J. Math. Anal. Appl. 454 (2017) 585–616

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Well-posedness of stochastic second grade fluids

Nikolai Chemetov [a], Fernanda Cipriano [b,*]

[a] *Departamento de Matemática, Faculdade de Ciências da Universidade de Lisboa, Portugal*
[b] *Departamento de Matemática, Faculdade de Ciências e Tecnologia da Universidade Nova de Lisboa and Centro de Matemática e Aplicações, Portugal*

A R T I C L E   I N F O

Article history:
Received 22 April 2016
Available online 5 May 2017
Submitted by A. Mazzucato

Keywords:
Second grade fluid
Solvability
Stability
Stochastic

A B S T R A C T

The theory of turbulent Newtonian fluids shows that the choice of the boundary condition is a relevant issue because it can modify the behavior of a fluid by creating or avoiding a strong boundary layer. In this study, we consider stochastic second grade fluids filling a two-dimensional bounded domain with the Navier-slip boundary condition (with friction). We prove the well-posedness of this problem and establish a stability result. Our stochastic model involves a multiplicative white noise and a convective term with third order derivatives, which significantly complicate the analysis.

1. Introduction

The study considers stochastic incompressible fluids of second grade, which are a special class of non-Newtonian fluids. Unlike Newtonian fluids where only the stretching tensor appears in the characterization of the stress response to a deformation fluid, the Cauchy stress tensor $\mathbb{T}$ of non-Newtonian fluids is defined by

$$\mathbb{T} = -\pi\mathbb{I} + \nu A_1 + \alpha_1 A_2 + \alpha_2\, A_1^2,$$

where the first term $-\pi\mathbb{I}$ is due to the incompressibility of the fluid and $\mathbb{A}_1$, $\mathbb{A}_2$ are the two first Rivlin–Ericksen tensors (cf. [35])

$$\mathbb{A}_1(y) = \nabla y + (\nabla y)^\top \quad \text{and} \quad \mathbb{A}_2(y) = \dot{\mathbb{A}}_1(y) + \mathbb{A}_1(y)\nabla y + (\nabla y)^\top\, \mathbb{A}_1(y),$$

where y denotes the velocity of the fluid, the superposed dot is the material time derivative, ν is the kinematic viscosity of the fluid, and α_1, α_2 are constant material moduli. A previous study [18] showed

* Corresponding author.
E-mail addresses: nvchemetov@fc.ul.pt, nvchemetov@gmail.com (N. Chemetov), cipriano@fct.unl.pt (F. Cipriano).

http://dx.doi.org/10.1016/j.jmaa.2017.04.060
0022-247X/

that the thermodynamic laws and stability principles impose $\alpha_1 \geq 0$ and $\alpha_1 + \alpha_2 = 0$. We set $\alpha = \alpha_1$ and assume that $\alpha_1 > 0$.

It is well known that small random perturbations in turbulent fluids can produce relevant macroscopic effects. Therefore, the incorporation of stochastic white noise force in the Navier–Stokes equations [3] is widely recognized as important for understanding turbulence phenomena. Thus, in [2] (see Lemma 2.2), the stochastic Navier–Stokes equations were deduced from fundamental principles by showing that the stochastic Navier–Stokes equations are a real physical model. The stochastic Navier–Stokes equations are now quite well understood (e.g., see [16,20,30,36], and the references therein). However, few results have been reported regarding stochastic non-Newtonian fluids [17,32–34]. In this study, we consider the stochastic second grade equations with multiplicative noise given by

$$\begin{cases} \frac{\partial}{\partial t}(Y - \alpha\Delta Y) = \nu\Delta Y - \mathrm{curl}\,(Y - \alpha\Delta Y) \times Y - \nabla\pi + U + G(t, Y)\,\dot{W}_t, \\ \mathrm{div}\, Y = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathcal{O} \times (0, T), \end{cases} \tag{1.1}$$

where U is a body force, $G(t, Y)\,\dot{W}_t$ is a multiplicative white noise, and $\mathcal{O}$ is a bounded domain of $\mathbb{R}^2$ with a boundary Γ.

Studying this system requires suitable boundary conditions on the boundary Γ of the domain. The Dirichlet boundary condition given by

$$Y = 0 \qquad \text{on } \Gamma$$

is accepted as an appropriate boundary condition and it is the most usual. Another physical relevant boundary condition considered in previous studies is the Navier boundary condition

$$Y \cdot \mathrm{n} = 0, \qquad [2(\mathrm{n} \cdot DY) + \gamma Y] \cdot \tau = 0 \qquad \text{on } \Gamma, \tag{1.2}$$

where $\mathrm{n} = (n_1, n_2)$ and $\tau = (-n_2, n_1)$ are the unit normal and tangent vectors, respectively, to the boundary Γ, $DY = \frac{\nabla Y + (\nabla Y)^\top}{2}$ is the symmetric part of the velocity gradient, and $\gamma > 0$ is a friction coefficient on Γ.

The stochastic partial differential equations (1.1) with the Dirichlet boundary condition were studied by [32] and [34]. In the former study, tightness arguments were used together with the Skorohod theorem to prove the existence of a weak stochastic solution in the sense that the Brownian motion, which is part of the solution, was not given in advance; whereas in the second study, the existence and uniqueness of a strong stochastic solution was proved. In pioneering studies [31] and [13] (see also [12]), the deterministic second grade equations with the Dirichlet boundary condition were studied mathematically for the first time, while [6] investigated the deterministic equations with a particular Navier boundary condition (without friction, i.e., when $\gamma = 0$). The physical interpretations of these second grade equations were given by [8,18,19, 21,23], and [24]. It is relevant to recall that the deterministic methods are based on the Faedo–Galerkin approximation method and a priori estimates. Then, compactness arguments can be used to pass to the limit of the respective approximate equations in the distributional sense. Unfortunately, for the stochastic partial differential equations, a priori estimates are not sufficient to pass to the limit of the approximate equations due to the lack of regularity on the time and stochastic variables. Thus, in order to obtain a strong stochastic solution, we should verify that the sequence of the Galerkin approximations converges strongly in some adequate topology.

We should note that even if the Dirichlet boundary condition is widely accepted as an appropriate boundary condition at the surface of the contact between a fluid and a solid, it is also a source of many problems because it attaches fluid particles to the boundary, thereby creating a strong boundary layer (cf. [15,25,26, 28]). In addition, the Navier boundary condition allows the slippage of the fluid on the boundary, which makes it possible to address important problems, such as the boundary layer problem when the viscosity ν and/or the elastic response α tend to zero (cf. [7,11,9,10,14,27,29]). However, even if the Navier-slip boundary condition allows us to solve interesting problems, technically, when compared with the Dirichlet boundary condition, it requires a more careful mathematical analysis to show the well-posedness of system (1.1)–(1.2) and to establish the stability properties for the solution because the boundary terms obtained by integrating by parts for the convective term do not vanish and they should be estimated in an appropriate manner.

To the best of our knowledge, we investigate stochastic second grade fluid equations with the Navier boundary condition for the first time in this study. To show the well-posedness, we follow the Faedo–Galerkin approximation method by taking an appropriate basis, as employed in previous studies. First, we deduce uniform estimates for the approximate solutions that allow us to pass to the limit with respect to the weak topology. In order to show that the limit process is a solution, we employ the methods developed in [4] to study the stochastic Navier–Stokes equations. In particular, we show that the approximate solutions already converge strongly up to a certain stopping time, and thus we establish the existence and uniqueness results for the solution of system (1.1)–(1.2) as a stochastic process with values in H^3. We should note that analogous reasoning was employed by [34] to handle stochastic second grade fluid equations with homogeneous Dirichlet boundary conditions.

The remainder of this paper is organized as follows. In Section 2, we state the functional setting and introduce useful notations. In Section 3, we present some well-known results and lemmas related to the nonlinear term of $(1.1)_1$, which is applied in the following sections. The main result concerning the existence of a strong stochastic solution is established in Section 4. Finally, in Section 5, we study the stability property.

2. Functional setting and notations

We consider the stochastic second grade fluid model in a bounded and simply connected domain $\mathcal{O}$ of $\mathbb{R}^2$ with a sufficiently regular boundary Γ

$$\begin{cases} d(\upsilon(Y)) = (\nu\Delta Y - \mathrm{curl}(\upsilon(Y)) \times Y - \nabla\pi + U)\,dt + G(t,Y)\,dW_t, & \\ \mathrm{div}\,Y = 0 & \text{in } \mathcal{O}\times(0,T), \\ Y\cdot \mathrm{n} = 0, \qquad [2(\mathrm{n}\cdot DY) + \gamma Y]\cdot\tau = 0 & \text{on } \Gamma\times(0,T), \\ Y(0) = Y_0 & \text{in } \mathcal{O}, \end{cases} \tag{2.1}$$

where $\nu > 0$ is the constant viscosity of the fluid, $\alpha > 0$ is a constant material modulus, the constant $\gamma > 0$ is the friction coefficient of Γ, Δ and ∇ denote the Laplacian and the gradient, respectively, $Y = (Y_1, Y_2)$ is a two-dimensional (2D) velocity field, and

$$\upsilon(Y) = Y - \alpha\Delta Y.$$

The function π represents the pressure, U is a distributed mechanical force, and the term

$$G(t,Y)\,dW_t = \sum_{k=1}^{m} G^k(t,Y)\,dW_t^k$$

corresponds to the stochastic perturbation, where $G(t,Y) = (G^1(t,Y),\ldots,G^m(t,Y))$ has suitable growth assumptions, as defined in the following, and $W_t = (W_t^1,\ldots,W_t^m)$ is a standard $\mathbb{R}^m$-valued Wiener process defined on a complete probability space $(\Omega,\mathcal{F},P)$ endowed with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. We assume that $\mathcal{F}_0$ contains every P-null subset of Ω.

Let X be a real Banach space endowed with the norm $\|\cdot\|_X$. We denote $L^p(0,T;X)$ as the space of X-valued measurable p-integrable functions y defined on $[0,T]$ for $p \geq 1$.

For $p, r \geq 1$, let $L^p(\Omega, L^r(0,T;X))$ be the space of the processes $y = y(\omega,t)$ with values in X defined on $\Omega \times [0,T]$, adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, and endowed with the norms

$$\|y\|_{L^p(\Omega,L^r(0,T;X))} = \left(\mathbb{E}\left(\int_0^T \|y\|_X^r \, dt\right)^{\frac{p}{r}}\right)^{\frac{1}{p}}$$

and

$$\|y\|_{L^p(\Omega,L^\infty(0,T;X))} = \left(\mathbb{E} \sup_{t\in[0,T]} \|y\|_X^p\right)^{\frac{1}{p}} \qquad \text{if } r = \infty,$$

where $\mathbb{E}$ is the mathematical expectation with respect to the probability measure P. As usual, in the notation for processes $y = y(\omega,t)$, we generally omit the dependence on $\omega \in \Omega$.

In Equation (2.1), the vector product $\times$ for 2D vectors $y = (y_1, y_2)$ and $z = (z_1, z_2)$ is calculated as $y \times z = (y_1, y_2, 0) \times (z_1, z_2, 0)$. The curl of the vector y is equal to $\operatorname{curl} y = \frac{\partial y_2}{\partial x_1} - \frac{\partial y_1}{\partial x_2}$ and the vector product of $\operatorname{curl} y$ with the vector z is understood as

$$\operatorname{curl} y \times z = (0, 0, \operatorname{curl} y) \times (z_1, z_2, 0).$$

Given two vectors $y, z \in \mathbb{R}^2$, $y \cdot z = \sum_{i=1}^2 y_i z_i$ denotes the usual scalar product in $\mathbb{R}^2$, and given two matrices A, B, we denote $A \cdot B = \sum_{ij=1}^2 A_{ij} B_{ij}$.

Let us introduce the following Hilbert spaces

$$\begin{aligned}
&H(\mathrm{curl};\mathcal{O}) = \left\{y \in L^2(\mathcal{O}) \mid \operatorname{curl} y \in L^2(\mathcal{O}), \quad \operatorname{div} y = 0 \text{ in } \mathcal{O}\right\},\\
&H = \left\{y \in L^2(\mathcal{O}) \mid \operatorname{div} y = 0 \text{ in } \mathcal{O} \text{ and } y \cdot \mathrm{n} = 0 \text{ on } \Gamma\right\},\\
&V = \left\{y \in H^1(\mathcal{O}) \mid \operatorname{div} y = 0 \text{ in } \mathcal{O} \text{ and } y \cdot \mathrm{n} = 0 \text{ on } \Gamma\right\},\\
&W = \left\{y \in V \cap H^2(\mathcal{O}) \mid [2(\mathrm{n} \cdot Dy) + \gamma y] \cdot \tau = 0 \text{ on } \Gamma\right\},\\
&\widetilde{W} = W \cap H^3(\mathcal{O}).
\end{aligned} \tag{2.2}$$

We denote $(\cdot,\cdot)$ as the inner product in $L^2(\mathcal{O})$ and $\|\cdot\|_2$ as the associated norm. The norm in the space $H^p(\mathcal{O})$ is denoted by $\|\cdot\|_{H^p}$. Let us note that $H(\mathrm{curl};\mathcal{O})$ is a subspace of $H^1(\mathcal{O})$. Let us denote

$$(Dy, Dz) = \int_{\mathcal{O}} Dy \cdot Dz.$$

On the space V, we consider the following inner product

$$(y,z)_V = (\upsilon(y), z) = (y,z) + 2\alpha\,(Dy, Dz) + \alpha\gamma \int_\Gamma y \cdot z$$

and the corresponding norm $\|\cdot\|_V$. We can verify that the norms $\|\cdot\|_{H^1}$ and $\|\cdot\|_V$ are equivalent because of the Korn inequality

$$\|y\|_{H^1} \leq C\left(\|Dy\|_2 + \|y\|_2\right), \qquad \forall y \in H^1(\mathcal{O}). \tag{2.3}$$

C denotes a generic positive constant, which may depend only on the domain $\mathcal{O}$, the regularity of the boundary Γ, the physical constants ν, α, γ, and K, as defined in (2.5).

Let B be a given Hilbert space with inner product $(\cdot,\cdot)_B$. For a vector

$$h = (h^1, \dots, h^m) \in B^m = \overbrace{B \times ... \times B}^{m\text{-times}},$$

we introduce the norm

$$\|h\|_B = \sum_{i=1}^{m} \|h_i\|_B$$

and we define the absolute value of the inner product of h with a fixed $v \in B$ as

$$|(h, v)_B| = \left(\sum_{k=1}^{m} \left(h^k, v\right)_B^2 \right)^{1/2}. \tag{2.4}$$

Assume that $G(t, y) : [0, T] \times V \to V^m$ is Lipschitz on y, and it satisfies linear growth, i.e., a positive constant K exists such that

$$\begin{aligned} \|G(t, y) - G(t, z)\|_V^2 &\le K \|y - z\|_V^2, \\ \|G(t, y)\|_V &\le K (1 + \|y\|_V), \qquad \forall y, z \in V,\ t \in [0, T]. \end{aligned} \tag{2.5}$$

3. Preliminary results

Let us introduce the Helmholtz projector $\mathbb{P} : L^2(\mathcal{O}) \longrightarrow H$, which is the linear bounded operator defined by $\mathbb{P}y = \tilde{y}$, where $\tilde{y} \in H$ is characterized by the Helmholtz decomposition

$$y = \tilde{y} + \nabla\phi, \qquad \phi \in H^1(\mathcal{O}).$$

We recall some useful inequalities, i.e., the Poincaré inequality

$$\|y\|_2 \le C \|\nabla y\|_2 \qquad \text{for all } y \in V$$

and the Sobolev inequality

$$\|y\|_4 \le C \|\nabla y\|_2 \qquad \text{for all } y \in V.$$

Now, we present the first result in this section, which is a well known and very important property concerning the Navier boundary conditions (see Lemma 4.1 and Corollary 4.2 in [26]). Let k be the curvature of Γ. By parameterizing Γ by the arc length s, the following relation holds

$$\tfrac{\partial \mathrm{n}}{\partial \tau} = \tfrac{d\mathrm{n}}{ds} = k\tau.$$

Lemma 3.1. *Let $y \in H^2(\mathcal{O}) \cap V$ be a vector field that verifies the Navier boundary condition. Then,*

$$\operatorname{curl} y = g(y) \qquad \text{on } \Gamma \qquad \text{with } \ g(y) = (2k - \gamma)\, y \cdot \tau. \tag{3.1}$$

Proof. Let us first note that the anti-symmetric tensor $Ay = \nabla y - (\nabla y)^\top$ can be written in the form

$$Ay = \operatorname{curl} y \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The symmetry of Dy and the anti-symmetry of Ay imply that

$$(Dy)\,\tau \cdot \mathrm{n} = (Dy)\,\mathrm{n} \cdot \tau \quad \text{and} \quad (Ay)\,\tau \cdot \mathrm{n} = -\,(Ay)\,\mathrm{n} \cdot \tau.$$

It follows that

$$(\nabla y)\tau \cdot \mathrm{n} = (Dy)\,\mathrm{n} \cdot \tau - \tfrac{1}{2}\,(Ay)\,\mathrm{n} \cdot \tau,$$

which is equivalent to

$$\operatorname{curl} y = -2(\nabla y)\tau \cdot \mathrm{n} + 2(Dy)\mathrm{n} \cdot \tau. \tag{3.2}$$

By taking the derivative of the expression $y \cdot \mathrm{n} = 0$ in the direction of the tangent vector τ, we deduce that

$$(\nabla y)\tau \cdot \mathrm{n} = -k\, y \cdot \tau. \tag{3.3}$$

The conclusion is then a consequence of (3.2) and (3.3). □

Now, we state a formula that can be derived easily via integration by parts

$$-\int_{\mathcal{O}} \triangle y \cdot z = -\int_{\Gamma} 2(Dy)n \cdot z + \int_{\mathcal{O}} 2\, Dy \cdot Dz, \tag{3.4}$$

which holds for any $y \in H^2(\mathcal{O}) \cap V$ and $z \in H^1(\mathcal{O})$. Using the boundary conditions, this gives the relation

$$-\int_{\mathcal{O}} \triangle y \cdot z = \gamma \int_{\Gamma} y \cdot z + \int_{\mathcal{O}} 2\, Dy \cdot Dz \qquad \text{for any } y \in W \text{ and } z \in V, \tag{3.5}$$

which is used throughout this study.

Let us consider the following modified Stokes system with Navier boundary conditions

$$\begin{cases} h - \alpha \Delta h + \nabla p = f, & \operatorname{div} h = 0 \qquad \text{in } \mathcal{O}, \\ h \cdot \mathrm{n} = 0, & [2(\mathrm{n} \cdot Dh) + \gamma h] \cdot \tau = 0 \qquad \text{on } \Gamma. \end{cases} \tag{3.6}$$

Next, we state a lemma concerning the regularity properties of the solution of this system.

Lemma 3.2. *Suppose that $f \in H^m(\mathcal{O})$, $m = 0, 1$. Then, system (3.6) has a solution $(h, p) \in H^{m+2}(\mathcal{O}) \times H^{m+1}(\mathcal{O})$, and the following estimates hold*

$$\|h\|_{H^2} \le C\|f\|_2, \tag{3.7}$$

$$\|h\|_{H^3} \le C\|f\|_{H^1}. \tag{3.8}$$

Proof. If we suppose that $f \in L^2(\mathcal{O})$, then the existence of the solution (h, p) with h in $H^1(\mathcal{O})$ is given by the Lax–Milgram lemma. Multiplying $(3.6)_1$ by h, we derive

$$\|h\|_2^2 + \alpha \left(2\,\|Dh\|_2^2 + \gamma\|h\|^2_{L^2(\Gamma)}\right) = (f, h) \leq \|f\|_2\,\|h\|_2\,,$$

which gives

$$\|h\|_{H^1} \leq C\,\|f\|_2\,. \tag{3.9}$$

In addition, after applying the operator curl to system (3.6), we derive the following system for $u = \operatorname{curl} h$

$$\begin{cases} u - \alpha\Delta u = \operatorname{curl} f & \text{in } \mathcal{O}, \\ u = g(h) = (2k-\gamma)\ h\cdot\tau & \text{on } \Gamma. \end{cases} \tag{3.10}$$

Let us denote the extension of the unit exterior normal n (and the tangent $\tau = (-n_2, n_1)$) on the whole domain $\overline{\mathcal{O}}$ by the same notation, n (and τ). Then, the function $z = u - (2k-\gamma)\ h\cdot\tau$ solves the system

$$\begin{cases} z - \alpha\Delta z = \operatorname{curl} f - (2k-\gamma)\ h\cdot\tau + \alpha\Delta\left[(2k-\gamma)\ h\cdot\tau\right] & \text{in } \mathcal{O}, \\ z = 0 & \text{on } \Gamma. \end{cases} \tag{3.11}$$

Multiplying equation $(3.11)_1$ by z, integrating by parts, and using (3.9), we deduce that

$$\|z\|_2 + \alpha\|\nabla z\|_2 \leq C\left(\|f\|_2 + \|h\|_{H^1}\right) \leq C\|f\|_2,$$

which implies that

$$\|u\|_{H^1} \leq C\left(\|f\|_2 + \|h\|_{H^1}\right) \leq C\|f\|_2. \tag{3.12}$$

In addition, estimate (2.3.3.7), p. 110 in [22] for system (3.10) gives

$$\begin{aligned} \|u\|_{H^2} &\leq C\left(\|\operatorname{curl} f\|_2 + \|(2k-\gamma)\ h\cdot\tau\|_{H^{2-\frac12}(\Gamma)}\right) \\ &\leq C\left(\|f\|_{H^1} + \|h\|_{H^2}\right). \end{aligned} \tag{3.13}$$

h solves system (3.6), so a stream function φ exists such that $h = \nabla^\perp\varphi$, which satisfies the system

$$\begin{cases} \Delta\varphi = u & \text{in } \mathcal{O}, \\ \varphi = 0 & \text{on } \Gamma \end{cases} \tag{3.14}$$

and the estimate

$$\|\varphi\|_{H^{2+m}} \leq \|u\|_{H^m}, \qquad m \in \mathbb{N}_0, \tag{3.15}$$

by Theorem 2.5.1.1, p. 128 in [22].

By combining (3.12) and (3.15) with $m = 1$, we deduce that

$$\|\varphi\|_{H^3} \leq C\|u\|_{H^1} \leq C\|f\|_2;$$

hence, $h = \nabla^\perp\varphi \in H^2$ and (3.7) hold. Moreover, (3.13) and (3.15) with $m = 2$ imply that

$$\|\varphi\|_{H^4} \leq \|u\|_{H^2} \leq C\left(\|f\|_{H^1} + \|h\|_{H^2}\right).$$

Given (3.7), we conclude that $h = \nabla^\perp\varphi \in H^3$ and (3.8) hold. $\square$

Let us recall that the space W introduced in (2.2) is naturally endowed with the Sobolev norm $\|\cdot\|_{H^2}$. The next result follows directly from Lemma 5 in [5] and it helps to introduce W as an equivalent norm used to analyze the stability in Section 5.

Lemma 3.3. *For each $y \in W$, we have*

$$\|\upsilon(y) - \mathbb{P}\upsilon(y)\|_2 \le C\,\|y\|_{H^1}\,, \tag{3.16}$$

$$\|\upsilon(y) - \mathbb{P}\upsilon(y)\|_{H^1} \le C\,\|y\|_{H^2}\,. \tag{3.17}$$

The next regularity result is fundamental for establishing the well-posedness of the velocity equation (see Propositions 6 in [6] and Lemma 2.1 in [12] for similar results).

Lemma 3.4. *Let $y \in \widetilde{W}$. Then, the following estimates hold*

$$\|y\|_{H^2} \le C\,(\|\mathbb{P}\,\upsilon(y)\|_2 + \|y\|_{H^1})\,, \tag{3.18}$$

$$\|y\|_{H^3} \le C\,(\|\mathrm{curl}\,\upsilon(y)\|_2 + \|y\|_{H^1})\,. \tag{3.19}$$

Proof. Considering system (3.6) with $f = \upsilon(y)$, then the pair $(y, 0)$ is obviously the solution of this system. Hence, the estimate (3.7) yields

$$\|y\|_{H^2} \le C\,\|\upsilon(y)\|_2 \le C\,(\|\upsilon(y) - \mathbb{P}\upsilon(y)\|_2 + \|\mathbb{P}\upsilon(y)\|_2)\,.$$

By applying (3.16), we can obtain (3.18).

$\mathrm{curl}\,\upsilon(y) \in L^2(\mathcal{O})$ and $\nabla\cdot(\mathrm{curl}\,\upsilon(y)) = 0$, so a unique vector-potential $\psi \in H^1(\mathcal{O})$ exists such that

$$\begin{cases} \mathrm{curl}\,\psi = \mathrm{curl}\,\upsilon(y), \qquad \mathrm{div}\,\psi = 0 & \text{in } \mathcal{O}, \\ \psi\cdot \mathrm{n} = 0 & \text{on } \Gamma \end{cases}$$

and

$$\|\psi\|_{H^1} \le C\,\|\mathrm{curl}\,\upsilon(y)\|_2\,. \tag{3.20}$$

It follows that $\mathrm{curl}\,(y - \alpha\Delta y - \psi) = 0$ and $\pi \in L^2(\mathcal{O})$ exists such that

$$y - \alpha\Delta y - \psi + \nabla\pi = 0.$$

Hence, y is the solution of the Stokes system (3.6), where f is replaced by ψ.

As a consequence of (3.8), we have

$$\|y\|_{H^3} \le C\,(\|\psi\|_{H^1} + \|y\|_{H^1})\,. \tag{3.21}$$

Using (3.20), we obtain the claimed result (3.19). $\square$

In order to define the solution of equation $(2.1)_1$ in the distributional sense, we introduce a trilinear functional, which is well known in the context of Navier–Stokes equations

$$b(\phi, z, y) = (\phi\cdot\nabla z, y)\,, \qquad \forall \phi, z, y \in V.$$

In the following, we often employ the property

$$b(\phi, z, y) = -b(\phi, y, z), \tag{3.22}$$

which is obtained via integration by parts given that ϕ is divergence-free and $(\phi \cdot n) = 0$ on Γ.

Straightforward computations yield the following relation

$$(\operatorname{curl} \upsilon(y) \times z, \phi) = b(\phi, z, \upsilon(y)) - b(z, \phi, \upsilon(y)) \qquad \forall y \in \widetilde{W},\ z, \phi \in V. \tag{3.23}$$

In the next lemma, we deduce crucial estimates that are important for establishing the well-posedness of system (2.1), as well as for proving the stability property of their solutions. We should note that some estimates follow from an adaptation of the method considered in [6] to prove the uniqueness.

Lemma 3.5. *Let $y,\ z,\ \phi \in \widetilde{W}$. Then,*

$$|(\operatorname{curl} \upsilon(y) \times z, \phi)| \leq C \|y\|_{H^3} \|z\|_{H^1} \|\phi\|_{H^3}, \tag{3.24}$$

$$|(\operatorname{curl} \upsilon(y) \times z, \phi)| \leq C \|y\|_{H^1} \|z\|_{H^3} \|\phi\|_{H^3}, \tag{3.25}$$

$$|(\operatorname{curl} \upsilon(y) \times z, y)| \leq C \|y\|_{H^1}^2 \|z\|_{H^3}. \tag{3.26}$$

Proof. *First step. The proof of estimate (3.24).* We can directly estimate

$$|(\operatorname{curl} \upsilon(y) \times z, \phi)| \leq \|\phi\|_\infty \|z\|_2 \|\operatorname{curl} \upsilon(y)\|_2 \leq \|\phi\|_{H^3} \|z\|_2 \|y\|_{H^3}$$

by Sobolev's embedding $H^3(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O})$. Hence, we have (3.24).

Second step. The proof of estimate (3.25). Equality (3.23) gives

$$(\operatorname{curl} \upsilon(y) \times z, \phi) = b(\phi, z, y) - b(z, \phi, y) - \alpha \left(b(\phi, z, \Delta y) - b(z, \phi, \Delta y)\right). \tag{3.27}$$

Using Sobolev's embedding $H^1(\mathcal{O}) \hookrightarrow L^4(\mathcal{O})$, it is easy to see that

$$\begin{aligned} |b(\phi, z, y) - b(z, \phi, y)| &= |b(\phi, z, y) + b(z, \phi, y)| \\ &\leq \|\phi\|_4 \|\nabla z\|_2 \|y\|_4 + \|z\|_4 \|\nabla \phi\|_2 \|y\|_4 \\ &\leq C \|\phi\|_{H^1} \|z\|_{H^1} \|y\|_{H^1}. \end{aligned} \tag{3.28}$$

After integrating by parts and using the boundary conditions, we derive

$$\begin{aligned} b(\phi, z, \Delta y) &= \sum_{i,j=1}^{2} \int_{\mathcal{O}} \phi_i \frac{\partial z_j}{\partial x_i} \Delta y_j = \sum_{i,j,k=1}^{2} \int_{\mathcal{O}} \phi_i \frac{\partial z_j}{\partial x_i} \frac{\partial}{\partial x_k} \left(\frac{\partial y_j}{\partial x_k} - \frac{\partial y_k}{\partial x_j} \right) = \sum_{i,j,k=1}^{2} \int_{\mathcal{O}} \phi_i \frac{\partial z_j}{\partial x_i} \frac{\partial}{\partial x_k} (A_{jk}(y)) \\ &= \sum_{i,j,k=1}^{2} \int_{\Gamma} \phi_i \frac{\partial z_j}{\partial x_i} A_{jk}(y) n_k - \sum_{i,j,k=1}^{2} \int_{\mathcal{O}} \frac{\partial}{\partial x_k} \left(\phi_i \frac{\partial z_j}{\partial x_i} \right) A_{jk}(y) \\ &= \sum_{i,j=1}^{2} \int_{\Gamma} \phi_i \frac{\partial z_j}{\partial x_i} g(y) \tau_j - \sum_{i,j,k=1}^{2} \int_{\mathcal{O}} \frac{\partial \phi_i}{\partial x_k} \frac{\partial z_j}{\partial x_i} A_{jk}(y) - \sum_{i,j,k=1}^{2} \int_{\mathcal{O}} \phi_i \frac{\partial^2 z_j}{\partial x_k \partial x_i} A_{jk}(y) \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{3.29}$$

Again, after integrating by parts, it follows that

$$
\begin{aligned}
I_1 &= \sum_{i,j=1}^{2} \int_{\Gamma} \phi_i \tfrac{\partial z_j}{\partial x_i}\, g(y)\tau_j = \sum_{i,j=1}^{2} \int_{\mathcal{O}} \phi_i \tfrac{\partial z_j}{\partial x_i}\, g(y)\tau_j \operatorname{div} \mathrm{n} + \sum_{i,j,k=1}^{2} \int_{\mathcal{O}} \tfrac{\partial}{\partial x_k} \big(\phi_i \tfrac{\partial z_j}{\partial x_i}\, g(y)\tau_j\big) n_k \\
&= b\left(\phi, z, \operatorname{div} \mathrm{n}\, g(y)\tau\right) + b\left((\mathrm{n}\cdot\nabla)\,\phi, z,\, n_k\, g(y)\tau\right) + \sum_{k=1}^{2} b\left(\phi,\, \tfrac{\partial z}{\partial x_k}, n_k\, g(y)\tau\right) \\
&\quad + b\left(\phi, z,\, (\mathrm{n}\cdot\nabla)\,\big(g(y)\tau\big)\right).
\end{aligned}
$$

Then, using Sobolev's embedding $H^2(\mathcal{O}) \hookrightarrow L^\infty(\mathcal{O})$, we can easily derive

$$
\begin{aligned}
&|b\left(\phi, z, \Delta y\right)| \le |b\left(\phi, z, \operatorname{div} \mathrm{n}\, g(y)\tau\right)| + |b\left((\mathrm{n}\cdot\nabla)\,\phi, z,\; g(y)\tau\right)| + \left|b\left(\phi, z,\, (\mathrm{n}\cdot\nabla)\,\big(g(y)\tau\big)\right)\right| \\
&\quad + \sum_{k=1}^{2} \left(\left|b\left(\phi,\, \tfrac{\partial z}{\partial x_k},\, n_k\, g(y)\tau\right)\right| + \left|b\left(\tfrac{\partial \phi}{\partial x_k}, z, A_{\cdot\, k}(y)\right)\right| + \left|b\left(\phi,\, \tfrac{\partial z}{\partial x_k}, A_{\cdot\, k}(y)\right)\right|\right) \\
&\le C\|z\|_{H^3}\|\phi\|_{H^3}\|y\|_{H^1}.
\end{aligned} \tag{3.30}
$$

By symmetry, it follows that

$$
|b\left(z, \phi, \Delta y\right)| \le C\|z\|_{H^3}\|\phi\|_{H^3}\|y\|_{H^1}. \tag{3.31}
$$

Then, (3.25) follows from (3.27)–(3.31).

Third step. The proof of estimate (3.26). As given in the computations above for (3.29), we obtain

$$
\begin{aligned}
b\left(z, y, \Delta y\right) &= \sum_{i,j=1}^{2} \int_{\Gamma} z_i \tfrac{\partial y_j}{\partial x_i}\, g(y)\tau_j - \sum_{i,j,k=1}^{2} \int_{\mathcal{O}} \tfrac{\partial z_i}{\partial x_k} \tfrac{\partial y_j}{\partial x_i} A_{jk}(y) - \sum_{i,j,k=1}^{2} \int_{\mathcal{O}} z_i \tfrac{\partial^2 y_j}{\partial x_k \partial x_i} A_{jk}(y) \\
&= J_1 + J_2 + J_3,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &= b\left(z, y, \operatorname{div} \mathrm{n}\, g(y)\tau\right) + b\left((\mathrm{n}\cdot\nabla)\, z, y,\; g(y)\tau\right) + \sum_{k=1}^{2} b\left(z,\, \tfrac{\partial y}{\partial x_k}, n_k\, g(y)\tau\right) \\
&\quad + b\left(z, y, (\mathrm{n}\cdot\nabla)\,\big(g(y)\tau\big)\right), \\
J_2 &= -\sum_{k=1}^{2} b\left(\tfrac{\partial z}{\partial x_k}, y, A_{\cdot\, k}(y)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
J_3 &= -\sum_{i,j,k=1}^{2} \int_{\mathcal{O}} z_i \tfrac{\partial}{\partial x_i}\left(\tfrac{\partial y_j}{\partial x_k}\right)\left(\tfrac{\partial y_j}{\partial x_k} - \tfrac{\partial y_k}{\partial x_j}\right) = \sum_{i,j,k=1}^{2} \int_{\mathcal{O}} z_i \tfrac{\partial}{\partial x_i}\left(\tfrac{\partial y_j}{\partial x_k}\right) \tfrac{\partial y_k}{\partial x_j} \\
&= \frac{1}{2} \sum_{i,j,k=1}^{2} \int_{\mathcal{O}} z_i \left[\tfrac{\partial}{\partial x_i}\left(\tfrac{\partial y_j}{\partial x_k}\right) \tfrac{\partial y_k}{\partial x_j} + \tfrac{\partial}{\partial x_i}\left(\tfrac{\partial y_k}{\partial x_j}\right) \tfrac{\partial y_j}{\partial x_k}\right] = \sum_{i,j,k=1}^{2} \int_{\mathcal{O}} z_i \tfrac{\partial}{\partial x_i}\left(\tfrac{\partial y_j}{\partial x_k} \tfrac{\partial y_k}{\partial x_j}\right) = 0.
\end{aligned}
$$

Therefore, we derive

$$|b(z,y,\Delta y)| \le |b(z,y,\operatorname{div}\mathrm{n}\,g(y)\tau)| + |b((\mathrm{n}\cdot\nabla)\,z,y,\,g(y)\tau)| + \sum_{k=1}^{2}\left|b\left(z,\,n_k\,g(y)\tau,\,\frac{\partial y}{\partial x_k}\right)\right|$$

$$+\left|b\left(z,y,(\mathrm{n}\cdot\nabla)\left(g(y)\tau\right)\right)\right| + \sum_{k=1}^{2}\left|b\left(\frac{\partial z}{\partial x_k},y,A_{\cdot k}(y)\right)\right| \le C\|z\|_{H^2}\|y\|_{H^1}^2, \tag{3.32}$$

where we use

$$\sum_{k=1}^{2} b\left(z,\,\frac{\partial y}{\partial x_k},n_k\,g(y)\tau\right) = -\sum_{k=1}^{2} b\left(z,\,n_k\,g(y)\tau,\,\frac{\partial y}{\partial x_k}\right)$$

by (3.22).

By taking $\phi = y$ in (3.29), we have

$$b(y,z,\Delta y) = \sum_{i,j=1}^{2}\int_{\Gamma} y_i \frac{\partial z_j}{\partial x_i} g(y)\tau_j - \sum_{k=1}^{2} b\left(\frac{\partial y}{\partial x_k},z,A_{\cdot k}(y)\right) - \sum_{k=1}^{2} b\left(y,\frac{\partial z}{\partial x_k},A_{\cdot k}(y)\right).$$

Considering the embedding theorems $H^2(\mathcal{O}) \hookrightarrow C(\overline{\mathcal{O}})$, $H^1(\mathcal{O}) \hookrightarrow L^2(\Gamma)$ and $H^1(\mathcal{O}) \hookrightarrow L^4(\mathcal{O})$, we have

$$\begin{aligned}|b(y,z,\Delta y)| &\le \sum_{i,j=1}^{2}\int_{\Gamma}\left|y_i \frac{\partial z_j}{\partial x_i} g(y)\tau_j\right| + \sum_{k=1}^{2}\left(\left|b\left(\frac{\partial y}{\partial x_k},z,A_{\cdot k}(y)\right)\right| + \left|b\left(y,\frac{\partial z}{\partial x_k},A_{\cdot k}(y)\right)\right|\right) \\ &\le C\|y\|_{L^2(\Gamma)}^2\,\|\nabla z\|_{C(\overline{\mathcal{O}})} + C\,\|\nabla y\|_2^2\,\|\nabla z\|_\infty + \sum_{i,j=1}^{2}\|y\|_4\left\|\frac{\partial^2 z}{\partial x_i\partial x_k}\right\|_4\|\nabla y\|_2 \\ &\le C\,\|y\|_{H^1}^2\,\|z\|_{H^3}\,. \end{aligned} \tag{3.33}$$

Then, (3.26) is a consequence of (3.28) and (3.32)–(3.33). □

4. Existence of the strong solution

In this section, we establish the existence of a strong solution for system (2.1) in the probabilistic sense.

Definition 4.1. Let

$$U \in L^2(\Omega\times(0,T);H(\mathrm{curl};\mathcal{O})), \qquad Y_0 \in L^2(\Omega,\widetilde{W}).$$

A stochastic process $Y \in L^2(\Omega, L^\infty(0,T;\widetilde{W}))$ is a strong solution of (2.1) if for a.e.-P and a.e. $t \in (0,T)$, the following equation holds

$$\begin{aligned}(v(Y(t)),\phi) = &\int_0^t\left[-2\nu\,(DY(s),D\phi) - \nu\gamma\int_{\Gamma} y\cdot\phi\,dx - (\operatorname{curl} v(Y(s))\times Y(s),\phi)\right]ds \\ &+ (v(Y(0)),\phi) + \int_0^t (U(s),\phi)\,ds + \int_0^t (G(s,Y(s)),\phi)\,dW_s \end{aligned} \tag{4.1}$$

for all $\phi \in V$, where the nonlinear term should be understood in the sense of

$$(\operatorname{curl} \upsilon(Y(t)) \times Y(t), \phi) = b(\phi, Y(t), \upsilon(Y(s))) - b(Y(t), \phi, \upsilon(Y(s)))$$

and the stochastic integral is defined by

$$\int_0^t (G(s, Y(s)), \phi)\, dW_s = \sum_{k=1}^m \int_0^t \left(G^k(s, Y(s)), \phi\right)\, dW_s^k.$$

We now formulate our main existence and uniqueness result in this section.

Theorem 4.2. *Assume that*

$$U \in L^p(\Omega \times (0,T); H(\operatorname{curl}; \mathcal{O})), \qquad Y_0 \in L^p(\Omega, V) \cap L^2(\Omega, \widetilde{W}) \qquad \textit{for some} \;\; 4 \le p < \infty.$$

Then, a unique solution Y to equation (4.1) exists that belongs to

$$L^2(\Omega, L^\infty(0,T; \widetilde{W})) \cap L^p(\Omega, L^\infty(0,T; V)).$$

Moreover, the following estimates hold

$$\frac{1}{2}\mathbb{E} \sup_{s\in[0,t]} \|Y(s)\|_V^2 + \mathbb{E}\int_0^t \left(4\nu \|DY\|_2^2 + 2\nu\gamma \|Y\|_{L^2(\Gamma)}^2\right) ds \le C\left(\mathbb{E}\|Y_0\|_V^2 + \mathbb{E}\|U\|_{L^2(0,t;L^2)}^2 + 1\right),$$

$$\mathbb{E} \sup_{s\in[0,t]} \|\operatorname{curl} \upsilon(Y(s))\|_2^2 \le C\left(\mathbb{E}\|\operatorname{curl} \upsilon(Y_0)\|_2^2 + \mathbb{E}\|U\|_{L^2(0,t;H^1)}^2 + 1\right).$$

The proof of the theorem is obtained by *Galerkin's approximation method.* We consider the inner product of $\widetilde{W}$ defined by

$$(y,z)_{\widetilde{W}} = (\operatorname{curl}\upsilon(y), \operatorname{curl}\upsilon(z)) + (y,z)_V\,. \tag{4.2}$$

Considering (2.3) and (3.19), the norm $\|\cdot\|_{\widetilde{W}}$ induced by this inner product is equivalent to $\|\cdot\|_{H^3}$. The injection operator $I : \widetilde{W} \to V$ is a compact operator, so a basis $\{e_i\} \subset \widetilde{W}$ of eigenfunctions exists

$$(y, e_i)_{\widetilde{W}} = \lambda_i (y, e_i)_V\,, \qquad \forall y \in \widetilde{W},\; i \in \mathbb{N}, \tag{4.3}$$

which is an orthonormal basis for V and the corresponding sequence $\{\lambda_i\}$ of eigenvalues verifies $\lambda_i > 0$, $\forall i \in \mathbb{N}$ and $\lambda_i \to \infty$ as $i \to \infty$. We note that the ellipticity of Equation (4.3) increases the regularity of the solutions. Hence, without loss of generality, we can consider $\{e_i\} \subset H^4$ (see [6]).

In this section, we consider this basis and introduce the Faedo–Galerkin approximation of system (2.1). Let $W_n = \operatorname{span}\{e_1, \ldots, e_n\}$ and define

$$Y_n(t) = \sum_{j=1}^n c_j^n(t) e_j$$

as the solution of the stochastic differential equation

$$\begin{cases} d(\upsilon(Y_n), \phi) = ((\nu\Delta Y_n - \operatorname{curl}(\upsilon(Y_n)) \times Y_n + U), \phi)\, dt + (G(t, Y_n), \phi)\, dW_t, \\ Y_n(0) = Y_{n,0}, \qquad\qquad \forall \phi \in W_n, \end{cases} \tag{4.4}$$

where $Y_{n,0}$ denotes the projection of the initial condition Y_0 onto the space W_n.

We note that $\{\widetilde{e}_j = \frac{1}{\sqrt{\lambda_j}} e_j\}_{j=1}^{\infty}$ is an orthonormal basis for $\widetilde{W}$ and

$$Y_{n,0} = \sum_{j=1}^{n} (Y_0, e_j)_V \, e_j = \sum_{j=1}^{n} (Y_0, \widetilde{e}_j)_{\widetilde{W}} \, \widetilde{e}_j,$$

and thus the Parseval's identity gives

$$\|Y_n(0)\|_V \leq \|Y_0\|_V \quad \text{and} \quad \|Y_n(0)\|_{\widetilde{W}} \leq \|Y_0\|_{\widetilde{W}} \,. \tag{4.5}$$

Equation (4.4) defines a system of stochastic ordinary differential equations in $\mathbb{R}^n$ with locally Lipschitz nonlinearities. Hence, a local-in-time solution Y_n exists as an adapted process in the space $C([0, T_n]; W_n)$. The global-in-time existence of Y_n follows from the uniform estimates based on $n = 1, 2, ...$, which are deduced in the next lemma (similar reasoning was employed by [1,34]).

Lemma 4.3. *Assume that*

$$U \in L^2(\Omega \times (0,T); H(\text{curl}; \mathcal{O})), \qquad Y_0 \in L^2(\Omega, \widetilde{W}).$$

Then, problem (4.4) admits a unique solution $Y_n \in L^2(\Omega, L^\infty(0,T; \widetilde{W}))$*. Furthermore, for any* $t \in [0,T]$*, the following estimates hold*

$$\frac{1}{2}\mathbb{E} \sup_{s\in[0,t]} \|Y_n(s)\|_V^2 + \mathbb{E}\int_0^t \left(4\nu \|DY_n\|_2^2 + 2\nu\gamma \|Y_n\|_{L^2(\Gamma)}^2\right) ds$$
$$\leq C\left(1 + \mathbb{E}\,\|Y_0\|_V^2 + \mathbb{E}\|U\|_{L^2(0,t;L^2)}^2\right), \tag{4.6}$$

$$\frac{1}{2}\mathbb{E} \sup_{s\in[0,t]} \|\text{curl}\, v\,(Y_n(s))\|_2^2 + \frac{2\nu}{\alpha}\mathbb{E}\int_0^t \|\text{curl}\, v\,(Y_n)\|_2^2 \, ds \leq \mathbb{E}\,\|\text{curl}\, v\,(Y_0)\|_2^2$$
$$+ C\mathbb{E}\int_0^t \|\text{curl}\, U\|_2^2 \, ds + C\mathbb{E}\int_0^t \left(1 + \|Y_n\|_V^2\right) ds \tag{4.7}$$

and

$$\mathbb{E} \sup_{s\in[0,t]} \|Y_n(s)\|_{\widetilde{W}}^2 \leq C(\mathbb{E}\,\|Y_0\|_{\widetilde{W}}^2 + \mathbb{E}\,\|\,U\|_{L^2(0,t;H(\text{curl};\mathcal{O}))}^2), \tag{4.8}$$

where C *are positive constants that are independent of* n *(and they may depend on the data considered for our problem in the domain* $\mathcal{O}$*, the regularity of* Γ*, and the physical constants* ν*,* α*,* γ*).*

Proof. For each $n \in \mathbb{N}$, let us consider the sequence $\{\tau_N^n\}_{N\in\mathbb{N}}$ of the stopping times

$$\tau_N^n = \inf\{t \geq 0 : \|Y_n(t)\|_{H^3} \geq N\} \wedge T_n.$$

In order to simplify the notation, let us introduce the function

$$f(Y_n) = (\nu\Delta Y_n - \text{curl}\,(v\,(Y_n)) \times Y_n + U) \in H^1(\mathcal{O}).$$

By taking $\phi = e_i$ for each $i = 1, \dots, n$ in Equation (4.4), we obtain

$$d\,(Y_n, e_i)_V = (f(Y_n), e_i)\; dt + (G(t, Y_n), e_i)\; dW_t. \tag{4.9}$$

Step 1. In the space V for Y_n, obtain an estimate depending on the stopping times τ_N^n.

The Itô formula gives

$$d\,(Y_n, e_i)_V^2 = 2\,(Y_n, e_i)_V\,(f(Y_n), e_i)\; dt + 2\,(Y_n, e_i)_V\,(G(t, Y_n), e_i)\; dW_t + |(G\,(t, Y_n)\,, e_i)|^2\; dt,$$

where the module in the last term is defined by (2.4). Summing these equalities over $i = 1, \dots, n$, we obtain

$$d\,\|Y_n\|_V^2 = 2\,(f(Y_n), Y_n)\; dt + 2\,(G(t, Y_n), Y_n)\; dW_t + \sum_{i=1}^{n} |(G\,(t, Y_n)\,, e_i)|^2\; dt.$$

We know that

$$\begin{aligned}(f(Y_n), Y_n) &= -2\nu\,\|DY_n\|_2^2 - \nu\gamma\,\|Y_n\|_{L^2(\Gamma)}^2 + \alpha\,(b\,(Y_n, Y_n, \Delta Y_n) - b\,(Y_n, Y_n, \Delta Y_n))\\ &\quad - b\,(Y_n, Y_n, Y_n) + (U, Y_n)\\ &= -2\nu\,\|DY_n\|_2^2 - \nu\gamma\,\|Y_n\|_{L^2(\Gamma)}^2 + (U, Y_n)\,,\end{aligned} \tag{4.10}$$

and thus

$$\begin{aligned}d\,\|Y_n\|_V^2 &= 2\left(-2\nu\,\|DY_n\|_2^2 - \nu\gamma\,\|Y_n\|_{L^2(\Gamma)}^2 + (U, Y_n)\right)\; dt\\ &\qquad + 2\,(G(t, Y_n), Y_n)\; dW_t + \sum_{i=1}^{n} |(G\,(t, Y_n)\,, e_i)|^2\; dt.\end{aligned} \tag{4.11}$$

Let $\tilde{G}_n$ be the solution of (3.6) for $f = G(t, Y_n)$. Then,

$$(\tilde{G}_n, e_i)_V = (G(t, Y_n), e_i) \quad \text{for } i = 1, \dots, n$$

which implies that

$$\sum_{i=1}^{n} |(G\,(t, Y_n)\,, e_i)|^2 = \|\widetilde{G}_n\|_V^2 \le C\,\|G\,(t, Y_n)\|_2^2 \le C(1 + \|Y_n\|_V^2), \tag{4.12}$$

where we use the fact that $\widetilde{G}_n$ solves the elliptic type problem (3.6) for $f = G(t, Y_n)$ and assumption $(2.5)_2$.

Let us take $t \in [0, T]$, and after integrating over the time interval $(0, s)$, $0 \le s \le \tau_N^n \wedge t$ of equality (4.11) and estimating (4.12), we have

$$\begin{aligned}\|Y_n(s)\|_V^2 + \int_0^s (4\nu\,\|DY_n\|_2^2 \; + 2\nu\gamma\,\|Y_n\|_{L^2(\Gamma)}^2)dr &\le \|Y_n(0)\|_V^2 + C(1 + \int_0^s \|U\|_2^2\; dr)\\ &\quad + \int_0^s \|Y_n\|_V^2\; dr + 2\int_0^s (G\,(r, Y_n)\,, Y_n)\; dW_r.\end{aligned} \tag{4.13}$$

The Burkholder–Davis–Gundy inequality gives

$$\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \left| \int_0^s (G(r,Y_n), Y_n)\, dW_r \right| \leq \mathbb{E} \left(\int_0^{\tau_N^n\wedge t} |(G(s,Y_n), Y_n)|^2\, ds \right)^{\frac{1}{2}}$$

$$\leq \mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \|Y_n(s)\|_2 \left(\int_0^{\tau_N^n\wedge t} \|G(s,Y_n)\|_2^2\, ds \right)^{\frac{1}{2}}$$

$$\leq \varepsilon\, \mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \|Y_n(s)\|_2^2 + C_\varepsilon \mathbb{E} \int_0^{\tau_N^n\wedge t} (1+\|Y_n\|_V^2)\, ds.$$

By substituting the last inequality with the selected $\varepsilon = \frac{1}{2}$ in (4.13) and considering (4.5), we derive

$$\frac{1}{2}\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \|Y_n(s)\|_V^2 + \mathbb{E} \int_0^{\tau_N^n\wedge t} \left(4\nu \|DY_n\|_2^2 + 2\nu\gamma \|Y_n\|_{L^2(\Gamma)}^2 \right) ds \leq \mathbb{E}\|Y_0\|_V^2$$

$$+ C\mathbb{E} \int_0^t (1+\|U\|_2^2)\, ds + C\mathbb{E} \int_0^{\tau_N^n\wedge t} \|Y_n\|_V^2\, ds.$$

Hence, if we denote $1_{[0,\tau_N^n]}$ as the characteristic function of the interval $[0,\tau_N^n]$, then the function

$$f(t) = \mathbb{E} \sup_{s\in[0,t]} 1_{[0,\tau_N^n]} \|Y_n(s)\|_V^2$$

fulfills the Gronwall type inequality

$$\frac{1}{2} f(t) \leq C \int_0^t f(s) ds + \mathbb{E}\|Y_n(0)\|_V^2 + C\mathbb{E} \int_0^t (1+\|U\|_2^2)\, ds,$$

which implies that

$$\frac{1}{2}\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \|Y_n(s)\|_V^2 + \mathbb{E} \int_0^{\tau_N^n\wedge t} \left(4\nu \|DY_n\|_2^2 + 2\nu\gamma \|Y_n\|_{L^2(\Gamma)}^2 \right) ds$$

$$\leq C \left(1 + \mathbb{E}\|Y_0\|_V^2 + \mathbb{E}\|U\|_{L^2(0,t;L^2)}^2 \right). \tag{4.14}$$

Step 2. L^2 estimate for $\operatorname{curl} v(Y_n)$ *depending on the stopping times* τ_N^n.

The deduction of this estimate is quite long. First, let us consider the solutions $\tilde{f}_n$ and $\tilde{G}_n$ of (3.6) for $f = f(Y_n)$ and $f = G(t,Y_n)$, respectively. Then, the following relations hold

$$(\tilde{f}_n, e_i)_V = (f(Y_n), e_i), \qquad (\tilde{G}_n, e_i)_V = (G(t,Y_n), e_i). \tag{4.15}$$

If we use these relations in Equality (4.9), we obtain

$$d\,(Y_n, e_i)_V = \left(\tilde{f}_n, e_i\right)_V dt + \left(\tilde{G}_n, e_i\right)_V dW_t.$$

Multiplying the last identity by λ_i and using (4.3) in the resulting equation yields

$$d\,(Y_n,e_i)_{\widetilde{W}}=(\tilde f_n,e_i)_{\widetilde{W}}\,dt+(\tilde G_n,e_i)_{\widetilde{W}}\,dW_t.$$

In addition, the Itô formula gives

$$d\,(Y_n,e_i)_{\widetilde{W}}^2=2\,(Y_n,e_i)_{\widetilde{W}}\,(\tilde f_n,e_i)_{\widetilde{W}}\,dt+2\,(Y_n,e_i)_{\widetilde{W}}\,(\tilde G_n,e_i)_{\widetilde{W}}\,dW_t+|(\tilde G_n,e_i)_{\widetilde{W}}|^2\,dt.$$

After multiplying this equality by $\frac{1}{\lambda_i}$ and summing over $i=1,\ldots,n$, we obtain

$$d\,\|Y_n\|_{\widetilde{W}}^2=2(\tilde f_n,Y_n)_{\tilde W}\,dt+2(\tilde G_n,Y_n)_{\widetilde{W}}\,dW_t+\sum_{i=1}^n\frac{1}{\lambda_i}|(\tilde G_n,e_i)_{\widetilde{W}}|^2\,dt,$$

i.e.,

$$\begin{aligned}d(\|\operatorname{curl} \upsilon\,(Y_n)\,\|_2^2\;+\|Y_n\|_V^2)&=2\left((\operatorname{curl} \upsilon(\tilde f_n),\operatorname{curl} \upsilon(Y_n))+(\tilde f_n,Y_n)_V\right)\,dt\\&\quad+2\left((\operatorname{curl} \upsilon(\tilde G_n),\operatorname{curl} \upsilon(Y_n))+(\tilde G_n,Y_n)_V\right)\,dW_t+\sum_{i=1}^n\lambda_i|(G(t,Y_n),e_i)_V|^2\,dt\end{aligned}$$

by the definition of the inner product (4.2). The definitions of $\tilde f_n$ and $\tilde G_n$ as solutions of (3.6) imply that

$$\begin{aligned}d(\|\operatorname{curl} \upsilon\,(Y_n)\,\|_2^2\;+\|Y_n\|_V^2)&=2\,((\operatorname{curl} f(Y_n),\operatorname{curl} \upsilon(Y_n))+(f(Y_n),Y_n))\,dt\\&\quad+2\,((\operatorname{curl} G(t,Y_n),\operatorname{curl} \upsilon(Y_n))+(G(t,Y_n),Y_n))\,dW_t+\sum_{i=1}^n\lambda_i|(G(t,Y_n),e_i)_V|^2\,dt,\end{aligned}$$

which reduces to

$$\begin{aligned}d\|\operatorname{curl} \upsilon\,(Y_n)\,\|_2^2&=2\,((\operatorname{curl} f(Y_n),\operatorname{curl} \upsilon(Y_n)))\,dt\\&\quad+2\,((\operatorname{curl} G(t,Y_n),\operatorname{curl} \upsilon(Y_n)))\,dW_t+\sum_{i=1}^n(\lambda_i-1)|(G(t,Y_n),e_i)|^2\,dt,\end{aligned}\tag{4.16}$$

considering Equality (4.11).

Since

$$\operatorname{curl}\,(\operatorname{curl}\,(\upsilon(Y_n))\times Y_n)=(Y_n\cdot\nabla)\operatorname{curl} \upsilon(Y_n)\quad\text{and}\quad((Y_n\cdot\nabla)\operatorname{curl} \upsilon(Y_n),\operatorname{curl} \upsilon(Y_n))=0,$$

then we have

$$\begin{aligned}(\operatorname{curl} f(Y_n),\operatorname{curl} \upsilon(Y_n))&=(\nu\operatorname{curl}\Delta Y_n+\operatorname{curl} U,\operatorname{curl} \upsilon(Y_n))\\&=\left(-\frac{\nu}{\alpha}\operatorname{curl} \upsilon(Y_n)+\frac{\nu}{\alpha}\operatorname{curl} Y_n+\operatorname{curl} U,\operatorname{curl} \upsilon(Y_n)\right).\end{aligned}$$

By substituting the last relation into (4.16), we derive

$$\begin{aligned}d\,\|\operatorname{curl} \upsilon\,(Y_n)\|_2^2\;+\frac{2\nu}{\alpha}\,\|\operatorname{curl} \upsilon\,(Y_n)\|_2^2\,dt&=2\left(\frac{\nu}{\alpha}\operatorname{curl} Y_n+\operatorname{curl} U,\operatorname{curl} \upsilon(Y_n)\right)\,dt\\&\quad+2\,((\operatorname{curl} G(t,Y_n),\operatorname{curl} \upsilon(Y_n)))\,dW_t+\sum_{i=1}^n(\lambda_i-1)|(G(t,Y_n),e_i)|^2\,dt.\end{aligned}\tag{4.17}$$

Let us take $t\in[0,T]$. By integrating over the time interval $(0,s)$, $0\le s\le\tau_N^n\wedge t$, taking the supremum and the expectation, we obtain

$$\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \|\operatorname{curl} \upsilon\,(Y_n(s))\|_2^2 + \frac{2\nu}{\alpha}\mathbb{E}\int_0^{\tau_N^n\wedge t} \|\operatorname{curl} \upsilon\,(Y_n)\|_2^2\, ds \leq \mathbb{E}\,\|\operatorname{curl} \upsilon\,(Y_n(0))\|_2^2$$
$$+ 2\mathbb{E}\int_0^{\tau_N^n\wedge t} \left|\left(\frac{\nu}{\alpha}\operatorname{curl} Y_n + \operatorname{curl} U, \operatorname{curl} \upsilon(Y_n)\right)\right| ds$$
$$+ 2\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \left|\int_0^s (\operatorname{curl} G(r,Y_n), \operatorname{curl} \upsilon(Y_n))\; dW_r\right|$$
$$+ \mathbb{E}\int_0^{\tau_N^n\wedge t} \sum_{i=1}^n |\lambda_i - 1|\,|(G(s,Y_n), e_i)|^2\, ds. \tag{4.18}$$

Moreover,

$$2\mathbb{E}\int_0^{\tau_N^n\wedge t} \left|\left(\frac{\nu}{\alpha}\operatorname{curl} Y_n + \operatorname{curl} U, \operatorname{curl} \upsilon(Y_n)\right)\right| ds$$
$$\leq \mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \|\operatorname{curl} \upsilon\,(Y_n(s))\|_2 \int_0^{\tau_N\wedge t} \left(\frac{\nu}{\alpha}\,\|\operatorname{curl} Y_n\|_2 + \|\operatorname{curl} U\|_2\right) ds$$
$$\leq \varepsilon\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \|\operatorname{curl} \upsilon\,(Y_n(s))\|_2^2 + C_\varepsilon\mathbb{E}\int_0^{\tau_N^n\wedge t} (\|\operatorname{curl} Y_n\|_2^2 + \|\operatorname{curl} U\|_2^2)\, ds,$$

i.e.,

$$2\mathbb{E}\int_0^{\tau_N^n\wedge t} \left|\left(\frac{\nu}{\alpha}\operatorname{curl} Y_n + \operatorname{curl} U, \operatorname{curl} \upsilon(Y_n)\right)\right| ds$$
$$\leq \varepsilon\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \|\operatorname{curl} \upsilon\,(Y_n(s))\|_2^2 + C_\varepsilon\mathbb{E}\int_0^{\tau_N^n\wedge t} (\|\operatorname{curl} Y_n\|_2^2 + \|\operatorname{curl} U\|_2^2)\, ds. \tag{4.19}$$

The Burkholder–Davis–Gundy inequality and estimate (2.5)$_2$ imply that

$$2\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \left|\int_0^s (\operatorname{curl} G\,(r,Y_n)\,, \operatorname{curl} \upsilon(Y_n))\; dW_r\right| \leq 2\mathbb{E}\left(\int_0^{\tau_N^n\wedge t} |(\operatorname{curl} G\,(s,Y_n)\,, \operatorname{curl} \upsilon(Y_n))|^2\; ds\right)^{\frac{1}{2}}$$
$$\leq 2\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \|\operatorname{curl} \upsilon(Y_n(s))\|_2 \left(\int_0^{\tau_N^n\wedge t} \|\operatorname{curl} G\,(s,Y_n)\|_2^2\; ds\right)^{\frac{1}{2}}$$
$$\leq \varepsilon\,\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \|\operatorname{curl} \upsilon(Y_n)\|_2^2 + C_\varepsilon\mathbb{E}\int_0^{\tau_N^n\wedge t} \|G\,(s,Y_n)\|_V^2\; ds$$
$$\leq \varepsilon\,\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \|\operatorname{curl} \upsilon(Y_n)\|_2^2 + C_\varepsilon\mathbb{E}\int_0^{\tau_N^n\wedge t} \left(1 + \|\,Y_n(s)\|_V^2\right) ds,$$

i.e.,

$$2\mathbb{E}\sup_{s\in[0,\tau_N^n\wedge t]}\left|\int_0^s\left(\operatorname{curl}G\left(r,Y_n\right),\operatorname{curl}v(Y_n)\right)\,dW_r\right|$$
$$\leq\varepsilon\,\mathbb{E}\sup_{s\in[0,\tau_N^n\wedge t]}\|\operatorname{curl}v(Y_n)\|_2^2+C_\varepsilon\mathbb{E}\int_0^{\tau_N^n\wedge t}\left(1+\|\,Y_n\|_V^2\right)\,ds. \tag{4.20}$$

After substituting (4.19)–(4.20) into (4.18) and choosing $\varepsilon=\frac{1}{4}$, we obtain

$$\frac{1}{2}\mathbb{E}\sup_{s\in[0,\tau_N^n\wedge t]}\|\operatorname{curl}v\left(Y_n(s)\right)\|_2^2+\frac{2\nu}{\alpha}\mathbb{E}\int_0^{\tau_N^n\wedge t}\|\operatorname{curl}v\left(Y_n\right)\|_2^2\,ds\leq\mathbb{E}\,\|\operatorname{curl}v\left(Y_n(0)\right)\|_2^2$$
$$+\,C\mathbb{E}\int_0^t\|\operatorname{curl}U\|_2^2\,ds+C\mathbb{E}\int_0^{\tau_N^n\wedge t}\left(1+\|Y_n(s)\|_V^2\right)\,ds. \tag{4.21}$$

Step 3. The limit transition as $N\to\infty$ in estimates (4.14) and (4.21).

Since

$$\mathbb{E}\,\|\operatorname{curl}v\left(Y_n(0)\right)\|_2^2\leq C\mathbb{E}\,\|Y_n(0)\|_{H^3}^2\leq C\mathbb{E}\,\|Y_0\|_{H^3}^2\leq C$$

and

$$\mathbb{E}\int_0^{\tau_N^n\wedge t}\left(1+\|Y_n(s)\|_V^2\right)\,ds\leq C$$

by (4.5) and (4.14), then we obtain

$$\mathbb{E}\sup_{s\in[0,\tau_N^n\wedge t]}\|\operatorname{curl}v\left(Y_n(s)\right)\|_2^2\leq C.$$

Therefore, estimates (3.19), (4.14) imply that

$$\mathbb{E}\sup_{s\in[0,\tau_N^n\wedge T]}\|Y_n(s)\|_{H^3}^2\leq C,$$

where C is a constant that is independent of N and n. Let us fix $n\in\mathbb{N}$, and by writing

$$\mathbb{E}\sup_{s\in[0,\tau_N^n\wedge T]}\|Y_n(s)\|_{H^3}^2=\mathbb{E}\left(\sup_{s\in[0,\tau_N^n\wedge T]}1_{\{\tau_N^n<T\}}\;\|Y_n(s)\|_{H^3}^2\right)$$
$$+\,\mathbb{E}\left(\sup_{s\in[0,\tau_N^n\wedge T]}1_{\{\tau_N^n\geq T\}}\;\|Y_n(s)\|_{H^3}^2\right)$$
$$\geq\mathbb{E}\left(\max_{s\in[0,\tau_N^n]}1_{\{\tau_N^n<T\}}\;\|Y_n(s)\|_{H^3}^2\right)\geq N^2P\left(\tau_N^n<T\right), \tag{4.22}$$

we deduce that $P\left(\tau_N^n<T\right)\to 0$, as $N\to\infty$. This means that $\tau_N^n\to T$ in probability as $N\to\infty$. Then, a subsequence $\{\tau_{N_k}^n\}$ of $\{\tau_N^n\}$ (which may depend on n) exists such that

$$\tau_{N_k}^n(\omega) \to T \qquad \text{for a.e.} \;\; \omega \in \Omega \quad \text{as } k \to \infty.$$

Since $\tau_{N_k}^n \leq T_n \leq T$, then we deduce that $T_n = T$, so Y_n is a global-in-time solution of the stochastic differential equation (4.4). In addition, the sequence $\{\tau_N^n\}$ of the stopping times is monotone on N for each fixed n, so we can apply the monotone convergence theorem in order to pass to the limit in Inequalities (4.14) and (4.21) as $N \to \infty$, thereby deducing estimates (4.6) and (4.7).

Step 4. Estimate in the space $\widetilde{W}$ for Y_n. By substituting estimate (4.6) into (4.7) and using Lemma 3.4, we immediately derive the main estimate (4.8) of this lemma. □

In the next lemma, by assuming better integrability for the data U, Y_0, we improve the integrability properties for the solution Y_n of problem (4.4).

Lemma 4.4. *Assume that*

$$U \in L^p(\Omega \times (0,T); H), \qquad Y_0 \in L^p(\Omega, V) \qquad \text{for some} \;\; 4 \leq p < \infty.$$

Then, the solution Y_n of problem (4.4) belongs to $L^p(\Omega, L^\infty(0,T;V))$ and verifies the estimate

$$\mathbb{E} \sup_{s\in[0,t]} \|Y_n(t)\|_V^p \leq C\mathbb{E}\,\|Y_0\|_V^p + C\,(1 + \mathbb{E}\int_0^t \|U\|_2^p \, ds), \tag{4.23}$$

where C is a positive constant that is independent of n.

Proof. For each $n \in \mathbb{N}$, let us define a suitable sequence $\{\tau_N^n\}_{N\in\mathbb{N}}$ for the stopping times

$$\tau_N^n = \inf\{t \geq 0 : \|Y_n(t)\|_V \geq N\} \wedge T.$$

After applying the Itô formula for the function $\theta(x) = x^q$, $q \geq 1$, to process (4.11), we have

$$\begin{aligned} d\,\|Y_n\|_V^{2q} &= q\,\|Y_n\|_V^{2q-2}\left[-\left(4\nu\,\|DY_n\|_2^2 + 2\nu\gamma\,\|Y_n\|_{L^2(\Gamma)}^2\right) + 2\,(U, Y_n) + \sum_{i=1}^n |(G\,(t, Y_n)\,, e_i)|^2\right] dt \\ &\quad + 2q\,\|Y_n\|_V^{2q-2}\,(G(t,Y_n), Y_n)\; dW_t + 2q(q-1)\,\|Y_n\|_V^{2q-4}\,|(G(t,Y_n),Y_n)|^2 \; dt. \end{aligned}$$

Let us take $t \in [0,T]$. By integrating over the time interval $[0,s]$, $0 \leq s \leq \tau_N^n \wedge t$, we obtain

$$\begin{aligned} \|Y_n(s)\|_V^{2q} \leq \|Y_n(0)\|_V^{2q} &+ q\int_0^s \|Y_n\|_V^{2q-2}\left|2\,(U, Y_n) + \sum_{i=1}^n |(G\,(r, Y_n)\,, e_i)|^2\right| dr \\ &+ 2q\left|\int_0^s \|Y_n\|_V^{2q-2}\,(G(r,Y_n),Y_n)\; dW_r\right| \\ &+ 2q(q-1)\int_0^s \|Y_n\|_V^{2q-4}\,|(G(r,Y_n),Y_n)|^2 \; dr. \end{aligned} \tag{4.24}$$

From estimate (4.12), we have

$$\sum_{i=1}^{n} |(G(t, Y_n), e_i)|^2 \le C(1 + \|Y_n\|_V^2).$$

By taking the supremum on $s \in [0, \tau_N^n \wedge t]$, the expectation in (4.24), applying Burkholder–Davis–Gundy's and Young's inequalities, and proceeding analogously to (4.13), we obtain

$$\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \|Y_n(s)\|_V^{2q} \le \mathbb{E}\, \|Y_n(0)\|_V^{2q} + C_q\, \mathbb{E} \int_0^t \|U\|_2^{2q}\, ds$$
$$+ \frac{1}{2}\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \|Y_n(s)\|_V^{2q} + C_q(1 + \mathbb{E} \int_0^{\tau_N\wedge t} \|Y_n\|_V^{2q}\, ds).$$

Using Gronwall's inequality, we deduce that

$$\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge t]} \|Y_n(s)\|_V^{2q} \le C\mathbb{E}\, \|Y_0\|_V^{2q} + C\,(1 + \mathbb{E} \int_0^t \|U\|_2^{2q}\, ds) \tag{4.25}$$

for any $q \ge 1$ and $t \in [0, T]$. Using the fact that

$$\mathbb{E} \sup_{s\in[0,\tau_N^n\wedge T]} \|Y_n(s)\|_V^{2q} \le C$$

with a C that is independent of n and N, we may reason as given in the proof of Lemma 4.3 in order to verify that for each n, $\tau_N^n \to T$ in probability, as $N \to \infty$. Then, a subsequence $\{\tau_{N_k}^n\}$ of $\{\tau_N^n\}$ (which may depend on n) exists such that $\tau_{N_k}^n \to T$ for a.e. $\omega \in \Omega$, as $k \to \infty$. Now, let us consider $q = \frac{p}{2}$. Using the monotone convergence theorem, we pass to the limit in (4.25) as $k \to \infty$, thereby deriving estimate (4.23). □

Proof of Theorem 4.2. Existence. The proof is divided into four steps.

Step 1. Estimates and convergence related to the projection operator.

Let $P_n : \widetilde{W} \to W_n$ be the orthogonal projection defined by

$$P_n y = \sum_{j=1}^{n} \widetilde{c}_j \widetilde{e}_j \quad \text{with} \quad \widetilde{c}_j = (y, \widetilde{e}_j)_{\widetilde{W}}, \qquad \forall y \in \widetilde{W},$$

where $\{\widetilde{e}_j = \frac{1}{\sqrt{\lambda_j}} e_j\}_{j=1}^{\infty}$ is the orthonormal basis of $\widetilde{W}$. It is easy to check that

$$P_n y = \sum_{j=1}^{n} c_j e_j \quad \text{with} \quad c_j = (y, e_j)_V, \qquad \forall y \in \widetilde{W}.$$

By Parseval's identity, we have

$$\|P_n y\|_V \le \|y\|_V, \quad \forall y \in V,$$
$$\|P_n y\|_{\widetilde{W}} \le \|y\|_{\widetilde{W}} \qquad \text{and} \qquad P_n y \longrightarrow y \qquad \text{strongly in } \widetilde{W}, \quad \forall y \in \widetilde{W}.$$

Considering an arbitrary $Z \in L^2(\Omega \times (0, T); \widetilde{W})$, we have

$$\|P_n Z\|_{\widetilde{W}} \leq \|Z\|_{\widetilde{W}} \qquad \text{and} \qquad P_n Z(\omega,t) \to Z(\omega,t) \qquad \text{strongly in } \ \widetilde{W},$$

which are valid for P-a.e. $\omega \in \Omega$ and a.e. $t \in (0,T)$. Hence, Lebesgue's dominated convergence theorem and the inequality

$$\|Z\|_V \leq C\|Z\|_{\widetilde{W}} \qquad \text{for any } Z \in \widetilde{W}$$

imply that

$$\begin{aligned} P_n Z &\longrightarrow Z \qquad \text{strongly in } \ L^2(\Omega \times (0,T); \widetilde{W})), \\ P_n Z &\longrightarrow Z \qquad \text{strongly in } \ L^2(\Omega \times (0,T); V)). \end{aligned} \tag{4.26}$$

Step 2. Passing to the limit in the weak sense.

We have

$$\mathbb{E} \sup_{t\in[0,T]} \|Y_n(t)\|^2_{\widetilde{W}} \leq C, \qquad \mathbb{E} \sup_{t\in[0,T]} \|Y_n(t)\|^p_V \leq C \tag{4.27}$$

for some constants C that are independent of the index n by estimates (4.8) and (4.23). Therefore, a suitable subsequence Y_n exists, which is indexed by the same index n to simplify the notation, such that

$$\begin{aligned} Y_n &\rightharpoonup Y \qquad \text{*-weakly in } \ L^2(\Omega, L^\infty(0,T;\widetilde{W})), \\ Y_n &\rightharpoonup Y \qquad \text{*-weakly in } \ L^p(\Omega, L^\infty(0,T;V)). \end{aligned} \tag{4.28}$$

Moreover, we have

$$\begin{aligned} P_n Y &\longrightarrow Y \qquad \text{strongly in } \ L^2(\Omega \times (0,T); \widetilde{W})), \\ P_n Y &\longrightarrow Y \qquad \text{strongly in } \ L^2(\Omega \times (0,T); V)). \end{aligned} \tag{4.29}$$

Let us introduce the operator $B : \widetilde{W} \times V \to \widetilde{W}^*$, defined as

$$(B\,(y,z)\,,\phi) = (\operatorname{curl} v(y) \times z, \phi) \qquad \text{for any } \ y,\phi \in \widetilde{W} \qquad \text{and } \ z \in V,$$

and we state some useful properties. Relation (3.23) gives

$$(B\,(y,z)\,,\phi) = -\,(B\,(y,\phi)\,,z)\,, \qquad (B\,(y,z)\,,z) = 0, \tag{4.30}$$

and (3.24), (3.25) yield

$$\|B\,(y,z)\,\|_{\widetilde{W}^*} \leq C\,\|z\|_V\,\|y\|_{\widetilde{W}}\,, \tag{4.31}$$

$$\|B\,(y,z)\,\|_{\widetilde{W}^*} \leq C\,\|y\|_V\,\|z\|_{\widetilde{W}}\,. \tag{4.32}$$

According to (3.26), a fixed constant C_1 exists such that

$$\|B\,(y,y)\,\|_{\widetilde{W}^*} \leq C_1\,\|y\|^2_V\,, \tag{4.33}$$

and thus

$$\|B(y,y)\|_{L^2(\Omega\times(0,T);\widetilde{W}^*))} \leq C_1\,\|y\|^2_{L^4(\Omega,L^4(0,T;V))}\,. \tag{4.34}$$

In addition, considering (2.5), (4.28), the operators $B^*(t)$ and $G^*(t)$ exist such that

$$\begin{aligned} B(Y_n, Y_n) &\rightharpoonup B^*(t) \qquad \text{weakly in } \ L^2(\Omega\times(0,T);\widetilde{W}^*), \\ G(t, Y_n) &\rightharpoonup G^*(t) \qquad \text{weakly in } \ L^2(\Omega\times(0,T);V^m). \end{aligned} \tag{4.35}$$

By passing to the limit $n\to\infty$ in Equation (4.4), we find that the limit function Y satisfies the stochastic differential equation

$$d\left(\upsilon\left(Y\right),\phi\right) = \left[(\nu\Delta Y + U, \phi) - \langle B^*(t), \phi\rangle\right] dt + \left(G^*(t), \phi\right) dW_t, \qquad \forall \phi\in\widetilde{W}. \tag{4.36}$$

Step 3. Deduction of strong convergence as $n\to\infty$ depending on the stopping times τ_M.

In order to prove that the limit process Y satisfies Equation (4.1), we adapt the methods given by [4] (also see [34]). Let us introduce a sequence (τ_M), $M\in\mathbb{N}$, of stopping times defined by

$$\tau_M(\omega) = \inf\{t\geq 0 : \|Y(t)\|_{\widetilde{W}}(\omega)\geq M\}\wedge T, \qquad \omega\in\Omega.$$

By taking the difference between (4.4) and (4.36), we deduce that

$$\begin{aligned} d\left(P_nY - Y_n, e_i\right)_V = &\left[(\nu\Delta(Y-Y_n), e_i) + \langle B(Y_n,Y_n) - B^*(t), e_i\rangle\right] dt \\ &- \left(G(t,Y_n) - G^*(t), e_i\right) dW_t, \end{aligned} \tag{4.37}$$

which is valid for any $e_i\in W_n$, $i=1,...,n$.

By applying Itô's formula, Equation (4.37) gives

$$\begin{aligned} d(P_nY - Y_n, e_i)_V^2 = &\, 2\left(P_nY - Y_n, e_i\right)_V \left[(\nu\Delta(Y-Y_n), e_i) + \langle B(Y_n,Y_n) - B^*(t), e_i\rangle\right] dt \\ &- 2\left(P_nY - Y_n, e_i\right)_V \left(G(t,Y_n) - G^*(t), e_i\right) dW_t + \left|(G(t,Y_n) - G^*(t), e_i)\right|^2 dt, \end{aligned}$$

and by summing over the index i from 1 to n, we derive

$$\begin{aligned} &d\left(\|P_nY - Y_n\|_V^2\right) + \left(4\nu\|D(P_nY - Y_n)\|_2^2 + 2\nu\gamma\left\|P_nY - Y_n\right\|_{L^2(\Gamma)}^2\right) dt \\ &= 2\nu(\Delta(P_nY - Y), P_nY - Y_n)\, dt \\ &\quad + 2\langle B(Y_n,Y_n) - B^*(t), P_nY - Y_n\rangle\, dt \\ &\quad + \sum_{i=1}^{n}\left|(G(t,Y_n) - G^*(t), e_i)\right|^2 dt - 2\left(G(t,Y_n) - G^*(t), P_nY - Y_n\right) dW_t. \end{aligned} \tag{4.38}$$

Let us note that

$$\begin{aligned} \langle B(Y_n,Y_n) \ - B^*(t), P_nY - Y_n\rangle &= \langle B(Y_n,Y_n) - B(P_nY,P_nY), P_nY - Y_n\rangle \\ &\quad + \langle B(P_nY,P_nY) - B(Y,Y), P_nY - Y_n\rangle + \langle B(Y,Y) - B^*(t), P_nY - Y_n\rangle \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{4.39}$$

Using (4.30), we derive

$$\begin{aligned} I_1 &= \langle B(Y_n,Y_n) - B(P_nY,P_nY), P_nY - Y_n\rangle \\ &= \langle B(Y_n,Y_n) - B(Y_n,P_nY) + B(Y_n,P_nY) - B(P_nY,P_nY), P_nY - Y_n\rangle \\ &= -\langle B(Y_n, P_nY - Y_n), P_nY - Y_n\rangle - \langle B(P_nY - Y_n, P_nY), P_nY - Y_n\rangle \\ &= \langle B(P_nY - Y_n, P_nY - Y_n), P_nY\rangle, \end{aligned}$$

which together with (4.33) implies that

$$|I_1| \leq C_1 \|Y\|_{\widetilde{W}} \|P_n Y - Y_n\|_V^2 . \tag{4.40}$$

For the term I_2, we have

$$\begin{aligned}|I_2| &= |\langle B(P_nY, P_nY) - B(Y,Y), P_nY - Y_n\rangle| \\ &\leq \|B(P_nY, P_nY) - B(Y,Y)\|_{\widetilde{W}^*} \|P_nY - Y_n\|_{\widetilde{W}} ,\end{aligned}$$

and for every $\phi \in W$, from (4.31) and (4.32), it follows that

$$\begin{aligned}\|B(P_nY, P_nY) - B(Y,Y)\|_{\widetilde{W}^*} &\leq \|B(P_nY - Y, P_nY)\|_{\widetilde{W}^*} + \|B(Y, P_nY - Y\|_{\widetilde{W}^*} \\ &\leq C\|Y\|_{\widetilde{W}}\|P_nY - Y\|_V,\end{aligned}$$

and thus we obtain

$$|I_2| \leq C \|Y\|_{\widetilde{W}} \|P_nY - Y\|_V \|P_nY - Y_n\|_{\widetilde{W}} . \tag{4.41}$$

In addition, by denoting $\widetilde{G}_n$, $\widetilde{G}$, and $\widetilde{G}^*$ as the solutions of the Stokes system (3.6) for $f = G(t, Y_n)$, $f = G(t, Y)$, and $f = G^*(t)$, respectively, we have

$$(G(t, Y_n) - G^*(t), e_i) = (\widetilde{G}_n - \widetilde{G}^*, e_i)_V, \qquad i = 1, 2, \ldots, n.$$

Then,

$$\sum_{i=1}^{n} |(G(t, Y_n) - G^*(t), e_i)|^2 = \|P_n\widetilde{G}_n - P_n\widetilde{G}^*\|_V^2.$$

The standard relation $x^2 = (x-y)^2 - y^2 + 2xy$ allows us to write

$$\begin{aligned}\|P_n\widetilde{G}_n - P_n\widetilde{G}^*\|_V^2 = \|P_n\widetilde{G}_n - P_n\widetilde{G}\|_V^2 - \|P_n\widetilde{G} - P_n\widetilde{G}^*\|_V^2 \\ + 2(P_n\widetilde{G}_n - P_n\widetilde{G}^*, P_n\widetilde{G} - P_n\widetilde{G}^*)_V.\end{aligned}$$

From the properties of the solutions of the Stokes system (3.6) and (2.5), we have

$$\|P_n\widetilde{G}_n - P_n\widetilde{G}\|_V^2 \leq \|\widetilde{G}_n - \widetilde{G}\|_V^2 \leq \|G(t, Y_n) - G(t,Y)\|_{L^2}^2 \leq K \|Y_n - Y\|_V^2 ,$$

and thus for the fixed constant $C_2 = 2K$, we have

$$\begin{aligned}\|P_n\widetilde{G}_n - P_n\widetilde{G}^*\|_V^2 \leq K \|Y_n - Y\|_V^2 &- \|P_n\widetilde{G} - P_n\widetilde{G}^*\|_V^2 \\ &+ 2(P_n\widetilde{G}_n - P_n\widetilde{G}^*, P_n\widetilde{G} - P_n\widetilde{G}^*)_V \\ &\leq C_2 \|Y_n - P_nY\|_V^2 + C \|P_nY - Y\|_V^2 - \|P_n\widetilde{G} - P_n\widetilde{G}^*\|_V^2 \\ &+ 2(P_n\widetilde{G}_n - P_n\widetilde{G}^*, P_n\widetilde{G} - P_n\widetilde{G}^*)_V.\end{aligned} \tag{4.42}$$

The positive constants C_1 and C_2 in (4.40) and (4.42), are independent of n and they may depend on the data, i.e., the domain $\mathcal{O}$, the regularity of Γ, and the physical constants ν, α, γ, K.

According to the convergence results (4.26)–(4.29), (4.35), we note that by passing to the limit in Equation (4.38) in a suitable manner, as $n \to \infty$, then all the terms containing $P_nY - Y$ will vanish on the right-hand side of Equality (4.38), according to Relations (4.26), (4.41), and (4.42), but the terms with $Y_n - P_nY$ will remain. Fortunately, these terms can be eliminated by introducing the auxiliary function

$$\xi(t) = e^{-C_2 t - 2C_1 \int_0^t \|Y\|_{\widetilde{W}} ds}.$$

Now, by applying Itô's formula to Equality (4.38), we obtain

$$\begin{aligned}
& d\left(\xi(t)\|P_nY - Y_n\|_V^2\right) + \xi(t)\left(4\nu\|D(P_nY - Y_n)\|_2^2 + 2\nu\gamma\,\|P_nY - Y_n\|_{L^2(\Gamma)}^2\right) dt \\
&= 2\nu\xi(t)(\Delta(P_nY - Y), P_nY - Y_n)\, dt \\
&\quad + 2\xi(t)\langle B(Y_n, Y_n) - B^*(t), P_nY - Y_n\rangle\, dt + \xi(t)\sum_{i=1}^{n} \left|(G(t, Y_n) - G^*(t), e_i)\right|^2\, dt \\
&\quad - 2\xi(t)\left(G(t, Y_n) - G^*(t), P_nY - Y_n\right) dW_t \\
&\quad - C_2\xi(t)\|P_nY - Y_n\|_V^2\, dt - 2C_1\xi(t)\,\|Y\|_{\widetilde{W}}\,\|P_nY - Y_n\|_V^2\, dt. \qquad (4.43)
\end{aligned}$$

By integrating this over the time interval $(0, \tau_M(\omega))$, taking the expectation, and applying estimates (4.39), (4.40), and (4.42), we deduce that

$$\begin{aligned}
& \mathbb{E}\left(\xi(\tau_M)\|P_nY(\tau_M) - Y_n(\tau_M)\|_V^2\right) + \mathbb{E}\int_0^{\tau_M} \xi(s)\|P_n\widetilde{G} - P_n\widetilde{G}^*\|_V^2 ds \\
&\quad + \mathbb{E}\int_0^{\tau_M} \xi(s)\left(4\nu\|D(P_nY - Y_n)\|_2^2 + 2\nu\gamma\,\|P_nY - Y_n\|_{L^2(\Gamma)}^2\right) ds \\
&\leq 2\nu\mathbb{E}\int_0^{\tau_M} \xi(s)(\Delta(P_nY - Y), P_nY - Y_n)\, ds \\
&\quad + 2\mathbb{E}\int_0^{\tau_M} \xi(s)I_2\, ds + 2\mathbb{E}\int_0^{\tau_M} \xi(s)I_3\, ds \\
&\quad + \mathbb{E}\int_0^{\tau_M} \xi(s)\left[C\,\|P_nY - Y\|_V^2 + 2(P_n\widetilde{G}_n - P_n\widetilde{G}^*, P_n\widetilde{G} - P_n\widetilde{G}^*)_V\right] ds \\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

In the following, we show that for each $M \in \mathbb{N}$, the right-hand side of this inequality tends to zero as $n \to \infty$.

Using (4.27)–(4.28) and the properties of the projection P_n, we have

$$\begin{aligned}
|J_1| &= \left|2\nu\mathbb{E}\int_0^T \xi(s)(1_{[0,\tau_M]}(s)\Delta(P_nY - Y), P_nY - Y_n)\, ds\right| \\
&\leq C\|P_nY - Y\|_{L^2(\Omega\times(0,T);H^2)}\|P_nY - Y_n\|_{L^2(\Omega\times(0,T);H^2)} \\
&\leq C\|P_nY - Y\|_{L^2(\Omega\times(0,T);H^2)}\left(\|Y\|_{L^2(\Omega\times(0,T);H^2)} + \|Y_n\|_{L^2(\Omega\times(0,T);H^2)}\right) \\
&\leq C\|P_nY - Y\|_{L^2(\Omega\times(0,T);H^2)},
\end{aligned}$$

which tends to zero as $n \to \infty$ by (4.29). Considering estimates (4.8), (4.41) and given that $1_{[0,\tau_M]}(s)\,\|Y(s)\|_{\widetilde{W}} \leq M$, P-a.e. in Ω, then we deduce that

$$
\begin{aligned}
|J_2| &\leq 2\mathbb{E}\left|\int_0^{\tau_M} \xi(s) I_2\,ds\right| \\
&\leq 2\mathbb{E}\int_0^T \xi(s)1_{[0,\tau_M]}(s)\,\|Y\|_{\widetilde{W}}\,\|P_nY-Y\|_{\widetilde{W}}\left(\|Y\|_{\widetilde{W}}+\|Y_n\|_{\widetilde{W}}\right)ds \\
&\leq CM\|P_nY-Y\|_{L^2(\Omega\times(0,T);\widetilde{W})}\left(\|Y\|_{L^2(\Omega\times(0,T);\widetilde{W})}+\|Y_n\|_{L^2(\Omega\times(0,T);\widetilde{W})}\right) \\
&\leq CM\|P_nY-Y\|_{L^2(\Omega\times(0,T);\widetilde{W})},
\end{aligned}
$$

which also converges to zero by (4.29).

The convergence of (4.28) and (4.29) shows that

$$
P_nY - Y_n \to 0 \qquad \text{weakly in } L^2(\Omega\times(0,T),\widetilde{W}),
$$

and thus for any operator $A \in L^2(\Omega\times(0,T),\widetilde{W^*})$ we have

$$
\mathbb{E}\int_0^T \langle A, P_nY-Y_n\rangle\,ds \to 0 \qquad \text{as } n\to\infty.
$$

The function $1_{[0,\tau_M]}(s)\xi(s)$ is bounded and independent of the space variable, so we have

$$
\begin{aligned}
&\|1_{[0,\tau_M]}(s)\xi(s)\left(B(Y,Y)-B^*\right)\|^2_{L^2(\Omega\times(0,T),\widetilde{W}^*)} \\
&\leq C\left(\|B(Y,Y)\|^2_{L^2(\Omega\times(0,T),\widetilde{W}^*)}+\|B^*\|^2_{L^2(\Omega\times(0,T),\widetilde{W}^*)}\right)\leq C,
\end{aligned}
$$

by (4.27), (4.34), and (4.35). Therefore,

$$
\begin{aligned}
J_3 &= 2\mathbb{E}\int_0^{\tau_M}\xi(s)I_3\,ds \\
&= 2\mathbb{E}\int_0^T \langle 1_{[0,\tau_M]}(s)\xi(s)\left(B(Y,Y)-B^*(s)\right), P_nY-Y_n\rangle\,ds \to 0 \qquad \text{as } n\to\infty.
\end{aligned}
$$

We write

$$
\begin{aligned}
J_4 &= \mathbb{E}\int_0^{\tau_M}\xi(s)\left[C\,\|P_nY-Y\|_V^2+2(P_n\widetilde{G}_n-P_n\widetilde{G}^*, P_n\widetilde{G}-P_n\widetilde{G}^*)_V\right]ds \\
&= C\mathbb{E}\int_0^T 1_{[0,\tau_M]}(s)\xi(s)\,\|P_nY-Y\|_V^2\,ds \\
&\quad + C\mathbb{E}\int_0^T 1_{[0,\tau_M]}(s)\xi(s)(\widetilde{G}_n-\widetilde{G}^*, P_n\widetilde{G}-P_n\widetilde{G}^*)_V\,ds.
\end{aligned}
$$

Due to (4.29), we have

$$\left|\mathbb{E}\int_0^T 1_{[0,\tau_M]}(s)\xi(s)\,\|P_nY-Y\|_V^2\,ds\right|\le \mathbb{E}\int_0^T \|P_nY-Y\|_V^2\,ds\to 0,\ \text{ as } n\to\infty.$$

Now, for each stochastic process $Z\in L^2(\Omega\times(0,T),H)$, let us denote $\widetilde{Z}$ as the solution of the modified Stokes problem (3.6). We recall that the operator

$$A: Z\to \widetilde{Z}$$

is a linear and continuous operator from $L^2(\Omega\times(0,T),V)$ onto $L^2(\Omega\times(0,T),V)$. By applying Proposition A.2 in [4] (also see the references therein), it follows that A is continuous for the weak topology, i.e., if $Z_n\rightharpoonup Z$ weakly in $L^2(\Omega\times(0,T),V)$, then $\widetilde{Z}_n\rightharpoonup\widetilde{Z}$ weakly in $L^2(\Omega\times(0,T),V)$. Due to this property and the convergence result (4.35), we obtain

$$\widetilde{G}_n-\widetilde{G}^*\rightharpoonup\widetilde{G}-\widetilde{G}^*\qquad \text{weakly in } L^2(\Omega\times(0,T),V^m). \tag{4.44}$$

Moreover, we have $\widetilde{G}-\widetilde{G}^*\in\widetilde{W}^m$ and

$$\begin{aligned}
&P_n(\widetilde{G}-\widetilde{G}^*)\to\widetilde{G}-\widetilde{G}^*\qquad \text{strongly in } L^2(\Omega\times(0,T),\widetilde{W}^m),\\
&P_n(\widetilde{G}-\widetilde{G}^*)\to\widetilde{G}-\widetilde{G}^*\qquad \text{strongly in } L^2(\Omega\times(0,T),V^m).
\end{aligned}$$

Then, we can verify that

$$1_{[0,\tau_M]}(s)\xi(s)\,P_n(\widetilde{G}-\widetilde{G}^*)\to 1_{[0,\tau_M]}(s)\xi(s)\,(\widetilde{G}-\widetilde{G}^*)\qquad \text{strongly in } L^2(\Omega\times(0,T),V^m). \tag{4.45}$$

As a consequence of (4.44) and (4.45), we have

$$\begin{aligned}
&\mathbb{E}\int_0^T 1_{[0,\tau_M]}(s)\xi(s)(\widetilde{G}_n-\widetilde{G}^*,P_n\widetilde{G}-P_n\widetilde{G}^*)_V\,ds\\
&=\mathbb{E}\int_0^T(\widetilde{G}_n-\widetilde{G}^*,1_{[0,\tau_M]}(s)\xi(s)P_n(\widetilde{G}-\widetilde{G}^*))_V\,ds\to 0,\ \text{ as } n\to\infty.
\end{aligned}$$

After combining all the convergence results, we obtain the following strong convergences depending on the stopping times τ_M,

$$\begin{aligned}
\lim_{n\to\infty}\mathbb{E}\left(\xi(\tau_M)\|P_nY(\tau_M)-Y_n(\tau_M)\|_V^2\right)&=0,\\
\lim_{n\to\infty}\mathbb{E}\int_0^{\tau_M}\xi(s)\|P_n\widetilde{G}-P_n\widetilde{G}^*\|_V^2ds&=0,\\
\lim_{n\to\infty}\mathbb{E}\int_0^{\tau_M}\xi(s)\left(4\nu\|D(P_nY-Y_n)\|_2^2+2\nu\gamma\,\|P_nY-Y_n\|_{L^2(\Gamma)}^2\right)ds&=0,
\end{aligned}$$

for each $M\in\mathbb{N}$. A strictly positive constant μ exists such that $\mu\le 1_{[0,\tau_M]}(s)\xi(s)\le 1$, so it follows that

$$\lim_{n\to\infty}\mathbb{E}\int_0^{\tau_M}\|P_nY-Y_n\|_V^2ds=0 \quad \text{implying} \quad \lim_{n\to\infty}\mathbb{E}\int_0^{\tau_M}\|Y-Y_n\|_V^2ds=0$$

by (4.29). In addition, considering (4.26), we have

$$\mathbb{E}\int_0^{\tau_M}\|\widetilde{G}-\widetilde{G}^*\|_V^2ds=0. \tag{4.46}$$

Step 4. Identification of $B^*(t)$ with $B(Y,Y)$ and $G^*(t)$ with $G(t,Y)$.

Now, we can show that the limit function Y satisfies Equation (4.1). By integrating Equation (4.36) on the time interval $(0,\tau_M\wedge t)$, we derive

$$\begin{aligned}(\upsilon(Y(\tau_M\wedge t)),\phi)-(\upsilon(Y_0),\phi)=&\int_0^{\tau_M\wedge t}[(\nu\Delta Y+U,\phi)-\langle B^*(s),\phi\rangle]\,ds\\&+\int_0^{\tau_M\wedge t}(G^*(s),\phi)\,dW_s\end{aligned} \tag{4.47}$$

for any $\phi\in\widetilde{W}$.

From (4.46), it follows that

$$1_{[0,\tau_M]}(t)\widetilde{G}=1_{[0,\tau_M]}(t)\widetilde{G}^* \qquad \text{a.e. in } \Omega\times(0,T),$$

which implies that

$$1_{[0,\tau_M]}(t)G(t,Y)=1_{[0,\tau_M]}(t)G^*(t) \qquad \text{a.e. in } \Omega\times(0,T) \tag{4.48}$$

by (3.6). Since $B(Y_n,Y_n)-B(Y,Y)=B(Y_n,Y_n-Y)-B(Y_n-Y,Y)$, then by using (4.31)–(4.32), we have

$$\|B(Y_n,Y_n)-B(Y,Y)\|_{\widetilde{W}^*}\le C\left(\|Y_n\|_{\widetilde{W}}+\|Y\|_{\widetilde{W}}\right)\|Y_n-Y\|_V.$$

Then, for any $\varphi\in L^\infty(\Omega\times(0,T);\widetilde{W})$, by using (4.27), (4.28),

$$\begin{aligned}&\Big|\mathbb{E}\int_0^T 1_{[0,\tau_M]}(s)\langle B(Y_n,Y_n)-B(Y,Y),\ \varphi\rangle\,ds\Big|\\&\qquad\le C\mathbb{E}\int_0^T 1_{[0,\tau_M]}(s)\left(\|Y_n\|_{\widetilde{W}}+\|Y\|_{\widetilde{W}}\right)\|Y_n-Y\|_V\|\varphi\|_{\widetilde{W}}\,ds\\&\qquad\le C\|\varphi\|_{L^\infty(\Omega\times(0,T),\widetilde{W})}\mathbb{E}\int_0^T\left(\|Y_n\|_{\widetilde{W}}+\|Y\|_{\widetilde{W}}\right)\|Y_n-Y\|_V\,ds\\&\qquad\le C\|\varphi\|_{L^\infty(\Omega\times(0,T),\widetilde{W})}\left(\mathbb{E}\int_0^{\tau_M}\|Y_n-Y\|_V^2\,ds\right)^{\frac{1}{2}}\to 0, \qquad \text{as } n\to\infty.\end{aligned}$$

Considering $(4.35)_1$ and that the space $L^\infty(\Omega\times(0,T);\widetilde{W})$ is dense in $L^2(\Omega\times(0,T);\widetilde{W})$, we obtain

$$1_{[0,\tau_M]}(s)B^*(s) = 1_{[0,\tau_M]}(s)B(Y,Y) \qquad \text{a.e. in } \Omega \times (0,T). \tag{4.49}$$

By introducing identities (4.48), (4.49) into Equation (4.47), it follows that

$$\begin{aligned}(\upsilon\,(Y(\tau_M \wedge t))\,,\phi) - (\upsilon\,(Y_0)\,,\phi) = &\int_0^{\tau_M\wedge t} [(\nu\Delta Y + U, \phi) - \langle B(Y,Y), \phi\rangle]\, ds \\ &+ \int_0^{\tau_M\wedge t} (G(s,Y),\phi)\, dW_s.\end{aligned} \tag{4.50}$$

Now, by reasoning as given in (4.22) we have $\tau_M \to T$ a.e. in Ω. We can pass to the limit in each term of Equation (4.50) in $L^1(\Omega \times (0,T))$ as $M \to \infty$ by applying the Lebesgue-dominated convergence theorem and the Burkholder–Davis–Gundy inequality to the last (stochastic) term, thereby deriving Equation (4.1) a.e. in $\Omega \times (0,T)$.

We note that the estimates for Y_n in Lemmas 4.3 and 4.4 are also valid for the limit process Y due to convergence (4.28).

The uniqueness of the solution Y follows from the stability result, as shown in the next section. □

5. Stability result for the solutions

In this section, we establish a stability property for solutions of the stochastic second grade fluid model (2.1). Although we have an existence result with H^3 space regularity, the difference between the two solutions can only be estimated (with respect to the initial data) in the space H^2. It is convenient to introduce the following norm on the space W

$$\|y\|_W = \|y\|_V + \|\mathbb{P}\upsilon(y)\|_2, \qquad y \in W.$$

As a consequence of (3.18) and (2.3), this norm $\|\cdot\|_W$ is equivalent to $\|\cdot\|_{H^2}$.

Theorem 5.1. *Assume that for some $4 \le p < \infty$,*

$$U_1, U_2 \in L^p(\Omega, L^p(0,T; H(\mathrm{curl}; \mathcal{O}))), \qquad Y_{1,0}, Y_{2,0} \in L^p(\Omega, V) \cap L^2(\Omega, \widetilde{W})$$

and

$$Y_1, Y_2 \in L^2(\Omega, L^\infty(0,T; \widetilde{W})) \cap L^p(\Omega, L^\infty(0,T; V))$$

are the corresponding solutions of (2.1) in the sense of the variational equality (4.1).

Then, the strictly positive constants C_3 and C exist that depend only on the data (the domain $\mathcal{O}$, the regularity of Γ, and the physical constants ν, α, γ, K), which satisfy the following estimate

$$\mathbb{E} \sup_{s\in[0,t]} \xi(s)\, \|Y_1(s) - Y_2(s)\|_W^2 \le C(\mathbb{E}\, \|Y_{1,0} - Y_{2,0}\|_W^2 + \mathbb{E}\int_0^t \xi(s)\, \|U_1(s) - U_2(s)\|_2^2\, ds) \tag{5.1}$$

with the function ξ defined as

$$\xi(t) = e^{-C_3 \int_0^t (\|Y_1\|_{H^3} + \|Y_2\|_{H^3})ds}.$$

Proof. The process $Y = Y_1 - Y_2$ satisfies the system

$$\begin{cases} dv(Y) = (\nu\Delta Y - \operatorname{curl} v(Y) \times Y_2 - \operatorname{curl} v(Y_1) \times Y - \nabla\pi + U)\, dt \\ \qquad\qquad + (G(t, Y_1) - G(t, Y_2))\, dW_t \\ \nabla \cdot Y = 0 & \text{in } \mathcal{O} \times (0, T), \\ Y \cdot \mathrm{n} = 0, \qquad [2(\mathrm{n} \cdot DY) + \gamma Y] \cdot \tau = 0 & \text{on } \Gamma \times (0, T), \\ Y(0) = Y_0 = Y_{1,0} - Y_{2,0} & \text{in } \mathcal{O}, \end{cases} \tag{5.2}$$

where $\pi = \pi_1 - \pi_2$ and $U = U_1 - U_2$. By applying the operator $(I - \alpha\mathbb{P}\Delta)^{-1}$ to Equation $(5.2)_1$, we deduce a stochastic differential equation for Y and using Itô's formula, we obtain

$$\begin{aligned} d\,\|Y\|_V^2 &= 2\left((\nu\Delta Y - \operatorname{curl} v(Y) \times Y_2 - \operatorname{curl} v(Y_1) \times Y + U), Y\right) dt \\ &\quad + \|\widetilde{G}_1 - \widetilde{G}_2\|_V^2\, dt + 2\left(G(t, Y_1) - G(t, Y_2), Y\right) dW_t, \end{aligned} \tag{5.3}$$

where $\widetilde{G}_i$ are the solutions for the modified Stokes problem (3.6) with $f = G(t, Y_i)$, $i = 1, 2$. Hence, using assumption (2.5), we have

$$\|\widetilde{G}_1 - \widetilde{G}_2\|_V^2 \leq C\|G(t, Y_1) - G(t, Y_2)\|_2^2 \leq C\|Y\|_V^2.$$

Considering property (3.23), estimate (3.26), and Young's inequality, we derive

$$\begin{aligned} \|Y(t)\|_V^2 &+ \int_0^t \left(4\nu\,\|DY\|_2^2 + 2\nu\gamma\,\|Y\|_{L^2(\Gamma)}^2\right) ds \leq \|Y_0\|_V^2 + C\int_0^t \|Y_2\|_{H^3}\,\|Y\|_Y^2\, ds \\ &+ \int_0^t \|U\|_2^2\, ds + C\int_0^t \|Y\|_V^2\, ds + 2\int_0^t \left(G(s, Y_1) - G(s, Y_2), Y\right) dW_s \end{aligned} \tag{5.4}$$

The Itô formula also gives

$$\begin{aligned} d\,\|\mathbb{P}v\,(Y)\|_2^2 &= 2\left(\nu\Delta Y - \operatorname{curl} v(Y) \times Y_2 - \operatorname{curl} v(Y_1) \times Y + U, \mathbb{P}v\,(Y)\right) dt \\ &\quad + \|G(t, Y_1) - G(t, Y_2)\|_2^2\, dt + 2\left(G(t, Y_1) - G(t, Y_2), \mathbb{P}v(Y)\right) dW_t. \end{aligned}$$

By estimating the nonlinear term

$$|(\operatorname{curl} v(Y) \times Y_2 + \operatorname{curl} v(Y_1) \times Y, \mathbb{P}v(Y))| \leq C_3\left(\|Y_2\|_{H^3} + \|Y_1\|_{H^3}\right)\left(\|Y\|_V^2 + \|\mathbb{P}v(Y)\|_2^2\right)$$

and using (2.5), we deduce that

$$\begin{aligned} \|\mathbb{P}v\,(Y(t))\|_2^2 + \frac{2\nu}{\alpha}\int_0^t \|v\,(Y)\|_2^2\, ds &\leq \|\mathbb{P}v\,(Y_0)\|_2^2 + 2\int_0^t \left(\frac{\nu}{\alpha}Y + U, \mathbb{P}v\,(Y)\right) ds \\ &\quad + C_3\int_0^t \left(\|Y_2\|_{H^3} + \|Y_1\|_{H^3}\right)\left(\|Y\|_V^2 + \|\mathbb{P}v(Y)\|_2^2\right) ds \\ &\quad + C\int_0^t \|Y\|_V^2\, ds + 2\int_0^t \left(G(s, Y_1) - G(s, Y_2), \mathbb{P}v(Y)\right) dW_s. \end{aligned}$$

By summing this inequality with (5.4), we obtain

$$\|Y(t)\|_V^2 + \|\mathbb{P}\upsilon\,(Y(t))\|_2^2 \le \|Y_0\|_V^2 + \|\mathbb{P}\upsilon\,(Y_0)\|_2^2 + \int_0^t \|U\|_2^2\,ds + C\int_0^t \left(\|Y\|_V^2 + \|\mathbb{P}\upsilon\,(Y)\,\|_2^2\right)\,ds + C_3\int_0^t (\|Y_2\|_{H^3} + \|Y_1\|_{H^3})\left(\|Y\|_V^2 + \|\mathbb{P}\upsilon(Y)\|_2^2\right)\,ds + 2\int_0^t (G(s,Y_1) - G(s,Y_2), Y + \mathbb{P}\upsilon(Y))\,dW_s.$$

Taking $\xi(t) = e^{-C_3\int_0^t (\|Y_2\|_{H^3} + \|Y_2\|_{H^3})ds}$ and applying Itô's formula, it is then easy to obtain

$$\xi(t)\left(\|Y(t)\|_V^2 + \|\mathbb{P}\upsilon\,(Y(t))\|_2^2\right) \le \|Y_0\|_V^2 + \|\mathbb{P}\upsilon\,(Y_0)\|_2^2 + \int_0^t \xi(s)\|U\|_2^2\,ds + C\int_0^t \xi(s)\left(\|Y\|_V^2 + \|\mathbb{P}\upsilon\,(Y)\,\|_2^2\right)\,ds + 2\int_0^t \xi(s)\,(G(s,Y_1) - G(s,Y_2), Y + \mathbb{P}\upsilon(Y))\,dW_s. \quad (5.5)$$

The Burkholder–Davis–Gundy inequality gives

$$\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s \xi(r)\,(G(r,Y_1) - G(r,Y_2), Y + \mathbb{P}\upsilon(Y))\,dW_r\right| \le \mathbb{E}\left(\int_0^t \xi^2(s)\,\|Y\|_V^2\,\|Y + \mathbb{P}\upsilon(Y)\|_2^2\,ds\right)^{\frac{1}{2}} \le \varepsilon\mathbb{E}\sup_{s\in[0,t]}\xi(s)\,(\|Y\|_V + \|\mathbb{P}\upsilon(Y)\|_2) + C_\varepsilon\mathbb{E}\int_0^t \xi(s)\,(\|Y\|_V + \|\mathbb{P}\upsilon(Y)\|_2)\,ds.$$

By substituting this inequality with $\varepsilon = \frac{1}{2}$ in (5.5) and taking the supremum on the time interval $[0,t]$ and the expectation, we deduce that

$$\mathbb{E}\sup_{s\in[0,t]}\xi(s)\,\|Y(s)\|_W^2 \le \mathbb{E}\,\|Y_0\|_W^2 + \mathbb{E}\int_0^t \xi(s)\|U\|_2^2\,ds + C\mathbb{E}\int_0^t \xi(s)\|Y(s)\|_W^2\,ds.$$

Hence, Gronwall's inequality yields (5.1). □

Acknowledgments

We would like to thank the anonymous referees for useful suggestions and comments, which helped to improve this study.

F. Cipriano was supported partly by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project UID/MAT/00297/2013 (Centro de Matemática e Aplicações).

References

[1] S. Albeverio, Z. Brzeźniak, J.-L. Wu, Existence of global solutions and invariant measures for stochastic differential equations driven by Poisson type noise with non-Lipschitz coefficients, J. Math. Anal. Appl. 371 (2010) 309–322.
[2] A. Bensoussan, Stochastic Navier–Stokes equations, Acta Appl. Math. 38 (1995) 267–304.
[3] A. Bensoussan, R. Temam, Equations stochastiques du type Navier–Stokes, J. Funct. Anal. 13 (1973) 195–222.
[4] H.I. Breckner, Approximation and Optimal Control of the Stochastic Navier–Stokes Equations, Ph.D. Thesis, Halle (Saale), 1999.
[5] A.V. Busuioc, D. Iftimie, A non-Newtonian fluid with Navier boundary conditions, J. Dynam. Differential Equations 18 (2006) 357–379.
[6] A.V. Busuioc, T.S. Ratiu, The second grade fluid and averaged Euler equations with Navier-slip boundary conditions, Nonlinearity 16 (2003) 1119–1149.
[7] A.V. Busuioc, D. Iftimie, M.C. Lopes Filho, H.J. Nussenzveig Lopes, Incompressible Euler as a limit of complex fluid models with Navier boundary conditions, J. Differential Equations 252 (2012) 624–640.
[8] R. Camassa, D.D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (1993) 1661–1664.
[9] N.V. Chemetov, F. Cipriano, Boundary layer problem: Navier–Stokes equations and Euler equations, Nonlinear Anal. Real World Appl. 14 (6) (2013) 2091–2104.
[10] N.V. Chemetov, F. Cipriano, The inviscid limit for the Navier–Stokes equations with slip condition on permeable walls, J. Nonlinear Sci. 23 (5) (2013) 731–750.
[11] N.V. Chemetov, F. Cipriano, Inviscid limit for Navier–Stokes equations in domains with permeable boundaries, Appl. Math. Lett. 33 (2014) 6–11.
[12] D. Cioranescu, V. Girault, Weak and classical solutions of a family of second grade fluids, Int. J. Non-Linear Mech. 32 (1997) 317–335.
[13] D. Cioranescu, E.H. Ouazar, Existence and uniqueness for fluids of second grade, in: Nonlinear Partial Differential Equations and Their Applications, Collège de France Seminar, Paris, 1982/1983, vol. 4, Pitman, Boston, MA, 1984, pp. 178–197.
[14] F. Cipriano, I. Torrecilla, Inviscid limit for 2D stochastic Navier–Stokes equations, J. Stoch. Process. Appl. 125 (6) (2015) 2405–2426.
[15] T. Clopeau, A. Mikelic, R. Robert, On the vanishing viscosity limit for the $2D$ incompressible Navier–Stokes equations with the friction type boundary conditions, Nonlinearity 11 (1998) 1625–1636.
[16] G. Da Prato, A. Debussche, 2D stochastic Navier–Stokes equations with a time-periodic forcing term, J. Dynam. Differential Equations 20 (2) (2008) 301–335.
[17] G. Deugoue, M. Sango, Weak solutions to stochastic 3D Navier–Stokes-α model of turbulence: asymptotic behavior, J. Math. Anal. Appl. 384 (2011) 49–62.
[18] J.E. Dunn, R.L. Fosdick, Thermodynamics, stability and boundedness of fluids of complexity 2 and fluids of second grade, Arch. Ration. Mech. Anal. 56 (1974) 191–252.
[19] J.E. Dunn, K.R. Rajagopal, Fluids of differential type: critical review and thermodynamical analysis, Internat. J. Engrg. Sci. 33 (1995) 689–729.
[20] F. Flandoli, An introduction to 3D stochastic fluid dynamics, in: SPDE in Hydrodynamic: Recent Progress and Prospects, in: Lecture Notes in Mathematics, vol. 1942, Springer, Berlin/Heidelberg, 2008, pp. 51–150.
[21] C. Foias, D.D. Holm, E.S. Titi, The Navier–Stokes–alpha model of fluid turbulence, Physica D 152–153 (2001) 505–519.
[22] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman Publishing Inc, 1985.
[23] D.D. Holm, J.E. Marsden, R.S. Ratiu, Euler–Poincaré models of ideal fluids with nonlinear dissipation, Phys. Rev. Lett. 349 (1998) 4173.
[24] D.D. Holm, J.E. Marsden, R.S. Ratiu, The Euler–Poincaré equations and semi-direct products with applications to continuum theories, Adv. Math. 137 (1998) 1–81.
[25] D. Iftimie, G. Planas, Inviscid limits for the Navier–Stokes equations with Navier friction boundary conditions, Nonlinearity 19 (2006) 899–918.
[26] J.P. Kelliher, Navier–Stokes equations with Navier boundary conditions for a bounded domain in the plane, SIAM J. Math. Anal. 38 (2006) 210–232.
[27] J.S. Linshiz, E.S. Titi, On the convergence rate of the Euler-α, an inviscid second-grade complex fluid, model to Euler equations, J. Stat. Phys. 138 (2010) 305–332.
[28] M.C. Lopes Filho, H.J. Nussenzveig Lopes, G. Planas, On the inviscid limit for $2D$ incompressible flow with Navier friction condition, SIAM J. Math. Anal. 36 (2005) 1130–1141.
[29] M.C. Lopes Filho, H.J. Nussenzveig Lopes, E.S. Titi, A. Zang, Convergence of the $2D$ Euler-α to Euler equations in the Dirichlet case: indifference to boundary layers, Physica D 292–293 (1) (2015) 51–61.

[30] R. Mikulevicius, B.L. Rozovskii, Global L^2-solutions of stochastic Navier–Stokes equations, Ann. Probab. 33 (1) (2005) 137–176.
[31] E.H. Ouazar, Sur les Fluides de Second Grade, Thèse 3ème Cycle, Université Pierre et Marie Curie, 1981.
[32] P.A. Razafimandimby, M. Sango, Weak solutions of a stochastic model for two-dimensional second grade fluids, Bound. Value Probl. 2010 (2010) (electr. version), 47 pp.
[33] P.A. Razafimandimby, M. Sango, Asymptotic behaviour of solutions of stochastic evolution equations for second grade fluids, C. R. Acad. Sci. Paris, Ser. I 348 (13–14) (2010) 787–790.
[34] P.A. Razafimandimby, M. Sango, Strong solution for a stochastic model of two-dimensional second grade fluids: existence, uniqueness and asymptotic behavior, Nonlinear Anal. 75 (2012) 4251–4270.
[35] R.S. Rivlin, J.L. Ericksen, Stress-deformation relations for isotropic materials, Arch. Ration. Mech. Anal. 4 (1955) 323–425.
[36] M. Sango, Density dependent stochastic Navier–Stokes equations with non Lipschitz random forcing, Rev. Math. Phys. 22 (6) (2010) 669–697.