J. Math. Anal. Appl. 435 (2016) 1552–1572

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Slater determinants of orthogonal polynomials ☆

Dimitar K. Dimitrov [a], Yuan Xu [b,*]

[a] Departamento de Matemática Aplicada, IBILCE, Universidade Estadual Paulista, 15054-000 Saõ José do Rio Preto, SP, Brazil

[b] Department of Mathematics, University of Oregon, Eugene, OR 97403-1222, United States

ARTICLE INFO

Article history:
Received 23 January 2015
Available online 18 November 2015
Submitted by M.J. Schlosser

Keywords:
Slater determinant
Orthogonal polynomials
Wronskian
Laplace transform

ABSTRACT

The symmetrized Slater determinants of orthogonal polynomials with respect to a non-negative Borel measure are shown to be represented by constant multiple of Hankel determinants of two other families of polynomials, and they can also be written in terms of Selberg type integrals. Applications include positive determinants of polynomials of several variables and Jensen polynomials and its derivatives for entire functions.

© 2015 Elsevier Inc. All rights reserved.

1. Introduction

Let $f = \{f_i\}$ be a sequence of functions. For any given pair of nonnegative integers n and m and for fixed $t_1, \ldots, t_m \in \mathbb{R}$, we consider the Slater determinant

$$S_{n,m}(f; t_1, \ldots, t_m) := \det \begin{bmatrix} f_n(t_1) & \ldots & f_{n+m-1}(t_1) \\ \vdots & \ldots & \vdots \\ f_n(t_m) & \ldots & f_{n+m}(t_m) \end{bmatrix}$$

and the symmetrized Slater determinant

$$W_{n,m}(f; t_1, \ldots, t_m) := \frac{S_{n,m}(f; t_1, \ldots, t_m)}{V(t_1, \ldots, t_m)}, \tag{1.1}$$

where $V(t_1, \ldots, t_m)$ is the Vandermond determinant of $t_1, \ldots, t_m$. When the variables coincide, the symmetrized Slater determinant becomes, up to a constant, the Wronskian determinant

☆ Research supported by the Brazilian foundations FAPESP under Grants 2009/13832-9 and 2014/08328-8, CNPq under Grant 307183/2013-0 and by the NSF under Grant DMS-1510296.

* Corresponding author.
E-mail addresses: dimitrov@ibilce.unesp.br (D.K. Dimitrov), yuan@uoregon.edu (Y. Xu).

http://dx.doi.org/10.1016/j.jmaa.2015.11.039
0022-247X/© 2015 Elsevier Inc. All rights reserved.

$$W(f_n,\dots,f_{n+m-1};x) := \det\begin{bmatrix} f_n(x) & f_{n+1}(x) & \dots & f_{n+m-1}(x) \\ f'_n(x) & f'_{n+1}(x) & \dots & f'_{n+m-1}(x) \\ \dots & \dots & \dots & \dots \\ f_n^{(m-1)}(x) & f_{n+1}^{(m-1)}(x) & \dots & f_{n+m-1}^{(m-1)}(x) \end{bmatrix}.$$

Slater determinants are wave functions of multi-particle fermion systems in Quantum Mechanics [2,14]. For most of the models that are well understood, the wave functions are related to the classical orthogonal polynomials. The main purpose of this paper is to study properties of Slater determinants for orthogonal polynomials.

Let $d\mu$ be a positive Borel measure on $\mathbb{R}$ for which orthogonal polynomials $p = \{p_n\}$ exist. We shall denote by $S_{n,m}(t_1,\dots,t_m)$ and $W_{n,m}(t_1,\dots,t_m)$ the Slater and symmetrized Slater determinant $S_{n,m}(p;t_1,\dots,t_m)$ and $W_{n,m}(p;t_1,\dots,t_m)$ throughout this paper. Let μ_n be the n-th moment of $d\mu$,

$$\mu_n := \int_{\mathbb{R}} t^n d\mu(t), \qquad n = 0,1,2,\dots,$$

and let M_n be the Hankel matrix of the moments defined by

$$M_n := [\mu_{i+j}]_{i,j=0}^n = \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \dots & \dots & \ddots & \dots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{bmatrix}.$$

It is known [15] that $\det M_n > 0$. The orthogonal polynomial p_n with respect to $d\mu$ can be defined by

$$p_n(x) = \det \left[\begin{array}{c|c} & \mu_n \\ & \mu_{n+1} \\ M_{n-1} & \vdots \\ & \mu_{2n-1} \\ \hline 1 x, \dots x^{n-1} & x^n \end{array}\right], \qquad n = 0,1,2,\dots, \tag{1.2}$$

which has the leading coefficient $\det M_{n-1}$.

Associated with the measure $d\mu$, we define two additional sequences of polynomials,

$$q_n(x) := \sum_{k=0}^n \mu_k \binom{n}{k} (-x)^{n-k}, \qquad n = 0,1,2,\dots. \tag{1.3}$$

$$r_{m,n}(x) := \sum_{k=0}^m \mu_{n+k} \binom{m}{k} (-x)^{m-k}, \qquad n,m = 0,1,2,\dots. \tag{1.4}$$

These polynomials may be viewed as shifted moments of $d\mu$ and moments of $(\cdot - x)^m d\mu$, respectively, because it is easy to see that

$$q_n(x) = \int_{\mathbb{R}} (t-x)^n d\mu(t), \quad \text{and} \quad r_{m,n}(x) = \int_{\mathbb{R}} t^n (t-x)^m d\mu(t). \tag{1.5}$$

We define their extension to several variables by

$$q_n(t_1,\dots,t_m;x) := \int_{\mathbb{R}} (t-x)^n (t-t_1)\cdots(t-t_m) d\mu(t), \tag{1.6}$$

$$r_n(t_1,\dots,t_m) := \int_{\mathbb{R}} t^n (t-t_1)\cdots(t-t_m) d\mu(t). \tag{1.7}$$

One of the main results in this paper shows that the symmetrized Slater determinant $W_{n,m}(t_1,\dots,t_m)$ of orthogonal polynomials can be represented by the Hankel determinant of either $q_n(t_1,\dots,t_{m-1};t_m)$ or $r_n(t_1,\dots,t_m)$, and it can also be represented as a Selberg type integral. The results may be considered as generalizations of results obtained in a long paper by Karlin and Szegő [10] and in a more recent paper of Leclerc [11]. The latter is our starting point. Indeed, when all variables coincide, our Theorem 2.1 below becomes

$$W(p_n,\dots,p_{n+m-1};x) = C_{n,m}\det\left[q_{m+i+j}(x)\right]_{i,j=0}^{n-1} = C_{n,m}\det\left[r_{m,i+j}(x)\right]_{i,j=0}^{n-1},$$

where $C_{n,m}$ is constant. The first identity is exactly Leclec's result, whereas the second one appears to be new. Another consequence of our results shows that the function

$$F(x_1,\dots,x_r) := \det\begin{bmatrix} p_n(x_1) & p_{n+1}(x_1) & \dots & p_{n+2r-1}(x_1) \\ p_n'(x_1) & p_{n+1}'(x_1) & \dots & p_{n+2r-1}'(x_1) \\ \vdots & \dots & \dots & \vdots \\ p_n(x_r) & p_{n+1}(x_r) & \dots & p_{n+2r-1}(x_r) \\ p_n'(x_r) & p_{n+1}'(x_r) & \dots & p_{n+2r-1}'(x_r) \end{bmatrix}$$

is nonnegative for all $n,r\in\mathbb{N}$ and $(x_1,\dots,x_r)\in\mathbb{R}^r$. For $r=1$ this is a consequence of the Christoffel–Darboux formula. For $r>1$ this appears to be new and it is a special case of an even more general result.

As an application, we show that the polynomials q_n and $r_{m,n}$ are closely related to the Jensen polynomials of entire functions in the Laguerre–Pólya class, and use our results to deduce new properties for the Jensen polynomials. We will also discuss an interplay between our principal results and the orthogonal polynomials that arise in the study of Toda lattices.

The paper is organized as follows: the main result for the Slater determinants of orthogonal polynomials is stated in the next section. The proof and further discussions are given in Section 3. The connection with Jensen polynomials and Toda lattices are discussed in Section 4. Examples on classical orthogonal polynomials are given in Section 5.

2. Main results on Slater determinants

To emphasize the dependence on the measure $d\mu$, we sometimes write $M_n(d\mu)=M_n$, $p_n(d\mu;x)=p_n(x)$ etc.

Theorem 2.1. *For every $n,m\in\mathbb{N}$, the symmetric Slater determinant $W_{n,m}(t_1,\dots,t_m)$ obeys the identities*

$$W_{n,m}(t_1,\dots,t_m) = B_{n,m}\det\left[q_{i+j+1}(t_1,\dots,t_{m-1};t_m)\right]_{i,j=0}^{n-1} \tag{2.1}$$

$$= B_{n,m}\det\left[r_{i+j}(t_1,\dots,t_{m-1},t_m)\right]_{i,j=0}^{n-1}, \tag{2.2}$$

where

$$B_{n,m} = (-1)^{nm}\prod_{k=1}^{m-1}\det M_{k+n-1}.$$

For $1\le k\le m$, let $\sigma_k(t_1,\dots,t_m)$ denote the elementary symmetric functions of $t_1,\dots,t_m$ defined by

$$\sigma_k(t_1,\dots,t_m) = \sum_{1\le j_1<j_2<\dots<j_k\le n} t_{j_1}\cdots t_{j_k}. \tag{2.3}$$

By the definition of $r_n(t_1,\dots,t_m)$,

$$r_n(t_1,\dots,t_m) = \sum_{k=0}^{m}(-1)^k \sigma_k(t_1,\dots,t_m)\, \mu_{n+m-k}(d\mu).$$

It follows form Theorem 2.1 that the symmetric Slater determinant can be written in a concise form in terms of the moments of $d\mu$. Indeed, with $t = (t_1,\dots,t_m)$,

$$W_{1,m}(t) = B_{1,m} r_0(t) = (-1)^m \prod_{k=1}^{m-1} \det M_k \sum_{k=0}^{m}(-1)^k \sigma_k(t)\mu_{m-k}(w),$$

which appears in [6], and the next case is

$$W_{2,m}(t) = B_{2,m}\left[r_2(t)r_0(t) - r_1(t)^2\right], \quad B_{2,m} = \prod_{k=1}^{m-1} \det M_{k+1}.$$

We regard $q_n(x)$ defined in (1.3) as the case $m = 0$ of $q_n(t_1,\dots,t_m;x)$ and, evidently, $r_{m,n}(x)$ defined in (1.4) is $r_n(t_1,\dots,t_m)$ with $t_1 = \ldots = t_m = x$. Setting $t_1 = \ldots = t_m = x$ in Theorem 2.1 gives the following corollary.

Corollary 2.2. *For all $m, n \in \mathbb{N}$,*

$$\begin{aligned} W(p_n,\dots,p_{n+m-1};x) &= C_{n,m} \det\left[q_{m+i+j}(x)\right]_{i,j=0}^{n-1} && (2.4)\\ &= C_{n,m} \det\left[r_{m,i+j}(x)\right]_{i,j=0}^{n-1}, && (2.5)\end{aligned}$$

where $C_{n,m}$ is a constant given by

$$C_{n,m} := (-1)^{nm} \prod_{k=1}^{m-1} k! \det M_{k+n-1}.$$

Identity (2.4) was proved by Leclerc [11]. Notice that it is easy to see that the determinant $W(p_n,\dots,p_{n+m-1};x)$ is a polynomial of degree mn, but this is not obvious for the determinant $[q_{m+i+j}(x)]_{i,j=0}^{n-1}$. It is clear, however, that $\det[r_{m,i+j}(x)]_{i,j=0}^{n-1}$ is a polynomial of degree mn, since each $r_{m,i+j}(x)$ is of degree m.

For a further generalization, we make the following definitions.

Let $m_1,\dots,m_r \in \mathbb{N}$ and $m := m_1 + \ldots + m_r$. We define the $m \times m$ determinant

$$S_n^{m_1,\dots,m_r}(t_1,\dots,t_r) := \det\begin{bmatrix} p_n(t_1) & p_{n+1}(t_1) & \ldots & p_{n+m-1}(t_1)\\ p_n'(t_1) & p_{n+1}'(t_1) & \ldots & p_{n+m-1}'(t_1)\\ \ldots & \ldots & \ldots & \ldots\\ p_n^{(m_1-1)}(t_1) & p_{n+1}^{(m_1-1)}(t_1) & \ldots & p_{n+m-1}^{(m_1-1)}(t_1)\\ \vdots & \vdots & \vdots & \vdots\\ p_n(t_r) & p_{n+1}(t_r) & \ldots & p_{n+m-1}(t_r)\\ p_n'(t_r) & p_{n+1}'(t_r) & \ldots & p_{n+m-1}'(t_r)\\ \ldots & \ldots & \ldots & \ldots\\ p_n^{(m_r-1)}(t_r) & p_{n+1}^{(m_r-1)}(t_r) & \ldots & p_{n+m-1}^{(m_r-1)}(t_r)\end{bmatrix}, \qquad (2.6)$$

its symmetrized verson

$$W_n^{m_1,\dots,m_r}(t_1,\dots,t_r) := \frac{S_n^{m_1,\dots,m_r}(t_1,\dots,t_r)}{\prod_{1\le i<j\le r}(t_j-t_i)^{m_i m_j}},$$

as well as the corresponding polynomials

$$q_n^{m_1,\dots,m_r}(t_1,\dots,t_r;x) := q_n\big(\overbrace{t_1,\dots,t_1}^{m_1},\dots,\overbrace{t_r,\dots,t_r}^{m_r};x\big), \tag{2.7}$$

$$r_n^{m_1,\dots,m_r}(t_1,\dots,t_r) := r_n\big(\overbrace{t_1,\dots,t_1}^{m_1},\dots,\overbrace{t_r,\dots,t_r}^{m_r}\big). \tag{2.8}$$

Theorem 2.3. *For every $n \in \mathbb{N}$, $m_1,\dots,m_r \in \mathbb{N}$ with $m := m_1+\ldots+m_r$,*

$$W_n^{m_1,\dots,m_r}(t_1,\dots,t_r) = C_n^{m_1,\dots,m_r}\det\left[q_{i+j+m_r}^{m_1,\dots,m_{r-1}}(t_1,\dots,t_{r-1};t_r)\right]_{i,j=0}^{n-1} \tag{2.9}$$

$$= C_n^{m_1,\dots,m_r}\det\left[r_{i+j}^{m_1,\dots,m_r}(t_1,\dots,t_r)\right]_{i,j=0}^{n-1}, \tag{2.10}$$

where

$$C_n^{m_1,\dots,m_r} := (-1)^{nm}\prod_{i=1}^{r}\prod_{j=1}^{m_i} j!\prod_{k=1}^{m-1}\det M_{k+n-1}.$$

The matrix in (2.10) is the moment matrix $M_{n-1}(w_{m_1,\dots,m_r}(t_1,\dots,t_r))$ for the measure $w_{m_1,\dots,m_r}(t_1,\dots,t_r;x) = (x-t_1)^{m_1}\cdots(x-t_r)^{m_r}d\mu(x)$. We also obtain an integral representation for the determinants in (2.6).

Theorem 2.4. *For $m_1,\dots,m_r \in \mathbb{N}$ and $n \in \mathbb{N}$,*

$$S_n^{m_1,\dots,m_r}(t_1,\dots,t_r) = C_n^{m_1,\dots,m_r}\prod_{1\le i<j\le r}(t_j-t_i)^{m_i m_j}$$
$$\times\frac{1}{n!}\int_{\mathbb{R}^n}\prod_{i=1}^{r}\prod_{j=1}^{n}(s_j-t_i)^{m_i}\prod_{1\le i,j\le n}(s_i-s_j)^2\prod_{j=1}^{n}d\mu(s_j).$$

The integral in the above is a special case of the Selberg integral when $d\mu = w(x)dx$ and w is a classical weight function. We refer to [8] for a beautiful account of the Selberg integrals. In the case of $m_1 = \ldots = m_r = 1$, this result appeared in [3] and is well known in the random matrix community.

An immediate consequence of our main result is the following remarkable corollary.

Corollary 2.5. *Let $n, m_1,\dots,m_r$ be positive integers. Then*

$$S_n^{2m_1,\dots,2m_r}(t_1,\dots,t_r) \ge 0 \qquad \textit{for every}\ \ (t_1,\dots,t_r)\in\mathbb{R}^r.$$

Furthermore, equality holds only if $t_i = t_j$ for some $i \ne j$.

In the case of $r = 1$, S_n^m is the Wronskian $W(p_n,\dots,p_{n+m-1})$ and it is nonnegative on the real line if m is even, as shown by Karlin and Szegő in [10]. Our explicit integral representation gives a direct proof of this classical result.

We end this section by mentioning another connection of the Slater determinant. For $\alpha = (\alpha_1,\dots,\alpha_m) \in \mathbb{N}_0^m$ with $0 \le \alpha_1 \le \ldots \le \alpha_m = n$, define

$$P_\alpha^n(u_1,\dots,u_m) = \frac{\det\left[p_{\alpha_{m-i+1}+i-1}(t_j)\right]_{i,j=1}^m}{V(t_1,\dots,t_m)},$$

where $u_i = \sigma_{m-i+1}(t_1,\dots,t_m)$, the elementary symmetric function of $t_1,\dots,t_m$ defined at (2.3), then P_α^n is a polynomial of degree n in $(u_1,\dots,u_m)$ and, moreover, the set $\{P_\alpha^n : 0 \le \alpha_1 \le \dots \le \alpha_m = n\}$ is a complete set of orthogonal polynomials in m variables [7, Section 5.4.1]. The Slater determinant corresponds to the case of $\alpha = (n,\dots,n)$.

3. Proofs of the main results and further results

We divide this section into subsections. The first subsection contains several lemmas on the polynomials p_n, q_n and r_n. Our main results on the determinants are proved in the second subsection. The last section contains other related results on Slater determinants.

3.1. Lemmas

We start with a fundamental tool in our work, a well-known identity that can be found in [13, p. 62].

Lemma 3.1. *Let f_i, g_j be functions such that $f_i g_j \in L^1(\mathbb{R})$ for $1 \le i,j \le n$. Then*

$$\det\left[\int_{\mathbb{R}} f_i(t)g_j(t)d\mu(t)\right]_{i,j=1}^n = \int_{\mathbb{R}^n} \det\left[f_j(t_i)\right]_{i,j=1}^n \det\left[g_j(t_i)\right]_{i,j=1}^n \prod_{i=1}^n d\mu(t_i). \tag{3.1}$$

Recall that the Vandermond determinant is given explicitly by

$$V(t_1,\dots,t_m) = \det\begin{bmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_m \\ \vdots & \dots & \dots & \vdots \\ t_1^{m-1} & t_2^{m-1} & \dots & t_m^{m-1} \end{bmatrix} = \prod_{1\le i<j\le m} (t_j - t_i).$$

One immediate consequence of the identity (3.1) is an integral expression of the orthogonal polynomial $p_n(d\mu)$.

Lemma 3.2. *For $n = 0,1,\dots,$*

$$p_n(d\mu;x) = (-1)^n \det\left[q_{i+j+1}(d\mu;x)\right]_{i,j=0}^{n-1}.$$

Proof. We use (3.1) with $f_j(t) = (x-t)^j$ and $g_j(t) = (x-t)^{j+1}$ to obtain, by (1.5), that

$$\begin{aligned} \det\left[q_{i+j+1}(x)\right]_{i,j=0}^{n-1} &= \frac{1}{n!}\int_{\mathbb{R}^n} \det\left[(t_i-x)^{j-1}\right]_{i,j=1}^n \det\left[(t_i-x)^j\right]_{i,j=1}^n \prod_{k=1}^n d\mu(t_k) \\ &= \frac{1}{n!}\int_{\mathbb{R}^n} (t_1-x)\cdots(t_n-x) \prod_{1\le i<j\le n} (t_i-t_j)^2 \prod_{k=1}^n d\mu(t_k) \\ &= (-1)^n p_n(x), \end{aligned} \tag{3.2}$$

where the last identity follows from a well-known expression for orthogonal polynomials; see, for example, [15, p. 27]. $\square$

The integral representation (3.2) is the special case $m = 1$ of (2.4), already established in [11]. We include the proof since it illustrates the strength of (3.1), which will be used several times in this section.

Lemma 3.3. *The polynomial* $q_n^{m_1,\ldots,m_r}(t_1,\ldots,t_r;\cdot)$ *satisfies*

(1) $q_n(t_1,\ldots,t_m;x) = q_{n+1}(t_1,\ldots,t_{m-1};x) + (x-t_m)q_n(t_1,\ldots,t_{m-1};x)$;
(2) $q_n(x,\ldots,x;x) = q_{n+m}(x)$.

Proof. The first item follows from (1.6) by writing $t - t_m = t - x + x - t_m$. The second item follows directly from (1.6). □

For given $t_1,\ldots,t_m$, let $d\mu_m$ be the measure defined by

$$d\mu_m(x) = d\mu_m(t_1,\ldots,t_m;x) := (t-t_1)\cdots(t-t_m)d\mu(x).$$

Lemma 3.4. *For* $m, n = 1, 2, \ldots$,

$$\det\left[q_{i+j+1}(t_1,\ldots,t_{m-1};t_m)\right]_{i,j=0}^{n-1} = \det\left[r_{i+j}(t_1,\ldots,t_m)\right]_{i,j=0}^{n-1}. \tag{3.3}$$

Proof. By the definition of w_m, we can write

$$q_{i+j+1}(t_1,\ldots,t_{m-1};t_m) = \int_{\mathbb{R}} (t-t_m)^{i+j} d\mu_m(t).$$

By (3.1) with $f_j(t) = g_j(t) = (t-t_m)^j$, we see that

$$\begin{aligned}\det\left[q_{i+j+1}(t_1,\ldots,t_{m-1};t_m)\right]_{i,j=0}^{n-1} &= \int_{\mathbb{R}^n} \left[V(s_1-t_m,\ldots,s_n-t_m)\right]^2 \prod_{i=1}^n d\mu_m(s_i)\\ &= \int_{\mathbb{R}^n} \left[V(s_1,\ldots,s_n)\right]^2 \prod_{i=1}^n d\mu_m(s_i),\end{aligned}$$

where we have used the closed form of the Vandermond determinant in the last step. Applying (3.1) with $f_j(t) = g_j(t) = t^j$, it follows that the last integral can be written as

$$\int_{\mathbb{R}^n} \left[V(s_1,\ldots,s_n)\right]^2 \prod_{i=1}^n d\mu_m(s_i) = \det\left[\int_{\mathbb{R}} s^{i+j} d\mu_m(s)\right]_{i,j=0}^{n-1}.$$

Directly from the definition, the integral on the right-hand side is $r_{i+j}(t_1,\ldots,t_m)$, which proves (3.3). □

We note that each r_{i+j} is a symmetric function of $t_1,\ldots,t_m$, so that the right-hand side of the identity (3.3) is a symmetric function, which is, however, not obvious from the left-hand side of (3.3) because $q_n(t_1,\ldots,t_{m-1};t_m)$ is not symmetric in $t_1,\ldots,t_m$.

Lemma 3.4 shows that we have established the identity (2.2) and, setting $t_1 = \ldots = t_m$, the identity (2.5). Thus, we only need to prove our main theorems in terms of q_n, that is, (2.1).

For $q_n^{m_1,\ldots,m_r}$, defined in (2.7), we can write its Hankel determinant as an integral.

Lemma 3.5. *For* $m_1, \dots, m_r \in \mathbb{N}$ *and* $n \in \mathbb{N}$,

$$\det \left[q_{i+j+1}^{m_1,\dots,m_r}(t_1, \dots, t_r; x)\right]_{i,j=0}^{n-1} = \frac{1}{n!} \int_{\mathbb{R}^n} \prod_{i=1}^{n}(s_i - x) \prod_{i=1}^{r}\prod_{j=1}^{n}(s_j - t_i)^{m_i} \prod_{1\le i,j\le n} (s_i - s_j)^2 \prod_{j=1}^{n} d\mu(s_j). \tag{3.4}$$

Proof. Let $m = m_1 + \dots + m_r$. If $m_1 = \dots = m_r = 1$, then $r = m$ and, by (3.2) and (3.7), it follows that

$$\det \left[q_{i+j+1}(u_1, \dots, u_m; x)\right]_{i,j=0}^{n-1} = (-1)^n p_n(d\nu_m; x) = \frac{1}{n!} \int_{\mathbb{R}^n} \prod_{i=1}^{n}(s_i - x) \prod_{1\le i,j\le n} (s_i - s_j)^2 \prod_{j=1}^{n}\prod_{i=1}^{m}(s_j - u_i) d\mu(s_j).$$

Setting $u_1 = \dots = u_{m_1} = t_1$, $u_{m_1+1} = \dots = m_{m_1+m_2} = t_2, \dots,$ in the above identity completes the proof. □

Our last lemma in this subsection is well known. We give a proof since the same procedure will be used later.

Lemma 3.6. *For* $n, m \in \mathbb{N}$,

$$W_{n,m}(x, \dots, x) = \prod_{j=1}^{m-1} j!\; W(p_n, \dots, p_{n+m-1}; x).$$

Proof. In the determinant $W_{n,m}(t_1, \dots, t_m)$, we set $t_j = t_1 + jh$ for $j = 1, \dots, m$ and rewrite the j-th row of the left hand side in terms of the forwarded difference $\triangle_h^{j-1} p_{n+i}(t_j)$. Since $\prod_{1\le i<j\le m}(t_j - t_i) = \prod_{j=1}^{m-1} j! h^{m(m-1)/2}$, taking the limit $h \to 0$ completes the proof. □

3.2. Slater determinants

We prove the following result from which Theorem 2.1 can be deduced.

Theorem 3.7. *For* $n \in \mathbb{N}$, $m_1, \dots, m_r \in \mathbb{N}$ *and* $m := m_1 + \dots + m_r$,

$$S_n^{m_1,\dots,m_r,1}(t_1, \dots, t_r, x) = B_n^{m_1,\dots,m_r} \prod_{i=1}^{r}(x - t_i)^{m_i} \prod_{1\le i<j\le r} (t_j - t_i)^{m_i m_j} \times \det \left[q_{i+j+1}^{m_1,\dots,m_r}(t_1, \dots, t_r; x)\right]_{i,j=0}^{n-1}, \tag{3.5}$$

where

$$B_n^{m_1,\dots,m_r} := (-1)^{n(m+1)} \prod_{i=1}^{r}\prod_{j=1}^{m_i} j! \prod_{k=1}^{m} \det M_{k+n-1}.$$

Proof. We first prove the case of $m_1 = \dots = m_r = 1$, for which $r = m$, by induction on m. That is, we prove

$$\det \begin{bmatrix} p_n(t_1) & p_{n+1}(t_1) & \ldots & p_{n+m}(t_1) \\ \vdots & \vdots & \vdots & \vdots \\ p_n(t_m) & p_{n+1}(t_m) & \ldots & p_{n+m}(t_m) \\ p_n(x) & p_{n+1}(x) & \ldots & p_{n+m}(x) \end{bmatrix} = B_{n,m} \prod_{i=1}^{m}(x-t_i) \prod_{1\le i<j\le m}(t_j-t_i)$$
$$\times \det\left[q_{i+j+1}(t_1,\ldots,t_m;x)\right]_{i,j=0}^{n-1}, \tag{3.6}$$

where $B_{n,m} = (-1)^{n(m+1)} \prod_{k=1}^{m} \det M_{k+n-1}$. For $m = 0$, (3.6) is the identity in the lemma. We now assume that (3.6) holds for a fixed $m-1$ and prove that it holds with $m-1$ replaced by m.

We can assume, without loss of generality, that $w(u)$ is supported on $[a,b]$ with a finite number a. Indeed, let $\chi_{[a,b]}$ denote the indicator function of the interval $[a,b]$; if w is supported on $(-\infty,b]$, then we can establish the identity for the truncated weight function $\chi_{[a,b]}(x)w(x)$, and then take the limit $a \to -\infty$. Since both p_k and q_k can be written as integrals against the weight function, as seen by (3.2) and (3.4), and the identity (3.6) contains finitely many such polynomials, the limit exists as $a \to -\infty$ exists.

Since (3.6) is a polynomial identity, we only need to establish it for $t_1,\ldots,t_m$ less than a. Then $d\mu_m(t) := (t-t_1)\ldots(t-t_m)d\mu(t)$ is a nonnegative weight function on $[a,b]$. It follows by the Lemma 3.2 that $p_n(d\nu_m)$ is given by

$$p_n(d\nu_m;x) = (-1)^n \det\left[q_{i+j+1}(t_1,\ldots,t_m;x)\right]_{i,j=0}^{n-1}. \tag{3.7}$$

By (3.3) the leading coefficient $\gamma_n(d\nu_m)$ of $p_n(d\nu_m;x)$ is given by

$$\gamma_n(d\nu_m) = \det M_{n-1}(d\nu_m) = \det\left[q_{i+j+1}(t_1,\ldots,t_{m-1};t_m)\right]_{i,j=0}^{n-1}. \tag{3.8}$$

Moreover, by the Christoffel formula [15, p. 30], $p_n(d\mu_m)$ can also be given by

$$p_n(d\nu_m;x) = \frac{A_{n,m}(t)}{\prod_{k=1}^{m}(x-t_j)} \det \begin{bmatrix} p_n(t_1) & p_{n+1}(t_1) & \cdots & p_{n+m}(t_1) \\ \vdots & \ldots & \ldots & \vdots \\ p_n(t_m) & p_{n+1}(t_m) & \cdots & p_{n+m}(t_m) \\ p_n(x) & p_{n+1}(x) & \cdots & p_{n+m}(x) \end{bmatrix}, \tag{3.9}$$

where $A_{n,m}(t) = A_{n,m}(t_1,\ldots,t_m)$ is independent of x. In particular, the leading coefficient of x^{n+m} in $\prod_{i=1}^{m}(x-t_i)p_n(d\nu_m;x)$, which is the same as $\gamma_n(d\nu_m)$, is given by

$$\gamma_n(d\nu_m) = A_{n,m}(t) \det M_{n+m-1} \det\left[p_{n+j-1}(t_i)\right]_{i,j=1}^{m},$$

where we have used (1.2), from which it follows, by the induction hypothesis, that

$$\gamma_n(d\nu_m) = A_{n,m}(t) \det M_{n+m-1} B_{n,m} \prod_{1\le i<j\le m}(t_j-t_i) \det\left[q_{i+j+1}(t_1,\ldots,t_{m-1};t_m)\right]_{i,j=0}^{n-1}.$$

Comparing the latter with (3.8) we obtain

$$\frac{1}{A_{n,m}(t)} = \det M_{n+m-1} B_{n,m} \prod_{1\le i<j\le m}(t_j-t_i).$$

Consequently, combining (3.7) and (3.9) proves (3.6) with $m-1$ replaced by m, where the constant $B_{n,m}$ satisfies the relation $B_{n,m+1} = (-1)^n \det M_{n+m-1} B_{n,m}$. This completes the induction and the proof of (3.6).

Now we apply the limit procedure in Lemma 3.6 on the identity (3.6). Setting $t_j = t_1 + jh$ for $j = 1,\ldots,m_1$ in the identity and using

$$\prod_{1\le i<j\le m}(t_i-t_j)=\prod_{1\le i<j\le m_1}(t_i-t_j)\prod_{i=1}^{m_1}\prod_{j=m_1+1}^{m}(t_i-t_j)\prod_{m_1+1\le i<j\le m}(t_i-t_j),$$

we take the limit $h\to 0$ to conclude that

$$W_n^{m_1,1\dots,1,1}(t_1,t_{m_r+1},\dots,t_m,x)=B^{m_1,1,\dots,1}\prod_{j=m_1+1}^{m}(t_1-t_j)^{m_1}\prod_{m_1+1\le i<j\le m}(t_i-t_j)$$
$$\times (x-t_1)^{m_1}\prod_{i=m_1+1}^{m}(x-t_i)\det\left[q_{i+j+1}^{m_1,1,\dots,1}(t_1,t_{m_1+1}\dots,t_m;x)\right]_{i,j=0}^{n-1},$$

where the constant is given by

$$B^{m_1,1,\dots,1}=B^{1,1,\dots,1}\prod_{k=1}^{m_1-1}k!.$$

Repeating the above process by setting $t_{m_1+1}=\dots=t_{m_1+m_2}=t_2$, so that

$$\prod_{j=m_1+1}^{m}(t_1-t_j)^{m_1}=(t_1-t_2)^{m_1m_2}\prod_{j=m_1+m_2+1}^{m}(t_i-t_j)^{m_i},$$

it follows that (3.5) holds for $S_n^{m_1,m_2,1,\dots,1,1}$. Continuing this process completes the proof of (3.5). □

We note that Theorem 3.7 is more general than Theorem 2.1. Indeed, if $m_1=\dots=m_r=1$, then (3.5) becomes (3.6), which is (2.1) after replacing x by t_{m+1} and then replacing m by $m-1$. Together with Lemma (3.4), this completes the proof of Theorem 2.1.

Proof of Theorem 2.3. Taking m_r-th derivative of (3.5) with respect to x and then setting $x=t_r$, the left-hand side becomes $S_n^{m_1,\dots,m_{r-1},m_r+1}(t_1,\dots,t_r)$, whereas the constant in the right-hand side becomes $m!B_n^{m_1,\dots,m_r}$ and the main term becomes

$$\prod_{1\le i<j\le r-1}(t_j-t_i)^{m_im_j}\prod_{i=1}^{r-1}(t_r-t_i)^{m_i(m_r+1)}\det\left[q_{i+j+1}^{m_1,\dots,m_r}(t_1,\dots,t_r;t_r)\right]_{i,j=0}^{n-1}.$$

By the definition of q_n, it is easy to see that

$$q_{i+j+1}^{m_1,\dots,m_r}(t_1,\dots,t_r;t_r)=q_{i+j+m_r+1}^{m_1,\dots,m_{r-1}}(t_1,\dots,t_{r-1};t_r).$$

Replacing m_r by m_r-1 in the resulting identity proves (2.9). Then (2.10) follows from (3.3). □

When $r=1$ and $t_1=x$, the identity (2.9) becomes (2.4).

Proof of Theorem 2.4. Combining (3.5) with (3.4), we obtain that

$$S_n^{m_1,\dots,m_r,1}(t_1,\dots,t_r,x)=B_n^{m_1,\dots,m_r}\prod_{i=1}^{r}(x-t_i)^{m_i}\prod_{1\le i<j\le r}(t_j-t_i)^{m_im_j}$$
$$\times\frac{1}{n!}\int_{\mathbb{R}^n}\prod_{i=1}^{n}(s_i-x)\prod_{i=1}^{r}\prod_{j=1}^{n}(s_j-t_i)^{m_i}\prod_{1\le i,j\le n}(s_i-s_j)^2\prod_{j=1}^{n}d\mu(s_j).$$

The integral representation of $S_n^{m_1,\ldots,m_r}(t_1,\ldots,t_r)$ is deduced from comparing the leading coefficient of x^{n+m} in the above identity. $\square$

We note that Corollary 2.5 follows immediately from Theorem 2.4 and it also follows from (2.10), since the right hand side of (2.10) is the determinant of the moment matrix $M_{n-1}(d\mu_{m_1,\ldots,m_r})$ of the weight function $d\mu_{m_1,\ldots,m_r}(t) = (t-t_1)^{m_1}\cdots(t-t_r)^{m_r}d\mu(t)$, which is nonnegative if $m_1,\ldots,m_r$ are even positive integers.

3.3. Further results on determinants

For positive integers $\ell_1,\ell_2,\ldots,\ell_n$, we define

$$F[q_{\ell_1},\ldots,q_{\ell_n}](t_1,\ldots,t_m;x) := \det\left[q_{\ell_i+j-1}(t_1,\ldots,t_m;x)\right]_{i,j=1}^n .$$

With this notation, the identity (3.6) becomes

$$\det\left[p_{n+j-1}(t_i)\right]_{i,j=1}^m = B_{n,m-1}\prod_{1\le i<j\le m}(t_j-t_i)F[q_1,\ldots,q_n](t_1,\ldots,t_{m-1};t_m).$$

Furthermore, for $1\le j\le n+1$, we define

$$F[q_1,\ldots,\widehat{q}_j,\ldots,q_{n+1}] := F[q_1,\ldots,q_{j-1},q_{j+1},\ldots,q_{n+1}].$$

Lemma 3.8. *For $m,n\in\mathbb{N}$,*

$$F[q_m,q_{m+1},\ldots,q_{m+n-1}](t;x) = \sum_{k=0}^n (x-t)^k F[q_m,\ldots,\widehat{q}_{m+k},\ldots,q_{m+n}](x).$$

Proof. By Lemma 3.3, $q_k(t;x) = q_{k+1}(x) + (x-t)q_k(x)$. Using this relation and writing the determinant $F(q_1,\ldots,q_n)(t,x)$ as a sum of two determinants according to the fist row, we obtain

$$\begin{aligned} &F[q_m,\ldots,q_{m+n-1}](t;x) \\ &= \begin{vmatrix} q_{m+1}(x) & \cdots & q_{n+m}(x) \\ q_{m+1}(t;x) & \cdots & q_{n+m}(t;x) \\ \vdots & \cdots & \vdots \\ q_{m+n}(t;x) & \cdots & q_{m+n}(t;x) \end{vmatrix} + (x-t)\begin{vmatrix} q_m(x) & \cdots & q_{n+m-1}(x) \\ q_{m+1}(t;x) & \cdots & q_{n+m}(t;x) \\ \vdots & \cdots & \vdots \\ q_{m+n}(t;x) & \cdots & q_{m+n}(t;x) \end{vmatrix}. \end{aligned}$$

Applying the relation $q_k(t;x) = q_{k+1}(x) + (x-t)q_k(x)$ multiple times, it is easy to see that the first determinant simplifies to $F[q_{m+1},\ldots,q_{m+n}](x)$. For the second determinant, we repeat the above procedure by splitting it into two determinants according to the second row, and simplify the first one to $F[q_m,\widehat{q}_{m+1},q_{m+2}\cdot\ldots,q_{m+n}](x)$. Continuing this process, it is easy to see that the last determinant is $F[q_m,\ldots,q_{m+n-1}](x)$. $\square$

Proposition 3.9. *For $m\le k\le n+m$,*

$$\begin{vmatrix} p_n(x) & \cdots & p_{n+m}(x) \\ \vdots & \cdots & \vdots \\ p_n^{(m-1)}(x) & \cdots & p_{n+m}^{(m-1)}(x) \\ p_n^{(k)}(x) & \cdots & p_{n+m}^{(k)}(x) \end{vmatrix} = (-1)^{k-m}C_n^{1,m}\,k!\,F[q_m,\ldots,\widehat{q}_k,\ldots,q_{m+n}](x). \tag{3.10}$$

Proof. Setting $r = 2$, $m_1 = 1$, $m_2 = m$, $t_1 = t$ and $t_2 = x$ in the identity (2.9), Lemma 3.8 yields

$$\begin{vmatrix} p_n(t) & \cdots & p_{n+m}(t) \\ p_n(x) & \cdots & p_{n+m}(x) \\ \vdots & \cdots & \vdots \\ p_n^{(m-1)}(x) & \cdots & p_{n+m}^{(m-1)}(x) \end{vmatrix} = C_n^{1,m}(x-t)^m F[q_m \ldots, q_{n+m-1}](t;x)$$

$$= C_n^{1,m} \sum_{j=0}^{n} (x-t)^{m+j} F[q_m, \ldots, \widehat{q}_{m+j}, \ldots, q_{m+n}](x).$$

Taking $k = m + j$ derivatives of the above identity with respect to t and setting $t = x$, we obtain (3.10) after changing the first row to the last row. $\square$

If we want more gaps in the derivatives of p_n in the determinant, we will need, by (3.6), an extension of Lemma 3.8 to more than two variables. For example, by (3.6) and an obvious extension of Lemma 3.8 to more variables,

$$\begin{vmatrix} p_n(x) & p_{n+1}(x) & p_{n+2}(x) \\ p_n(t_1) & p_{n+1}(t_1) & p_{n+2}(t_1) \\ p_n(t_2) & p_{n+1}(t_2) & p_{n+2}(t_2) \end{vmatrix} = B_{n,2}(t_2 - t_1)(t_2 - x)(t_1 - x) F[q_1, \ldots, q_n](t_1, t_2; x)$$

$$= B_{n,2}(x - t_1)(x - t_2)(t_2 - t_1) \sum_{k=0}^{n} (x - t_1)^k F[q_1, \ldots, \widehat{q}_{k+1}, \ldots q_{n+1}](t_2; x).$$

Writing $t_2 - t_1 = (x - t_1) - (x - t_2)$, then taking derivatives with respect to t_1 and setting $t_1 = x$, it follows that

$$\begin{vmatrix} p_n(x) & p_{n+1}(x) & p_{n+2}(x) \\ p_n''(x) & p_{n+1}''(x) & p_{n+2}''(x) \\ p_n(t_2) & p_{n+1}(t_2) & p_{n+2}(t_2) \end{vmatrix} = 2B_{n,2}(x - t_2)$$
$$\times \left(F[q_1, \ldots, q_n](t_2; x) - (x - t_2) F[q_2, \ldots, q_{n+1}](t_2; x)\right)$$

where we have used Lemma 3.8 in the second term. Consequently, expanding F in terms of the power of $x - t_2$ by using Lemma 3.8, we derive the following:

Proposition 3.10. *For $k \geq 2$,*

$$\begin{vmatrix} p_n(x) & p_{n+1}(x) & p_{n+2}(x) \\ p_n''(x) & p_{n+1}''(x) & p_{n+2}''(x) \\ p_n^{(k)}(x) & p_{n+1}^{(k)}(x) & p_{n+2}^{(k)}(t_2) \end{vmatrix} = 2k! B_{n,2}$$
$$\times \left(F[q_1, \ldots, \widehat{q}_k, \ldots, q_n](x) - F[q_2, \ldots, \widehat{q}_k, \ldots, q_{n+1}](x)\right).$$

For $j > 2$, however, the above discussion leads to

$$\begin{vmatrix} p_n(x) & p_{n+1}(x) & p_{n+2}(x) \\ p_n^{(j)}(x) & p_{n+1}^{(j)}(x) & p_{n+2}^{(j)}(x) \\ p_n(t_2) & p_{n+1}(t_2) & p_{n+2}(t_2) \end{vmatrix} = B_{n,2} j! (x - t_2)$$
$$\times \left(F[q_1, \ldots, \widehat{q}_{j-2}, \ldots q_{n+1}](t_2; x) - (x - t_2) F[q_1, \ldots, \widehat{q}_{j-1}, \ldots q_{n+1}](t_2; x)\right).$$

In order to continue the above procedure, we have to expand the right-hand side in powers of $x - t$, for which we need a formula such as

$$F[q_1, \ldots, \widehat{q}_k, \ldots, q_{n+1}](t; x) = \sum_{i=1}^{k} (x-t)^{i-1} \sum_{j=k+1}^{n+2} (x-t)^{j-k-1} \times F[q_1, \ldots, \widehat{q}_i, \ldots, \widehat{q}_j, \ldots, q_{n+2}](x).$$

Taking derivatives with respect to t_2 and setting $t_2 = x$, we can then write

$$\begin{vmatrix} p_n(x) & p_{n+1}(x) & p_{n+2}(x) \\ p_n^{(j)}(x) & p_{n+1}^{(j)}(x) & p_{n+2}^{(j)}(x) \\ p_n^{(k)}(x) & p_{n+1}^{(k)}(x) & p_{n+2}^{(k)}(x) \end{vmatrix}$$

as a sum of the determinants of the form $F[q_1, \ldots, \widehat{q}_i, \ldots, \widehat{q}_j, \ldots, q_{n+2}](x)$.

4. Laguerre–Pólya class of entire functions and Toda lattices

4.1. The Laguerre–Pólya class

The real entire function $\psi(x)$ is said to belong to the Laguerre–Pólya class $\mathcal{LP}$ if it can be represented as

$$\psi(x) = cx^m e^{-ax^2+bx} \prod_{k=1}^{\infty} (1 + x/x_k) e^{-x/x_k},$$

where c, b and x_k are real, $x_k \neq 0$, $a \geq 0$, $m \in \mathbb{N}_0$ and $\sum x_k^{-2} < \infty$. The functions in $\mathcal{LP}$, and only these, obey the property that they are local uniform limits, that is, uniform limits on the compact subsets of $\mathbb{C}$, of polynomials with only real zeros. Such polynomials are usually called hyperbolic ones. The Laguerre–Pólya class has been studied extensively since the Riemann hypothesis is equivalent to the fact that the Riemann ξ-function, the one that Titchmarsh denotes by Ξ, belongs to $\mathcal{LP}$. We refer to [4–6] and the references therein.

Laguerre gave a necessary condition for a function ψ to be in the Laguerre–Pólya class $P \in \mathcal{LP}$: if $\psi \in \mathcal{LP}$ then

$$L(\psi; x) = [\psi'(x)]^2 - \psi(x)\psi''(x) \geq 0 \qquad \forall x \in \mathbb{R}. \tag{4.1}$$

Jensen established a necessary and sufficient condition. If $\psi \in \mathcal{LP}$ and its Maclaurin expansion is

$$\psi(x) = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \tag{4.2}$$

then its Jensen polynomials are defined by

$$g_n(x) = g_n(\psi; x) := \sum_{j=0}^{n} \binom{n}{j} \gamma_j x^j, \qquad n = 0, 1, \ldots.$$

Jensen himself established the following fundamental theorem in [9] (see [12]):

Proposition 4.1. *A function ψ with the Maclaurin expansion (4.2) belongs to $\mathcal{LP}$ if and only if all its Jensen polynomials $g_n(\psi; x)$, $n \in \mathbb{N}$, are hyperbolic. Moreover, the sequence $\{g_n(\psi; x/n)\}$ converges locally uniformly to $\psi(x)$.*

The generalized Jensen polynomials are defined by

$$g_{n,k}(x) = g_{n,k}(\psi; x) := \sum_{j=0}^{n} \binom{n}{j} \gamma_{k+j} x^j, \quad n, k = 0, 1, \ldots.$$

It is evident that $g_{n,0}(x) = g_n(x)$ and $g_{n,k}(\psi; x) = g_n(\psi^{(k)}; x)$, which shows, in particular, that $g_{n,k}(\psi; x/n) \to \psi^{(k)}(x)$ locally uniformly. Furthermore, it is easy to verify that $g_{n+k}^{(k)}(\psi; x) = \frac{(n+k)!}{k!} g_{n,k}(x)$. Consequently, it follows from (4.1) that, if $\psi \in \mathcal{LP}$ then

$$L_n(\psi; x) := (n+2)[g_{n+1}'(\psi; x)]^2 - (n+1) g_n(\psi; x)\, g_n''(\psi; x) \geq 0, \qquad \forall x \in \mathbb{R}. \tag{4.3}$$

Let us call $L_n(\psi; x)$ the Laguerre determinant of Jensen polynomials.

Craven and Csordas [4] (see also [5]) gave another criterion in terms of the Turán determinant of Jensen polynomials:

Proposition 4.2. *Let the Maclaurin coefficients of the real entire function ψ be such that $\gamma_{k-1}\gamma_{k+1} < 0$ whenever $\gamma_k = 0$, $k = 1, 2, \ldots$. Then $\psi \in \mathcal{LP}$ if and only if*

$$T_n(\psi; x) := g_n^2(\psi; x) - g_{n-1}^2(\psi; x) g_{n+1}^2(\psi; x) > 0, \quad \forall x \in \mathbb{R} \setminus \{0\} \;\; \textit{and } n \in \mathbb{N}. \tag{4.4}$$

Our main result on the determinant shows that if ψ is a Laplace transform of a non-negative measure, then the Laguerre polynomial inequalities and the Turán inequalities are equivalent. Let us consider the bilateral Laplace transform

$$\mathcal{L}_\mu(z) := \int_{\mathbb{R}} e^{-zt} d\mu(t), \qquad z \in \mathbb{C},$$

for a real nonnegative measure $d\mu$ and it formal Maclaurin expansion

$$\mathcal{L}_\mu(z) = \sum_{k=0}^{\infty} \frac{\mu_k}{k!} (-z)^k, \qquad \mu_k := \int_0^{\infty} t^k d\mu(t).$$

Then its Jensen polynomials g_n and $g_{n,k}$ are given by, with $\gamma_k = (-1)^k \mu_k$,

$$g_n(\mathcal{L}_\mu; z) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j \mu_j z^j \quad \text{and} \quad g_{n,k}(\mathcal{L}_\mu; z) = \sum_{j=0}^{n} \binom{n}{j} (-1)^{j+k} \mu_{j+k} z^j.$$

A direct verification shows that

$$g_n^{(j)}(\mathcal{L}_\mu; z) = \frac{n!}{(n-j)!} g_{n-j,j}(\mathcal{L}_\mu; z), \qquad 0 \leq j \leq n. \tag{4.5}$$

It turns out that g_n is related to our q_n and $g_{n,k}$ is related to our $r_{k,n}$.

Lemma 4.3. *For $0 \leq k \leq n$,*

$$g_n(\mathcal{L}\mu; x) = (-x)^n q_n\left(d\mu; \frac{1}{x}\right) \quad \textit{and} \quad g_{n,k}(\mathcal{L}\mu; x) = (-1)^{n+k} x^n r_{n,k}\left(d\mu; \frac{1}{x}\right). \tag{4.6}$$

Proof. These follow directly from the definitions of Jensen's polynomials. $\square$

Theorem 4.4. *For $m, n \in \mathbb{N}$,*

$$\det[g_{m+i+j}(x)]_{i,j=0}^{n-1} = x^{n(n-1)} \det\left[\frac{m!}{(m+i+j)!} g_{m+i+j}^{(i+j)}(x)\right]_{i,j=0}^{n-1}. \tag{4.7}$$

Proof. By (4.6), it is easy to see that

$$\det[g_{m+i+j}(x)]_{i,j=0}^{n-1} = x^{nm+n(n-1)} \det[q_{m+i+j}(1/x)]_{i,j=0}^{n-1},$$

$$\det\left[\frac{m!}{(m+i+j)!} g_{m+i+j}^{(i+j)}(x)\right]_{i,j=0}^{n-1} = x^{nm} \det\left[r_{m,i+j}(1/x)\right]_{i,j=0}^{n-1},$$

so that (4.7) follows from (2.5). □

In particular, when $n = 2$, the identity (4.7) becomes

$$[g_{m+1}(x)]^2 - g_m(x) g_{m+2}(x)$$
$$= \frac{x^2}{(m+2)(m+1)^2} \left((m+2)[g'_{m+1}(x)]^2 - (m+1) g_m(x) g''_{m+2}(x)\right),$$

and it was observed by Craven and Csordas [4]. Identity (4.7) gives a direct relation between the Turán determinants and the Laguerre determinants of any order.

4.2. Toda lattices and orthogonal polynomials

The Toda lattice is a model for a nonlinear one-dimensional crystal that describes the motion of a chain of N particles with nearest neighbor interactions. The Hamiltonian of the Toda lattice is

$$H(\mathbf{p}, \mathbf{q}) = \sum_{k=1}^{N} \left(\frac{p_k^2(t)}{2} + e^{-(q_{k+1}(t)-q_k(t))}\right),$$

where p_k is the moment of the k-th particle and q_k is its displacement from the equilibrium. With the change of variables of Flashka and Moser

$$a_k = \frac{1}{2} e^{-(q_{k+1}-q_k)/2}, \quad b_k = -\frac{1}{2} p_k,$$

the equations of motion become

$$a'_k(t) = a_k(t)(b_k(t) - b_k(t)) \quad \text{and} \quad a'_k(t) = 2(a_k^2(t) - a_{k-1}^2(t)).$$

Let $L = (l_{i,j})$ be the Jacobi matrix with diagonal entries $l_{k,k} = b_k$ and off-diagonal ones $l_{k,k+1} = l_{k+1,k} = a_k$, and let $B = (b_{i,j})$ whose only non-zero elements are the off-diagonal entries $b_{k,k+1} = -b_{k+1,k} = a_k$. Then the Lax form of the equations of motion is

$$\frac{d}{dt} L = [B, L].$$

The matrix L is naturally associated with the sequence of orthonormal polynomials, with the time variable as a parameter, which satisfy the three term recurrence relation

$$a_n(t)\, p_{n+1}(x;t) = (x - b_n(t))\, p_n(x;t) - a_{n-1}(t)\, p_{n-1}(x;t).$$

These are in fact the characteristic polynomials of the principal minors of L and they are orthogonal with respect to the measure $d\mu_t(x) = e^{tx}d\mu_0(x)$, where $d\mu_0$ corresponds to $t = 0$. Once the direct problem with the initial data at $t = 0$ is solved and the polynomials $p_n(x, 0)$ are obtained, one needs to solve the inverse problem. A fundamental problem is to construct $p_n(x, t)$. In order to do so, as it is seen from (1.2), it suffices to calculate the moments $m_k(t) = \int x^k e^{tx} d\mu_0(x)$. These moments are obtained by successive differentiation of $m_0(t)$ because $dm_k(t)/dt = m_{k+1}(t)$. Therefore, the principal task in solving the inverse problem is to determine

$$m_0(t) = \int e^{tx} d\mu_0(x).$$

One possible approach is to approximate this formal Laplace transform by the Wronskians of the orthogonal polynomials $p_n(x; 0) = p_n(d\mu; x)$.

If $d\mu$ is a non-negative measure supported on $[0, \infty)$, then $\mathcal{L}\mu(x)$ is the Laplace transform of μ. Let $(\mathcal{L}\mu)^{(k)}$ be the k-th derivative of $\mathcal{L}\mu$. Then, for $x \geq 0$,

$$(\mathcal{L}\mu)^{(k)}(x) = \int_{\mathbb{R}} (-t)^k e^{-xt} d\mu(t) = (-1)^k \mu_k^{(x)},$$

where $\mu_n^{(x)}$ is the k-th moment of the measure $d\mu^{(x)} := e^{-xt}d\mu(t)$. By (4.5) and $g_n((\mathcal{L}\mu)^{(k)}; x) = g_{n,k}(\mathcal{L}\mu; x)$, we can rewrite (2.5) as

$$\det[g_{m+i+j}(\mathcal{L}\mu; x)]_{i,j=0}^{n-1} = x^{n(n-1)} \det \left[g_m((\mathcal{L}\mu)^{(i+j)}; x) \right]_{i,j=0}^{n-1}. \tag{4.8}$$

Furthermore, by (4.6) and (2.4), we conclude that

$$C_{n,m} \det \left[g_m((\mathcal{L}\mu)^{(i+j)}; x) \right]_{i,j=0}^{n-1} = x^{nm} W(p_n, \ldots, p_{n+m-1}; 1/x).$$

In particular, when $n = 1$, we obtain the following corollary:

Corollary 4.5. *For $m \in \mathbb{N}$, $x \in \mathbb{R}$,*

$$g_m(\mathcal{L}\mu; x) = (-x)^{nm} \frac{W(p_1, \ldots, p_m; 1/x)}{\prod_{k=1}^{m-1} k! \det M_k(d\mu)} \to \mathcal{L}\mu(x), \quad m \to \infty.$$

As another corollary of these relations, we deduce the following result:

Corollary 4.6. *Let μ be a nonnegative Borel measure and assume that its Laplace transform $\mathcal{L}\mu$ is real analytic on $[0, \infty)$. Then for $n = 1, 2, \ldots,$*

$$\det \left[(\mathcal{L}\mu)^{(i+j)}(x) \right]_{i,j=0}^{n-1} \geq 0, \quad x \in (0, \infty).$$

Proof. From (4.6) and Corollary 2.5, the right hand side of (4.8) is nonnegative if m is an even positive integer. By its definition, it is easy to see that

$$g_m(\mathcal{L}\mu; x) = \int_0^\infty (1 - tx)^m d\mu(t).$$

It follows from dominant convergence theorem that

$$\lim_{m\to\infty} g_{2m}\left((\mathcal{L}\mu)^{(i+j)}; \frac{x}{2m}\right) = \lim_{m\to\infty} \int_0^\infty \left(1-\frac{tx}{2m}\right)^{2m} t^{i+j} d\mu(t) = (\mathcal{L}\mu)^{(i+j)}(x).$$

For fixed n, the above limit carries over to the determinant in the right hand side of (4.8). This completes the proof. □

In fact, since the determinant in the corollary is that of the moment matrix for $d\mu^{(x)}$, it is positive.

5. Examples

In this section we illustrate our main results in the case of classical orthogonal polynomials. We recall that if $\widetilde{p}_n(d\mu)$ denotes the orthonormal polynomial of degree n with respect to the measure $d\mu$, then

$$\widetilde{p}_n(d\mu; x) = \sqrt{\frac{\det M_{n-1}(d\mu)}{\det M_n(d\mu)}} x^n + \dots, \tag{5.1}$$

which can be used to determine the determinant of $M_n(w)$ for the classical orthogonal polynomials.

5.1. Hermite polynomials

For the weight function $w(x) = e^{-x^2}dx/\sqrt{\pi}$, which is normalized so that $\mu_0 = 1$, its moments μ_k are given by

$$\mu_{2k} = \frac{(2k)!}{2^{2k}k!} = \left(\frac{1}{2}\right)_k \quad \text{and} \quad \mu_{2k+1} = 0,$$

where $(a)_k := a(a+1)\dots(a+n-1)$ is the Pochhammer symbol. The corresponding orthogonal polynomials are the Hermite polynomials $H_n(x)$,

$$H_n(x) = \sum_{0\le k\le n/2} (-1)^k \frac{n!}{k!(n-2k)!}(2x)^{n-2k},$$

normalized by $H_n(x) = 2^n x^n + \dots$. The orthonormal Hermite polynomial is $\widetilde{H}_n(x) = H_n(x)/\sqrt{2^n n!}$. Hence, it follows from (5.1) that the determinant of the moment matrix M_n for $w(x)$ satisfies

$$\det M_n = \frac{n!}{2^n} \det M_{n-1} = \dots = \prod_{k=1}^n \frac{k!}{2^k}.$$

Since the orthogonal polynomial p_n which appears in (2.4) is given by

$$p_n(x) = \det M_{n-1} \frac{H_n(x)}{2^n},$$

it follows that in this case

$$\det\left[p_{n+j-1}(t_i)\right]_{i,j=1}^m = \prod_{j=1}^m \frac{\det M_{n+j-2}}{2^{n+j-1}} \det\left[H_{n+j-1}(t_i)\right]_{i,j=1}^m.$$

Let $w(t_1,\dots,t_m;x) = (x-t_1)\dots(x-t_m)\,e^{-x^2}$. According to (2.2), we obtain that

$$\frac{\det\left[H_{n+j-1}(t_i)/2^{n+j-1}\right]_{i,j=1}^{m}}{V(t_1,\dots,t_m)} = (-1)^{nm}\frac{\det M_{n-1}(w(t_1,\dots,t_m))}{\det M_{n-1}}, \tag{5.2}$$

where $M_{n-1}(w(t_1,\dots,t_m))$ is the moment matrix of $w(t_1,\dots,t_m)$. Furthermore, (1.6) shows that

$$q_{n+1}(t_1,\dots,t_{m-1};t_m)w(t)dt = \int_{\mathbb{R}} (t-t_m)^n w(t_1,\dots,t_m;t)dt$$

is the shifted moment of $w(t_1,\dots,t_m)$. According to (2.1), we can replace the term $\det M_{n-1}(w(t_1,\dots,t_m))$ in (5.2) by the determinant of these shifted moments of w_m.

By the definition of (1.3), it follows readily that

$$q_n(x) = \sum_{j=0}^{n} \mu_j \binom{n}{j}(-x)^{n-j} = i^n \frac{H_n(ix)}{2^n}.$$

Furthermore, it is easy to see that, if n is even, then

$$r_{m,n}(x) = \left(\frac{1}{2}\right)_{\frac{n}{2}} (-x)^m {}_3F_1\left(\begin{matrix}\frac{-m}{2}, \frac{-m+1}{2}, \frac{n+1}{2} \\ \frac{1}{2}\end{matrix}; -x^{-2}\right),$$

and if n is odd, then

$$r_{m,n}(x) = -m\left(\frac{1}{2}\right)_{\frac{n+1}{2}} (-x)^{m+1} {}_3F_1\left(\begin{matrix}\frac{-m+1}{2}, \frac{-m+2}{2}, \frac{n+2}{2} \\ \frac{3}{2}\end{matrix}; -x^{-2}\right).$$

Now, if $u_k = b^k v_k$, then it is easy to verify that

$$\det\left[u_{\ell_i+j-1}\right]_{i,j=1}^{n} = b^{\sum_{i=1}^{n}(\ell_i+i-1)} \det\left[b_{\ell_i+j-1}\right]_{i,j=1}^{n}.$$

Using this identity, it follows that (2.4) becomes, for the Hermite polynomials,

$$\begin{aligned}\det\left[H_{n+j-1}^{(k-1)}(x)\right]_{k,j=1}^{m} &= \frac{(-1)^{mn} 2^{\frac{m(m-1)}{2}}\, i^{n(n+m-1)}}{2^{\frac{n(n-1)}{2}} \prod_{k=m}^{n-1} k!} \det\left[H_{m+k+j}(ix)\right]_{k,j=0}^{n-1} \\ &= \frac{(-1)^{mn} 2^{\frac{(m+n)(m+n-1)}{2}}}{\prod_{k=m}^{n-1} k!} \det\left[r_{m,k+j}(x)\right]_{k,j=0}^{n-1}.\end{aligned}$$

The first equality appeared in [11, (33)].

5.2. Laguerre polynomials

For $\alpha > -1$, the weight function $w_\alpha(t) = t^\alpha e^{-t}/\Gamma(\alpha+1)$ has the moments $\mu_k(w_\alpha) = (\alpha+1)_k$, $k = 0, 1, \dots$. The Laguerre polynomials are orthogonal with respect to w_α and they are explicitly given by

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!} \sum_{j=0}^{n} \frac{(-n)_j x^j}{(\alpha+1)_j j!} = \gamma_n x^n + \dots, \qquad \gamma_n := \frac{(-1)^n}{n!}.$$

The leading coefficient of the orthonormal Laguerre polynomial of degree n is given by $1/\sqrt{n!(\alpha+1)_n}$, so that, by (5.1),

$$M_n(w_\alpha) = \prod_{k=1}^{n} \frac{M_k(w_\alpha)}{M_{k-1}(w_\alpha)} = \prod_{k=1}^{n} k!(\alpha+1)_k.$$

Let $w_\alpha(t_1,\dots,t_m;x) := (x-t_1)\dots(x-t_m)x^\alpha e^{-x}$. According to (2.2), we obtain that

$$\frac{\det\left[L_{n+j-1}^{\alpha}(t_i)/\gamma_{n+j-1}\right]_{i,j=1}^{m}}{V(t_1,\dots,t_m)} = (-1)^{nm}\frac{\det M_{n-1}(w_\alpha(t_1,\dots,t_m))}{\det M_{n-1}(w_\alpha)},$$

where $M_{n-1}(w_\alpha(t_1,\dots,t_m))$ is the moment matrix of $w_\alpha(t_1,\dots,t_m)$.

By the definition of $q_n = q_n(w_\alpha)$ in (1.3), we obtain that

$$\begin{aligned} q_n(w_\alpha;x) &= \sum_{j=0}^{n}\binom{n}{j}(\alpha+1)_{n-j}(-x)^j \\ &= (-1)^n(-n-\alpha)_n \sum_{j=0}^{n}\frac{(-n)_j(-x)^j}{(-n-\alpha)_j j!} = (-1)^n n! L_n^{(-n-\alpha-1)}(-x). \end{aligned}$$

This polynomial appeared in [11] as a constant multiple of $L_n^{-n-2\alpha}(x)$, but $-n-2\alpha$ should be $-n-\alpha-1$. Furthermore, by the definition of $r_{m,n} = r_{m,n}(w_\alpha)$ in (1.4) and using $(a)_{j+k} = (a)_j(a+j)_k$,

$$\begin{aligned} r_{m,n}(w_\alpha;x) &= \sum_{j=0}^{n}\binom{n}{j}(\alpha+1)_{n+m-j}(-x)^j = (\alpha+1)_n q_m(w_{n+\alpha};x) \\ &= (-1)^m m!(\alpha+1)_n L_m^{(-n-m-\alpha-1)}(-x). \end{aligned}$$

It follows that (2.4) becomes, for the Laguerre polynomials,

$$\begin{aligned} \det\left[(L_{n+j-1}^{\alpha}(x))^{(i-1)}\right]_{i,j=1}^{m} &= A_{m,n}\det\left[(m+i+j)!L_{m+i+j}^{-m-i-j-\alpha-1}(-x)\right]_{i,j=0}^{n-1} \\ &= A_{m,n}(m!)^n \det\left[(\alpha+1)_{i+j}L_m^{-m-i-j-\alpha-1}(-x)\right]_{i,j=0}^{n-1}, \end{aligned}$$

where

$$A_{m,n} = \frac{(-1)^{m(m-1)/2}\prod_{k=1}^{m-1}k!}{\prod_{j=1}^{m}(n+j-1)!\det M_{n-1}(w_\alpha)}.$$

Notice that the two determinants in the second identity have the Laguerre polynomials of the same parameters but different degrees.

5.3. Gegenbauer polynomials

For $\lambda > -1/2$, the weight function $w_\lambda(t) = c_\lambda(1-t^2)^{\lambda-1/2}$, where $c_\lambda = \Gamma(\lambda+1)/(\Gamma(\frac{1}{2})\Gamma(\lambda+\frac{1}{2}))$ is chosen so that $\mu_0 = 1$ and the moments are given by

$$\mu_{2k} = \frac{(\frac{1}{2})_k}{(\lambda+1)_k} \quad \text{and} \quad \mu_{2k+1} = 0, \quad k = 0,1,\dots.$$

The Gegenbauer polynomials C_n^λ are orthogonal with respect to w_λ, they satisfy

$$c_\lambda \int_{-1}^{1} C_n^\lambda(t) C_m^\lambda(t) w_\lambda(t) dt = h_n^\lambda \delta_{n,m}, \quad h_n^\lambda := \frac{\lambda(2\lambda)_n}{(n+\lambda)n!},$$

and those polynomials are given explicitly as

$$C_n^\lambda(x) = \gamma_n^\lambda x^n {}_2F_1\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2} \\ 1-n-\lambda \end{matrix}; \frac{1}{x^2}\right), \qquad \gamma_n^\lambda := \frac{(\lambda)_n 2^n}{n!}. \tag{5.3}$$

The leading coefficient γ_n^λ of C_n^λ divided by $\sqrt{h_n^\lambda}$ is the leading coefficient of the orthonormal Gegenbauer polynomial of degree n. Hence, by (5.1),

$$M_n(w_\lambda) = \prod_{k=1}^{n} \frac{M_k(w_\lambda)}{M_{k-1}(w_\lambda)} = \prod_{k=1}^{n} \frac{\lambda(2\lambda)_k k!}{(k+\lambda)(\lambda)_k^2 2^{2k}} = \frac{\lambda^n}{(\lambda+1)_n} \prod_{k=1}^{n} \frac{(2\lambda)_k k!}{(\lambda)_k^2 2^{2k}}.$$

Let $w_\lambda(t_1,\ldots,t_m;x) := (x-t_1)\ldots(x-t_m)(1-x)^{\lambda-1/2}$. According to (2.2), we obtain that

$$\frac{\det\left[C_{n+j-1}^\lambda(t_i)/\gamma_{n+j-1}^\lambda\right]_{i,j=1}^m}{V(t_1,\ldots,t_m)} = (-1)^{nm} \frac{\det M_{n-1}(w_\lambda(t_1,\ldots,t_m))}{\det M_{n-1}(w_\lambda)}, \tag{5.4}$$

where $M_{n-1}(w_\lambda(t_1,\ldots,t_m))$ is the moment matrix of $w_\lambda(t_1,\ldots,t_m)$.

Lemma 5.1. *For w_λ and $n = 0,1,\ldots$, $q_n = q_n(w_\lambda)$ is given by*

$$q_n(x) = \frac{n!}{2^n(\lambda+1)_n} C_n^{-n-\lambda}(x) = (x^2-1)^{n/2} \frac{C_n^{\lambda+1/2}\left(-x/\sqrt{x^2-1}\right)}{C_n^{\lambda+1/2}(1)}. \tag{5.5}$$

Proof. Directly from the definition of (1.3), it is easy to see that

$$q_n(x) = \sum_{j=0}^{n} \binom{n}{j} \mu_j (-x)^{n-j} = (-x)^n {}_2F_1\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2} \\ \lambda+1 \end{matrix}; \frac{1}{x^2}\right). \tag{5.6}$$

Writing $\lambda+1 = 1-n-(-n-\lambda)$ and using $(-\lambda-n)_n = (-1)^n(\lambda+1)_n$, the first expression for q_n follows from (5.3). Applying the identity [1, (2.3.14)],

$${}_2F_1\left(\begin{matrix} -n, b \\ c \end{matrix}; x\right) = \frac{(c-b)_n}{(c)_n} {}_2F_1\left(\begin{matrix} -n, b \\ b+1-n-c \end{matrix}; 1-x\right)$$

to the right hand side of (5.6), it is easy to see that we obtain

$$\begin{aligned} q_n(x) &= (-x)^n \frac{(\lambda+\frac{1}{2})_n 2^n}{(2\lambda+1)_n} {}_2F_1\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2} \\ \frac{1}{2}-n-\lambda \end{matrix}; 1-\frac{1}{x^2}\right) \\ &= \frac{n!}{(2\lambda+1)_n} (x_2-1)^{n/2} C_n^{\lambda+1/2}\left(-x/\sqrt{x^2-1}\right) \end{aligned}$$

by (5.3), which is the second representation of q_n since $C_n^{\lambda+1/2}(1) = (2\lambda+1)_n/n!$. $\square$

As in the case of the Hermite polynomials, we have the following formulas for $r_{m,n}(x)$. If n is even then

$$r_{m,n}(x) = \frac{(\frac{1}{2})_{\frac{n}{2}}}{(\lambda+1)_{\frac{n}{2}}} (-x)^{m-1} {}_3F_2\left(\begin{matrix} \frac{-m+1}{2}, \frac{-m+2}{2}, \frac{n+2}{2} \\ \frac{3}{2}, \lambda+1+\frac{n+1}{2}, \end{matrix}; -x^{-2}\right),$$

where as if n is odd, then

$$r_{m,n}(x) = -m\frac{\left(\frac{1}{2}\right)_{\frac{n+1}{2}}}{(\lambda+1)_{\frac{n+1}{2}}}(-x)^m {}_3F_2\left(\begin{matrix}\frac{-m}{2}, \frac{-m+1}{2}, \frac{n+1}{2} \\ \frac{1}{2}, \lambda+1+\frac{n}{2},\end{matrix}; -x^{-2}\right).$$

It follows that (2.4) becomes, for the Gegenbauer polynomials,

$$\begin{aligned}\det\left[(C^\lambda_{n+j-1}(x))^{(i-1)}\right]_{i,j=1}^m &= \frac{(-1)^{mn}\prod_{k=1}^{m-1} k!\prod_{j=1}^m \gamma^\lambda_{n+j-1}}{\det M_{n-1}(w_\lambda)}(x^2-1)^{m(m+n-1)/2} \\ &\quad \times \det\left[\frac{C^{\lambda+1/2}_{m+i+j}(-x/\sqrt{x^2-1})}{C^{\lambda+1/2}_{m+i+j}(1)}\right]_{i,j=0}^{n-1} \\ &= \frac{(-1)^{mn}\prod_{k=1}^{m-1} k!\prod_{j=1}^m \gamma^\lambda_{n+j-1}}{\det M_{n-1}(w_\lambda)}\det\left[r^\lambda_{m,i+j}(x)\right]_{i,j=0}^{n-1}.\end{aligned}$$

The first equality already appeared in [11].

References

[1] G. Andrew, R. Askey, R. Roy, Special Functions, Encyclopedia of Mathematics and Its Applications, vol. 71, Cambridge Univ. Press, Cambridge, 1999.
[2] P. Atkins, R. Friedman, Molecular Quantum Mechanics, 4th ed., Oxford University Press Inc., New York, 2005.
[3] E. Brézin, S. Hikami, Characteristic polynomials of real symmetric random matrices, Comm. Math. Phys. 223 (2001) 363–382.
[4] T. Craven, G. Csordas, Jensen polynomials and the Turán and Laguerre inequalities, Pacific J. Math. 136 (1989) 241–260.
[5] G. Csordas, R. Varga, Necessary and sufficient conditions and the Riemann hypothesis, Adv. in Appl. Math. 11 (1990) 328–357.
[6] D.K. Dimitrov, Lee–Yang measures and wave functions, arXiv:1311.0596.
[7] C. Dunkl, Y. Xu, Orthogonal Polynomials of Several Variables, 2nd edition, Encyclopedia of Mathematics and Its Applications, vol. 155, Cambridge Univ. Press, Cambridge, 2014.
[8] P.J. Forrester, S.O. Warnaar, The importance of the Selberg integral, Bull. Amer. Math. Soc. (N.S.) 45 (2008) 489–534.
[9] J.L.W.V. Jensen, Recherches sur la théorie des équations, Acta Math. 36 (1913) 181–195.
[10] S. Karlin, G. Szegő, On certain determinants whose elements are orthogonal polynomials, J. Anal. Math. 8 (1960) 1–157.
[11] B. Leclerc, On certain formulas of Karlin and Szegő, Sém. Lothar. Combin. 41 (1998), Art. B41d.
[12] N. Obrechkoff, Zeros of Polynomials, Bulgarian Acad. Sci., Sofia, 1963 (in Bulgarian); English translation, Marin Drinov Acad. Publ. House, 2003.
[13] G. Pólya, G. Szegő, Problems and Theorems in Analysis, vol. I, Springer-Verlag, Berlin and New York, 1972.
[14] J.C. Slater, The theory of complex spectra, Phys. Rev. 34 (1929) 1293–1322.
[15] G. Szegő, Orthogonal Polynomials, 4th edition, Amer. Math. Soc., Providence, RI, 1975.