widehat{\nu_k}(\xi+\gamma)\right|^2 = 1 \qquad \text{and} \qquad \sum_{\gamma\in\Gamma} \left|\widehat{\mu}_{b,\mathcal{D},\{n_k\}}(\xi+\gamma)\right|^2 \leq 1 \qquad (\xi \in \mathbb{R}).$$

Fix $\xi \in (-1, 1)$ and set

$$f_k(\gamma) = \begin{cases} |\widehat{\nu}_k(\xi+\gamma)|^2, & \text{if } \gamma \in \Gamma_k; \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad f(\gamma) = \begin{cases} |\widehat{\mu}_{b,\mathcal{D},\{n_k\}}(\xi+\gamma)|^2, & \text{if } \gamma \in \Gamma; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\lim_{k\to\infty} f_k(\gamma) = f(\gamma)$ for $\gamma \in \Gamma$. For any $\gamma \in \Gamma_k$, we have $b^{-(k+1)L}\gamma \in T(b^L, \widetilde{\mathcal{C}})$ and

$$\begin{aligned} |\widehat{\mu}_{b,\mathcal{D},\{n_k\}}(\xi+\gamma)|^2 &= \left|\widehat{\nu}_k(\xi+\gamma)\right|^2 \prod_{j>k} \left|\widehat{\delta}_{b^{-jL}\widetilde{\mathcal{D}}_j}(\xi+\gamma)\right|^2 \\ &\geq \left|\widehat{\nu}_k(\xi+\gamma)\right|^2 \prod_{j\geq 1} \left|\widehat{\delta}_{b^{-kL-j}\mathcal{D}}(\xi+\gamma)\right|^2 \\ &= \left|\widehat{\nu}_k(\xi+\gamma)\right|^2 \left|\widehat{\mu}_{b,\mathcal{D}}(b^{-kL}(\xi+\gamma))\right|^2. \end{aligned}$$

From Corollary 4.2, we choose δ, ϵ to satisfy the following condition

$$|\widehat{\mu}_{b,\mathcal{D}}(\xi)|^2 \geq \epsilon, \quad \text{for all} \ \ \xi \in (T(b^L, \widetilde{C}))_\delta, \tag{4.2}$$

where $(T(b^L, \widetilde{C}))_\delta$ is the δ-neighborhood of $T(b^L, \widetilde{C})$. Whence $f(\gamma) \geq \epsilon f_k(\gamma)$ if $b^{-kL} < \delta/2$. Hence by Lebesgue's dominated convergence theorem and Proposition 2.1, $\mu_{b,\mathcal{D},\{n_k\}}$ is a spectral measure. The proof of Theorem 4.3 is complete. □

In the end of this section, we give a proposition and an example to illustrate Theorem 4.3. The following proposition shows that the cardinality of the digit set $\mathcal{D}$ is no more than $\lfloor \frac{1}{2}(b+1) \rfloor$ under the assumption of Theorem 4.3. Here, the symbol $\lfloor x \rfloor$ stands for the lower integer part of the real number x.

Proposition 4.4. *Let $b \geq 2$ be a positive integer, $0 \in \mathcal{C} \subseteq \{0, 1, \cdots, b-1\}$, and every point in $T(b, \mathcal{C} \cup (-\mathcal{C}))$ has a unique radix expansion in base b, then $2\#\mathcal{C} \leq b+1$.*

Proof. We first have $\#\mathcal{A} = 2\#\mathcal{C} - 1$. Suppose that $\#\mathcal{A} > b$ and fix $n \geq 1$. Then the set

$$E_n := \{\sum_{j=1}^{n} a_j b^{-j} : a_j \in \mathcal{A}, a_n \neq 0\}$$

contains at least $(b+1)^{n-1}b$ different expansions, since each element of $E_n(\subseteq T(b, \mathcal{A}))$ has a unique radix expansion in base b. However, for any $w = \sum_{j=1}^{n} a_j b^{-j} \in E_n$, we have

$$|b^n w| \leq (b-1)(b^{n-1} + \cdots + b + 1) \leq b^n - 1.$$

Hence there is no more than $2(b^n - 1) + 1 = 2b^n - 1$ possible different values in E_n. Note that $(b+1)^{n-1}b > b^n - 1$, hence some number in E_n must have at least two different expansions in base b. It is a contradiction. Which completes the proof. □

Example 4.5. Let $b = 8$, and let $\mathcal{D} = \{0, d_1, d_2, d_3\}$, where $d_j \in \mathcal{D}_j = j + 4\mathbb{Z}$ for $j = 1, 2, 3$. Then the measure $\mu_{b,\mathcal{D},\{n_k\}}$ is a spectral measure for any increasing sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$.

Proof. Let $\mathcal{C} = \{0, 2, 4, 6\}$. Then the matrix in (2.4) is

$$H_{8^{-1}\mathcal{D},\mathcal{C}} := \frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}$$

is unitary so $(8, \mathcal{D})$ is admissible. By Theorem 4.3, it is enough to show that each element in $T(b, \mathcal{C} \cup (-\mathcal{C}))$ has a unique radix expansion in base b.

Indeed, it follows from (1.3) that the self-similar set $K := T(b, \mathcal{C} \cup (-\mathcal{C}))$ of IFS $\{\varphi_a(x) = b^{-1}(x + a) : a \in \mathcal{C} \cup (-\mathcal{C})\}$ is contained in the interval $I := [-6/7, 6/7]$. It is easy to check that $\varphi_a(I) \cap \varphi_{a'}(I) = \emptyset$ for distinct $a, a' \in \mathcal{C} \cup (-\mathcal{C})$. Hence $\varphi_a(K) \cap \varphi_{a'}(K)(\subseteq \varphi_a(I) \cap \varphi_{a'}(I)) = \emptyset$ for distinct $a, a' \in \mathcal{C} \cup (-\mathcal{C})$. That is, the IFS $\{\varphi_a\}$ satisfies the strong separation condition (SSC). As we all know, SSC implies that each element in $T(b, \mathcal{C} \cup (-\mathcal{C}))$ has a unique radix expansion in base b. The proof is complete. □

5. Ultimately periodic sequences

In the present section we will deal with a class of sequences called ultimately periodic sequence, which is a generalization of arithmetic procession.

Definition 5.1. A sequence $\{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N}$ is *ultimately periodic* (respectively *periodic*) if there exist constants $K,\ m,\ T \geq 1$ such that

$$n_{k+T} = n_k + m, \qquad k \geq K \qquad (\text{respectively} \ \ n_{k+T} = n_k + m,\ k \geq 1).$$

If $K = T = 1$, then the sequence $\{n_k\}_{k=1}^{\infty}$ is an *arithmetic procession.*

For example, the sequence $\{1, 4, 5, 7, 8, 10, 11, \cdots\}$ is ultimately periodic, where $K = 2$, $T = 2$, $m = 3$. Remove the first term, the sequence $\{4, 5, 7, 8, 10, 11, \cdots\}$ is periodic.

Theorem 5.2. *Let $b \geq 2$ be a positive integer, and let $\mathcal{D}$ be a finite subset of $\mathbb{Z}$ such that $(b, \mathcal{D})$ is admissible. Suppose that the sequence $\{n_k\}_{k=1}^{\infty}$ is ultimately periodic. Then the measure $\mu_{b,\mathcal{D},\{n_k\}}$ is a spectral measure.*

In order to prove Theorem 5.2, we need the following fundamental theorem.

Theorem A. *(See [19].) Let $b \geq 2$ be a positive integer, $\mathcal{D}$ and $\mathcal{C}$ be two finite subsets of $\mathbb{Z}$ with the same cardinality such that $(b^{-1}\mathcal{D}, \mathcal{C})$ is a compatible pair. Then the self-similar measure $\mu_{b,\mathcal{D}}$ is a spectral measure. Moreover, if $0 \in \mathcal{C} \subseteq [-b+2, b-2]$, then $\mu_{b,\mathcal{D}}$ is a spectral measure with a spectrum*

$$\Lambda(b, \mathcal{C}) = \{\sum_{j=1}^{n} b^{j-1} c_j : c_j \in \mathcal{C}, \quad \textit{for all} \ \ n \in \mathbb{N}\}.$$

Proof of Theorem 5.2. According to the definition of ultimately periodic sequence, there exist positive integers K, T, m with $T \leq m$ such that

$$(n_{K+jT}, \cdots, n_{K+(j+1)T-1}) = (n_K, \cdots, n_{K+T-1}) + jm, \quad j \in \mathbb{N}.$$

Then the infinite convolution product measure

$$\mu = \delta_{b^{-n_{K+T}}\mathcal{D}} * \delta_{b^{-n_{K+T+1}}\mathcal{D}} * \cdots$$

exists in the weak convergence. Hence $\mu_{b,\mathcal{D},\{n_k\}} = \mu_{K+T-1} * \mu$. It is sufficient to show that μ is a spectral measure with a spectrum

$$\Lambda' = b^{n_{K+T}-1}\mathcal{C} + b^{n_{K+T+1}-1}\mathcal{C} + \cdots, \qquad \text{when} \ \ \mathcal{C} \subseteq [-b+2, b-2].$$

In fact, by Proposition 2.3, we first have

$$\sum_{\dot{\lambda} \in \Lambda_{K+T-1}(b,\mathcal{C},\{n_k\})} \left|\widehat{\mu}_{K+T-1}(\xi + \dot{\lambda})\right|^2 = 1 \qquad (\xi \in \mathbb{R}). \tag{5.1}$$

In view of that fact that (μ, Λ') is a spectral pair if and only if

$$\sum_{\ddot{\lambda} \in \Lambda'} \left|\widehat{\mu}(\xi + \ddot{\lambda})\right|^2 = 1 \qquad (\xi \in \mathbb{R}), \tag{5.2}$$

it follows from (5.1), (5.2) and the identity $\widehat{\mu}_{b,\mathcal{D},\{n_k\}}(\xi) = \widehat{\mu}_{K+T-1}(\xi)\widehat{\mu}(\xi)$ that

$$\begin{aligned}
&\sum_{\lambda\in\Lambda(b,\mathcal{C},\{n_k\})} \left|\widehat{\mu}_{b,\mathcal{D},\{n_k\}}(\xi+\lambda)\right|^2 \\
&= \sum_{\dot{\lambda}\in\Lambda_{K+T-1}(b,\mathcal{C},\{n_k\})} \sum_{\ddot{\lambda}\in\Lambda'} \left|\widehat{\mu}_{b,\mathcal{D},\{n_k\}}(\xi+\dot{\lambda}+\ddot{\lambda})\right|^2 \\
&= \sum_{\dot{\lambda}\in\Lambda_{K+T-1}(b,\mathcal{C},\{n_k\})} \sum_{\ddot{\lambda}\in\Lambda'} \left|\widehat{\mu}_{K+T-1}(\xi+\dot{\lambda}+\ddot{\lambda})\right|^2 \left|\widehat{\mu}(\xi+\dot{\lambda}+\ddot{\lambda})\right|^2 \\
&= \sum_{\dot{\lambda}\in\Lambda_{K+T-1}(b,\mathcal{C},\{n_k\})} \left|\widehat{\mu}_{K+T-1}(\xi+\dot{\lambda})\right|^2 \sum_{\ddot{\lambda}\in\Lambda'} \left|\widehat{\mu}(\xi+\dot{\lambda}+\ddot{\lambda})\right|^2 \\
&= \sum_{\dot{\lambda}\in\Lambda_{K+T-1}(b,\mathcal{C},\{n_k\})} \left|\widehat{\mu}_{K+T-1}(\xi+\dot{\lambda})\right|^2 \\
&= 1 \qquad (\xi\in\mathbb{R}),
\end{aligned}$$

which shows that $(\mu_{b,\mathcal{D},\{n_k\}}, \Lambda(b,\mathcal{C},\{n_k\}))$ is a spectral pair.

Next we show that (μ,Λ') is a spectral pair. Set

$$\widetilde{D} = b^{n_{K+T-1}} \sum_{i=K}^{K+T-1} b^{-n_i}\mathcal{D} \quad \text{and} \quad \widetilde{C} = b^{m-n_{K+T-1}} \sum_{i=K}^{K+T-1} b^{n_i-1}\mathcal{C}.$$

Then both $\widetilde{D}$ and $\widetilde{C}$ are subsets of $\mathbb{Z}$, and $(b^{-m}\widetilde{D},\widetilde{C})$ forms a compatible pair. Since the sequence $\{n_k\}_{k=K+T}^{\infty}$ is periodic, we have

$$\mu = \delta_{b^{-n_{K+T-1}}b^{-m}\widetilde{D}} * \delta_{b^{-n_{K+T-1}}b^{-2m}\widetilde{D}} * \cdots$$

in the weak convergence and

$$\Lambda' = b^{n_{K+T-1}}(\widetilde{C}+b^m\widetilde{C}+\cdots) =: b^{n_{K+T-1}}\Lambda(b^m,\widetilde{C}). \tag{5.3}$$

With an easy calculation, $\widetilde{C} \subseteq [2-b^m, b^m-2]$. So by Theorem A, $\Lambda(b^m,\widetilde{C})$ is a spectrum for the self-similar measure

$$\mu_{b^m,\widetilde{D}} = \delta_{b^{-m}\widetilde{D}} * \delta_{b^{-2m}\widetilde{D}} * \cdots.$$

That is,

$$\sum_{\lambda\in\Lambda(b^m,\widetilde{C})} \left|\widehat{\mu}_{b^m,\widetilde{D}}(\xi+\lambda)\right|^2 = 1 \qquad (\xi\in\mathbb{R}). \tag{5.4}$$

It follows from (5.3), (5.4) and the identity $\widehat{\mu}(\xi) = \widehat{\mu}_{b^m,\widetilde{D}}(b^{-n_{K+T-1}}\xi)$ that

$$\sum_{\lambda'\in\Lambda'} \left|\widehat{\mu}(\xi+\lambda')\right|^2 = \sum_{\lambda\in\Lambda(b^m,\widetilde{C})} \left|\widehat{\mu}_{b^m,\widetilde{D}}(b^{-n_{K+T-1}}\xi+\lambda)\right|^2 = 1 \qquad (\xi\in\mathbb{R}),$$

which shows that (μ,Λ') is a spectral pair. The proof of Theorem 5.2 is complete. □

Remark 5.3. If the sequence $\{n_k\}_{k=1}^{\infty}$ satisfies that $n_{k+1} = n_k + N$ for $k\in\mathbb{N}$ where N chosen as in Theorem 3.2, then Theorem 5.2 and Theorem 3.2 are consistent.

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