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The Bishop–Phelps–Bollobás property for numerical radius of operators on $L_1(\mu)$ [☆]

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ABSTRACT

In this paper, we introduce the notion of the Bishop–Phelps–Bollobás property for numerical radius (BPBp-ν) for a subclass of the space of bounded linear operators. Then, we show that certain subspaces of $\mathcal{L}(L_1(\mu))$ have the BPBp-ν for every finite measure μ. As a consequence we deduce that the subspaces of finite-rank operators, compact operators and weakly compact operators on $L_1(\mu)$ have the BPBp-ν.

1. Introduction

In this paper, we provide a version of Bishop–Phelps–Bollobás theorem for numerical radius for operators. To recall such result we introduce some notation. For a Banach space X, B_X and S_X will be the closed unit ball and the unit sphere of X, respectively. We will denote by X^* the topological dual of X and by $\mathcal{L}(X)$ the space of bounded linear operators on X endowed with the operator norm. The symbols $\mathcal{F}(X)$, $\mathcal{K}(X)$ and $\mathcal{WC}(X)$ denote the spaces of finite-rank operators, compact operators and weakly compact operators on X, respectively. It is well known that $\mathcal{F}(X) \subset \mathcal{K}(X) \subset \mathcal{WC}(X)$. Throughout this paper the normed spaces will be either real or complex.

Bishop–Phelps–Bollobás theorem states that for any Banach space X, given $0 < \varepsilon < 1$, and $(x, x^*) \in B_X \times S_{X^*}$ such that $|x^*(x) - 1| < \frac{\varepsilon^2}{2}$, there is a pair $(y, y^*) \in S_X \times S_{X^*}$ satisfying

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$$\|y - x\| < \varepsilon,\ \|y^* - x^*\| < \varepsilon \quad \text{and} \quad y^*(y) = 1$$

(see for instance [4], [5, Theorem 16.1] or [6, Corollary 2.4]).

After some interesting papers about denseness of the set of norm attaining operators, in 2008 it was initiated the study of versions of Bishop–Phelps–Bollobás Theorem for operators [1]. More recently it was considered the problem of obtaining versions of such results for numerical radius of operators (see [10, Definition 1.2]). We just mention that the numerical radius of an operator is a continuous semi-norm in the space $\mathcal{L}(X)$ for every Banach space X.

Guirao and Kozhushkina proved that the spaces c_0 and ℓ_1 satisfy the Bishop–Phelps–Bollobás property for numerical radius (BPBp-ν) in the real case as well as in the complex case [10]. Falcó showed the same result for $L_1(\mathbb{R})$ in the real case [9, Theorem 9]. Choi, Kim, Lee and Martín extended the previous result to $L_1(\mu)$ for any positive measure μ [11, Theorem 9]. Avilés, Guirao and Rodríguez provided sufficient conditions on a compact Hausdorff space K in order that $C(K)$ has the BPBp-ν in the real case [3, Theorem 2.2]. For instance, a metrizable space K satisfies the previous condition [3, Theorem 3.2]. It is an open problem whether or not such result is satisfied for any compact Hausdorff space K in the real case. In the complex case there are no results until now for $C(K)$ spaces.

In this paper, motivated by Definition 1.2 of [10], we introduce the notion of the BPBp-ν for subspaces of the space of bounded linear operators (see Definition 2.1). A Banach space X satisfies the BPBp-ν, introduced in [10], if and only if the space $\mathcal{M} = \mathcal{L}(X)$ satisfies the BPBp-ν. Then, we give some sufficient conditions on a subspace $\mathcal{M}$ of $\mathcal{L}(L_1(\mu))$ to satisfy the BPBp-ν, for any finite measure μ. More precisely, we show that $\mathcal{M}$ has the BPBp-ν if $\mathcal{M}$ contains the space of finite-rank operators on $L_1(\mu)$, is contained in the class of representable operators on $L_1(\mu)$ (see Definition 2.2) and $T_{|A} \in \mathcal{M}$ for every $T \in \mathcal{M}$ and any measurable set A, where $T_{|A}$ is the operator on $L_1(\mu)$ given by $T_{|A}(f) = T(f\chi_A)$ for all $f \in L_1(\mu)$. As a consequence of the main result we obtain that for any σ-finite measure μ, the spaces of finite-rank operators, compact operators and weakly compact operators on $L_1(\mu)$ have the BPBp-ν. The results are valid in the real as well as in the complex case.

2. Bishop–Phelps–Bollobás theorem for numerical radius for some classes of operators on $L_1(\mu)$

If X is a Banach space and $T \in \mathcal{L}(X)$, we recall that the *numerical radius* of T, $\nu(T)$, is defined by

$$\nu(T) = \sup\big\{|x^*(T(x))| : x \in S_X, x^* \in S_{X^*}, x^*(x) = 1\big\}.$$

In general the numerical radius is a semi-norm on $\mathcal{L}(X)$ satisfying $\nu(T) \le \|T\|$ for each $T \in \mathcal{L}(X)$. The numerical index of X, $n(X)$ is defined by

$$n(X) = \inf\{\nu(T) : T \in S_{\mathcal{L}(X)}\}.$$

Hence, $n(X)$ is the greatest constant t such that $t\|T\| \le \nu(T)$ for each $T \in \mathcal{L}(X)$. It is always satisfied that $0 \le n(X) \le 1$ and, in case that $n(X) = 1$, it is said that X has *numerical index equal to* 1. In such case it is satisfied that $\nu(T) = \|T\|$ for each $T \in \mathcal{L}(X)$. It is well known that the spaces $L_1(\mu)$ and $C(K)$ have numerical index equal to 1 for any measure μ and any compact Hausdorff space K [8, Theorem 2.2].

Guirao and Kozhushkina [10] introduced the definition of the BPBp-ν. We will use a little different concept by admitting subclasses of the space of bounded linear operators on a Banach space X.

Definition 2.1. Let X be a Banach space and $\mathcal{M}$ a subspace of $\mathcal{L}(X)$. We will say that $\mathcal{M}$ has the *Bishop–Phelps–Bollobás property for numerical radius* (BPBp-ν) if for every $0 < \varepsilon < 1$, there is $\eta(\varepsilon) > 0$ such that whenever $S \in \mathcal{M}$, $\nu(S) = 1$, $x_0 \in S_X$ and $x_0^* \in S_{X^*}$ are such that $x_0^*(x_0) = 1$ and $|x_0^*(S(x_0))| > 1 - \eta(\varepsilon)$, there are $T \in \mathcal{M}$, $x_1 \in S_X$ and $x_1^* \in S_{X^*}$ such that

i) $x_1^*(x_1) = 1$,
ii) $|x_1^*(T(x_1))| = \nu(T) = 1$,
iii) $\nu(T - S) < \varepsilon$, $\|x_1 - x_0\| < \varepsilon$ and $\|x_1^* - x_0^*\| < \varepsilon$.

Let us notice that for spaces with numerical index equal to one, Definition 2.1 can be reformulated by using the usual norm of the space $\mathcal{L}(X)$ instead of the numerical radius.

The following simple technical lemmas will be useful. Next lemma is a straightforward consequence of [1, Lemma 3.3].

Lemma 2.1. *Assume that* $\{z_k : k \in \mathbb{N}\} \subset \{z \in \mathbb{C} : |z| \leq 1\}$ *and* $\{\beta_k : k \in \mathbb{N}\} \subset \mathbb{C}$ *satisfies that* $\sum_{k=1}^{\infty} |\beta_k| = 1$. *If* $0 < \varepsilon < 1$ *and* $\mathrm{Re}\,(\sum_{k=1}^{\infty} \beta_k z_k) > 1 - \varepsilon^2$, *then*

$$\sum_{k \in B} |\beta_k| > 1 - \varepsilon,$$

where $B = \{k \in \mathbb{N} : \mathrm{Re}\,(\beta_k z_k) > (1 - \varepsilon)|\beta_k|\}$.

Next result is a generalization of Lemma 2.1 to $L_1(\mu)$. Also it extends [10, Lemma 2.3] where the authors state the analogous result for the sequence space ℓ_1.

Lemma 2.2. *Let* (Ω, Σ, μ) *be a measure space. Assume that* $0 < \varepsilon < 1$, $f \in B_{L_1(\mu)}$ *and* $g \in B_{L_\infty(\mu)}$ *are such that*

$$1 - \varepsilon^2 < \mathrm{Re} \int_\Omega fg \, d\mu.$$

Then the set C *given by*

$$C = \{t \in \Omega : \mathrm{Re}\, f(t)g(t) > (1 - \varepsilon)|f(t)|\},$$

satisfies that

$$\mathrm{Re} \int_C fg \, d\mu > 1 - \varepsilon.$$

Proof. It is clear that the set C is measurable. By assumption we have

$$\begin{aligned} 1 - \varepsilon^2 < \mathrm{Re} \int_\Omega fg \, d\mu &\leq \mathrm{Re} \int_C fg \, d\mu + (1 - \varepsilon) \int_{\Omega \setminus C} |f| \, d\mu \\ &\leq \varepsilon \mathrm{Re} \int_C fg \, d\mu + (1 - \varepsilon)\Big(\int_C |f| \, d\mu + \int_{\Omega \setminus C} |f| \, d\mu \Big) \leq \varepsilon \mathrm{Re} \int_C fg \, d\mu + 1 - \varepsilon. \end{aligned}$$

Hence,

$$\mathrm{Re} \int_C fg \, d\mu > 1 - \varepsilon. \qquad \square$$

Lemma 2.3. *Let z be a complex number, $0 < \varepsilon < 1$ and assume that*

$$\operatorname{Re} z > (1-\varepsilon)|z|.$$

Then

$$|z - |z|| < \sqrt{2\varepsilon}|z|.$$

Proof. We write $z = x + iy$, where $x, y \in \mathbb{R}$. Since $x^2 + y^2 = |z|^2$ and $x = \operatorname{Re} z > (1-\varepsilon)|z|$, we have $y^2 \leq |z|^2 - (1-\varepsilon)^2|z|^2 = (2\varepsilon - \varepsilon^2)|z|^2$. It follows that

$$\big|z - |z|\big|^2 = (|z| - x)^2 + y^2 < (\varepsilon|z|)^2 + (2\varepsilon - \varepsilon^2)|z|^2 = 2\varepsilon|z|^2. \qquad \square$$

We recall the following notion (see for instance [7, Definition III.3]).

Definition 2.2. Let (Ω, Σ, μ) be a finite measure space and Y a Banach space. An operator $T \in \mathcal{L}(L_1(\mu), Y)$ is called *Riesz representable* (or simply *representable*) if there is $h \in L_\infty(\mu, Y)$ such that $T(f) = \int_\Omega hf\, d\mu$ for all $f \in L_1(\mu)$. We say that the function h is a representation of T.

We will use the following identification.

Proposition 2.1. ([7, Lemma III.4, p. 62]) *Let (Ω, Σ, μ) be a finite measure space and Y be a Banach space. There is a linear isometry Φ from the space $\mathcal{R}$ of representable operators in $\mathcal{L}(L_1(\mu), Y)$ into $L_\infty(\mu, Y)$ such that if $T \in \mathcal{R}$ and $\Phi(T) = h$, then it is satisfied that*

$$T(f) = \int_\Omega hf\, d\mu, \qquad \textit{for all} \quad f \in L_1(\mu).$$

It is known that $\mathcal{WC}(L_1(\mu))$ is a subset of the representable operators into $L_1(\mu)$ whenever μ is any finite measure (see for instance [7, Theorem III.12, p. 75]). We will write $\mathcal{R}(L_1(\mu))$ for the space of representable operators into $L_1(\mu)$. Given $T \in \mathcal{L}(L_1(\mu))$ and a measurable subset A of Ω, we will denote by $T_{|A}$ the operator on $L_1(\mu)$ given by $T_{|A}(f) = T(f\chi_A)$ for all $f \in L_1(\mu)$.

In [2, Theorem 2.3] it was proved that a subspace of $\mathcal{L}(L_1(\mu), Y)$ that contains the subspace of finite-rank operators and is contained in the space of representable operators and that satisfies also an additional assumption has the Bishop–Phelps–Bollobás property for operators whenever Y has the so called AHSp, a property satisfied by $L_1(\mu)$. Now we will prove a parallel result for numerical radius for subspaces of $\mathcal{L}(L_1(\mu))$. Of course, such proof is more involved since we have to approximate one pair of elements (x, x^*) in the product of $S_{L_1(\mu)} \times S_{(L_1(\mu))^*}$ instead of one element in the unit sphere of $L_1(\mu)$.

In the proof of the next result we will write $g(f)$ instead of $\int_\Omega g(t)f(t)\, d\mu$ for each element $f \in L_1(\mu)$ and $g \in L_\infty(\mu)$.

Theorem 2.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and let $\mathcal{M}$ be a subspace of $\mathcal{L}(L_1(\mu))$ such that $\mathcal{F}(L_1(\mu)) \subseteq \mathcal{M} \subseteq \mathcal{R}(L_1(\mu))$. Assume also that for each measurable subset A of Ω and each $T \in \mathcal{M}$ it is satisfied $T_{|A} \in \mathcal{M}$. Then $\mathcal{M}$ has the BPBp-ν, and the function η satisfying Definition 2.1 is independent from the measure space and also from $\mathcal{M}$.*

Proof. Let us fix $0 < \varepsilon < 1$. We take $\eta\big(= \eta(\varepsilon)\big) = \frac{\varepsilon^8}{2^{33}}$. Assume that $T_0 \in S_{\mathcal{M}}$, $f_0 \in S_{L_1(\mu)}$ and $g_0 \in S_{L_\infty(\mu)}$ satisfy $g_0(f_0) = 1$ and $|g_0(T_0(f_0))| > 1 - \eta$. Let λ_0 be a scalar with $|\lambda_0| = 1$ and such that $|g_0(T_0(f_0))| = \operatorname{Re} \lambda_0 g_0(T_0(f_0))$. By changing T_0 by $\lambda_0 T_0$ we may assume that $\operatorname{Re} g_0(T_0(f_0)) = |g_0(T_0(f_0))|$. In view of

Proposition 2.1 there is a function $h_0 \in S_{L_\infty(\mu, L_1(\mu))}$ associated to the operator T_0. Since the proof is long we divided it into five steps.

Step 1. In this step we will approximate the pair of functions (f_0, g_0) by a new pair (f_1, g_1) such that f_1 and g_1 take a countable set of values and also there are subsets where f_1, g_1 are constant and h_0 has small oscillation on these subsets.

More concretely, we will show that there are functions $f_1 \in S_{L_1(\mu)}$ and $g_1 \in S_{L_\infty(\mu)}$ and a countable family $\{D_k : k \in J\} \subset \Omega$ of pairwise disjoint measurable sets such that $\mu(D_k) > 0$ for all $k \in J$, $\mu(\Omega \backslash \bigcup_{k\in J} D_k) = 0$ and such that the following conditions are satisfied

$$\|f_1 - f_0\|_1 < \frac{\varepsilon}{4}, \quad \|g_1 - g_0\|_\infty < \frac{\varepsilon}{4}, \tag{2.1}$$

$$\operatorname{Re} g_1(f_1) > 1 - \eta, \quad \operatorname{Re} g_1(T_0(f_1)) > 1 - \eta, \tag{2.2}$$

$$\text{for each } k \in J, \; f_1 \text{ and } g_1 \text{ are constant on } D_k \tag{2.3}$$

$$\sup\{\|h_0(s) - h_0(t)\|_1 : s, t \in D_k\} \le \eta, \quad \forall k \in J, \tag{2.4}$$

and

$$1 = \|h_0\|_\infty = \sup\{\|h_0(t)\|_1 : t \in \cup_{k\in J} D_k\}. \tag{2.5}$$

Since the set of simple functions is dense in both $L_1(\mu)$ and $L_\infty(\mu)$, there are simple functions $f_1 \in S_{L_1(\mu)}$ and $g_1 \in S_{L_\infty(\mu)}$ satisfying (2.1) and (2.2).

On the other hand, by [7, Theorem II.2, p. 42] there is a measurable subset E_1 of Ω such that $\mu(E_1) = 0$ and $h_0(\Omega \setminus E_1)$ is a separable subset of $L_1(\mu)$. Suppose that the set $\{y_i : i \in \mathbb{N}\}$ is dense in $h_0(\Omega \setminus E_1)$. Since f_1 and g_1 are simple functions, we can assume that $\operatorname{Im}(f_1) = \{a_r : r = 1, \dots, n\}$ and $\operatorname{Im}(g_1) = \{b_l : l = 1, \dots, m\}$. Now, for $i \in \mathbb{N}$, $r \in \{1, \dots, n\} = N$ and $l \in \{1, \dots, m\} = M$ we consider the following subsets of Ω

$$A_{(1,r,l)} = h_0^{-1}(B_{\frac{\eta}{2}}(y_1)) \cap (\Omega \setminus E_1) \cap f_1^{-1}(a_r) \cap g_1^{-1}(b_l)$$

and

$$A_{(i,r,l)} = (h_0^{-1}(B_{\frac{\eta}{2}}(y_i)) \setminus \cup_{e=1}^{i-1} h_0^{-1}(B_{\frac{\eta}{2}}(y_e))) \cap (\Omega \setminus E_1) \cap f_1^{-1}(a_r) \cap g_1^{-1}(b_l), \quad \forall i \ge 2.$$

It is clear that the elements of the family $\{A_{(i,r,l)} : (i, r, l) \in \mathbb{N} \times N \times M\}$ are measurable subsets of Ω and pairwise disjoint. Now, let $W = \{(i, r, l) \in \mathbb{N} \times N \times M : \mu(A_{(i,r,l)}) = 0\}$ and $E_2 = \bigcup_{(i,r,l)\in W} A_{(i,r,l)}$. By the definition of W it is trivially satisfied that E_2 is measurable and $\mu(E_2) = 0$. On the other hand there exists a measurable subset E_3 of $\Omega \setminus (E_1 \cup E_2)$ such that $\mu(E_3) = 0$ and $\|h\|_\infty = \sup\{\|h(t)\|_1 : t \in \Omega \backslash E_3\}$. Assume that $\{D_k : k \in J\}$ is the family of pairwise disjoint measurable subsets obtained by indexing the set $\{A_{(i,r,l)} \setminus E_3 : (i, r, l) \in (\mathbb{N} \times N \times M) \setminus W\}$. Then, we have that $\mu(D_k) > 0$ for all $k \in J$, $\mu(\Omega \setminus \bigcup_{k\in J} D_k) = 0$ and also the family $\{D_k : k \in J\}$ satisfies the conditions (2.3), (2.4) and (2.5). Therefore, by (2.3) there are sets of scalars $\{\alpha_k : k \in J\}$ and $\{\gamma_k : k \in J\}$ such that

$$f_1 = \sum_{k\in J} \alpha_k \frac{\chi_{D_k}}{\mu(D_k)}, \quad \sum_{k\in J} |\alpha_k| = 1, \qquad g_1 = \sum_{k\in J} \gamma_k \chi_{D_k}, \quad |\gamma_k| \le 1, \qquad \forall k \in J. \tag{2.6}$$

Step 2. In this step we will define another simple function $f_2 \in S_{L_1(\mu)}$ which is an approximation of f_1, and can be expressed as a finite sum instead of the countable sum appearing in the expression of f_1 given in (2.6).

By (2.6) and (2.2) there is a finite subset F of J such that

$$\sum_{k\in F}|\alpha_k| > 1-\eta > 0, \qquad \operatorname{Re} g_1\Big(\sum_{k\in F}\alpha_k\frac{\chi_{D_k}}{\mu(D_k)}\Big) > 1-\eta, \tag{2.7}$$

and also

$$\operatorname{Re} g_1\Big(T_0\Big(\sum_{k\in F}\alpha_k\frac{\chi_{D_k}}{\mu(D_k)}\Big)\Big) > 1-\eta. \tag{2.8}$$

For each $k\in F$ we put $\beta_k = \frac{\alpha_k}{\sum_{k\in F}|\alpha_k|}$ and define $f_2 = \sum_{k\in F}\beta_k\frac{\chi_{D_k}}{\mu(D_k)}$. In view of (2.7) and (2.8) we have that

$$\operatorname{Re} g_1(f_2) = \operatorname{Re} g_1\Big(\sum_{k\in F}\beta_k\frac{\chi_{D_k}}{\mu(D_k)}\Big) > 1-\eta \tag{2.9}$$

and

$$\operatorname{Re} g_1(T_0(f_2)) = \operatorname{Re} g_1\Big(T_0\Big(\sum_{k\in F}\beta_k\frac{\chi_{D_k}}{\mu(D_k)}\Big)\Big) > 1-\eta. \tag{2.10}$$

Clearly $f_2\in S_{L_1(\mu)}$ and by (2.6), (2.7) we have that

$$\begin{aligned}\|f_2-f_1\|_1 &= \Big\|\sum_{k\in F}\beta_k\frac{\chi_{D_k}}{\mu(D_k)} - \sum_{k\in J}\alpha_k\frac{\chi_{D_k}}{\mu(D_k)}\Big\|_1\\ &= \Big\|\sum_{k\in F}\beta_k\frac{\chi_{D_k}}{\mu(D_k)} - \sum_{k\in F}\alpha_k\frac{\chi_{D_k}}{\mu(D_k)} - \sum_{k\in J\setminus F}\alpha_k\frac{\chi_{D_k}}{\mu(D_k)}\Big\|_1\\ &\le \sum_{k\in F}|\beta_k-\alpha_k| + \sum_{k\in J\setminus F}|\alpha_k| = 1-\sum_{k\in F}|\alpha_k| + \sum_{k\in J\setminus F}|\alpha_k|\\ &= 2\Big(1-\sum_{k\in F}|\alpha_k|\Big) < 2\eta < \frac{\varepsilon}{4}.\end{aligned} \tag{2.11}$$

Step 3. Now, we approximate the function h_0 by a new one h_2 such that for each $k\in F$ the new function is constant on each D_k. So we also approximate the operator T_0 by a new one.

For this aim we choose an element t_k in D_k, for any $k\in F$, put $\psi_k = h_0(t_k)\in L_1(\mu)$ and define $h_1\in L_\infty(\mu, L_1(\mu))$ by

$$h_1 = h_0\chi_{\Omega\setminus(\bigcup_{k\in F}D_k)} + \sum_{k\in F}\psi_k\chi_{D_k}.$$

By (2.5) we have that $\|h_1\|_\infty\le 1$. If $T_1\in\mathcal{L}(L_1(\mu))$ is the operator associated to h_1, then T_1 is the sum of $T_{0|\Omega\setminus(\bigcup_{k\in F}D_k)}$ and a finite-rank operator, so $T_1\in B_{\mathcal{M}}$. By using (2.4), we clearly have

$$\begin{aligned}\|T_1-T_0\| = \|h_1-h_0\|_\infty &\le \sup\{\|\psi_k-h_0(t)\|_1 : t\in D_k, k\in F\}\\ &= \sup\{\|h_0(t_k)-h_0(t)\|_1 : t\in D_k, k\in F\}\le\eta.\end{aligned} \tag{2.12}$$

Since $\|T_0\|=1$ we get that $0<1-\eta\le\|T_1\|\le 1$. Now we define $T_2=\frac{T_1}{\|T_1\|}$ and so we have that

$$\|T_2-T_1\| = 1-\|T_1\|\le\eta.$$

In view of the previous inequality and (2.12) we obtain that

$$\|T_2 - T_0\| \le \|T_2 - T_1\| + \|T_1 - T_0\| \le 2\eta < \frac{\varepsilon}{4}. \tag{2.13}$$

From (2.10) and (2.13) we get that

$$\mathrm{Re}\; g_1(T_2(f_2)) \ge \mathrm{Re}\; g_1(T_0(f_2)) - \|T_2 - T_0\| > 1 - 3\eta. \tag{2.14}$$

On the other hand, it is clear that

$$T_1(f_2) = \int_\Omega h_1 f_2 \, d\mu = \int_{\Omega \setminus \bigcup_{k\in F} D_k} h_1 f_2 \, d\mu + \sum_{k\in F} \int_{D_k} h_1 f_2 \, d\mu = \sum_{k\in F} \beta_k \psi_k.$$

For simplicity, for each $k \in F$, put $\phi_k = \dfrac{\psi_k}{\|T_1\|}$. So we have that

$$T_2(f_2) = \sum_{k\in F} \beta_k \phi_k.$$

It is clear that $\phi_k \in B_{L_1(\mu)}$ for every $k \in F$. From (2.9) and (2.14) we obtain that

$$\mathrm{Re}\; g_1\left(\sum_{k\in F} \frac{\beta_k}{2} \left(\frac{\chi_{D_k}}{\mu(D_k)} + \phi_k \right) \right) = \mathrm{Re}\; g_1\left(\frac{f_2 + T_2(f_2)}{2} \right) > 1 - 2\eta.$$

Step 4. In this step we will obtain approximations f_3, T_3 of f_2 and T_2, respectively. We will check in the final step that T_3 attains its norm at f_3, a necessary condition for our purpose. In fact f_3 and T_3 are the final approximations to f_0 and T_0.

Define the set G as follows

$$G = \left\{ k \in F : \mathrm{Re}\; g_1\left(\frac{\beta_k}{2} \left(\frac{\chi_{D_k}}{\mu(D_k)} + \phi_k \right) \right) > \left(1 - \sqrt{2\eta}\right) |\beta_k| \right\}.$$

In view of Lemma 2.1 we have that

$$\sum_{k\in G} |\beta_k| > 1 - \sqrt{2\eta} = 1 - \frac{\varepsilon^4}{2^{16}}. \tag{2.15}$$

It is immediate that

$$\mathrm{Re}\; \beta_k g_1\left(\frac{\chi_{D_k}}{\mu(D_k)} \right) > \left(1 - 2\sqrt{2\eta}\right)|\beta_k| = \left(1 - \frac{\varepsilon^4}{2^{15}}\right)|\beta_k|, \qquad \forall k \in G.$$

So, for each $k \in G$ we have

$$\mathrm{Re}\; \beta_k \gamma_k = \mathrm{Re}\; \beta_k g_1\left(\frac{\chi_{D_k}}{\mu(D_k)} \right) > \left(1 - \frac{\varepsilon^4}{2^{15}}\right)|\beta_k| \ge \left(1 - \frac{\varepsilon^4}{2^{15}}\right)|\beta_k \gamma_k|.$$

Hence, we obtain that $\beta_k \neq 0$ for $k \in G$ and also that

$$|\gamma_k| > 1 - \frac{\varepsilon^4}{2^{15}} > 0, \qquad \forall k \in G. \tag{2.16}$$

By using also Lemma 2.3 we get

$$|\beta_k\gamma_k - |\beta_k\gamma_k|| < \frac{\varepsilon^2}{2^7}|\beta_k\gamma_k|.$$

Hence,

$$\left|\beta_k - \frac{|\beta_k\gamma_k|}{\gamma_k}\right| < \frac{\varepsilon^2}{2^7}|\beta_k| \text{ and } \left|\gamma_k - \frac{|\beta_k\gamma_k|}{\beta_k}\right| < \frac{\varepsilon^2}{2^7}|\gamma_k|, \qquad \forall k \in G, \tag{2.17}$$

so

$$\left|\frac{\gamma_k}{|\gamma_k|} - \frac{|\beta_k|}{\beta_k}\right| < \frac{\varepsilon^2}{2^7}, \qquad \forall k \in G. \tag{2.18}$$

The element f_3 given by

$$f_3 = \frac{1}{\sum_{k\in G}|\beta_k|}\sum_{k\in G}\frac{|\beta_k\gamma_k|}{\gamma_k}\frac{\chi_{D_k}}{\mu(D_k)}$$

belongs to the unit sphere of $L_1(\mu)$. Now, by using (2.15) and (2.17) we get that

$$\begin{aligned}
\|f_3 - f_2\|_1 &= \left\|\frac{1}{\sum_{k\in G}|\beta_k|}\sum_{k\in G}\frac{|\beta_k\gamma_k|}{\gamma_k}\frac{\chi_{D_k}}{\mu(D_k)} - \sum_{k\in F}\beta_k\frac{\chi_{D_k}}{\mu(D_k)}\right\|_1 \\
&= \left\|\frac{1}{\sum_{k\in G}|\beta_k|}\sum_{k\in G}\frac{|\beta_k\gamma_k|}{\gamma_k}\frac{\chi_{D_k}}{\mu(D_k)} - \sum_{k\in G}\beta_k\frac{\chi_{D_k}}{\mu(D_k)} - \sum_{k\in F\setminus G}\beta_k\frac{\chi_{D_k}}{\mu(D_k)}\right\|_1 \\
&\le \sum_{k\in G}\left|\frac{1}{\sum_{k\in G}|\beta_k|}\frac{|\beta_k\gamma_k|}{\gamma_k} - \beta_k\right| + \sum_{k\in F\setminus G}|\beta_k| \\
&\le \sum_{k\in G}\left|\frac{1}{\sum_{k\in G}|\beta_k|}\frac{|\beta_k\gamma_k|}{\gamma_k} - \frac{|\beta_k\gamma_k|}{\gamma_k}\right| + \sum_{k\in G}\left|\frac{|\beta_k\gamma_k|}{\gamma_k} - \beta_k\right| + \sum_{k\in F\setminus G}|\beta_k| \\
&\le 1 - \sum_{k\in G}|\beta_k| + \sum_{k\in G}\frac{\varepsilon^2}{2^7}|\beta_k| + \sum_{k\in F\setminus G}|\beta_k| \\
&\le 2\Big(1 - \sum_{k\in G}|\beta_k|\Big) + \frac{\varepsilon^2}{2^7} \le \frac{\varepsilon}{8}.
\end{aligned} \tag{2.19}$$

In view of (2.1), (2.11) and (2.19), we obtain that

$$\|f_3 - f_0\|_1 \le \|f_3 - f_2\|_1 + \|f_2 - f_1\|_1 + \|f_1 - f_0\|_1 < \frac{\varepsilon}{8} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon. \tag{2.20}$$

Now notice obviously that

$$\operatorname{Re}\,\beta_k g_1(\phi_k) > \big(1 - 2\sqrt{2\eta}\big)|\beta_k| > \Big(1 - \frac{\varepsilon^4}{2^{14}}\Big)|\beta_k|, \qquad \forall k \in G.$$

For each $k \in G$, define P_k as follows

$$P_k = \Big\{t \in \Omega : \operatorname{Re}\,\beta_k g_1(t)\phi_k(t) > \Big(1 - \frac{\varepsilon^2}{2^7}\Big)|\beta_k\phi_k(t)|\Big\}.$$

Clearly P_k is a measurable set. According to Lemma 2.2, for each $k \in G$ we have

$$\mathrm{Re} \int_{P_k} \beta_k g_1 \phi_k \, d\mu > \Big(1 - \frac{\varepsilon^2}{2^7}\Big)|\beta_k|,$$

so

$$\int_{P_k} |\phi_k| d\mu > 1 - \frac{\varepsilon^2}{2^7} > 0. \tag{2.21}$$

Let us fix $k \in G$ and $t \in P_k$. Notice that $\beta_k g_1(t) \neq 0$. By Lemma 2.3 it follows

$$\big|\beta_k g_1(t)\phi_k(t) - |\beta_k g_1(t)\phi_k(t)|\,\big| < \frac{\varepsilon}{2^3}|\beta_k g_1(t)\phi_k(t)|,$$

so

$$\left|\phi_k(t) - \frac{|\beta_k g_1(t)\phi_k(t)|}{\beta_k g_1(t)}\right| < \frac{\varepsilon}{2^3}|\phi_k(t)|, \qquad \forall k \in G, t \in P_k. \tag{2.22}$$

For each $k \in G$ we can define the element φ_k in $L_1(\mu)$ by

$$\varphi_k = \frac{\gamma_k}{|\gamma_k|} \frac{|\phi_k|}{\int_{P_k} |\phi_k| \, d\mu} \frac{|g_1|}{g_1} \chi_{P_k}.$$

It is immediate that $\varphi_k \in S_{L_1(\mu)}$. From (2.21) and (2.22), for each $k \in G$ we have

$$\begin{aligned}
\|\varphi_k - \phi_k\|_1 &\leq \|\varphi_k - \phi_k \chi_{P_k}\|_1 + \|\phi_k \chi_{\Omega \setminus P_k}\|_1 \\
&< \Big\|\varphi_k - \phi_k \chi_{P_k}\Big\|_1 + \frac{\varepsilon^2}{2^7} \\
&\leq \Big\|\varphi_k - \frac{\gamma_k}{|\gamma_k|}|\phi_k|\frac{|g_1|}{g_1}\chi_{P_k}\Big\|_1 + \Big\|\frac{\gamma_k}{|\gamma_k|}|\phi_k|\frac{|g_1|}{g_1}\chi_{P_k} - \frac{|\beta_k|}{\beta_k}|\phi_k|\frac{|g_1|}{g_1}\chi_{P_k}\Big\|_1 \\
&+ \Big\|\frac{|\beta_k|}{\beta_k}|\phi_k|\frac{|g_1|}{g_1}\chi_{P_k} - \phi_k \chi_{P_k}\Big\|_1 + \frac{\varepsilon^2}{2^7} \\
&\leq \Big\|\varphi_k - \frac{\gamma_k}{|\gamma_k|}|\phi_k|\frac{|g_1|}{g_1}\chi_{P_k}\Big\|_1 + \Big|\frac{\gamma_k}{|\gamma_k|} - \frac{|\beta_k|}{\beta_k}\Big| + \frac{\varepsilon}{2^3} + \frac{\varepsilon^2}{2^7} \\
&\leq \Big\|\varphi_k - \frac{\gamma_k}{|\gamma_k|}|\phi_k|\frac{|g_1|}{g_1}\chi_{P_k}\Big\|_1 + \frac{\varepsilon}{4} \quad \text{(by (2.18))} \\
&= \Big\|\frac{\gamma_k}{|\gamma_k|}\frac{|\phi_k|}{\int_{P_k}|\phi_k| \, d\mu}\frac{|g_1|}{g_1}\chi_{P_k} - \frac{\gamma_k}{|\gamma_k|}|\phi_k|\frac{|g_1|}{g_1}\chi_{P_k}\Big\|_1 + \frac{\varepsilon}{4} \\
&= 1 - \int_{P_k} |\phi_k| \, d\mu + \frac{\varepsilon}{4} < \frac{\varepsilon^2}{2^7} + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}.
\end{aligned} \tag{2.23}$$

Let the function h_3 be defined as follows

$$h_3 = \frac{h_1}{\|h_1\|_\infty} \chi_{\Omega \setminus \bigcup_{k \in G} D_k} + \sum_{k \in G} \varphi_k \chi_{D_k}.$$

It is easy to see that h_3 belongs to the unit sphere of $L_\infty(\mu, L_1(\mu))$. Let $T_3 \in S_{\mathcal{L}(L_1(\mu))}$ be the operator associated to the function h_3 in view of Proposition 2.1. Since G is a finite set, $\mathcal{F}(L_1(\mu)) \subset \mathcal{M}$ and $T_1 \in \mathcal{M}$, by using the assumptions on $\mathcal{M}$ we know that $T_3 \in S_{\mathcal{M}}$.

We also have that

$$\begin{aligned}
\|T_3 - T_2\| &= \left\| h_3 - \frac{h_1}{\|h_1\|_\infty} \right\|_\infty \\
&= \left\| h_3 \chi_{\Omega\setminus(\cup_{k\in G} D_k)} + \sum_{k\in G} h_3 \chi_{D_k} - \frac{h_1}{\|h_1\|_\infty} \chi_{\Omega\setminus(\cup_{k\in G} D_k)} - \sum_{k\in G} \frac{h_1}{\|h_1\|_\infty} \chi_{D_k} \right\|_\infty \\
&= \left\| \frac{h_1}{\|h_1\|_\infty} \chi_{\Omega\setminus(\cup_{k\in G} D_k)} + \sum_{k\in G} \varphi_k \chi_{D_k} - \frac{h_1}{\|h_1\|_\infty} \chi_{\Omega\setminus(\cup_{k\in G} D_k)} - \sum_{k\in G} \phi_k \chi_{D_k} \right\|_\infty \\
&= \left\| \sum_{k\in G} \big(\varphi_k - \phi_k\big)\, \chi_{D_k} \right\|_\infty = \sup_{k\in G} \big\|\varphi_k - \phi_k\big\|_1 \le \frac{\varepsilon}{2} \qquad \text{(by (2.23))}.
\end{aligned}$$

By the previous inequality and (2.13) we obtain

$$\|T_3 - T_0\| \le \|T_3 - T_2\| + \|T_2 - T_0\| < \varepsilon. \tag{2.24}$$

Step 5. Finally, we are going to find an approximation of g_1 and complete our proof.

We put $A = \left\{t \in \Omega : |g_1(t)| \ge 1 - \frac{\varepsilon^2}{2^7}\right\}$ and let the function g_2 be defined by $g_2 = \frac{g_1}{|g_1|}\chi_A + g_1\chi_{\Omega\setminus A}$. Since $g_1 \in S_{L_\infty(\mu)}$, we have that $g_2 \in S_{L_\infty(\mu)}$. It is also clear that

$$\|g_2 - g_1\|_\infty \le \frac{\varepsilon^2}{2^7}. \tag{2.25}$$

By using (2.1) and (2.25) we also have that

$$\|g_2 - g_0\|_\infty \le \|g_2 - g_1\|_\infty + \|g_1 - g_0\|_\infty \le \frac{\varepsilon^2}{2^7} + \frac{\varepsilon}{4} < \varepsilon. \tag{2.26}$$

By (2.16) we know that $|\gamma_k| > 1 - \frac{\varepsilon^4}{2^{15}}$ for each $k \in G$. Since $G \subset J$, in view of (2.6), the restriction of g_1 to D_k coincides with γ_k and so $D_k \subset A$ for all $k \in G$. Hence,

$$g_{2|D_k} = \frac{\gamma_k}{|\gamma_k|}, \qquad \forall k \in G.$$

Therefore, we deduce that

$$\begin{aligned}
g_2(f_3) &= g_2\left(\frac{1}{\sum_{k\in G} |\beta_k|} \sum_{k\in G} \frac{|\beta_k \gamma_k|}{\gamma_k} \frac{\chi_{D_k}}{\mu(D_k)}\right) \\
&= \frac{1}{\sum_{k\in G} |\beta_k|} \sum_{k\in G} \frac{|\beta_k \gamma_k|}{\gamma_k} \frac{1}{\mu(D_k)} g_2\big(\chi_{D_k}\big) \\
&= \frac{1}{\sum_{k\in G} |\beta_k|} \sum_{k\in G} \frac{|\beta_k \gamma_k|}{\gamma_k} \frac{\gamma_k}{|\gamma_k|} = 1.
\end{aligned} \tag{2.27}$$

For each $k \in G$, from the definition of P_k and A, we deduce that $P_k \subset A$, so

$$g_2(\varphi_k) = \int\limits_{P_k} \frac{\gamma_k}{|\gamma_k|} \frac{|\phi_k|}{\int_{P_k} |\phi_k|\, d\mu}\, d\mu = \frac{\gamma_k}{|\gamma_k|}. \tag{2.28}$$

Since

$$T_3(f_3) = \int\limits_{\Omega} h_3 f_3 \, d\mu = \frac{1}{\sum_{k\in G} |\beta_k|} \sum_{k\in G} \frac{|\beta_k \gamma_k|}{\gamma_k} \varphi_k,$$

by using (2.28) we have that

$$\begin{aligned} g_2(T_3(f_3)) &= \frac{1}{\sum_{k\in G} |\beta_k|} \sum_{k\in G} \frac{|\beta_k \gamma_k|}{\gamma_k} g_2(\varphi_k) \\ &= \frac{1}{\sum_{k\in G} |\beta_k|} \sum_{k\in G} \frac{|\beta_k \gamma_k|}{\gamma_k} \frac{\gamma_k}{|\gamma_k|} = 1. \end{aligned} \tag{2.29}$$

We have shown that there are elements $T_3 \in S_{\mathcal{M}}$, $f_3 \in S_{L_1(\mu)}$ and $g_2 \in S_{L_\infty(\mu)}$ that in view of (2.20), (2.24), (2.26), (2.27) and (2.29) satisfy

$$\|T_3 - T_0\| < \varepsilon, \qquad \|f_3 - f_0\|_1 < \varepsilon, \qquad \|g_2 - g_0\|_\infty < \varepsilon$$

and also

$$g_2(f_3) = g_2(T_3(f_3)) = 1.$$

So we showed that $\mathcal{M}$ has the BPBp-ν with the function η given by $\eta(\varepsilon) = \frac{\varepsilon^8}{2^{33}}$. $\square$

In case that μ is a σ-finite measure, there is a finite measure ζ and a linear isometry Φ from $L_1(\mu)$ onto $L_1(\zeta)$. From this fact we deduce the following result which generalizes Theorem 2.1 for some well-known classes of operators.

Corollary 2.1. *Let (Ω, Σ, μ) be a σ-finite measure space. The following subspaces of $\mathcal{L}(L_1(\mu))$ have the BPBp-ν and the function η satisfying Definition 2.1 is independent from the measure space.*

1) *The subspace of all finite-rank operators on $L_1(\mu)$.*
2) *The subspace of all compact operators on $L_1(\mu)$.*
3) *The subspace of all weakly compact operators on $L_1(\mu)$.*

In case that μ is finite, then the subspace of all representable operators on $L_1(\mu)$ also has the BPBp-ν.

Proof. Assume first that μ is a finite measure. It is known that $\mathcal{F}(L_1(\mu)) \subset \mathcal{K}(L_1(\mu)) \subset \mathcal{WC}(L_1(\mu)) \subset \mathcal{R}(L_1(\mu))$ and $T_{|A}(B_{L_1(\mu)}) \subset T(B_{L_1(\mu)})$ for each $T \in \mathcal{L}(L_1(\mu))$ and every measurable subset A of Ω. Also, it is clear that $T_{|A} \in \mathcal{R}(L_1(\mu))$ for any $T \in \mathcal{R}(L_1(\mu))$ and every measurable subset A of Ω. Therefore, the spaces $\mathcal{F}(L_1(\mu))$, $\mathcal{K}(L_1(\mu))$, $\mathcal{WC}(L_1(\mu))$ and $\mathcal{R}(L_1(\mu))$ satisfy the assumptions of Theorem 2.1, and so the above statements hold in case that μ is finite.

Now, let μ be a σ-finite measure. We will show that the space $\mathcal{F}(L_1(\mu))$ satisfies the BPBp-ν. There is a finite measure ζ and a surjective linear isometry Φ from $L_1(\mu)$ into $L_1(\zeta)$. The mapping Φ induces a surjective linear isometry from $\mathcal{F}(L_1(\mu))$ into $\mathcal{F}(L_1(\zeta))$ given by $T \mapsto \Phi \circ T \circ \Phi^{-1}$. Since Φ is an isometry, it follows that $\nu(T) = \nu(\Phi \circ T \circ \Phi^{-1})$ for every $T \in \mathcal{F}(L_1(\mu))$. On the other hand, it is satisfied that $(f, g) \in \Pi(L_1(\mu))$ if and only if $(\Phi(f), (\Phi^{-1})^t(g)) \in \Pi(L_1(\zeta))$. Also $(\Phi^{-1})^t(g)(\Phi \circ T \circ \Phi^{-1}(\Phi(f))) = g(T(f))$ for every $T \in \mathcal{F}(L_1(\mu))$. Since $\mathcal{F}(L_1(\zeta))$ has the BPBp-ν we deduce the same property for $\mathcal{F}(L_1(\mu))$.

The proofs of the statements 2) and 3) are analogous. $\square$

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References

[1] M.D. Acosta, R.M. Aron, D. García, M. Maestre, Bishop–Phelps–Bollobás property for operators, J. Funct. Anal. 254 (2008) 2780–2799.
[2] M.D. Acosta, J. Becerra-Guerrero, D. García, S.K. Kim, M. Maestre, Bishop–Phelps–Bollobás property for certain spaces of operators, J. Math. Anal. Appl. 414 (2014) 532–545.
[3] A. Avilés, A.J. Guirao, O. Kozhushkina, The Bishop–Phelps–Bollobás property for numerical radius in $C(K)$ spaces, J. Math. Anal. Appl. 419 (1) (2014) 395–421.
[4] B. Bollobás, An extension to the theorem of Bishop and Phelps, Bull. Lond. Math. Soc. 2 (1970) 181–182.
[5] F.F. Bonsall, J. Duncan, Numerical Ranges II, Lecture Note Series, vol. 10, London Math. Soc., Cambridge University Press, Cambridge, 1973.
[6] M. Chica, V. Kadets, M. Martín, S. Moreno-Pulido, F. Rambla-Barreno, Bishop–Phelps–Bollobás moduli of a Banach space, J. Math. Anal. Appl. 412 (2014) 697–719.
[7] J. Diestel, J.J. Uhl Jr., Vector Measures, Math. Surveys, vol. 15, Amer. Math. Soc., Providence, RI, 1977.
[8] J. Duncan, C.M. McGregor, J.D. Pryce, A.J. White, The numerical index of a normed space, J. Lond. Math. Soc. 2 (2) (1970) 481–488.
[9] J. Falcó, The Bishop–Phelps–Bollobás property for numerical radius on L_1, J. Math. Anal. Appl. 414 (1) (2014) 125–133.
[10] A.J. Guirao, O. Kozhushkina, The Bishop–Phelps–Bollobás property for numerical radius in $\ell_1(\mathbb{C})$, Studia Math. 218 (1) (2013) 41–54.
[11] S.K. Kim, H.J. Lee, M. Martín, On the Bishop–Phelps–Bollobás property for numerical radius, Abstr. Appl. Anal. 2014 (2014) 479208.