J. Math. Anal. Appl. 434 (2016) 729–764

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The (logarithmic) Sobolev inequalities along geometric flow and applications

Shouwen Fang [a,1], Tao Zheng [b,*,2]

[a] School of Mathematical Science, Yangzhou University, Yangzhou, Jiangsu 225002, PR China
[b] School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, PR China

ARTICLE INFO

Article history:
Received 6 February 2015
Available online 25 September 2015
Submitted by H.R. Parks

Keywords:
Geometric flow
Twisted Kähler–Ricci flow
Lorentzian mean curvature flow
Logarithmic Sobolev inequality
Sobolev inequality

ABSTRACT

For some class of geometric flows, we obtain the (logarithmic) Sobolev inequalities and their equivalence up to different factors directly and also obtain the long time non-collapsing and non-inflated properties, which generalize the results in the case of Ricci flow or List–Ricci flow or harmonic-Ricci flow. As applications, for mean curvature flow in Lorentzian space with nonnegative sectional curvature and twisted Kähler–Ricci flow on Fano manifolds, we get the results above.

1. Introduction

The role played by Sobolev inequality in analysis and geometry is well known and a fair amount of work has been devoted to its study. Let (M, g) be an n-dimensional ($n \geq 3$) compact Riemannian manifold. Aubin [1] proved the following Sobolev inequality

$$\left(\int_M |f|^{\frac{2n}{n-2}} \mathrm{d}\mu\right)^{\frac{n-2}{n}} \leq \alpha \int_M |\nabla f|^2 \mathrm{d}\mu + \beta \int_M f^2 \mathrm{d}\mu, \quad \forall f \in W^{1,2}(M),$$

where

$$\alpha = [K(n)]^2 + \varepsilon,\ \varepsilon > 0$$

* Corresponding author.
E-mail addresses: shwfang@163.com (S. Fang), zhengtao08@amss.ac.cn (T. Zheng).

[1] The work is supported by NSFC grant Nos. 11401514, 11371310 and the University Science Research Project of Jiangsu Province grant No. 13KJB110029.
[2] The work is supported by the fundamental fund of Beijing Institute of Technology Nos. 20131742009 and 20141742002, Postdoctoral fund Nos. 2014M550620 and 2015T80040, NSFC grant No. 11401023, and partially supported by NSFC grant No. 11471180.

http://dx.doi.org/10.1016/j.jmaa.2015.09.034

and β depends on bounds on the injectivity radius, sectional curvature and its derivatives and $K(n)$ is the best constant in the Sobolev inequality for $\mathbb{R}^n$ (see [36]). Hebey [19] proved that β can depend only on ε, the injective radius and the lower bound of the Ricci curvature. Hebey and Vaugon [21] proved that we can take $\varepsilon = 0$ but β still depends on the derivatives of curvature tensor.

Assume that $Ric \geq -Kg$, where K is a nonnegative constant. We consider Sobolev inequality like

$$\left(\int_M |f - f_M|^{\frac{2n}{n-2}} \mathrm{d}\mu\right)^{\frac{n-2}{n}} \leq S(M) \int_M |\nabla f|^2 \mathrm{d}\mu, \quad \forall\, f \in C^\infty(M, \mathbb{R}). \tag{1.1}$$

Gallot [15] proved

$$S(M) \leq e^{C_n(1+\sqrt{K}\mathrm{diam}(M))}[\mathrm{diam}(M)]^2[\mathrm{Vol}_g(M)]^{-\frac{2}{n}}. \tag{1.2}$$

Apart form the dimensional constant, the estimate above is sharp.

Let $B := B(x, r) \subset M$ be a ball with center x and radius r. Then in view of (1.1) and (1.2), it is natural to conjecture that

$$\left(\int_B |f - f_B|^{\frac{2n}{n-2}} \mathrm{d}\mu\right)^{\frac{n-2}{n}} \leq e^{C_n(1+\sqrt{K}r)} r^2 [\mathrm{Vol}_g(B)]^{-\frac{2}{n}} \int_B |\nabla f|^2 \mathrm{d}\mu, \quad \forall\, f \in C^\infty(B, \mathbb{R}),$$

where K is a nonnegative constant such that

$$Ric \geq -Kg, \quad \text{on } B(x, 2r).$$

Saloff-Coste [33] solved the conjecture partially. They proved that, for any $f \in C_0^\infty(B, \mathbb{R})$, if $n \geq 3$, there holds

$$\left(\int_B |f|^{\frac{2n}{n-2}} \mathrm{d}\mu\right)^{\frac{n-2}{n}} \leq e^{C_n(1+\sqrt{K}r)} r^2 [\mathrm{Vol}_g(B)]^{-\frac{2}{n}} \int_B \left(|\nabla f|^2 + r^{-2} f^2\right) \mathrm{d}\mu$$

and if $n \leq 2$, the above inequality holds with n replaced by any fixed $n' > 2$. More details about Sobolev inequality can be found in Aubin and Li [2], Biezuner [5] and the references therein.

In the case of Ricci flow

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t), \tag{1.3}$$

(logarithmic) Sobolev inequalities also play an important role in its analysis. One motivation for the $\mathcal{W}$-entropy comes from the log-Sobolev inequality of Gross [16] (see also Topping [37]). Due to the importance of (logarithmic) Sobolev inequality in the analysis of geometric flow, it is key to have a uniform control on the constants α and β.

Sesum and Tian [35] proved a uniform Sobolev imbedding for certain Kähler–Ricci flow with Ricci curvature bounded from below.

However, in general the constant β cannot be controlled uniformly along the Ricci flow. By making use of the (generalized) Perelman's $\mathcal{W}$ entropy [31], Zhang [42,43] and Ye [40,41,39,38] (see also Hsu [23]) proved (logarithmic) Sobolev inequalities along Ricci flow, from which and the method of [7] (see also Lemma 2.2 in [20] and its proof or Lemma 6.1 in [40]) they established long time non-collapsing result generalizing the

Perelman's short time result [31]. Zhang [44] also proved the long time non-inflated result for the normalized Kähler–Ricci flow on Fano manifolds.

In this paper, we consider the geometric flow

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2\mathcal{S}_{ij}(x,t) \tag{1.4}$$

on $M \times [0, T)$ for some (finite or infinite) $T > 0$ with a given initial Riemannian metric $g(0) = g_0$, where $\mathcal{S}_{ij}(x,t)$ is the component of a time-dependent symmetric 2-tensor $\mathcal{S}$. Motivated by [31], we define the $\mathcal{F}$ functional and $\mathcal{W}$ functional and prove their monotonicity under some assumptions. Next we obtain the (logarithmic) Sobolev inequalities and their equivalence up to different factors. We also prove the long time non-collapsing and non-inflated. As applications, for mean curvature flow in Lorentzian space and twisted Kähler–Ricci flow on Fano manifolds, we get the results above.

In the following, we denote the volume element of $g(t)$ by $\mathrm{d}\mu(t)$, the trace of $\mathcal{S}_{ij}(t)$ by S_t or $S(x,t) = \sum_{i,j=1}^{n} g^{ij}(t)\mathcal{S}_{ij}(t)$ (sometimes also by S simply without confusion), the volume of M with respect to $g(t)$ by $\mathrm{Vol}_{g(t)}(M)$, the first eigenvalue of $-\Delta_{g_t} + \frac{S_t}{4}$ by $\lambda_0(g(t))$ and the norm of the gradient of $u \in W^{1,2}(M)$ with respect to $g(t)$ by $|\nabla u|_t$.

For convenience, we define an evolving tensor quantity $\mathcal{D}_2$ associated to the tensor $\mathcal{S}$ (see for example [14] and the references therein).

Definition 1.1. Let $g(x,t)$ be a smooth solution to the geometric flow (1.4) on $M \times [0,T)$. Then for any $X \in \mathfrak{X}(M)$, we define

$$\begin{aligned}\mathcal{D}_2(\mathcal{S}, X) := \frac{\partial S}{\partial t} - \Delta_{g(t)} S - 2|\mathcal{S}|^2_{g(t)} \\ + 4(\nabla^i \mathcal{S}_{ij})X^j - 2X^i \nabla_i S + 2R_{ij}X^iX^j - 2\mathcal{S}_{ij}X^iX^j,\end{aligned} \tag{1.5}$$

where ∇ and R_{ij} are the Levi-Civita connection and Ricci curvature respectively with respect to the Riemannian metric $g(t)$. In particular, if for any vector field $X \in \mathfrak{X}(M)$ there holds $\mathcal{D}_2(\mathcal{S}, X) \geq 0$ on $[0, T)$, then we call $\mathcal{D}_2(\mathcal{S}, \cdot)$ nonnegative.

Theorem 1.1. *Assume that $g(x,t)$ is a smooth solution to the geometric flow (1.4) in $M \times [0,T)$ and that $\mathcal{D}_2(\mathcal{S}, \cdot)$ defined in (1.5) is nonnegative. For each $\sigma > 0$ and each $t \in [0, T)$, we have*

$$\int_M u^2 \ln u^2 \mathrm{d}\mu(t) \leq \sigma \int_M \left(|\nabla u|_t^2 + \frac{S_t}{4}u^2\right) \mathrm{d}\mu(t) - \frac{n}{2}\ln\sigma + A_1\left(t + \frac{\sigma}{4}\right) + A_2 \tag{1.6}$$

for any $u \in W^{1,2}(M)$ with $\int_M u^2 \mathrm{d}\mu(t) = 1$, where

$$A_1 = \frac{4}{C_S(M,g_0)^2 \mathrm{Vol}_{g_0}(M)^{\frac{2}{n}}} - \min S_0, \tag{1.7}$$

$$A_2 = n \ln C_S(M, g_0) + \frac{n}{2}(\ln n - 1), \tag{1.8}$$

and where $C_S(M, g_0)$ is the Sobolev constant defined in (2.1).

Therefore, we can deduce

$$\int_M u^2 \ln u^2 \mathrm{d}\mu(t) \leq \frac{n}{2} \ln\left[\alpha_I \left\{\int_M \left(|\nabla u|_t^2 + \frac{S_t}{4}u^2\right) \mathrm{d}\mu(t) + \frac{A_1}{4}\right\}\right] \tag{1.9}$$

for any $u \in W^{1,2}(M)$ *satisfying* $\int_M u^2 \mathrm{d}\mu(t) = 1$, *where*

$$\alpha_I = \frac{2e}{n} e^{\frac{2(A_1 t + A_2)}{n}}.$$

Theorem 1.2. *Assume that* $g(x,t)$ *is a smooth solution to the geometric flow (1.4) in* $M \times [0,T)$ *and that* $\mathcal{D}_2(\mathcal{S},\cdot)$ *defined in (1.5) is nonnegative. If* $\lambda_0(g_0)$ *is positive, then for each* $t \in [0,T)$ *and each* $\sigma > 0$ *there holds*

$$\int_M u^2 \ln u^2 \mathrm{d}\mu(t) \le \sigma \int_M \left(|\nabla u|_t^2 + \frac{S_t}{4} u^2 \right) \mathrm{d}\mu(t) - \frac{n}{2} \ln \sigma + C \tag{1.10}$$

for all $u \in W^{1,2}(M)$ *with* $\int_M u^2 \mathrm{d}\mu(t) = 1$, *where* C *depends only on the dimension* n, $\mathrm{Vol}_{g_0}(M)$, $C_S(M, g_0)$, $\lambda_0(g_0)$ *and the lower bound for* S_0.

Therefore, there holds for each $t \in [0,T)$

$$\int_M u^2 \ln u^2 \mathrm{d}\mu(t) \le \frac{n}{2} \ln \left[\alpha_{II} \int_M \left(|\nabla u|_t^2 + \frac{S_t}{4} u^2 \right) \mathrm{d}\mu(t) \right] \tag{1.11}$$

for all $u \in W^{1,2}(M)$ *with* $\int_M u^2 \mathrm{d}\mu(t) = 1$, *where*

$$\alpha_{II} = \frac{2e}{n} e^{\frac{2C}{n}}.$$

Theorem 1.3. *Assume that* $g(x,t)$ *is a smooth solution to the geometric flow (1.4) in* $M \times [0,T)$ *and that* $\mathcal{D}_2(\mathcal{S},\cdot)$ *defined in (1.5) is nonnegative. There hold*

(1) *if* $\lambda_0(g_0) > 0$, *for* $t \in [0, T)$ *and* $u \in W^{1,2}(M)$, *there holds*

$$\left(\int_M |u|^{\frac{2n}{n-2}} \mathrm{d}\mu(t) \right)^{\frac{n-2}{n}} \le A \int_M \left(|\nabla u|_t^2 + \frac{S_t}{4} u^2 \right) \mathrm{d}\mu(t), \tag{1.12}$$

where A *is a positive defined in (4.8), depending only on* n, $\mathrm{Vol}_{g_0}(M)$, $C_S(M, g_0)$, $\lambda_0(g_0)$ *and the lower bound of* S_0.

(2) *if* $T < \infty$, *for* $t \in [0, T)$ *and* $u \in W^{1,2}(M)$, *there holds*

$$\left(\int_M |u|^{\frac{2n}{n-2}} \mathrm{d}\mu(t) \right)^{\frac{n-2}{n}} \le A \int_M \left(|\nabla u|_t^2 + \frac{S_t}{4} u^2 \right) \mathrm{d}\mu(t) + B \int_M u^2 \mathrm{d}\mu(t), \tag{1.13}$$

where A *and* B *are defined in (4.9) and (4.10) respectively, depending only on* n, $\mathrm{Vol}_{g_0}(M)$, $C_S(M, g_0)$, $\lambda_0(g_0)$ *and the upper bound* T.

Remark 1.1. From the Jensen's inequality, we can deduce logarithmic Sobolev inequality from Sobolev inequality (see for example [40]). Now from the proof of Theorem 1.3, we know that Sobolev inequality implies also logarithmic Sobolev inequality with different constants. Therefore, we can say that (logarithmic) Sobolev inequalities are equivalent to each other up to constant factors. In the case of Ricci flow (1.3), the equivalence proved by making use of estimates on heat kernel can be found in Ye [40] and Zhang [43].

Remark 1.2. In the case of Ricci flow (1.3), the results in Theorem 1.1, Theorem 1.2 and Theorem 1.3 can be found in Zhang [42,43], Ye [40] and Hsu [23].

Remark 1.3. In the case of extended Ricci flow (so-called List–Ricci flow [26])

$$\begin{cases} \dfrac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + 4\mathrm{d}\phi(x,\,t) \otimes \mathrm{d}\phi(x,\,t), \\ \dfrac{\partial}{\partial t}\phi(x,\,t) = \Delta_{g(x,\,t)}\phi(x,\,t), \end{cases}$$

where $\phi \in C^\infty(M \times \mathbb{R},\, \mathbb{R})$, the Sobolev inequalities were obtained by Liu and Wang [28].

Remark 1.4. In the case of harmonic-Ricci flow (see [3,30,46])

$$\begin{cases} \dfrac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + 2\alpha(t)\nabla\psi \otimes \nabla\psi, \\ \dfrac{\partial}{\partial t}\psi(x,t) = \tau_{g(x,t)}\psi(x,t), \end{cases}$$

where $\psi(\cdot,t) : (M, g(\cdot,t)) \to (N,h)$ is a family of smooth maps between two Riemannian manifolds, both $g(\cdot,t)$ and h are Riemannian metrics, $\alpha(t)$ is a positive non-increasing function, and $\tau_g\psi$ denotes the intrinsic Laplacian of ψ, the Sobolev inequalities can be found in [13].

Given the (logarithmic) Sobolev inequalities, we can prove the κ-noncollapsing property and the so-called κ-noninflated property and also give some examples as applications.

The rest of the paper is organized as follows. In Section 2, we prove the equivalence between Sobolev inequality and logarithmic Sobolev inequality up to a different factor, which also holds in the case of geometric flow (1.4). In Section 3, we define $\mathcal{F}$ functional and $\mathcal{W}$ entropy and prove their monotonicity and prove the lower bound of S, assuming that $\mathcal{D}_2(\mathcal{S},\cdot)$ is nonnegative. In Section 4, we prove (logarithmic) Sobolev inequalities under geometric flow (1.4), i.e., Theorem 1.1, Theorem 1.2 and Theorem 1.3. In Section 5, we give the κ-noncollapsing property along geometric flow (1.4). In Section 6, based on a series of properties of fundamental solution to conjugate heat equation along geometric flow (1.4), we prove the so-called κ noninflated property. In Section 7, as applications, we consider Lorentzian mean curvature flow (7.1) on ambient Lorentzian manifold with nonnegative sectional curvature and twisted Kähler–Ricci flow (7.2) on Fano manifolds and obtain the results mentioned in the first six sections along these two geometric flows.

2. The (logarithmic) Sobolev inequalities on Riemannian manifolds and their relations

In this section, first we give some (logarithmic) Sobolev inequalities and lemmas which will be useful in the following sections.

Let $(M,\, g)$ be an n-dimensional ($n \geq 3$) compact Riemannian manifold. Then the Sobolev constant of $(M,\, g)$ (for the exponent 2) is defined to be

$$C_S(M,\,g) = \sup\left\{ \|u\|_{\frac{2n}{n-2}} - \frac{1}{\mathrm{Vol}_g(M)^{\frac{1}{n}}}\|u\|_2 : \quad u \in C^1(M),\, \|\nabla u\|_2 = 1 \right\}. \tag{2.1}$$

Therefore, the Sobolev inequality (for the exponent 2) is

$$\|u\|_{\frac{2n}{n-2}} \leq C_S(M,\,g)\|\nabla u\|_2 + \frac{1}{\mathrm{Vol}_g(M)^{\frac{1}{n}}}\|u\|_2, \quad \forall\, u \in W^{1,2}(M). \tag{2.2}$$

We need the following fundamental results (see for example [40]).

Theorem 2.1. *Let (M, g) be an n-dimensional $(n \geq 3)$ compact Riemannian manifold and $\mathcal{S}$ be any symmetric 2-tensor with trace $S = \sum_{i,j=1}^{n} g^{ij}\mathcal{S}_{ij}$. Then for any $u \in W^{1,2}(M)$ with $\|u\|_2 = 1$, there hold*

$$\int_M u^2 \ln u^2 \mathrm{d}\mu \leq n \ln \left(C_S(M, g)\|\nabla u\|_2 + \frac{1}{\mathrm{Vol}_g(M)^{\frac{1}{n}}} \right), \tag{2.3}$$

$$\int_M u^2 \ln u^2 \mathrm{d}\mu \leq \frac{n\alpha C_S(M,g)^2}{2} \int_M \left(|\nabla u|^2 + \frac{S}{4}u^2 \right) \mathrm{d}\mu - \frac{n}{2}(\ln \alpha - \ln 2 + 1)$$
$$+ \frac{n\alpha}{2}\left(\frac{1}{\mathrm{Vol}_g(M)^{\frac{2}{n}}} - \frac{\min S^-}{4} C_S(M,g)^2 \right), \tag{2.4}$$

where α is any positive real number and $S^- = \min\{S, 0\}$.

Moreover, if the first eigenvalue $\lambda_0 = \lambda_0(g)$ of the operator $-\Delta_g + \frac{S}{4}$ is positive, we can deduce

$$\int_M u^2 \ln u^2 \mathrm{d}\mu \leq \frac{nAC_S(M, g)^2}{2} \int_M \left(|\nabla u|^2 + \frac{S}{4}u^2 \right) - \frac{n}{2}\ln A + \frac{n}{2}\ln 2 + \sigma_0, \tag{2.5}$$

where

$$\delta_0 = \delta_0(g) = \left(\lambda_0 C_S(M, g)^2 + \frac{1}{\mathrm{Vol}_g(M)^{\frac{2}{n}}} - C_S(M, g)^2 \frac{\min S^-}{4} \right)^{-1}, \tag{2.6}$$

$$\sigma_0 = \sigma_0(g) = \frac{n}{2}\left[\ln \left(\lambda_0 C_S(M, g)^2 + \frac{1}{\mathrm{Vol}_g(M)^{\frac{2}{n}}} - C_S(M, g)^2 \frac{\min S^-}{4} \right) \right.$$
$$\left. - \ln(\lambda_0 C_S(M, g)^2) - 1 \right] \tag{2.7}$$

and A is any positive real number satisfying $A \geq \delta_0$.

Now we give some fundamental materials which will be useful in the proof of Logarithmic Sobolev inequality implying Sobolev inequality. The ideas come from [4] and the references therein.

Let $(M, \mathcal{E}, \mu)$ be a measurable space with a nonnegative σ-finite measure μ. For convenience, let $\mathcal{F}^+$ be nonnegative function on M and be contained in all L^p-space with respect to the measure μ.

Let $W(f)$ be a given norm or semi-norm on $\mathcal{F}^+$ which will be determined later. For $\rho > 1$, $k \in \mathbb{Z}$, define

$$f_{\rho,\, k} = \min\{(f - \rho^k)^+,\, \rho^k(\rho - 1)\},$$

where $(f - \rho^k)^+ = \max\{f - \rho^k,\, 0\}$.

For any $f \in \mathcal{F}^+$, define

$$a_{f,p,k,\rho} = \rho^{pk}\mu(f \geq \rho^k).$$

Lemma 2.2. *For any $f \in \mathcal{F}^+$ and any $\rho > 1$, we have*

$$\frac{\rho^p - 1}{\rho^p} \sum_{k\in\mathbb{Z}} a_{f,p,k,\rho} \leq \|f\|_p^p \leq (\rho^p - 1) \sum_{k\in\mathbb{Z}} a_{f,p,k,\rho}. \tag{2.8}$$

Proof. From

$$\begin{aligned}\int_M f^p \mathrm{d}\mu &= \sum_{k\in\mathbb{Z}} \int_{\rho^k \le f \le \rho^{k+1}} f^p \mathrm{d}\mu \\ &\le \sum_{k\in\mathbb{Z}} \rho^{p(k+1)} \Big(\mu(f \ge \rho^k) - \mu(f \ge \rho^{k+1})\Big) \\ &= \rho^p \sum_{k\in\mathbb{Z}} a_{f,p,k,\rho} - \sum_{k\in\mathbb{Z}} a_{f,p,k+1,\rho} \\ &= (\rho^p - 1) \sum_{k\in\mathbb{Z}} a_{f,p,k,\rho}\end{aligned}$$

and

$$\begin{aligned}\int_M f^p \mathrm{d}\mu &= \sum_{k\in\mathbb{Z}} \int_{\rho^k \le f \le \rho^{k+1}} f^p \mathrm{d}\mu \\ &\ge \sum_{k\in\mathbb{Z}} \rho^{pk} \Big(\mu(f \ge \rho^k) - \mu(f \ge \rho^{k+1})\Big) \\ &= \sum_{k\in\mathbb{Z}} a_{f,p,k,\rho} - \frac{1}{\rho^p} \sum_{k\in\mathbb{Z}} a_{f,p,k+1,\rho} \\ &= \frac{\rho^p - 1}{\rho^p} \sum_{k\in\mathbb{Z}} a_{f,p,k,\rho},\end{aligned}$$

we can deduce (2.8). □

Lemma 2.3. *For $f \in \mathcal{F}^+$ and $1 \le p \le +\infty$, we have*

$$\left(\frac{\rho-1}{\rho}\right)^p \frac{1}{\rho^p - 1} \|f\|_p^p \le \sum_{k\in\mathbb{Z}} \|f_{\rho,\,k}\|_p^p \le \left(\frac{\rho-1}{\rho}\right)^{p-1} \|f\|_p^p. \tag{2.9}$$

Proof. Since

$$\begin{aligned}\int_M |f_{\rho,\,k}|^p \mathrm{d}\mu &= p \int_0^{\rho^{k+1}-\rho^k} t^{p-1} \mu(f - \rho^k \ge t) \mathrm{d}t \\ &= p \int_{\rho^k}^{\rho^{k+1}} (s - \rho^k)^{p-1} \mu(f \ge s) \mathrm{d}s,\end{aligned}$$

for $p \ge 1$, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \int_M |f_{\rho,\,k}|^p \mathrm{d}\mu &= p \sum_{k\in\mathbb{Z}} \int_{\rho^k}^{\rho^{k+1}} (s - \rho^k)^{p-1} \mu(f \ge s) \mathrm{d}s \\ &\le \left\{ \sup_{k\in\mathbb{Z}} \sup_{s\in[\rho^k,\,\rho^{k+1}]} \left(\frac{s-\rho^k}{s}\right)^{p-1} \right\} \left\{ p \sum_{k\in\mathbb{Z}} \int_{\rho^k}^{\rho^{k+1}} s^{p-1} \mu(f \ge s) \mathrm{d}s \right\} \\ &= \left(\frac{\rho-1}{\rho}\right)^{p-1} \int_M f^p \mathrm{d}\mu.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\int_M |f_{\rho,\,k}|^p \mathrm{d}\mu &= p\sum_{k\in\mathbb{Z}}\int_{\rho^k}^{\rho^{k+1}} (s-\rho^k)^{p-1}\mu(f\geq s)\mathrm{d}s \\ &\geq \sum_{k\in\mathbb{Z}}\mu(f\geq \rho^{k+1})p\int_{\rho^k}^{\rho^{k+1}} (s-\rho^k)^{p-1}\mathrm{d}s \\ &= \left(\frac{\rho-1}{\rho}\right)^p \sum_{k\in\mathbb{Z}} a_{f,p,k+1,\rho} \\ &\geq \left(\frac{\rho-1}{\rho}\right)^p \frac{1}{\rho^p-1}\|f\|_p^p.\end{aligned}$$

Thus, we can obtain (2.9). □

For $p, s \in (0, +\infty]$ and $\vartheta \in (0, 1]$, assume that there holds

$$\|f\|_p \leq (CW(f))^{\vartheta}\|f\|_s^{1-\vartheta}, \tag{$S^{\vartheta}_{p,s}$}$$

where the associated parameter $q \in (-\infty, 0) \cup (0, +\infty) \cup \{\infty\}$ by setting

$$\frac{1}{p} = \frac{\vartheta}{q} + \frac{1-\vartheta}{s}. \tag{2.10}$$

Lemma 2.4. *For a function $f \in \mathcal{F}^+$, define*

$$\varphi: \ u \longmapsto \ln \|f\|_{\frac{1}{u}}.$$

Then $\varphi''(u) \geq 0$.

Proof. Since

$$\varphi'(u) = -\|f\|_{\frac{1}{u}}^{-\frac{1}{u}}\int_M f^{\frac{1}{u}}\ln\left(\frac{f}{\|f\|_{\frac{1}{u}}}\right)^{\frac{1}{u}},$$

define

$$\phi(r) := -\|f\|_r^{-r}\int_M f^r \ln\left(\frac{f}{\|f\|_r}\right)^r.$$

Then we have

$$\phi'(r) = \frac{r}{\|f\|_r^{2r}}\left\{\left[\int_M f^r\ln f\mathrm{d}\mu\right]^2 - \left(\int_M f^r\mathrm{d}\mu\right)\left(\int_M f^r(\ln f)^2\mathrm{d}\mu\right)\right\}$$

$$
\begin{aligned}
&\le \frac{r}{\|f\|_r^{2r}} \left\{ \left[\int_M \left(f^{\frac{r}{2}}\right)^2 \mathrm{d}\mu \right] \left[\int_M \left(f^{\frac{r}{2}} \ln f\right)^2 \mathrm{d}\mu \right] \right. \\
&\qquad \left. - \left(\int_M f^r \mathrm{d}\mu \right) \left(\int_M f^r (\ln f)^2 \mathrm{d}\mu \right) \right\} = 0.
\end{aligned}
$$

Thus

$$
\varphi''(u) = -\frac{1}{u^2}\phi'\left(\frac{1}{u}\right) \ge 0. \qquad \square
$$

Theorem 2.5. *(See Theorem 10.2 in [4].) If for any $f \in \mathcal{F}^+$, we have logarithmic Sobolev inequality*

$$
\int_M \left[f^p \ln \left(\frac{f}{\|f\|_p} \right)^p \mathrm{d}\mu \right] \le \left(\frac{1}{p} - \frac{1}{q} \right)^{-1} \|f\|_p^p \ln \left(\frac{CW(f)}{\|f\|_p} \right), \tag{LS_p^q}
$$

then we can deduce $(S_{p,s}^{\vartheta})$ for all $0 < s < p$ and vise versa.

Proof. From Lemma 2.4, the function

$$
\psi(u) = \frac{\varphi(u) - \varphi\left(\frac{1}{p}\right)}{u - \frac{1}{p}}
$$

is increasing of u, where we can define $\psi\left(\frac{1}{p}\right) = \varphi'\left(\frac{1}{p}\right)$.

Therefore, from (LS_p^q), for $0 < s < p$ we can deduce (noticing that $\frac{1}{p} > \frac{1}{q}$)

$$
\begin{aligned}
-\psi(s) \le \psi\left(\frac{1}{p}\right) &= \varphi'\left(\frac{1}{p}\right) \\
&= \|f\|_p^{-p} \int_M f^p \ln \left(\frac{f}{\|f\|_p} \right)^p \\
&\le \left(\frac{1}{p} - \frac{1}{q} \right)^{-1} \ln \left(\frac{CW(f)}{\|f\|_p} \right),
\end{aligned}
$$

which is $(S_{p,s}^{\vartheta})$ exactly.

Now assume $(S_{p,s}^{\vartheta})$ holds for any $0 < s < p$. Rewrite $(S_{p,s}^{\vartheta})$ as

$$
\left(\frac{\|f\|_p}{\|f\|_s} \right)^{\left(\frac{1}{s} - \frac{1}{p}\right)^{-1}} \le \left(\frac{CW(f)}{\|f\|_s} \right)^{\left(\frac{1}{s} - \frac{1}{q}\right)^{-1}}.
$$

Taking logarithms, we have

$$
(\ln \|f\|_p - \ln \|f\|_s) \left(\frac{1}{s} - \frac{1}{p} \right)^{-1} \le \left(\frac{1}{s} - \frac{1}{q} \right)^{-1} \ln \left(\frac{CW(f)}{\|f\|_s} \right).
$$

Letting $s \longrightarrow p$, we get (LS_p^q). $\square$

Lemma 2.6. *If for $\alpha > 0$, there holds*

$$\left(\sum_{k\in\mathbb{Z}} W(f_{\rho,k})^{\alpha}\right)^{\frac{1}{\alpha}} \leq A(\alpha,\rho)W(f), \quad \forall\, f \in \mathcal{F}^+,$$

where $A(\alpha,\rho)$ is a constant depending on α and ρ, then $(S^{\vartheta}_{p,s})$ implies

$$\|f\|_q \leq (\rho^q - 1)^{\frac{1}{q}} \rho^{\frac{q-s}{p-s}} \frac{CA(p,\rho)}{\rho - 1} W(f). \tag{$S_{q,p,s}$}$$

Proof. From $(S^{\vartheta}_{p,s})$, we have

$$\int\limits_M f^p_{\rho,k} \mathrm{d}\mu \leq (CW(f_{\rho,k}))^{p\vartheta} \left(\int\limits_M f^s_{\rho,k} \mathrm{d}\mu\right)^{\frac{p(1-\vartheta)}{s}}. \tag{2.11}$$

Since

$$\begin{aligned} &\int\limits_M f^s_{\rho,k} \mathrm{d}\mu \leq \rho^{sk}(\rho-1)^s \mu(f \geq \rho^k) \\ &\int\limits_M f^p_{\rho,k} \mathrm{d}\mu \geq \rho^{pk}(\rho-1)^p \mu(f \geq \rho^{k+1}), \end{aligned} \tag{2.12}$$

we can deduce

$$a_{f,q,k+1,\rho} \leq \rho^q (\rho-1)^{-p\vartheta} (CW(f_{\rho,k}))^{p\vartheta} a^{\frac{p(1-\vartheta)}{s}}_{f,q,k,\rho}.$$

Therefore, we have

$$\begin{aligned} \sum_{k\in\mathbb{Z}} a_{f,q,k,\rho} &= \sum_{k\in\mathbb{Z}} a_{f,q,k+1,\rho} \\ &\leq \sum_{k\in\mathbb{Z}} \rho^q(\rho-1)^{-p\vartheta}(CW(f_{\rho,k}))^{p\vartheta} a^{\frac{p(1-\vartheta)}{s}}_{f,q,k,\rho} \\ &\leq \rho^q(\rho-1)^{-p\vartheta} C^{p\vartheta} \left(\sum_{k\in\mathbb{Z}} W(f_{\rho,k})^p\right)^{\vartheta} \left(\sum_{k\in\mathbb{Z}} a^{\frac{p}{s}}_{f,q,k,\rho}\right)^{1-\vartheta} \\ &\leq \rho^q(\rho-1)^{-p\vartheta} C^{p\vartheta} \left(\sum_{k\in\mathbb{Z}} W(f_{\rho,k})^p\right)^{\vartheta} \left(\sum_{k\in\mathbb{Z}} a_{f,q,k,\rho}\right)^{\frac{p(1-\vartheta)}{s}}. \end{aligned}$$

Therefore, we have

$$\sum_{k\in\mathbb{Z}} a_{f,q,k,\rho} \leq \rho^{q\frac{q-s}{p-s}} (\rho-1)^{-q} C^q \left(\sum_{k\in\mathbb{Z}} W(f_{\rho,k})^p\right)^{\frac{q}{p}}. \tag{2.13}$$

Taking $p = q$ in (2.8), from (2.13), we can deduce

$$\begin{aligned}\int_M f^q \mathrm{d}\mu &\le (\rho^q - 1)\rho^{q\frac{q-s}{p-s}}(\rho - 1)^{-q}C^q \left(\sum_{k\in\mathbb{Z}} W(f_{\rho,k})^p\right)^{\frac{q}{p}} \\ &\le (\rho^q - 1)\rho^{q\frac{q-s}{p-s}}(\rho - 1)^{-q}C^q A(p,\rho)^q W(f)^q,\end{aligned}$$

which is $(S_{q,p,s})$ as desired. □

Let (M, g) be an n-dimensional Riemannian manifold. Then for $f \in \mathcal{F}^+$, define non-negative functional

$$W(f) = \left(\int_M (|\nabla f|^p + Sf^p)\mathrm{d}\mu + c\int_M f^p \mathrm{d}\mu\right)^{\frac{1}{p}},$$

where $S \in C^0(M, \mathbb{R})$ and c is a constant.

Lemma 2.7. *If $c + S \ge 0$ and $1 \le p < +\infty$, then we have*

$$\left(\sum_{k\in\mathbb{Z}} W(f_{\rho,k})^\alpha\right)^{\frac{1}{\alpha}} \le W(f),$$

where any $\alpha \ge p$ is constant.

Proof. Since $c + S \ge 0$, we can consider $(c + S)\mathrm{d}\mu$ as a new measure. Therefore, for $p \ge 1$, similar to Lemma 2.3, we can also deduce

$$\sum_{k\in\mathbb{Z}}\int_M (c+S)f_{\rho,\,k}^p \mathrm{d}\mu \le \left(\frac{\rho-1}{\rho}\right)^{p-1}\int_M (c+S)f^p\mathrm{d}\mu.$$

Obviously, there holds

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\int_M |\nabla f_{\rho,\,k}|^p\mathrm{d}\mu &= \sum_{k\in\mathbb{Z}}\int_{\rho^k\le f\le\rho^{k+1}} |\nabla f|^p\mathrm{d}\mu \\ &= \int_M |\nabla f|^p\mathrm{d}\mu.\end{aligned}$$

Therefore, for $\alpha \ge p$, we can get

$$\begin{aligned}\left(\sum_{k\in\mathbb{Z}} W(f_{\rho,k})^\alpha\right)^{\frac{1}{\alpha}} &= \left(\sum_{k\in\mathbb{Z}}\left(\int_M \left(|\nabla f_{\rho,\,k}|^p + Sf_{\rho,\,k}^p\right)\mathrm{d}\mu + c\int_M f_{\rho,\,k}^p\mathrm{d}\mu\right)^{\frac{\alpha}{p}}\right)^{\frac{1}{\alpha}} \\ &\le \left(\sum_{k\in\mathbb{Z}}\int_M |\nabla f_{\rho,\,k}|^p\mathrm{d}\mu + \sum_{k\in\mathbb{Z}}\int_M (c+S)f_{\rho,\,k}^p\mathrm{d}\mu\right)^{\frac{1}{p}}\end{aligned}$$

$$\leq \left(\int_M |\nabla f|^p \mathrm{d}\mu + \left(\frac{\rho-1}{\rho}\right)^{p-1} \int_M (c+S) f^p \mathrm{d}\mu \right)^{\frac{1}{p}}$$
$$\leq W(f). \qquad \square$$

3. Preliminaries of geometric flow

In this section, we give some fundamental properties about the geometric flow (1.4). Let (M, g) be an n-dimensional compact Riemannian manifold. Motivated by [31], fixing a real-valued function $S \in C^\infty(M, \mathbb{R})$, we can define, for any $h \in C^\infty(M, \mathbb{R})$ with $\int_M e^{-h} \mathrm{d}\mu = 1$,

$$\mathcal{F}(g, h) = \int_M (S + |\nabla h|^2) e^{-h} \mathrm{d}\mu$$

and

$$\mathcal{W}(g, f, \tau) = \int_M \left[\tau(S + |\nabla f|^2) + f - n\right] \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} \mathrm{d}\mu, \tag{3.1}$$

where τ is a positive number and $f \in C^\infty(M, \mathbb{R})$ satisfies

$$\int_M \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} \mathrm{d}\mu = 1. \tag{3.2}$$

Let $v = e^{-\frac{h}{2}}$ and

$$u = \frac{e^{-\frac{f}{2}}}{(4\pi\tau)^{\frac{n}{4}}}. \tag{3.3}$$

Then we have

$$\mathcal{F}(g, h) = \mathcal{F}^*(g, v) = \int_M (4|\nabla v|^2 + S v^2) \mathrm{d}\mu, \quad \int_M v^2 \mathrm{d}\mu = 1$$

and

$$\mathcal{W}(g, f, \tau) = \mathcal{W}^*(g, u, \tau) - \frac{n}{2} \ln \tau - \frac{n}{2} \ln(4\pi) - n \tag{3.4}$$

where

$$\mathcal{W}^*(g, u, \tau) = \int_M \left[\tau(4|\nabla u|^2 + S u^2) - u^2 \ln u^2\right] \mathrm{d}\mu, \quad \int_M u^2 \mathrm{d}\mu = 1.$$

We define

$$4\lambda_0(g) := \inf_{\int_M v^2 \mathrm{d}\mu = 1} \mathcal{F}^*(g, v)$$

and

$$\mu^*(g,\tau) := \inf_{\int_M u^2 \mathrm{d}\mu=1} \mathcal{W}^*(g,u,\tau).$$

In the case of geometric flow (1.4), we take the function S as $S(x,t)$, the trace of time-dependent symmetric 2-tensor $\mathcal{S}$ with respect to Riemannian metric $g(x,t)$.

Lemma 3.1. *Assume that $g(x,t)$ is a smooth solution to the geometric flow (1.4) in $M \times [0,T)$. Let h be a positive solution to the backward heat equation*

$$\frac{\partial}{\partial t}h(x,\,t) = -\Delta_{g(x,\,t)}h + |\nabla h|^2_{g(x,\,t)} - S(x,\,t).$$

Then we have

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = \int\limits_M \Big(2|h_{ij}+\mathcal{S}_{ij}|^2 + \mathcal{D}_2(\mathcal{S},\nabla h)\Big)e^{-h}\mathrm{d}\mu(t)$$

and

$$\frac{\mathrm{d}\mathcal{W}}{\mathrm{d}t} = \int\limits_M \tau\left[2\left|f_{ij}+\mathcal{S}_{ij}-\frac{1}{2\tau}g_{ij}\right|^2 + \mathcal{D}_2(\mathcal{S},\nabla f)\right]\frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}\mathrm{d}\mu(t), \tag{3.5}$$

where

$$\frac{\partial}{\partial t}f(x,\,t) = -\Delta_{g(x,\,t)}f(x,\,t) + |\nabla f|^2_{g(x,\,t)} - S(x,\,t) + \frac{n}{2\tau(t)} \tag{3.6}$$

and for any $\sigma > 0$ and $0 \le t^* < T$,

$$\tau(t) = t^* + \sigma - t.$$

In particular, both $\mathcal{F}$ entropy and $\mathcal{W}$ entropy are non-decreasing in t if $\mathcal{D}_2(\mathcal{S},\cdot)$ is nonnegative and all times $t \in [0,T)$, from which we can get that $\lambda_0(g(t))$ is non-decreasing of t and

$$\mu^*(g(t),\sigma) \ge \mu^*(g(0),t+\sigma) + \frac{n}{2}\ln\frac{\sigma}{t+\sigma} \tag{3.7}$$

for all $t \in [0,T)$ and $\sigma > 0$ (the case $t = 0$ is trivial).

Proof. The proof here is just direct computation. We use the method in [14]. Set

$$P = 2\Delta h - |\nabla h|^2 + S$$

By Lemma 2.1 in [14], let us take $\alpha = 2, \beta = 1, \lambda = 0, a = 1, b = d = 0, c = -1$. Then we can get

$$\begin{aligned}\frac{\partial P}{\partial t} &= -\Delta P + 2\nabla P\cdot\nabla h + 2|h_{ij}+S_{ij}|^2 + \frac{\partial S}{\partial t} - \Delta S - 2|S_{ij}|^2 \\ &\quad - 2\nabla h\cdot\nabla S + 4h_i\nabla_j\mathcal{S}_{ij} - 2\mathcal{S}_{ij}h_ih_j + 2R_{ij}h_ih_j \\ &= -\Delta P + 2\nabla P\cdot\nabla h + 2|h_{ij}+\mathcal{S}_{ij}|^2 + \mathcal{D}_2(\mathcal{S},\nabla h).\end{aligned} \tag{3.8}$$

Combining (3.8) and the definition of $\mathcal{F}$ entropy, we derive

$$\begin{aligned}\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_M Pe^{-h}\mathrm{d}\mu(t) = \int_M \left(\frac{\partial P}{\partial t} - P\frac{\partial h}{\partial t} - PS\right)e^{-h}\mathrm{d}\mu(t) \\ &= \int_M \Big[-\Delta P + 2\nabla P\cdot\nabla h + 2|h_{ij}+\mathcal{S}_{ij}|^2 + \mathcal{D}_2(\mathcal{S},\nabla h) \\ &\qquad + P(\Delta h - |\nabla h|^2 + S) - PS\Big]e^{-h}\mathrm{d}\mu(t) \\ &= \int_M \Big[-e^h\Delta(Pe^{-h}) + 2|h_{ij}+\mathcal{S}_{ij}|^2 + \mathcal{D}_2(\mathcal{S},\nabla h)\Big]e^{-h}\mathrm{d}\mu(t) \\ &= \int_M \Big(2|h_{ij}+\mathcal{S}_{ij}|^2 + \mathcal{D}_2(\mathcal{S},\nabla h)\Big)e^{-h}\mathrm{d}\mu(t).\end{aligned}$$

Hence, it follows that $\mathcal{F}$ entropy is non-decreasing.

The monotonicity of $\mathcal{W}$ entropy had been proved in Theorem 3.1 of [24] (see also [14,17]).

Since $\mathcal{D}_2(\mathcal{S},\cdot)$ is nonnegative, from (3.4) and (3.5), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}^*(g,u,\tau) \geq \frac{n}{2}\frac{\mathrm{d}}{\mathrm{d}t}\ln\tau,$$

where

$$u = u(t) = \frac{e^{-\frac{f(t)}{2}}}{(4\pi\tau(t))^{\frac{n}{4}}},$$

which satisfies the equation

$$\frac{\partial u}{\partial t} = -\Delta u - \frac{|\nabla u|^2}{u} + \frac{S}{2}u.$$

It follows that

$$\mu^*(g(t_1),\tau(t_1)) \leq \mu^*(g(t_2,\tau(t_2)) + \frac{n}{2}\ln\frac{\tau(t_1)}{\tau(t_2)}.$$

Choosing $t_1 = 0$ and $t_2 = t^*$ we can obtain

$$\mu^*(g(0),t^*+\sigma) \leq \mu^*(g(t^*),\sigma) + \frac{n}{2}\ln\frac{t^*+\sigma}{\sigma}. \tag{3.9}$$

Since $0 < t^* < T$ is arbitrary, (3.9) can be rewritten as (3.7).

Similarly, we can get that $\lambda_0(g(t))$ is non-decreasing of t. $\square$

Remark 3.1. The authors would like to thank Professor Hong Huang for pointing out the references [17,24].

Lemma 3.2. *Assume that $g(x,t)$ is a smooth solution to the geometric flow (1.4) in $M\times[0,T)$ and that $\mathcal{D}_2(\mathcal{S},\cdot)$ is nonnegative. We have*

$$\min_{x\in M} S(x,t) \geq \min_{x\in M} S(x,0). \tag{3.10}$$

Moreover, we have either

$$S(x,t) \geq 0, \tag{3.11}$$

or

$$\min_{x\in M} S(x,t) \geq \frac{1}{\frac{1}{\min\limits_{x\in M} S(x,0)} - \frac{2t}{n}}. \tag{3.12}$$

Proof. Since $\mathrm{D}_2(\mathcal{S}, \cdot)$ is nonnegative, taking $X = 0$, we have

$$\frac{\partial S}{\partial t} - \Delta S - 2|\mathcal{S}_{ij}|^2_{g(t)} \geq 0,$$

from which we can get

$$\frac{\partial S}{\partial t} - \Delta S - \frac{2}{n}S^2 \geq 0.$$

From the maximum principle, we have (3.10).

If $\min\limits_{x\in M} S(x,0) \geq 0$, we have (3.11). Otherwise, at the minimal point of $S(x,t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\min_{x\in M} S(x,t)\right) - \frac{2}{n}\left[\min_{x\in M} S(x,t)\right]^2 \geq 0. \tag{3.13}$$

From the theory of ordinary differential equation, by (3.13), we can get (3.12). □

4. Proofs of theorems about (logarithmic) Sobolev inequalities

We will also need the following elementary lemma (see for example [40]).

Lemma 4.1. *Let $a > 0$ and b be constants. Then the minimum of the function $y = a\sigma - \frac{n}{2}\ln\sigma + b$ for $\sigma > 0$ is $\frac{n}{2}\ln(\alpha a)$, where*

$$\alpha = \frac{2e}{n}e^{\frac{2b}{n}}. \tag{4.1}$$

Proof of Theorem 1.1. For $u \in W^{1,2}(M)$ with $\int_M u^2 \mathrm{d}\mu(0) = 1$, taking

$$\alpha = \frac{8(t+\sigma)}{nC_S(M,g_0)^2}, \quad S = S_0$$

in (2.4), we have

$$\begin{aligned}\int_M u^2 \ln u^2 \mathrm{d}\mu(0) \leq (t+\sigma)\int_M \left(4|\nabla u|_0^2 + S_0 u^2\right)\mathrm{d}\mu(0) \\ + \frac{n}{2}(2\ln C_S(M,\, g_0) + \ln n - 2\ln 2 - 1) \\ - \frac{n}{2}\ln(t+\sigma) + (t+\sigma)\left(\frac{4}{C_S(M,\, g_0)^2 \mathrm{Vol}_{g_0}(M)^{\frac{2}{n}}} - \min S_0\right).\end{aligned}$$

It follows that

$$\mu^*(g(0), t+\sigma) \geq \frac{n}{2}\ln(t+\sigma) - (t+\sigma)\left(\frac{4}{C_S(M,\, g_0)^2 \mathrm{Vol}_{g_0}(M)^{\frac{2}{n}}} - \min S_0\right) - \frac{n}{2}(2\ln C_S(M,\, g_0) + \ln n - 2\ln 2 - 1). \tag{4.2}$$

From (3.7) and (4.2), we can deduce

$$\mu^*(g(t), \sigma) \geq \frac{n}{2}\ln\sigma - (t+\sigma)\left(\frac{4}{C_S(M,\, g_0)^2 \mathrm{Vol}_{g_0}(M)^{\frac{2}{n}}} - \min S_0\right) - \frac{n}{2}(2\ln C_S(M,\, g_0) + \ln n - 2\ln 2 - 1),$$

or

$$\mu^*\left(g(t), \frac{\sigma}{4}\right) \geq \frac{n}{2}\ln\sigma - \left(t+\frac{\sigma}{4}\right)\left(\frac{4}{C_S(M,\, g_0)^2 \mathrm{Vol}_{g_0}(M)^{\frac{2}{n}}} - \min S_0\right) - \frac{n}{2}(2\ln C_S(M,\, g_0) + \ln n - 1)$$

which is equivalent to (1.6).

Taking

$$a = \int_M \left(|\nabla u|_t^2 + \frac{S_t}{4}u^2\right) \mathrm{d}\mu(t) + \frac{A_1}{4} > 0$$

and $b = A_1 t + A_2$ in Lemma 4.1, from (1.6), we can get (1.9). □

Before prove Theorem 1.2, we need the following lemma.

Lemma 4.2. *Assume that $g(x,t)$ is a smooth solution to the geometric flow (1.4) in $M \times [0,T)$ and that $\mathcal{D}_2(\mathcal{S}, \cdot)$ defined in (1.5) is nonnegative. If $\lambda_0(g_0)$ is positive, then for any $\sigma > 0$ and $t \in [0,T)$ satisfying $t+\sigma \geq \frac{n}{8}C_S(M,g_0)^2\delta_0$, there holds*

$$\int_M u^2 \ln u^2 \mathrm{d}\mu(t) \leq \sigma \int_M \left(|\nabla u|_t^2 + \frac{S_t}{4}u^2\right) \mathrm{d}\mu(t) - \frac{n}{2}\ln\sigma + \frac{n}{2}\ln n + n\ln C_S(M,g_0) + \sigma_0(g_0) \tag{4.3}$$

for any $u \in W^{1,2}(M)$ with $\int_M u^2 \mathrm{d}\mu(t) = 1$, where $C_S(M,g_0)$ is the Sobolev constant defined in (2.1), $\delta_0 = \delta_0(g_0)$ is the number defined in (2.6) and the number $\sigma_0(g_0)$ is defined in (2.7).

Proof. Assume $t+\sigma \geq \frac{n}{8}C_S(M,g_0)^2\delta_0(g_0)$. Choosing

$$A = \frac{8(t+\sigma)}{nC_S(M,g_0)^2} \geq \delta_0(g_0),$$

from (2.5), we can deduce

$$\int_M u^2 \ln u^2 \mathrm{d}\mu(0) \le 4(t+\sigma) \int_M \left(|\nabla u|_0^2 + \frac{S_0}{4} u^2 \right) \mathrm{d}\mu(0) - \frac{n}{2} \ln(t+\sigma)$$
$$+ \frac{n}{2}\left(2 \ln C_S(M, g_0) + \ln n - 2 \ln 2 \right) + \sigma_0(g_0),$$

where $u \in W^{1,2}(M)$ satisfying $\int_M u^2 \mathrm{d}\mu(0) = 1$.

It follows that

$$\mu^*(g_0, t+\sigma) \ge \frac{n}{2} \ln(t+\sigma) - \frac{n}{2}\left(2 \ln C_S(M, g_0) + \ln n - 2 \ln 2 \right) - \sigma_0(g_0). \tag{4.4}$$

From (3.7) and (4.4), we can deduce

$$\mu^*(g(t), \sigma) \ge \frac{n}{2} \ln \sigma - \frac{n}{2}\left(2 \ln C_S(M, g_0) + \ln n - 2 \ln 2 \right) - \sigma_0(g_0)$$

or

$$\mu^*\left(g(t), \frac{\sigma}{4} \right) \ge \frac{n}{2} \ln \sigma - \frac{n}{2}\left(2 \ln C_S(M, g_0) + \ln n \right) - \sigma_0(g_0),$$

which is equivalent to (4.3). □

Remark 4.1. In the case of Ricci flow (1.3), the result in Lemma 4.2 can be found in Ye [40].

Note that the proofs of Theorem 1.1 and Lemma 4.2 lead to the following general result. Indeed, Theorem 1.1 and Lemma 4.2 can be seen as its special examples.

Theorem 4.3. *Let $g(t)$ be a smooth solution of the geometric flow (1.4) on $M \times [0, T)$ for some (finite or infinite) $T > 0$ with $\mathcal{D}_2(\mathcal{S}, \cdot)$ defined in (1.5) nonnegative and let $h(\sigma)$ be a scalar function for $\sigma > 0$. Assume that the initial metric $g_0 = g(0)$ satisfies the logarithmic Sobolev inequality*

$$\int_M u^2 \ln u^2 \mathrm{d}\mu(0) \le \sigma \int_M \left(|\nabla u|_0^2 + \frac{S_0}{4} u^2 \right) \mathrm{d}\mu(0) + h(\sigma)$$

for each $\sigma > 0$ and all $u \in W^{1,2}(M)$ with $\int_M u^2 \mathrm{d}\mu(0) = 1$. Then there holds at each $t \in [0, T)$

$$\int_M u^2 \ln u^2 \mathrm{d}\mu(t) \le \sigma \int_M \left(|\nabla u|_t^2 + \frac{S_t}{4} u^2 \right) \mathrm{d}\mu(t) + h(4t+\sigma) - \frac{n}{2} \ln \frac{\sigma}{4t+\sigma}$$

for each $\sigma > 0$ and all $u \in W^{1,2}(M)$ with $\int_M u^2 \mathrm{d}\mu(t) = 1$.

Given Theorem 1.1 and Lemma 4.2, we can deduce Theorem 1.2.

Proof of Theorem 1.2. Let $t \in [0, T)$ and $\sigma > 0$. If $t+\sigma < \frac{n}{8} C_S(M, g_0)^2 \delta_0(g_0)$, we apply (1.6) in Theorem 1.1 and bound $t + \frac{\sigma}{4}$ in (1.6) by $\frac{n}{8} C_S(M, g_0)^2 \delta_0(g_0)$. Otherwise, we apply (4.3) in Lemma 4.2. Then we can deduce (1.10). Since the eigenvalue $\lambda_0(g(t))$ is non-decreasing and $\lambda_0(g_0) > 0$ we have $\lambda_0(g(t)) > 0$ for all t. Therefore, we can deduce $\int_M \left(|\nabla u|_t^2 + \frac{S_t}{4} u^2 \right) \mathrm{d}\mu(t) > 0$ for all t. From Lemma 4.1 by setting $a = \int_M \left(|\nabla u|_t^2 + \frac{S_t}{4} u^2 \right) \mathrm{d}\mu(t)$ and $b = C$, we can get (1.11). □

Here we give a special conclusion of Theorem 1.2.

Corollary 4.4. *Suppose that $g(t)$ is a smooth solution of the geometric flow (1.4) on $M \times [0, T)$ for some (finite or infinite) $T > 0$ with $\mathcal{D}_2(\mathcal{S}, \cdot)$ defined in (1.5) nonnegative. If $\lambda_0(g_0)$ is positive, then for $t \in [0, T)$, we have*

$$\mathrm{Vol}_{g(t)}(M) \geq e^{-C} \tag{4.5}$$

when $\hat{S}_t \leq 0$, and

$$\mathrm{Vol}_{g(t)}(M) \geq e^{-\frac{1}{4}-C}\hat{S}_t^{-\frac{n}{2}} \tag{4.6}$$

when $\hat{S}_t > 0$. Here C is the constant in Theorem 1.2 and $\hat{S}_t$ is the average of S_t

$$\hat{S}_t = \frac{\int_M S_t \mathrm{d}\mu(t)}{\mathrm{Vol}_{g(t)}(M)}.$$

Proof. Taking $u = \mathrm{Vol}_{g(t)}(M)^{-\frac{1}{2}}$ in (1.10), we get

$$\ln \frac{1}{\mathrm{Vol}_{g(t)}(M)} \leq \frac{\sigma}{4}\hat{S}_t - \frac{n}{2}\ln\sigma + C.$$

If $\hat{S}_t \leq 0$, then taking $\sigma = 1$, we get (4.5). If $\hat{S}_t > 0$, then taking $\sigma = \hat{S}_t^{-1}$, we get (4.6). $\square$

Remark 4.2. In the case of Ricci flow (1.3), the result in Corollary 4.4 specializes to the one in Ye [40].

Given the logarithmic Sobolev inequalities in Theorem 1.1 and Theorem 1.2, we can deduce the uniform Sobolev inequality along geometric flow (1.4).

Proof of Theorem 1.3. In the case $\lambda_0(g_0) > 0$, letting

$$f = \frac{u}{\left(\int_M u^2 \mathrm{d}\mu(t)\right)^{\frac{1}{2}}},$$

from (1.11), we have

$$\begin{aligned}
&\int_M \left[u^2 \ln\left(\frac{u^2}{\int_M u^2 \mathrm{d}\mu(t)}\right)\mathrm{d}\mu(t)\right] \\
&\leq n\left(\int_M u^2 \mathrm{d}\mu(t)\right)\ln\left(\frac{\alpha_{II}\int_M\left(|\nabla u|_t^2 + \frac{S_t}{4}u^2\right)\mathrm{d}\mu(t)}{\int_M u^2}\right)^{\frac{1}{2}} \\
&\leq n\left(\int_M u^2 \mathrm{d}\mu(t)\right)\ln\left(\frac{\alpha_{II}\int_M\left(|\nabla u|_t^2 + \left(\frac{S_t - S_0^-}{4}\right)u^2\right)\mathrm{d}\mu(t)}{\int_M u^2 \mathrm{d}\mu(t)}\right)^{\frac{1}{2}},
\end{aligned}$$

where $S_0^- = \min\{0, S_0\}$. Define

$$W(f) := \left\{\int_M \left[|\nabla u|_t^2 + \left(\frac{S_t - S_0^-}{4}\right)u^2\right]\mathrm{d}\mu(t)\right\}^{\frac{1}{2}}.$$

Then from Lemma 2.6 and Lemma 2.7(by taking $\rho = 2$, $p = 2$, $s = 1$, $q = \frac{2n}{n-2}$), we have

$$\left(\int_M u^{\frac{2n}{n-2}} \mathrm{d}\mu(t)\right)^{\frac{n-2}{2n}} \leq \left(2^{\frac{2n}{n-2}} - 1\right)^{\frac{n-2}{2n}} 2^{\frac{n+2}{n-2}} \alpha_{II} W(f). \tag{4.7}$$

Since

$$\begin{aligned}\int_M u^2 \mathrm{d}\mu(t) &= \frac{\lambda_0(g(t))}{\lambda_0(g(t))} \int_M u^2 \mathrm{d}\mu(t) \\ &\leq \frac{1}{\lambda_0(g(t))} \int_M \left(|\nabla u|_t^2 + \frac{S_t}{4} u^2\right) \mathrm{d}\mu(t) \\ &\leq \frac{1}{\lambda_0(g_0)} \int_M \left(|\nabla u|_t^2 + \frac{S_t}{4} u^2\right) \mathrm{d}\mu(t),\end{aligned}$$

substituting the expression of α_{II} into (4.7), we have (1.12), where

$$A = (2^{\frac{2n}{n-2}} - 1)^{\frac{n-2}{n}} 2^{\frac{4n}{n-2}} e^{2+\frac{4C}{n}} \frac{\lambda_0(g_0) - S_0^-/4}{\lambda_0(g_0)}. \tag{4.8}$$

In case $T < \infty$, define

$$W(f) := \left\{\int_M \left(|\nabla u|_t^2 + \frac{S_t + A_1}{4} u^2\right) \mathrm{d}\mu(t)\right\}^{\frac{1}{2}}.$$

Then from (1.9), Lemma 2.6 and Lemma 2.7(by taking $\rho = 2$, $p = 2$, $s = 1$, $q = \frac{2n}{n-2}$), we have (1.13), where

$$A = \left(2^{\frac{2n}{n-2}} - 1\right)^{\frac{n-2}{n}} 2^{\frac{4n}{n-2}} \frac{e^2}{n^2} e^{\frac{4(A_1 T + A_2)}{n}}, \tag{4.9}$$

$$B = \left(2^{\frac{2n}{n-2}} - 1\right)^{\frac{n-2}{n}} 2^{\frac{4n}{n-2}} \frac{A_1 e^2}{4n^2} e^{\frac{4(A_1 T + A_2)}{n}}. \qquad \square \tag{4.10}$$

5. The κ-noncollapsing estimates under geometric flow

In the case of Ricci flow (1.3), the κ-noncollapsing property, the volume ratio between a geodesic ball and Euclidean ball with the same radius is bounded from below, is first proved by Perelman [31] under the assumption that curvature is bounded along the Ricci flow. Here we get the κ-noncollapsing estimates as follows.

Theorem 5.1. *Assume that $g(x,t)$ is a smooth solution to the geometric flow (1.4) in $M \times [0,T)$ and $\mathcal{D}_2(\mathcal{S}, \cdot)$ defined in (1.5) is nonnegative. There hold*

(1) *if $\lambda_0(g_0) > 0$ and $S_t \leq \frac{1}{r^2}$ holds on a geodesic ball $B(x,r)$, where $r > 0$, then for $t \in [0, T)$, there holds*

$$\mathrm{Vol}_{g(t)}(B(x,r)) \geq \left(\frac{1}{2^{n+3}A}\right)^{\frac{n}{2}} r^n,$$

where A is a positive constant defined in (4.8).

(2) *if* $T < \infty$ *and* $S_t \leq \frac{1}{r^2}$ *holds on a geodesic ball* $B(x, r)$ *with* $0 < r \leq L$*, then for* $t \in [0, T)$*, there holds*

$$\mathrm{Vol}_{g(t)}(B(x,r)) \geq \left(\frac{1}{2^{n+3}A + 2L^2B}\right)^{\frac{n}{2}} r^n,$$

where A *and* B *are defined in (4.9) and (4.10) respectively.*

Remark 5.1. In the case of Ricci flow, this version of the κ-noncollapsing property can be found in [40], which is particularly powerful and flexible and has important applications to Poincaré conjecture and the geometrization conjecture [32]. (More meanings and applications of the κ-noncollapsing property can be found in [40] and the references therein.)

The proof of Theorem 5.1 is a direct result of the following lemma.

Lemma 5.2. *Let* (M, g) *be an* n*-dimensional* $(n \geq 3)$ *compact Riemannian manifold and* $\mathcal{S}$ *be any symmetric* 2*-tensor with trace* $S = \sum_{i,j=1}^{n} g^{ij}\mathcal{S}_{ij}$. *Assume that for any* $u \in W^{1,2}(M)$*, there holds the Sobolev inequality*

$$\left(\int_M |u|^{\frac{2n}{n-2}} \mathrm{d}\mu\right)^{\frac{n-2}{n}} \leq A\int_M \left(|\nabla u|^2 + \frac{S}{4}u^2\right)\mathrm{d}\mu + B\int_M u^2 \mathrm{d}\mu.$$

If $S \leq \frac{1}{r^2}$ *holds on a geodesic ball* $B(x, r)$ *with* $0 < r \leq L$*, then there holds*

$$\mathrm{Vol}_g(B(x,r)) \geq \left(\frac{1}{2^{n+3}A + 2L^2B}\right)^{\frac{n}{2}} r^n.$$

Proof. The proof is very similar to the proof of Lemma 6.1 in [40]. Here we omit it. □

6. The κ-noninflated estimates under geometric flow

Except for κ non-collapsing property, the κ-noninflated property (the volume ratio between a geodesic ball and Euclidean ball with the same radius is bounded from above) is also very useful (in the case of Kähler–Ricci flow, the importance of upper bound of volume can be found in [34,8] and references therein).

To make the κ non-inflated property clear, we give a definition as follows.

Definition 6.1. A smooth, compact, n-dimensional geometric flow (1.4) is called κ non-inflated at the point (x_0, t_0) under scale ρ if the following statement holds.

(1) the geometric flow is defined in the space time cube

$$\left\{(x, t) : \mathrm{d}(x, x_0, t_0) < r,\ t \in [t_0 - r^2, t_0]\right\},$$

(2) for some positive constant α, $S(x,t) \leq \frac{\alpha}{t_0 - t}$ for all (x,t) in the above cube.

Then there exists a positive constant κ, which may depend on α such that

$$\mathrm{Vol}_{g(t_0)}(\mathrm{B}(x_0, r, t_0)) \leq \kappa r^n.$$

Remark 6.1. In the κ non-collapsing property, the condition $S(x, t) \le \frac{1}{r^2}$ on the $S(x, t)$ is included in the one $S(x,t) \le \frac{\alpha}{t_0 - t}$ of the κ non-inflated property in the same space time cube.

In the case of Ricci flow (1.3), our definition is the same as the one in Zhang [44].

Theorem 6.1. *Assume that $g(x,t)$ is a smooth solution to the geometric flow (1.4) in $M \times [0,T)$ and $\mathcal{D}_2(\mathcal{S},\cdot)$ defined in (1.5) and Ric $-\,\mathcal{S}$ are nonnegative. For any $x_0 \in M$, the geometric (1.4) is κ non-inflated at (x_0, t_0) under scale $\sqrt{t_0}$, where κ defined in (6.22) depends only on g_0, t_0 and α.*

Remark 6.2. The κ non-inflated property in Theorem 6.1 specializes to the one in Zhang [44] in the case of Ricci flow (1.3).

In order to prove the κ non-inflated property of geometric flow (1.4), we need the lemmas as follows.

Let $g(x,t)$ be a solution to the geometric flow (1.4) on $M \times [0, T)$, where M is a compact manifold and let ℓ, t be two moments in time such that $0 < \ell < t < T$, and x, $z \in M$. Let $G = G(z,\ell;x,t)$ be the fundamental solution of the conjugate heat equation

$$\partial_\ell f(z,\ell) + \Delta_{g(z,\ell)} f(z,\ell) - S(z,\ell) f(z,\ell) = 0$$

along the geometric flow (1.4). Fixing z, ℓ, we know that G, as a function of x and t, is the fundamental solution of heat equation (see for example Lemma 26.3 of Chapter 26 in [10])

$$\partial_t h(x,t) - \Delta_{g(x,t)} h(x,t) = 0. \tag{6.1}$$

Lemma 6.2. *Assume that $g(x,t)$ is a smooth solution to the geometric flow (1.4) in $M \times [0,T)$ and that $\mathcal{D}_2(\mathcal{S},\cdot)$ defined in (1.5) is nonnegative. We have*

$$\int_M G(z,\ell;x,t)\mathrm{d}\mu(x,t) \le 1 + C(1 + t - \ell)^{\frac{n}{2}}, \tag{6.2}$$

where C only depends on $\min_{x\in M} S(x,0)$. In particular, $C = 0$ when

$$S(x,t) \ge \min_{x\in M} S(x,0) \ge 0.$$

Proof. Since

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}} \int_M G(z,\ell;x,t)\mathrm{d}\mu(x,t) \\
&= \int_M \Big[\Delta_x G(z,\ell;x,t) - S(x,t)G(z,\ell;x,t)\Big]\mathrm{d}\mu(x,t) \\
&= -\int_M S(x,t)G(z,\ell;x,t)\mathrm{d}\mu(x,t),
\end{aligned} \tag{6.3}$$

from (3.11), (3.12) and (6.3), we have either

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_M G(z,\ell;x,t)\mathrm{d}\mu(x,t) \le 0$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M G(z,\ell;x,t)\mathrm{d}\mu(x,t) \le \frac{\int_M G(z,\ell;x,t)\mathrm{d}\mu(x,t)}{-\frac{1}{\min\limits_{x\in M} S(x,0)} + \frac{2t}{n}}.$$

Finally, we can deduce (6.2). □

Lemma 6.3. *Assume that $g(x,t)$ is a smooth solution to the geometric flow (1.4) in $M\times[0,T)$ and that $\mathcal{D}_2(\mathcal{S},\cdot)$ defined in (1.5) is nonnegative. We have*

$$G(z,\ell;x,t) \le \frac{\exp[L(t) - t\inf\limits_{y\in M} S^-(y,0)]}{(4(t-\ell))^{\frac{n}{2}}}, \tag{6.4}$$

where $0<\ell<t$ and

$$L(t) = 2A_1 t + A_2,$$

with A_1, A_2 the same as the ones defined in (1.7) and (1.8) up to adding constants depending only on n.

Moreover, if $S(x,\,0)\ge 0$, we have

$$G(z,\ell;x,t) \le \frac{e^C}{(4(t-\ell))^{\frac{n}{2}}}, \tag{6.5}$$

where C is the same as the ones defined in (1.10) up to adding constants depending only on n.

Proof. Let $f=f(x,\,t)$ be a positive solution to (6.1). Give $T_0>\ell$ and $t\in(\ell,\,T_0)$, defining

$$p(t) = \frac{T_0-\ell}{T_0-t},$$

we have $p(\ell)=1$ and $p(T_0)=+\infty$.

Applying the idea of Davies, we have

$$\begin{aligned}
\partial_t\|f\|_{p(t)} &= \partial_t\left[\left(\int_M f^{p(t)}\mathrm{d}\mu(x,t)\right)^{\frac{1}{p(t)}}\right] \\
&= -\frac{p'(t)}{p^2(t)}\|f\|_{p(t)}\ln\int_M f^{p(t)}\mathrm{d}\mu(x,t) + \frac{1}{p(t)}\left(\int_M f^{p(t)}\mathrm{d}\mu(x,t)\right)^{\frac{1}{p(t)}-1} \\
&\quad\times\left[\int_M f^{p(t)}(\ln f)p'(t)\mathrm{d}\mu(x,t)\right. \\
&\qquad\left.+\int_M f^{p(t)-1}(p(t)\Delta_x f(x,t) - f(x,t)S(x,t))\mathrm{d}\mu(x,t)\right]
\end{aligned}$$

multiplying both sides by $p(t)^2\|f\|_{p(t)}^{p(t)-1}$, we can deduce

$$
\begin{aligned}
p(t)^2\|f\|_{p(t)}^{p(t)-1}\partial_t\|f\|_{p(t)} = & -p'(t)\|f\|_{p(t)}^{p(t)}\ln\int_M f^{p(t)}\mathrm{d}\mu(x,t) \\
& + p(t)p'(t)\int_M f^{p(t)}(\ln f)\mathrm{d}\mu(x,t) \\
& - 4(p(t)-1)\int_M \left|\nabla\left(f^{\frac{p(t)}{2}}\right)\right|^2\mathrm{d}\mu(x,t) \\
& - p(t)\int_M \left(f^{\frac{p(t)}{2}}\right)^2 S(x,t)\mathrm{d}\mu(x,t)
\end{aligned}
$$

Define $v(x,\, t) = \frac{f^{\frac{p(t)}{2}}}{\left(\int_M f^{p(t)}\mathrm{d}\mu(x,t)\right)^{\frac{1}{2}}}$. Then we have

$$
\begin{aligned}
\|v\|_2 &= 1, \\
\int_M v^2\ln v^2 &= p(t)\int_M v^2\ln f - 2\int_M v^2\ln\|f^{\frac{p(t)}{2}}\|_2 \\
&= -2\ln\|f^{\frac{p(t)}{2}}\|_2 + p(t)\int_M v^2\ln f.
\end{aligned}
$$

Dividing both sides by $\|f\|_{p(t)}^{p(t)}$, we have

$$
\begin{aligned}
& p^2(t)\partial_t\ln\|f\|_{p(t)} \\
&= p'(t)\int_M v^2\ln v^2\mathrm{d}\mu(x,t) - 4(p(t)-1)\int_M|\nabla v|^2\mathrm{d}\mu(x,t) \\
&\quad - p(t)\int_M S(x,t)v^2\mathrm{d}\mu(x,t) \\
&= p'(t)\int_M v^2\ln v^2\mathrm{d}\mu(x,t) - \int_M S(x,t)v^2\mathrm{d}\mu(x,t) \\
&\quad - 4(p(t)-1)\int_M\left(|\nabla v|^2 + \frac{1}{4}S(x,t)v^2\right)\mathrm{d}\mu(x,t).
\end{aligned} \tag{6.6}
$$

From the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
\frac{4(p(t)-1)}{p'(t)} &= \frac{4(t-\ell)(T_0-t)}{T_0-\ell} \\
&\le \frac{(T_0-t+t-\ell)^2}{T_0-\ell} \\
&= T_0-\ell, \\
\frac{1}{p'(t)} &= \frac{(T_0-t)^2}{T_0-\ell} \le T_0-\ell.
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}
&p^2(t)\partial_t \ln \|f\|_{p(t)} \\
&= p'(t)\Bigg[\int_M v^2 \ln v^2 \mathrm{d}\mu(x,t) - \frac{1}{p'(t)}\int_M S(x,t)v^2 \mathrm{d}\mu(x,t) \\
&\qquad - \frac{4(p(t)-1)}{p'(t)}\int_M \left(|\nabla v|^2 + \frac{1}{4}S(x,t)v^2\right)\mathrm{d}\mu(x,t)\Bigg] \\
&\le p'(t)\Bigg[\int_M v^2 \ln v^2 \mathrm{d}\mu(x,t) - (T_0-\ell)\inf_{x\in M} S^-(x,t) \\
&\qquad - \frac{4(p(t)-1)}{p'(t)}\int_M \left(|\nabla v|^2 + \frac{1}{4}S(x,t)v^2\right)\mathrm{d}\mu(x,t)\Bigg].
\end{aligned} \tag{6.7}$$

Taking

$$\sigma = \frac{4(p(t)-1)}{p'(t)} \le T_0 - \ell$$

in (1.6), we can deduce

$$p^2(t)\partial_t \ln \|f\|_{p(t)} \le p'(t)\left(-n \ln \sqrt{\frac{4(p(t)-1)}{p'(t)}} + L(T_0) - (T_0-\ell)\inf_{x\in M} S^-(x,0)\right)$$

where, since $\sigma \le T_0 - \ell \le T_0$,

$$A_1\left(t + \frac{\sigma}{4}\right) + A_2 \le A_1 T_0 + A_2 = L(T_0)$$

and we also make use of (3.10) to obtain

$$-\inf_{x\in M} S^-(x,t) \le -\inf_{x\in M} S^-(x,0).$$

Since

$$\frac{p'(t)}{p^2(t)} = \frac{1}{T_0-\ell}$$

and

$$\frac{4(p(t)-1)}{p'(t)} = \frac{4(t-\ell)[T_0-\ell-(t-\ell)]}{T_0-\ell},$$

we can deduce

$$\begin{aligned}
\partial_t \ln \|f\|_{p(t)} \le \frac{1}{T_0-\ell}\Bigg\{ &-\frac{n}{2}\ln\left[\frac{4(t-\ell)[T_0-\ell-(t-\ell)]}{T_0-\ell}\right] \\
&+ L(T_0) - (T_0-\ell)\inf_{x\in M} S^-(x,0)\Bigg\}.
\end{aligned}$$

Integrating from $t = \ell$ to $t = T_0$, we can get

$$\ln \frac{\|f(\cdot, T_0)\|_\infty}{\|f(\cdot, \ell)\|_1} \le -\frac{n}{2} \ln[4(T_0 - \ell)] + L(T_0) - (T_0 - \ell) \inf_{x \in M} S^-(x, 0) + n.$$

Since

$$f(x, T_0) = \int_M G(z, \ell; x, T_0) f(z, \ell) \mathrm{d}\mu(z, \ell),$$

the above inequality implies that

$$G(z, \ell; x, T_0) \le \frac{\exp[L(T_0) - (T_0 - \ell) \inf_{x \in M} S^-(x, 0) + n]}{(4(T_0 - \ell))^{\frac{n}{2}}}.$$

Since $T_0 > \ell$ is arbitrary, we get (6.4) with maybe modified constants A_1 and A_2.

If $S(x, 0) \ge 0$, then we can use the logarithmic Sobolev inequality (1.10) in (6.7). Therefore, we can deduce (6.5) with a modified constant. □

Remark 6.3. We can also prove this lemma by Moser's Iteration. Here we follow [25] and just sketch it.

For $p \ge 1$, we have

$$\int_M f^p f_t \mathrm{d}\mu(t) - \int_M f^p \Delta f \mathrm{d}\mu(t) = 0,$$

that is,

$$\frac{1}{p+1} \partial_t \int_M f^{p+1} \mathrm{d}\mu(t) + \frac{1}{p+1} \int_M S_t f^{p+1} \mathrm{d}\mu(t) + \frac{4p}{(p+1)^2} \int_M |\nabla f^{\frac{p+1}{2}}|^2 \mathrm{d}\mu(t) = 0,$$

where we use the Stokes' theorem and that $\partial_t \mathrm{d}\mu(t) = -S_t \mathrm{d}\mu(t)$. Since $p \ge 1$, we have $4p \ge 2(p + 1)$. Therefore, we can deduce

$$\begin{aligned} \partial_t \int_M f^{p+1} \mathrm{d}\mu(t) + \int_M (S_t + C_0) f^{p+1} \mathrm{d}\mu(t) + 2 \int_M |\nabla f^{\frac{p+1}{2}}|^2 \mathrm{d}\mu(t) \\ \le C_0 \int_M f^{p+1} \mathrm{d}\mu(t), \end{aligned} \tag{6.8}$$

where

$$C_0 = \begin{cases} 0, & \min_x S_0 \ge 0, \\ \dfrac{4}{C_S(M, g_0)^2 \mathrm{Vol}_{g_0}(M)^{\frac{2}{n}}} - \min S_0, & \min_x S_0 < 0. \end{cases}$$

Define

$$\eta(t) = \begin{cases} 0, & 0 \le t \le \tau T, \\ \dfrac{t - \tau T}{(\theta - \tau) T}, & \tau T \le t \le \theta T, \\ 1, & \theta T \le t \le T. \end{cases}$$

Multiplying (6.8) by $\eta(t)$, we can deduce

$$\partial_t \left(\eta(t) \int_M f^{p+1} \mathrm{d}\mu(t) \right) + \frac{1}{2}\eta(t) \left(\int_M (S_t + C_0) f^{p+1} \mathrm{d}\mu(t) + 4 \int_M |\nabla f^{\frac{p+1}{2}}|^2 \mathrm{d}\mu(t) \right)$$
$$\leq (C_0 + \eta'(t)) \int_M f^{p+1} \mathrm{d}\mu(t).$$

Integrating this with respect to t gives

$$\sup_{\theta T \leq t \leq T} \int_M f^{p+1} \mathrm{d}\mu(t) + 2 \left\{ \int_{\theta T}^{T} \int_M \left(|\nabla f^{\frac{p+1}{2}}|^2 + \frac{S_t + C_0}{4} f^{p+1} \right) \mathrm{d}\mu(t) \mathrm{d}t \right\}$$
$$\leq 2 \left(\frac{1}{(\theta - \tau)T} + C_0 \right) \int_{\tau T}^{T} \int_M f^{p+1} \mathrm{d}\mu(t) \mathrm{d}t.$$

From Lemma 3.2, we know that $S_t + C_0 \geq 0$. From the proof of Theorem 1.3, we can have the Sobolev inequality

$$\left(\int_M u^{\frac{2n}{n-2}} \mathrm{d}\mu(t) \right)^{\frac{n-2}{n}} \leq A \int_M \left[|\nabla u|_t^2 + \frac{S_t + C_0}{4} u^2 \right] \mathrm{d}\mu(t), \tag{6.9}$$

where

$$A = \begin{cases} \left(2^{\frac{2n}{n-2}} - 1\right)^{\frac{n-2}{n}} 2^{\frac{4n}{n-2}} \left[C_S(M, g_0)\right]^4 e^{2 + \frac{4}{n}\sigma_0(g_0)}, & \inf\limits_{x \in M} S_0 \geq 0, \\ \dfrac{1}{n^2} \left(2^{\frac{2n}{n-2}} - 1\right)^{\frac{n-2}{n}} 2^{\frac{4n}{n-2}} e^{2 + \frac{4(A_1 t + A_2)}{n}}, & \inf\limits_{x \in M} S_0 < 0. \end{cases} \tag{6.10}$$

By making use of the Sobolev inequality above, we can get

$$\int_{\theta T}^{T} \int_M f^{(p+1)(1+\frac{2}{n})} \mathrm{d}\mu(t) \mathrm{d}t$$
$$\leq \int_{\theta T}^{T} \left(\int_M f^{p+1} d\mu(t) \right)^{\frac{2}{n}} \left(\int_M f^{(p+1)\frac{n}{n-2}} \mathrm{d}\mu(t) \right)^{\frac{n-2}{n}} \mathrm{d}t$$
$$\leq \sup_{\theta T \leq t \leq T} \left(\int_M f^{p+1} \mathrm{d}\mu(t) \right)^{\frac{2}{n}} A \int_{\theta T}^{T} \left(\int_M \left[(S_t + C_0) f^{p+1} d\mu(t) + 4 |\nabla f^{\frac{p+1}{2}}|^2 \right] \mathrm{d}\mu(t) \right) \mathrm{d}t$$
$$\leq 4A \left[C_0 + \frac{1}{(\theta - \tau)T} \right]^{1 + \frac{2}{n}} \left(\int_{\tau T}^{T} \int_M f^{p+1} \mathrm{d}\mu(t) \mathrm{d}t \right)^{1 + \frac{2}{n}}.$$

For $p \geq 2$, $0 < \tau < 1$, set

$$H(p,\tau) := \left(\int_{\tau T}^{T} \int_{M} f^p \mathrm{d}\mu(t) \mathrm{d}t \right)^{\frac{1}{p}}, \quad \chi = \frac{n+2}{n}.$$

Then for $0 < \tau < \theta < 1$, we have

$$H(p\chi, \theta) \leq (4A)^{\frac{1}{p\chi}} \left(C_0 + \frac{1}{(\theta-\tau)T} \right)^{\frac{1}{p}} H(p, \tau). \tag{6.11}$$

For $p_0 \geq 2$ fixed, defining

$$\gamma_i = p_0 \chi^{i-1}, \quad \theta_i = \theta - \frac{\theta - \tau}{2^{i-1}},$$

from (6.11), we have

$$H(\gamma_{k+1}, \theta_{k+1}) \leq (4A)^{\frac{1}{p_0 \chi^k}} \left(C_0 + \frac{2^k}{(\theta-\tau)T} \right)^{\frac{1}{p_0 \chi^{k-1}}} H(\gamma_k, \theta_k).$$

By iteration, we can deduce

$$\begin{aligned} H(\gamma_{k+1}, \theta_{k+1}) \leq {} & (4A)^{\frac{1}{p_0} \sum_{\ell=1}^{k} \frac{1}{\chi^\ell}} 2^{\frac{1}{p_0} \sum_{\ell=1}^{k} \frac{\ell}{\chi^{\ell-1}}} \\ & \left(C_0 + \frac{1}{(\theta-\tau)T} \right)^{\frac{1}{p_0} \sum_{\ell=1}^{k} \frac{1}{\chi^{\ell-1}}} H(\gamma_1, \theta_1). \end{aligned}$$

Letting $k \longrightarrow +\infty$, we have

$$\begin{aligned} \sup_{(x,t) \in M \times [\theta T, T]} |f(x,t)| \leq {} & (4A)^{\frac{n+2}{2p_0}} 2^{\frac{(n+2)^2}{4p_0}} \left(C_0 + \frac{1}{(\theta-\tau)T} \right)^{\frac{n+2}{2p_0}} \\ & \left(\int_{\tau T}^{T} \int_{M} f^{p_0} \mathrm{d}\mu(t) \mathrm{d}t \right)^{\frac{1}{p_0}}. \end{aligned}$$

For $0 < p < 2$, we set

$$h(\tau) = \sup_{(x,t) \in M \times [\tau T, T]} |f(x,t)|.$$

Then from the Young's inequality, we can get

$$h(\theta) \leq \frac{1}{2} h(\tau) + \frac{p}{2} \left[2^{\frac{(n+2)^2}{8}} (2-p) \right]^{\frac{2-p}{p}} (4A)^{\frac{n+2}{2p}} \left(C_0 + \frac{1}{(\theta-\tau)T} \right)^{\frac{n+2}{2p}} \left(\int_{\tau T}^{T} \int_{M} f^p \mathrm{d}\mu(t) \mathrm{d}t \right)^{\frac{1}{p}}$$

Then from Lemma 4.3 in [18], we get

$$h(\theta) \leq C A^{\frac{n+2}{2p}} \left(C_0 + \frac{1}{(\theta-\tau)T} \right)^{\frac{n+2}{2p}} \left(\int_{\tau T}^{T} \int_{M} f^p \mathrm{d}\mu(t) \mathrm{d}t \right)^{\frac{1}{p}}, \tag{6.12}$$

where C is a constant depending only on n and p.

Taking $p = 1$ in (6.12), from (6.2), we can get the estimates in the form of (6.4) and (6.5).

Lemma 6.4. *Assume that $g(x, t)$ is a smooth solution to the geometric flow (1.4) in $M \times [0, T)$ and that $\mathcal{D}_2(\mathcal{S}, \cdot)$ defined in (1.5) is nonnegative. For $0 \le \ell < t < T$ and any point x, we have*

$$G(x, \ell; x, t) \ge \frac{1}{(4\pi(t-\ell))^{\frac{n}{2}}} e^{-\frac{1}{2\sqrt{t-\ell}} \int_\ell^t \sqrt{t-s} S(x,s) \mathrm{d}s}. \tag{6.13}$$

Proof. For fixed (x, t), consider $G(z, \ell; x, t)$ as a function of (z, ℓ), $0 \le \ell < t$. Define $h(z, \ell)$ by

$$G(z, \ell; x, t) = \frac{e^{-h(z,\ell)}}{(4\pi(t-\ell))^{\frac{n}{2}}}.$$

Then we have

$$\partial_\ell h(z, \ell) + \Delta_{g(z,\ell)} h(z, \ell) - |\nabla h|^2_{g(z,\ell)} + S(z, \ell) - \frac{n}{2(t-\ell)} = 0. \tag{6.14}$$

If $\mathcal{D}_2(\mathcal{S}, \cdot)$ is nonnegative, Cao, Guo and Tran [6] proved

$$(t-\ell)\Big(2\Delta_{g(z,\ell)} h(z, \ell) - |\nabla h|^2_{g(z,\ell)} + S(z, \ell)\Big) + h(z, \ell) - n \le 0. \tag{6.15}$$

From (6.14) and (6.15), we have

$$-\partial_\ell h(z, \ell) \le \frac{1}{2} S(z, \ell) - \frac{1}{2} |\nabla h|^2_{g(z,\ell)} - \frac{h(z, \ell)}{2(t-\ell)}.$$

Thus, for any smooth curve $\gamma(\ell)$, we have

$$-\frac{\mathrm{d}}{\mathrm{d}\ell} h(\gamma(\ell), \ell) \le \frac{1}{2}\Big(S(\gamma(\ell), \ell) + |\dot\gamma(\ell)|^2_{g(\gamma(\ell),\ell)}\Big) - \frac{h(\gamma(\ell), \ell)}{2(t-\ell)}. \tag{6.16}$$

Taking $\gamma(\ell) \equiv x$, integrating from $\ell = t_2$ to $\ell = t_1$, we have

$$h(x, t_2)\sqrt{t-t_2} \le h(x, t_1)\sqrt{t-t_1} + \frac{1}{2}\int_{t_2}^{t_1} S(x, \ell)\sqrt{t-\ell}\,\mathrm{d}\ell,$$

where $0 \le t_2 < t_1 \le t$.

From Theorem 24.21 in [10], we know that $\lim_{t_1 \nearrow t} (t-t_1)^{\frac{n}{2}} G(x, t_1; x, t)$ is bounded. Thus, for any $0 \le \ell < t$, we have

$$h(x, \ell) \le \frac{1}{2\sqrt{t-\ell}} \int_\ell^t \sqrt{t-s} S(x, s)\mathrm{d}s.$$

Therefore, we can deduce (6.13). □

Lemma 6.5. *Assume that $g(x, t)$ is a smooth solution to the geometric flow (1.4) in $M \times [0, T)$ and that $\mathcal{D}_2(\mathcal{S}, \cdot)$ defined in (1.5) and $Ric - \mathcal{S}$ are nonnegative. Let $u(x, t)$ be the positive solution of*

$$\frac{\partial}{\partial t} u(x, t) = \Delta_{g(x,t)} u(x, t).$$

Then, for $\delta > 0$ and any x, $y \in M$, we have

$$u(x,t) \le U^{\frac{\delta}{1+\delta}} [u(y,t)]^{\frac{1}{1+\delta}} e^{\frac{\mathrm{dist}^2(x,y,t)}{4(t-s_0)\delta}}, \tag{6.17}$$

where $U = \sup_{(x,s)\in M\times[s_0,t]} u(x,s)$.

Proof. By Theorem 2.2 in [12], we know that for any $0 \le s_0 < t$ and $s \in [s_0, t]$

$$\frac{|\nabla u(x,s)|}{u(x,s)} \le \sqrt{\frac{1}{s-s_0}} \sqrt{\ln \frac{U}{u(x,s)}}. \tag{6.18}$$

Set $\psi(x,s) = \ln \frac{U}{u(x,s)}$, then inequality (6.18) yields

$$\left|\nabla \sqrt{\psi(x,s)}\right| = \frac{1}{2}\left|\frac{\nabla u}{u\sqrt{\psi}}\right| \le \frac{1}{\sqrt{4(s-s_0)}}.$$

Next, for any $x, y \in M$, let $\gamma : [0,1] \to M$ be a minimizing geodesic such that $\gamma(0) = x$ and $\gamma(1) = y$. Integrating the above inequality along the geodesic, we get

$$\sqrt{\ln \frac{U}{u(y,s)}} \le \sqrt{\ln \frac{U}{u(x,s)}} + \frac{\mathrm{dist}(x,y,s)}{\sqrt{4(s-s_0)}}.$$

Thus, for any $\delta > 0$, we have

$$\begin{aligned} \ln \frac{U}{u(y,s)} &\le \ln \frac{U}{u(x,s)} + \frac{\mathrm{dist}^2(x,y,s)}{4(t-s_0)} + \sqrt{\ln \frac{U}{u(x,s)}} \frac{\mathrm{dist}(x,y,s)}{\sqrt{s-s_0}} \\ &\le \ln \frac{U}{u(x,s)} + \frac{\mathrm{dist}^2(x,y,s)}{4(s-s_0)} + \delta \ln \frac{U}{u(x,s)} + \frac{\mathrm{dist}^2(x,y,s)}{4(s-s_0)\delta}. \end{aligned}$$

Taking exponential of both sides in the above inequality and taking $s = t$, we gives (6.17). □

Lemma 6.6. *Assume that $g(x,t)$ is a smooth solution to the geometric flow (1.4) in $M \times [0,T)$ and that $\mathcal{D}_2(\mathcal{S},\cdot)$ defined in (1.5) and $\mathrm{Ric} - \mathcal{S}$ are nonnegative. We have*

$$G(z,\ell;y,t) \ge \frac{c_1 J(t)}{(t-\ell)^{\frac{n}{2}}} e^{-\frac{\mathrm{dist}^2(z,y,t)}{t-\ell}} e^{-\frac{1}{\sqrt{t-\ell}}\int_\ell^t \sqrt{t-s}S(z,s)\mathrm{d}s}, \tag{6.19}$$

where c_1 depends only on n.

Proof. Set

$$u(x,t) = G(z,\ell;x,t), \quad s_0 = \frac{\ell+t}{2}, \quad K = \sup_{M\times[\frac{t+\ell}{2},t]} G(z,\ell;\cdot,\cdot).$$

From Lemma 6.5, we have

$$G(z,\ell;z,t) \le K^{\frac{\delta}{1+\delta}} [G(z,\ell;y,t)]^{\frac{1}{1+\delta}} e^{\frac{\mathrm{dist}^2(z,y,t)}{2(t-\ell)\delta}}. \tag{6.20}$$

Using (6.4), we know that

$$K \leq \frac{\exp[L(t) - (t-\ell) \inf\limits_{y \in M} S^-(y,0)]}{(4(t-\ell))^{\frac{n}{2}}}.$$

Denote $\exp[-L(t)+(t-\ell)\inf\limits_{y\in M} S^-(y,0)] = J(t)$. Then taking $\delta = 1$ in (6.20), from (6.13), we have (6.19). □

Proof of Theorem 6.1. Picking any $r \in (0, \sqrt{t_0})$, we consider geometric flow (1.4) in the space time cube

$$Q(x_0,t_0,r) = \left\{(x,s) | \mathrm{dist}(x,x_0,t_0) < r, \quad s \in [t_0 - r^2,\, t_0]\right\}.$$

For $r \in (0, \sqrt{t_0})$ and $x \in M$ with $\mathrm{dist}(x_0,x,t_0) \leq r$, from (6.19), we have

$$\begin{aligned} G(x_0, t_0 - r^2; x, t_0) &\geq \frac{c_1 J(t_0)}{r^n} e^{-1} e^{-\frac{1}{r}\int_{t_0-r^2}^{t_0} \sqrt{t_0-s}\, S(x_0,s)\mathrm{d}s} \\ &\geq \frac{c_1 J(t_0)}{r^n} e^{-1} e^{-\frac{1}{r}\int_{t_0-r^2}^{t_0} \sqrt{t_0-s}\frac{\alpha}{t_0-s}\mathrm{d}s} \\ &= \frac{c_1 J(t_0)}{r^n} e^{-1-2\alpha}. \end{aligned} \tag{6.21}$$

From (6.2) and (6.21), we deduce

$$\begin{aligned} 1 + C(1+r^2)^{\frac{n}{2}} &\geq \int\limits_M G(x_0, t_0 - r^2; x, t_0)\mathrm{d}\mu(x,t_0) \\ &\geq \int\limits_{\mathrm{dist}(x_0,x,t_0)\leq r} G(x_0, t_0 - r^2; x, t_0)\mathrm{d}\mu(x,t_0) \\ &\geq \frac{c_1 J(t_0)}{r^n} e^{-1-2\alpha} \int\limits_{\mathrm{dist}(x_0,x,t_0)\leq r} \mathrm{d}\mu(x,t_0). \end{aligned}$$

This implies

$$\mathrm{Vol}_{g(t_0)}(B(x_0,r,t_0))r^{-n} \leq \frac{\left[1 + C(1+t_0)^{\frac{n}{2}}\right] e^{1+2\alpha}}{cJ(t_0)}.$$

Taking

$$\kappa = \frac{\left[1 + C(1+t_0)^{\frac{n}{2}}\right] e^{1+2\alpha}}{cJ(t_0)}, \tag{6.22}$$

we obtain

$$\mathrm{Vol}_{g(t_0)}(B(x_0,r,t_0)) \leq \kappa r^n. \qquad \square$$

7. Applications

In this section, we will give some examples of the geometric flow (1.4). First, we will consider the Lorentzian mean curvature flow (see [22,29] and references therein).

Let M^n be a closed n-dimensional spacelike hypersurface in an ambient Lorentzian manifold L^{n+1} and let $F_0 : M^n \longrightarrow L^{n+1}$ be a smooth immersion of M^n into L^{n+1}. Consider a smooth one parameter family of immersions

$$F(\cdot, t) : M^n \longrightarrow L^{n+1}$$

satisfying $F(\cdot, 0) = F_0(\cdot)$ and

$$\frac{\partial F(p, t)}{\partial t} = H(p, t)\nu(p, t), \quad \forall\, (p, t) \in M \times [0, T),$$

where $H(p, t)$ and $\nu(p, t)$ denote the mean curvature and the future-oriented timelike normal vector for the hypersurface $M_t = F(M^n, t)$ at $F(p, t)$, respectively. It is easy to see that the induced metric solves the equation

$$\frac{\partial}{\partial t} g_{ij} = 2HA_{ij}, \tag{7.1}$$

where $A = (A_{ij})$ is the second fundamental form on M_t.

Theorem 7.1. *Let L^{n+1} be the ambient Lorentzian manifold with nonnegative sectional curvature. Then for evolution (7.1), Theorem 1.1, Theorem 1.2, Theorem 1.3, Lemma 4.2, Corollary 4.4, Theorem 5.1 and Theorem 6.1 hold.*

Proof. In this setting, we have $\mathcal{S}_{ij} = -HA_{ij}$ and $S = -H^2$. Marking the curvature with respect to the ambient Lorentzian manifold L^{n+1} with a bar, we have the Gauss equation

$$R_{ij} = \overline{R}_{ij} - HA_{ij} + A_{i\ell}A_{\ell j} + \overline{R}_{i0j0},$$

the Codazzi equation

$$\nabla_i A_{jk} - \nabla_k A_{ij} = \overline{R}_{0jki},$$

and the evolution equation for the mean curvature

$$\frac{\partial H}{\partial t} = \Delta H - H(|A|^2 + \overline{Ric}(\nu, \nu)),$$

where ν denotes the future-oriented timelike normal vector, represented by 0 in the index-notation. Using the three identities above, we get

$$\mathcal{D}_2(\mathcal{S}, X) = 2|\nabla H - A(X, \cdot)|^2 + 2\overline{Ric}(H\nu - X, H\nu - X) + 2\langle \overline{Rm}(X, \nu)\nu, X\rangle.$$

Since the ambient Lorentzian manifold L^{n+1} has nonnegative sectional curvature, the nonnegativity constraints of $\mathcal{D}_2(\mathcal{S}, X)$ holds naturally.

We also have

$$Ric(X, X) - \mathcal{S}(X, X) = \overline{Ric}(X, X) + X^i A_{i\ell}A_{\ell j}X^j + \langle \overline{Rm}(X, \nu)\nu, X\rangle \geq 0.$$

This completes the proof of Theorem 7.1. □

Second, let M be a real $n(= 2m)$ dimensional Fano manifold with Kähler form ω_0 associated to the Kähler metric g_0. We consider the twisted Kähler–Ricci flow (see [11,27,45] and the references therein)

$$\begin{cases} \dfrac{\partial}{\partial t} g_{i\bar{j}}(x,t) = -R_{i\bar{j}}(x,t) + \theta_{i\bar{j}}(x) + g_{i\bar{j}}(x,t), \\ g_{i\bar{j}}(x,0) = (g_0)_{i\bar{j}}(x), \end{cases} \tag{7.2}$$

where θ is a closed semi-positive $(1,1)$ form and

$$[2\pi c_1(M)] = [\omega(x,t) + \theta].$$

Here $\omega(x,t) = \sqrt{-1} g_{i\bar{j}}(x,t) \mathrm{d}z^i \wedge \mathrm{d}\overline{z^j}$ is the Kähler form of $g(x,t)$. We have

Theorem 7.2. *Let M be a real $n(=2m)$ dimensional Fano manifold with Kähler form ω_0 whose Kähler metric is denoted by g_0. Then for the twisted Kähler–Ricci flow (7.2) with the assumption above, there exists a positive constant $\kappa > 0$ depending only on the initial metric g_0 such that*

$$\mathrm{Vol}_{g(t)}\Big(B(x,r)\Big) \le \kappa r^n, \quad \forall\, (x,t) \in M \times (0,+\infty).$$

Remark 7.1. In the case of Kähler–Ricci flow ($\theta_{i\bar{j}} \equiv 0$), the conclusion in Theorem 7.2 is the one in Zhang [44] (see also [9]).

Remark 7.2. From the scaling transformation (7.3), it is not difficult to know that Theorem 1.1, Theorem 1.2, Theorem 1.3, Lemma 4.2, Corollary 4.4, Theorem 5.1 and Theorem 6.1 also hold for twisted Kähler–Ricci flow (7.2).

To avoid confusions, we give some preliminaries about Kähler geometry for special use in this paper. Let (M, ∇, g) be real n-dimensional ($n = 2m$) Kähler manifold, ∇ be the Levi-Civita connection (also Chern connection) and g be Riemannian metric which determines a unique Kähler metric and vise versa. So we can consider g itself as the Kähler metric. Assume that

$$z = (z^1, \cdots, z^m)$$

is the local coordinate system on M. The Käher form is

$$\omega = \sqrt{-1} \sum_{i,j=1}^m g_{i\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\overline{z^j},$$

where $g_{i\bar{j}} = g(\partial_{z^i}, \partial_{\overline{z^j}})$.

Let θ be a real $(1,1)$-form. Then we have

$$\overline{\theta_{i\bar{j}}} = \theta_{j\bar{i}},\ \mathrm{Tr}_g\theta = 2\sum_{i,j=1}^m g^{\bar{j}i}\theta_{i\bar{j}},\ |\theta|_g^2 = 2\sum_{i,j=1}^m g^{\bar{j}i} g^{\bar{q}p} \theta_{i\bar{q}} \theta_{p\bar{j}},$$

where $\sum_{j=1}^m g^{\bar{j}i} g_{k\bar{j}} = \delta_k^i$.

If θ is also closed, then we have

$$\frac{\theta_{i\bar{j}}}{\partial z^k} = \frac{\theta_{k\bar{j}}}{\partial z^i}, \quad \frac{\theta_{i\bar{j}}}{\partial \overline{z^\ell}} = \frac{\theta_{i\bar{\ell}}}{\partial \overline{z^j}},$$

which is equivalent to

$$\nabla_k \theta_{i\bar{j}} = \nabla_i \theta_{k\bar{j}}, \quad \nabla_{\bar{\ell}} \theta_{i\bar{j}} = \nabla_{\bar{j}} \theta_{i\bar{\ell}}.$$

For any $f \in C^\infty(M, \mathbb{R})$, we have

$$\Delta_g f = 2 \sum_{i,j=1}^{m} g^{\bar{j}i} \frac{\partial^2 f}{\partial z^i \partial \overline{z^j}}.$$

Proof of Theorem 7.2. For twisted Kähler–Ricci flow (7.2), define

$$\mathcal{S}_{i\bar{j}}(x,t) = R_{i\bar{j}}(x,t) - \theta_{i\bar{j}}(x).$$

By making use of scaling

$$t = -\ln(1-2s), \quad g_{i\bar{j}}(x,t) = \frac{1}{1-2s}\tilde{g}_{i\bar{j}}(x,s), \quad s \in [0, \frac{1}{2}), \tag{7.3}$$

we know that $\tilde{g}_{i\bar{j}}(x,s)$ satisfies the geometric flow equation

$$\frac{\partial}{\partial s}\tilde{g}_{i\bar{j}}(x,s) = -2\tilde{\mathcal{S}}_{i\bar{j}}(x,s),$$

where $\tilde{g}_{i\bar{j}}(x,0) = (g_0)_{i\bar{j}}(x)$ and

$$\tilde{\mathcal{S}}_{i\bar{j}}(x,s) = R_{i\bar{j}}(x, -\ln(1-2s)) - \theta_{i\bar{j}}(x).$$

Then we can get

$$\frac{\partial \tilde{\mathcal{S}}}{\partial s} - \Delta_{\tilde{g}}\tilde{S} - 2|\tilde{\mathcal{S}}|^2_{\tilde{g}} = 0, \tag{7.4}$$

where $\tilde{S} = \mathrm{Tr}_{\tilde{g}}\tilde{\mathcal{S}}$.

For any real-value vector $X \in \mathfrak{X}(M)$, it can be written as

$$X = \sum_{i=1}^{m} X^i \partial_{z^i} + \sum_{i=1}^{m} \overline{X^i \partial_{z^i}}.$$

Since θ is a real closed $(1,1)$-form, we have

$$\begin{aligned} 2\sum_{i,j=1}^{m} \tilde{\nabla}^i \theta_{i\bar{j}} \overline{X^j} &= 2\sum_{q,i,j=1}^{m} \tilde{g}^{\bar{q}i}\tilde{\nabla}_{\bar{q}}\theta_{i\bar{j}}\overline{X^j} \\ &= 2\sum_{q,i,j=1}^{m} \tilde{g}^{\bar{q}i}\tilde{\nabla}_{\bar{j}}\theta_{i\bar{q}}\overline{X^j} \\ &= \sum_{j=1}^{m} \left(\tilde{\nabla}_{\bar{j}}\mathrm{Tr}_{\tilde{g}}\theta\right)\overline{X^j}. \end{aligned} \tag{7.5}$$

From the second Bianchi identity, we also get

$$2\sum_{i,j=1}^{m} \tilde{\nabla}^i \tilde{R}_{i\bar{j}}\overline{X^j} = \sum_{j=1}^{m} \left(\tilde{\nabla}_{\bar{j}}\tilde{R}\right)\overline{X^j}. \tag{7.6}$$

Since θ is semi-positive, from (7.4), (7.5) and (7.6), we have

$$\mathcal{D}_2(\tilde{\mathcal{S}}, X) = 4 \sum_{i,j=1}^{m} \theta_{i\bar{j}} X^i \overline{X^j} \geq 0.$$

Collins and Székelyhidi [11] and Liu [27] proved that there exists a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^{m} g^{\bar{j}i} \left(R_{i\bar{j}}(x,t) - \theta_{i\bar{j}}(x) \right) \leq \alpha.$$

Therefore, we have

$$\tilde{S}(x,s) \leq \frac{\alpha}{\frac{1}{2} - s}, \quad s \in [0, \frac{1}{2}).$$

Choose $s_0 \in (0, \frac{1}{2})$ and $\tilde{r} \in [0, \sqrt{s_0})$. Then for $s \in [s_0 - \tilde{r}^2, s_0]$ and $x \in M$, we have

$$\tilde{S}(x,s) \leq \frac{\alpha}{s_0 - s}.$$

By Theorem 6.1, we have

$$\mathrm{Vol}_{\tilde{g}(s_0)}(B(x,\tilde{r})) \leq \kappa \tilde{r}^n. \tag{7.7}$$

From (7.3), we know that

$$\mathrm{dist}(x, y, \tilde{g}(s)) = \tilde{r}$$

implies

$$\mathrm{dist}(x, y, g(t)) = r$$

where

$$t = -\ln(1-2s), \quad r = \frac{\tilde{r}}{\sqrt{1-2s}}.$$

Therefore, from (7.7), we have

$$\mathrm{Vol}_{g(t_0)} \left[B\left(x, \frac{\tilde{r}}{\sqrt{1-2s_0}} \right) \right] \leq \kappa \left(\frac{\tilde{r}}{\sqrt{1-2s_0}} \right)^n,$$

that is, at any point $(x,t) \in M \times (0, +\infty)$, for the twisted Kähler–Ricci flow (7.2), we have

$$\mathrm{Vol}_{g(t)} \left[B\left(x, r\right) \right] \leq \kappa r^n, \tag{7.8}$$

where

$$r \in \left(0, \sqrt{\frac{e^t - 1}{2}} \right).$$

Since Collins and Székelyhidi [11] and Liu [27] proved that the diameter of $(M, g(t))$ is uniformly bounded, the above estimate (7.8) holds for all $r > 0$ with maybe a different constant κ. $\square$

Acknowledgments

This work was carried out while the authors were visiting Mathematics Department of Northwestern University. We would like to thank Professor Valentino Tosatti and Professor Ben Weinkove for hospitality and helpful discussions. The authors are also grateful to the anonymous referees and the editor for their careful reading and helpful suggestions which greatly improved the paper.

References

[1] T. Aubin, Problèmes isopérimétriques et espaces de Sobolev (French), J. Differential Geom. 11 (4) (1976) 573–598.
[2] T. Aubin, Y. Li, On the best Sobolev inequality, J. Math. Pures Appl. 78 (1999) 353–387.
[3] M. Băileşteanu, H. Tran, Heat kernel estimates under the Ricci-harmonic map flow, arXiv:1310.1619 [math.DG].
[4] D. Bakry, T. Coulhon, M. Ledoux, L. Saloff-Coste, Sobolev inequalities in disguise, Indiana Univ. Math. J. 44 (1995) 1033–1047.
[5] R.J. Biezuner, Best Constants, Optimal Sobolev Inequalities on Riemannian Manifolds and Applications, Rutgers, The State University of New Jersey, 2003.
[6] X. Cao, H. Guo, H. Tran, Harnack estimates for conjugate heat kernel on evolving manifolds, arXiv:1408.4155v1.
[7] G. Carron, Inégalités isopérimétriques de Faber–Krahn et conséquences, in: Actes de la Table Ronde de Géométrie Différentielle, Luminy, 1992, in: Sémin. Congr., vol. 1, Soc. Math. France, Paris, 1996, pp. 205–232.
[8] X. Chen, B. Wang, Space of Ricci flow (I), arXiv:0902.1545.
[9] X. Chen, B. Wang, On the conditions to extend Ricci flow (III), arXiv:1107.5110.
[10] B. Chow, S. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, L. Ni, The Ricci Flow: Techniques and Applications. Part III: Geometric-Analysis Aspects, Math. Surveys Monogr., vol. 163, American Mathematical Society, Providence, RI, 2010.
[11] T. Collins, G. Székelyhidi, The twisted Kähler–Ricci flow, arXiv:1207.5441v1.
[12] S. Fang, Differential Harnack estimates for heat equations with potentials under geometric flows, Arch. Math. 100 (2013) 179–189.
[13] S. Fang, T. Zheng, An upper bound of the heat kernel along the harmonic-Ricci flow, arXiv:1501.00639.
[14] S. Fang, P. Zhu, Differential Harnack estimates for backward heat equations with potentials under geometric flows, Commun. Pure Appl. Anal. 14 (2015) 793–809.
[15] S. Gallot, Inégalités isopérimetriques, courbure de Ricci et invariants géométriques. I, C. R. Math. Acad. Sci. Paris Sér. I 296 (1983) 333–336.
[16] L. Gross, Logarithmic Sobolev inequalities, Amer. J. Math. 97 (1975) 1061–1083.
[17] H. Guo, R. Philipowski, A. Thalmaier, Entropy and lowest eigenvalue on evolving manifolds, Pacific J. Math. 264 (2013) 61–81.
[18] Q. Han, F. Lin, Elliptic Partial Differential Equations, Courant Lect. Notes Math., vol. 1, Courant Institute of Mathematical Sciences, New York, 1997.
[19] E. Hebey, Optimal Sobolev inequalities on complete Riemannian manifolds with Ricci curvature bounded below and positive injectivity radius, Amer. J. Math. 118 (2) (1996) 291–300.
[20] E. Hebey, Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities, Courant Lect. Notes, 1999.
[21] E. Hebey, M. Vaugon, Meilleures constantes dans le thorme d'inclusion de Sobolev (French), Ann. Inst. H. Poincaré Anal. Non Linéaire 13 (1) (1996) 57–93.
[22] M. Holder, Contracting spacelike hypersurfaces by their inverse mean curvature, J. Aust. Math. Soc. Ser. A 68 (3) (2000) 285–300.
[23] S. Hsu, Uniform Sobolev inequalities for manifolds evolving by Ricci flow, arXiv:0708.0803v1.
[24] H. Huang, Optimal transportation and monotonic quantities on evolving manifolds, Pacific J. Math. 248 (2010) 305–316.
[25] W. Jiang, Bergman kernel along the Kähler–Ricci flow and Tian's conjecture, J. Reine Angew. Math. (2014), http://dx.doi.org/10.1515/crelle-2014-0015, Ahead of print.
[26] B. List, Evolution of an extended Ricci flow system, Comm. Anal. Geom. 16 (5) (2008) 1007–1048.
[27] J. Liu, The generalized Kähler Ricci flow, J. Math. Anal. Appl. 408 (2013) 751–761.
[28] X. Liu, K. Wang, A Gaussian upper bound of the conjugate heat equation along an extended Ricci flow, arXiv:1412.3200 [math.DG].
[29] R. Müller, Monotone volume formulas for geometric flows, J. Reine Angew. Math. 643 (2010) 39–57.
[30] R. Müller, Ricci flow coupled with harmonic map flow, Ann. Sci. Éc. Norm. Supér. (4) 45 (2012) 101–142.
[31] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math/0211159.
[32] G. Perelman, Ricci flow with surgery on three-manifolds, arXiv:math/0303109.
[33] L. Saloff-Coste, Uniformly elliptic operators on Riemannian manifolds, J. Differential Geom. 36 (1992) 417–450.
[34] N. Sesum, Convergence of a Kähler–Ricci flow, Math. Res. Lett. 12 (2005) 623–632.
[35] N. Sesum, G. Tian, Bounding scalar curvature and diameter along the Kähler–Ricci flow (after Perelman), J. Inst. Math. Jussieu 7 (2008) 575–587.
[36] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. 4 (1976) 353–372.
[37] P. Topping, Lectures on the Ricci Flow, vol. 325, Cambridge University Press, 2006.
[38] R. Ye, Sobolve inequalities, Riesz transforms and the Ricci flow, Commun. Math. Stat. 2 (2) (2014) 187–209.

[39] R. Ye, The logarithmic Sobolev inequality along the Ricci flow: the case $\lambda_0(g_0) = 0$, Commun. Math. Stat. 2 (3–4) (2014) 363–368.

[40] R. Ye, The logarithmic Sobolev inequality along the Ricci flow, arXiv:0707.2424v4.

[41] R. Ye, The logarithmic Sobolev inequality along the Ricci flow in dimension 2, arXiv:0708.2003.

[42] Q.S. Zhang, A uniform Sobolev inequality under Ricci flow, Int. Math. Res. Not. IMRN 17 (2007) 056, Erratum: Int. Math. Res. Not. IMRN 19 (2007) 096, Addendum: Int. Math. Res. Not. IMRN (2007) 138.

[43] Q.S. Zhang, Sobolev Inequalities, Heat Kernels under Ricci Flow, and the Poincaré Conjecture, CRC Press, Boca Raton, FL, 2011.

[44] Q.S. Zhang, Bounds on volume growth of geodesic balls under Ricci flow, Math. Res. Lett. 19 (1) (2012) 245–253.

[45] X. Zhang, X. Zhang, Generalized Kähler–Einstein metrics and energy functionals, Canad. J. Math. 66 (2014) 1413–1435.

[46] A. Zhu, Differential Harnack inequalities for the backward heat equation with potential under the harmonic-Ricci flow, J. Math. Anal. Appl. 406 (2013) 502–510.