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How strong a logistic damping can prevent blow-up for the minimal Keller–Segel chemotaxis system?

Tian Xiang

Institute for Mathematical Sciences, Renmin University of China, Beijing, 100872, China

A R T I C L E   I N F O

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A B S T R A C T

We study nonnegative solutions of parabolic–parabolic Keller–Segel minimal-chemotaxis-growth systems with prototype given by

$$\begin{cases} u_t = \nabla \cdot (d_1 \nabla u - \chi u \nabla v) + \kappa u - \mu u^2, & x \in \Omega, t > 0, \\ v_t = d_2 \Delta v - \beta v + \alpha u, & x \in \Omega, t > 0 \end{cases}$$

in a smooth bounded smooth but not necessarily *convex* domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ with nonnegative initial data u_0, v_0 and homogeneous Neumann boundary data, where $d_1, d_2, \alpha, \beta, \mu > 0$, $\chi, \kappa \in \mathbb{R}$. We provide quantitative and qualitative descriptions of the competition between logistic damping and other ingredients, especially, chemotactic aggregation to guarantee boundedness and convergence. Specifically, we first obtain an explicit formula $\mu_0 = \mu_0(n, d_1, d_2, \alpha, \chi)$ for the logistic damping rate μ such that the system has no blow-ups whenever $\mu > \mu_0$. In particular, for $\Omega \subset \mathbb{R}^3$, we get a clean formula for μ_0:

$$\mu_0(3, d_1, d_2, \alpha, \chi) = \begin{cases} \frac{3}{4d_1}\alpha\chi, & \text{if } d_1 = d_2, \chi > 0 \text{ and } \Omega \text{ is convex}, \\ \frac{3}{\sqrt{10}-2}(\frac{1}{d_1} + \frac{2}{d_2})\alpha|\chi|, & \text{otherwise.} \end{cases}$$

This offers a quantized effect of the logistic source on the prevention of blow-ups. Our result extends the fundamental boundedness principle by Winkler [42] with $d_1 = 1, d_2 = \alpha = \beta := 1/\tau$, Ω being convex and sufficiently large values of μ beyond a certain number not explicitly known (except the simple case $\tau = 1$ and $\chi > 0$) and quantizes the qualitative result of Yang et al. [52]. Besides, in non-convex domains, since $\mu_0(3, 1, 1, 1, \chi) = (7.743416\cdots)\chi$, the recent boundedness result, $\mu > 20\chi$, of Mu and Lin [25] is greatly improved. Then we derive another explicit formula:

$$\mu_1 = \mu_1(d_1, d_2, \alpha, \beta, \kappa, \chi) = \frac{\alpha|\chi|}{4}\sqrt{\frac{\kappa_+}{d_1 d_2 \beta}}$$

for the logistic damping rate so that convergence of bounded solutions is ensured and the respective convergence rates are explicitly calculated out whenever $\mu > \mu_1$. Recent convergence results of He and Zheng [9] are therefore complemented and refined.

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1. Introduction and main results

Chemotaxis, the tendency of cells, bacteria and similarly tiny organisms to orient the direction of movement (otherwise random) toward increasing or decreasing concentration of a signaling substance, has been attracting great attention in biological and mathematical community. A celebrated mathematical model, initially proposed by Keller–Segel [19,20], makes up of two parabolic equations reflecting chemotactic movement through a nonlinear advective–diffusive term as its most defining characteristic. Their pioneering works have initiated vast investigations of the K–S model and its various forms of variants since 1970. We refer to the beautiful survey papers [2,12,15,46], where a broad survey on the progress of various chemotaxis models and rich selection of references can be found.

If biological processes in which chemotaxis plays a role are modeled not only on small timescales, often the spontaneous growth of the population, whose density we will denote by u, should be incorporated. A prototypical choice to achieve this is the logistic type source $\kappa u - \mu u^2$ with birth and death rates κ and μ, respectively. Let us then begin with the most simplest perhaps the most interesting so-called minimal-chemotaxis-growth model

$$\begin{cases} u_t = \nabla \cdot (\nabla u - \chi u \nabla v) + \kappa u - \mu u^2, & x \in \Omega, t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n, n \geq 1$ is a smooth bounded domain, v denotes the concentration of the chemical signal, $\kappa \geq 0, \mu \geq 0, \tau \geq 0$ and $\chi \in \mathbb{R}$. The nonlinear term $\chi u \nabla v$, the defining term in chemotaxis models, is called chemotactic term: in the case $\chi > 0$, it models the cells movement toward the higher concentrations of the chemical signal, which is called (positive) chemotaxis, in the case $\chi < 0$, it models the cells movement away from the higher concentrations of the chemical signal, which is called negative chemotaxis.

Model (1.1) with $\kappa = 0, \mu = 0$ corresponds to the classical Keller–Segel minimal model [19,20], which and whose variants have been extensively explored since 1970. The striking feature of KS type models is the possibility of blow-up of solutions in a finite/infinite time (see, e.g., [2,15,46]), which strongly depends on the space dimension. A finite/infinite time blow-up never occurs in 1-dimension [11,30] (except in some extreme nonlinear degenerate diffusion model [6]), a critical mass blow-up occurs in 2-dimension: when the initial mass lies below the threshold solutions exist globally, while above the threshold solutions blow up in finite time [14,29,33], and generic blow-up in $\geq$ 3-D [41,46]. The knowledge about the classical KS type models appears to be rather complete, see the aforementioned surveys for more.

The blow up solution or a δ function is surely connected to the phenomenon of cell aggregation; on the other hand, various mechanisms proposed to the underlying model have manifested that blowup solutions are fully precluded while pattern formation arises [5,12,26,51]. Among those mechanisms (see the introduction in [51]), inclusion of a growth source of cells is a common choice. In particular, the presence of logistic source has been shown to have an effect of preventing ultimate growth of populations. Indeed, in the case $n = 1, 2$, even arbitrarily small $\mu > 0$ will be sufficient to suppress blow-up by ensuring all solutions to (1.1) are global-in-time and uniformly bounded for all reasonably initial data [11,30,31,50]. This is even true for a two-dimensional parabolic–elliptic chemotaxis system with singular sensitivity [8]. Whereas, in the case $n \geq 3$, the first boundedness and global existence were obtained for a parabolic–elliptic simplification of (1.1), i.e., $\tau = 0$, under the condition that $\mu > \frac{(n-2)}{n}\chi$ [37]. Recently, this result was improved to the borderline case $\mu \geq \frac{n-2}{n}\chi$ [10,18,39]. See also the existence of very weak solutions under more general conditions [40].

E-mail address: txiang@ruc.edu.cn.

For the full parabolic–parabolic minimal chemotaxis-growth mode, fundamental findings were obtained by Winkler [42]. Under the additional assumptions that Ω is convex and μ is beyond a certain number μ_0 not explicitly known (except the case $\tau = 1$ and $\chi > 0$, where $\mu > \frac{n}{4}\chi$ is sufficient to prevent blow-ups), he proved the existence and uniqueness of global, smooth, bounded solutions to (1.1). Recently, a progress on global boundedness to (1.1) with $\chi > 0$ was derived as long as $\mu > \theta_0\chi$ for some implicit positive constant θ_0 depending on Sobolev embedding constants [52]. In 2015, an explicit lower bound for a 3-D chemotaxis-fluid system with logistic source was obtained by Tao and Winkler [35], when applied to the chemotaxis system (1.1) with $\chi = \tau = 1$, their result states that $\mu \geq 23$ is enough to prevent blow-ups. This bound was further improved by Lin and Mu [25] (2016) in three dimensional settings, wherein they replaced the logistic source in (1.1) by the damping term $u - \mu u^r$ with $r \geq 2$ to derive the boundedness under

$$\mu^{\frac{1}{r-1}} > 20\chi. \tag{1.2}$$

Their arguments were done to the case that Ω is convex by remarking that they could be adapted to non-convex domains by virtue of the papers [17,28]. Of course, when $r > 2$, this result was already implied by [42,52]. Moreover, for the particular choices $\kappa = \tau = 1$, under certain largeness condition on the ration μ/χ, the stabilization of bounded solution (u, v) of the KS model (1.1) to the constant equilibrium $(1/\mu, 1/\mu)$ as $t \to \infty$ occurs [9,25,47,55]. While, for arbitrarily small $\mu > 0$, only existence of global weak solutions to (1.1) is available [22] in convex 3-D domains. Other dynamical properties of (1.1) can be found e.g. in [13,23,36,48,49]. Finally, we observe that enormous variants of (1.1) have been considered to provide conditions on diffusion, degradation, chemo-sensitivity and mostly on the growth source ensuring the boundedness of the proposed models [2,4,38,50,54] and the references therein, and that explosion of solutions is possible in chemotaxis systems despite logistic growth restriction [45]. Therefore, it is meaningful to detect more circumstances where no blow-up is allowed for the minimal KS model with logistic growth (1.1).

It is widely known that the KS minimal model (1.1) with $\kappa = \mu = 0$ admits both bounded and unbounded solutions, identified via the critical chemotactic sensitivity $u^{\frac{2}{n}}$ [7,16]. Therefore, the model (1.1) is simply a supercritical case with the balance of logistic damping and aggregation effects, for which the property of solutions should be not only qualitatively but also quantitatively determined by the parameters involved. Motivated by the works [25,35,42,52], we attempt to provide a quantitative description of the competition between logistic damping and other ingredients, especially, chemotactic aggregation, and, in particular, we aim to find a full picture on how the lower bound μ_0 of the logistic damping rate μ is affected by all the involving parameters so that no blow-up is allowed for $\mu > \mu_0$. Therefore, in this paper, we will consider a full parameter K–S minimal system with a growth source covering the standard logistic source as follows:

$$\begin{cases} u_t = \nabla \cdot (d_1 \nabla u - \chi u \nabla v) + f(u), & x \in \Omega, t > 0, \\ v_t = d_2 \Delta v - \beta v + \alpha u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n, n \geq 1$ is a smooth bounded domain but not necessarily convex, $d_1, d_2, \alpha, \beta > 0$, $\chi \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies $f(0) \geq 0$ as well as

$$f(s) \leq a - \mu s^2, \quad \forall s \geq 0 \tag{1.4}$$

for some $a \geq 0$ and $\mu > 0$.

With the aid of the boundedness criteria ([2,50]) on how the growth source affects the boundedness for a general class of chemotaxis-growth systems than (1.3), we provide a detailed algorithm to derive an explicit formula for the lower bound μ_0 of the logistic damping rate μ such that the system (1.3) admits only globally bounded solutions whenever $\mu > \mu_0$. For nonconvex domains, our procedure is mainly carried out in physically relevant settings ($n = 3, 4, 5$), where we have a clean and compact formula for μ_0. Precisely, our main quantitative findings in this regard read as follows:

Theorem 1.1 *(How strong a logistic damping can prevent blow-up for (1.3)). Let $\Omega \subset \mathbb{R}^n (n \geq 3)$ be a bounded smooth domain, the initial data (u_0, v_0) satisfy $u_0 \in C(\overline{\Omega})$ and $v_0 \in W^{1,p_0}(\Omega)$ with some $p_0 > n$ and let f satisfy the logistic condition (1.4) and $d_1, d_2, \alpha, \beta > 0$, $a \geq 0$ and $\chi \in \mathbb{R}$.*

(i) *For $n = 3$, let the lower logistic damping rate $\mu_0 = \mu_0(3, d_1, d_2, \alpha, \chi)$ of μ be explicitly given by*

$$\mu_0 = \begin{cases} \frac{3}{4d_1}\alpha\chi, & \text{if } d_1 = d_2, \chi > 0 \text{ and } \Omega \text{ is convex}, \\ \frac{3}{\sqrt{10}-2}(\frac{1}{d_1} + \frac{2}{d_2})\alpha|\chi|, & \text{otherwise}; \end{cases} \tag{1.5}$$

(ii) *for $n = 4, 5$, let the lower logistic damping rate $\mu_0 = \mu_0(n, d_1, d_2, \alpha, \chi)$ of μ be explicitly given by*

$$\mu_0 = \begin{cases} \frac{n}{4d_1}\alpha\chi, & \text{if } d_1 = d_2, \chi > 0 \text{ and } \Omega \text{ is convex }, \\ \max\Big\{\frac{1}{3}h(n, d_1, d_2), \frac{n}{\sqrt{2n+4}-2}(\frac{1}{d_1} + \frac{2}{d_2})\Big\}\alpha|\chi|, & \text{otherwise} \end{cases} \tag{1.6}$$

with

$$h(n, d_1, d_2) = \inf_{0<\epsilon<d_1, 0<\eta<d_2} \Big\{\sqrt{\tfrac{n}{18d_2\epsilon}} + \sqrt{\tfrac{1}{2\epsilon}(\tfrac{1}{\eta} + \tfrac{n}{2d_2})} + \sqrt{\tfrac{1}{(d_2-\eta)}(\tfrac{2}{\eta} + \tfrac{n}{2d_2})}\Big[\sqrt{2} + \tfrac{(d_1+d_2)}{2\sqrt{(d_1-\epsilon)(d_2-\eta)}}\Big]\Big\}.$$

Then, whenever $\mu > \mu_0$, the chemotaxis-growth system (1.3) has a unique global-in-time classical solution (u, v) for which both u and v are positive and uniformly bounded in $\Omega \times (0, \infty)$.

Remark 1.2 *(Notes on how strong a logistic damping can prevent blow-up).*

(P1) The explicit logistic damping rate μ_0 given in (1.5) or (1.6) exhibits the contributions of the degradation, creation and diffusion rates, etc in respective of boundedness of solutions of (1.3). That is, it shows how strong a logistic damping is needed to prevent blow-ups for (1.3).

(P2) The formula for μ_0 is not only explicitly expressible but also is independent of the degradation rate β of signals, the birth rate a of cells, the size of domain Ω, initial data u_0, v_0 and Sobolev embedding constants. This gives a quantized effect of the logistic source on preventing blow-ups, and hence improves the boundedness principles [42,50] and the qualitative result [52].

(P3) For $\chi = 0$ (no chemotaxis, cf. [50, Proposition 2.6]) or $\alpha = 0$ (decoupled), the boundedness and global existence are easily seen for any $\mu > 0$. In this sense, the form of $\mu_0(n, d_1, d_2, \alpha, \chi)$ captures and respects our common understanding.

(P4) The chemo-repulsion case, i.e., $\chi < 0$ is allowed as well in μ_0.

(P5) $\mu_0(n, d_1, d_2, \alpha, \chi) \to \infty$ as $d_1 \to 0$ or $d_2 \to 0$ (and it is decreasing in d_1 and d_2); thus small diffusion, especially, degenerate or nonlinear diffusion, enhances the possibility of the occurrence of blow-ups.

(P6) In the case that Ω is nonconvex, we have

$$\mu_0^{(\text{nonconvex})}(3, 1, 1, 1, \chi) = \frac{9}{\sqrt{10}-2}\chi = (7.743416\cdots)\chi.$$

Hence, the very recent boundedness result, $\mu > 20\chi$ obtained from (1.2) with $r = 2$, of Mu and Lin [25], and the byproduct boundedness, $\mu \geq \mu_0(3, 1, 1, 1, 1) = 23$ of Tao and Winkler [35], as their studied 3-D fluid system coupled with the minimal chemotaxis system (1.1), are greatly improved.

For mathematical completeness, one may wonder such an explicit formula is also available in n-D ($n \geq 6$). Indeed, our algorithm suggests that an explicit formula for μ_0 in $\geq$ 6-D of the form $\theta_0(n, d_1, d_2)\alpha|\chi|$, which is not clean but enjoys the first 5 properties (P1)–(P5), would be also available:

(Q1) For general $n \geq 6$, no blow-ups can occur to the minimal-chemotaxis-growth model (1.3) if $\mu > \mu_0(n, d_1, d_2, \alpha, \chi)$, where

$$\mu_0(n, d_1, d_2, \alpha, \chi) = \begin{cases} \frac{n}{4d_1}\alpha\chi, & \text{if } d_1 = d_2, \chi > 0 \text{ and } \Omega \text{ is convex}, \\ \theta_0(n, d_1, d_2)\alpha|\chi|, & \text{otherwise} \end{cases} \tag{1.7}$$

with some explicit (perhaps cumbersome) formula θ_0 in terms of n, d_1 and d_2 with the property that $\theta_0 \to \infty$ as either $d_1 \to 0$ or $d_2 \to 0$.

Indeed, the first case of (1.7) has been shown in [42] for the KS model (1.1) with $\tau = 1$, see Lemma 4.8 below for the full-parameter model (1.3). Based on [52] and a careful and painful inspection of our procedure, it is quite possible to trace out the formula θ_0 in (1.7). Here, we leave the rigorous justification for future investigations.

In nonconvex domains, the logistic damping rate $\mu_0(n, d_1, d_2, \alpha, \chi)$ is the smallest damping rate that we could obtain using this procedure. While, in convex domains, we have $\mu_0^{(\text{convex})}(3, 1, 1, 1, 1) = 0.75$. A comparison to $\mu_0^{(\text{nonconvex})}(3, 1, 1, 1, 1) = 7.743416\cdots$ as computed in (P3), indicates that convexity and equal diffusivity make a big difference in respective of boundedness and that the formula $\mu_0(n, d_1, d_2, \alpha, \chi)$ may not be optimal even through it meets the expected properties described in (P2) and (P3), on the other hand.

This discussion leads us to ask other challenging questions left for the minimal-chemotaxis-growth model (1.3):

(Q2) Does there exist a critical damping rate μ_0^c that distinguishes between occurrence and impossibility of blow-up for (1.3). That is, for a logistic source satisfying $f(u) \leq a - \mu u^2$, when $\mu < \mu_0^c$, blow-up occurs; whereas, when $\mu > \mu_0^c$, blow-up is impossible.

(Q3) What happens for small logistic damping $\mu < \mu_0$, boundedness or blow-up?

Questions akin to (Q2) and (Q3) may have been indicated by existing literature [42,50]. To explore them, a combination of the references [36,37,39,45,47,48] may be of some help. A complete (quantitative or qualitative) description of the competition between chemotactic aggregation and logistic damping is definitely worthwhile for future explorations.

When logistic damping wins over chemotactic aggregation, i.e., $\mu > \mu_0$ or equivalently $|\chi| < \frac{\mu}{\alpha\theta_0(n,d_1,d_2)}$, we wish to see the explicit effects of each term in (1.3) on the long time dynamical properties of bounded solutions. To this end, we study the large time behavior of solution for the minimal chemotaxis model with a standard logistic source:

$$\begin{cases} u_t = \nabla \cdot (d_1 \nabla u - \chi u \nabla v) + \kappa u - \mu u^2, & x \in \Omega, t > 0, \\ v_t = d_2 \Delta v - \beta v + \alpha u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & x \in \Omega. \end{cases} \tag{1.8}$$

For $d_1 = d_2 = \alpha = \beta = \kappa = 1$, the uniform convergence of bounded solutions to $(1/\mu, 1/\mu)$ is first proved in [47, Theorem 1.1] for $\mu/\chi > 0$ sufficiently large and then in [25, Theorem 1.3] for $\mu > 20\chi$; for $d_1 = d_2 = \alpha = \beta = 1$, He and Zheng [9] modified the energy functional method from [1] to obtain the stability of the constant equilibria $(0,0)$ and $(\kappa/\mu, \kappa/\mu)$ with convergence estimates [9, Theorem 3]. Next, for completeness and to see the role of other parameters in the large time behavior of the solutions, we combine the energy functional method from [1,9] to show the stability of the constant equilibria $(0,0)$ and $(\kappa/\mu, \alpha\kappa/(\beta\mu))$ for the full-parameter KS model (1.8). Our precise results on the large time limit of bounded solutions of (1.8) are collected in the following theorem.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded smooth domain, $u_0 \geq, \not\equiv 0$, $d_1, d_2, \alpha, \beta, \mu > 0$, $\kappa, \chi \in \mathbb{R}$ and, finally, let $\mu > \mu_0$ as obtained in Theorem 1.1 so that (u, v) be a global and bounded classical solution of (1.8).*

(i) *When $\kappa > 0$, assume additionally that*

$$\mu > \mu_1(d_1, d_2, \alpha, \beta, \kappa, \chi) = \frac{\alpha|\chi|}{4}\sqrt{\frac{\kappa}{d_1 d_2 \beta}}. \tag{1.9}$$

Then the solution (u, v) of (1.8) converges exponentially:

$$\|u(\cdot,t) - \frac{\kappa}{\mu}\|_{L^\infty(\Omega)} + \|v(\cdot,t) - \frac{\alpha\kappa}{\beta\mu}\ \|_{L^\infty(\Omega)} \leq Ce^{-\gamma t} \tag{1.10}$$

for all $t \geq 0$ and some large constant C independent of t and

$$\gamma = \frac{\min\Big\{(\mu - \frac{\alpha\kappa\chi^2}{4d_1 d_2 \mu}\epsilon_0), \frac{\kappa\chi^2}{4d_1 d_2\mu}(\beta - \frac{\alpha}{4\epsilon_0})\Big\}}{(n+2)\max\{\frac{\mu}{\kappa}, \frac{\kappa\chi^2}{8d_1 d_2 \mu}\}}, \quad \epsilon_0 = \frac{1}{2}(\frac{\alpha}{4\beta} + \frac{4d_1 d_2 \mu^2}{\alpha\kappa\chi^2}).$$

(ii) *When $\kappa = 0$, the solution (u, v) of (1.8) converges algebraically:*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \leq C(t+1)^{-\frac{1}{n+1}} \tag{1.11}$$

for all $t \geq 0$ and some large constant C independent of t.

(iii) *When $\kappa < 0$, the solution (u, v) of (1.8) converges exponentially:*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq Ce^{\frac{\kappa}{n+1}t}, \quad \|v(\cdot,t)\|_{L^\infty(\Omega)} \leq Ce^{-\frac{1}{2(n+1)}\min\{\beta,-\kappa\}t} \tag{1.12}$$

for all $t \geq 0$ and some large constant C independent of t.

Remark 1.4 *(Explicit effects on convergence and convergence rate).*

(P7) The formula $\mu_1(d_1, d_2, \alpha, \beta, \kappa, \chi)$ exhibits the explicit contributions of each gradient in (1.8) on convergence and it enjoys the property in (P3). Besides, our convergence complements and refines [9, Theorem 3 for $d_1 = d_2 = \alpha = \beta = 1$] by explicitly computing out the rates of convergence.

(P8) From Remark 1.2, we find that κ and β do not play any role in boundedness; while, they play big roles in the long time behavior as seen in (1.9), especially, the β-effect has not been detected yet in the existence literature. In particular, $\mu_1(d_1, d_2, \alpha, \beta, \kappa, \chi)$ is decreasing in d_1, d_2 and β, and $\mu_1(d_1, d_2, \alpha, \beta, \kappa, \chi) \to \infty$ as $d_1 \to 0$ or $d_2 \to 0$ or $\beta \to 0$. Therefore, small diffusion, especially, degenerate or small degradation, makes the stabilization harder. This together with (P5) may provide certain clues on how to produce blow-up solutions for Keller–Segel chemotaxis models with logistic source.

In chemotaxis-growth systems, the most challenging and interesting wide-open question is to detect the possibility of finite/infinite-time physical blow-ups ($n = 3$) (nonphysical radially symmetrical blow-ups has been demonstrated in [45] for $n \geq 5$). Remarks 1.2 and 1.4, especially, (P5) and (P8) suggest certain clues on how to produce unbounded solutions [3,7,24,43,44]. To attack such a challenging problem, one may try the following chemotaxis system and perhaps its simplified version:

$$\begin{cases} u_t = \nabla \cdot (\epsilon_1(u)\nabla u - u\nabla v) + \kappa u - \mu u^2, & x \in \Omega, t > 0, \\ \tau v_t = \epsilon_2 \Delta v - \epsilon_3 v + u, & x \in \Omega, t > 0 \end{cases} \tag{1.13}$$

with $\epsilon_i > 0 (i = 2, 3)$ being sufficiently small and either $\epsilon_1(u)$ being a sufficiently small positive constant or $\epsilon_1(u) \to 0$ as $u \to \infty$ (very slow diffusion at point of high densities). Indeed, for $\tau = 0, \epsilon_1(u) = 0$, $\epsilon_2 = \epsilon_3 = 1$ in (1.13), verifications can be found in [23,48,49]. We leave the challenging exploration of the possibility of blow-ups to (1.13) (and hence (Q2) and (Q3)) for our future studies.

2. Preliminaries and subtle inequalities for (1.1)

For convenience, we start with Young's inequality, which states, for any positive numbers p and q with $\frac{1}{p} + \frac{1}{q} = 1$, that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \qquad \forall a, b \geq 0.$$

This immediately implies the so-called Young's inequality with ϵ:

Lemma 2.1 *(Young' s inequality with ϵ). Let p and q be two given positive numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $\epsilon > 0$, it holds*

$$ab \leq \epsilon a^p + \frac{b^q}{(\epsilon p)^{\frac{q}{p}} q}, \qquad \forall a, b \geq 0.$$

The local solvability and extendibility of the parabolic–parabolic chemotaxis system (1.3) is well-established by using a suitable fixed point argument and standard parabolic regularity theory; see, for example, [16,42].

Lemma 2.2. *Let $d_1, d_2, \alpha, \beta > 0$, $a \geq 0$, $\chi \in \mathbb{R}$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. Suppose that the initial data (u_0, v_0) satisfies $u_0 \in C(\overline{\Omega})$ and $v_0 \in W^{1,p}(\Omega)$ with some $p > n$ and that $f \in W^{1,\infty}_{loc}(\mathbb{R})$ with $f(0) \geq 0$. Then there is a unique, nonnegative, classical maximal solution (u, v) of the IBVP (1.3) on some maximal interval $[0, T_{max})$ with $0 < T_{max} \leq \infty$ such that*

$$\begin{aligned} &u \in C(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})), \\ &v \in C(\overline{\Omega} \times [0, T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max})) \cap L^\infty_{loc}([0, T_{max}); W^{1,p}(\Omega)). \end{aligned}$$

In particular, if $T_{max} < \infty$, then

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,p}(\Omega)} \to \infty \qquad \text{as } t \to T^-_{max}. \tag{2.1}$$

For chemotaxis model without growth source, the total masses of cells are conserved. While, for chemotaxis system with logistic growth, this is not true but the L^1-norm is bounded by integrating the u-equation in (1.3). The following basic lemma has been well-known, cf. [42,50], for instance.

Lemma 2.3. *Let f satisfy the logistic condition (1.4). Then the solution (u, v) of the KS system (1.3) satisfies*

$$\int_\Omega u \leq \|u_0\|_{L^1} + (a + \frac{1}{4\mu})|\Omega|$$

and

$$\int_\Omega |\nabla v|^2 \leq \|\nabla v_0\|^2_{L^2} + \frac{2\alpha^2}{bd_2}\Big[\frac{a}{2\beta}|\Omega| + \|u_0\|_{L^1} + (a + \frac{1}{\mu})|\Omega|\Big]$$

for all $t \in [0, T_{max})$.

Next, we establish three subtle commonly used lemmas, which are refined results of the corresponding [42, Lemmas 2.2–2.4] even for Ω being convex. In such case, the ideas used to derive these inequalities are known [34,42].

Lemma 2.4. *Let f satisfy the logistic condition (1.4). Then, for any $p \geq 1$, the solution (u, v) of the KS system (1.3) satisfies*

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + (p-1)(d_1 - \epsilon)\int_\Omega u^{p-2}|\bigtriangledown u|^2 + \mu\int_\Omega u^{p+1} \leq \frac{(p-1)\chi^2}{4\epsilon}\int_\Omega u^p|\bigtriangledown v|^2 + a\int_\Omega u^{p-1} \tag{2.2}$$

for all $t \in (0, T_{max})$ and for any $\epsilon \in (0, d_1)$.

Proof. For any $p \geq 1$, multiplying the u-equation in (1.3) by u^{p-1} and integrating over Ω by parts, using Young's inequality with ϵ and the logistic condition (1.4), we conclude that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + (p-1)d_1\int_\Omega u^{p-2}|\bigtriangledown u|^2 \\
&= (p-1)\chi\int_\Omega u^{p-1}\bigtriangledown u \bigtriangledown v + \int_\Omega f(u)u^{p-1} \\
&\leq (p-1)\epsilon\int_\Omega u^{p-2}|\bigtriangledown u|^2 + \frac{(p-1)\chi^2}{4\epsilon}\int_\Omega u^p|\bigtriangledown v|^2 + \int_\Omega u^{p-1}(a - \mu u^2),
\end{aligned}$$

which gives the desired inequality (2.2). □

Lemma 2.5. *For $q \geq 1$, the solution (u, v) of the KS system (1.3) satisfies*

$$\begin{aligned}
&\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} + (q-1)(d_2 - \eta)\int_\Omega |\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2 + 2\beta\int_\Omega |\nabla v|^{2q} \\
&\leq \Big[\frac{(q-1)\alpha^2}{\eta} + \frac{n\alpha^2}{2d_2}\Big]\int_\Omega u^2|\nabla v|^{2(q-1)} + d_2\int_{\partial\Omega} |\nabla v|^{2(q-1)}\frac{\partial}{\partial\nu}|\nabla v|^2
\end{aligned} \tag{2.3}$$

for all $t \in (0, T_{max})$ and for any $\eta \in (0, d_2)$.

Proof. For any $q \geq 1$, by using the v-equation in (1.3) and integrating by parts, we deduce that

$$\begin{aligned}
\frac{1}{q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} &= 2\int_\Omega |\nabla v|^{2(q-1)}\nabla v\cdot(d_2\nabla\Delta v - \beta\nabla v + \alpha\nabla u)\\
&= d_2\int_\Omega |\nabla v|^{2(q-1)}\Delta|\nabla v|^2 - 2d_2\int_\Omega |\nabla v|^{2(q-1)}|D^2v|^2 - 2\beta\int_\Omega |\nabla v|^{2q}\\
&\quad + 2\alpha\int_\Omega \nabla u\nabla v|\nabla v|^{2(q-1)}\\
&= -(q-1)d_2\int_\Omega |\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2 + d_2\int_{\partial\Omega} |\nabla v|^{2(q-1)}\frac{\partial}{\partial\nu}|\nabla v|^2\\
&\quad - 2d_2\int_\Omega |\nabla v|^{2(q-1)}|D^2v|^2 - 2\beta\int_\Omega |\nabla v|^{2q} + 2\alpha\int_\Omega \nabla u\nabla v|\nabla v|^{2(q-1)},
\end{aligned}$$

where, from the first to the second line, we have used the point-wise identity

$$2\bigtriangledown v\cdot\bigtriangledown\bigtriangleup v = \bigtriangleup|\bigtriangledown v|^2 - 2|D^2v|^2,\quad |D^2v|^2 = \sum_{i,j=1}^n |v_{x_ix_j}|^2. \tag{2.4}$$

We use integration by parts, Young's inequality with ϵ and the fact that

$$|\bigtriangleup v|^2 = \Big(\sum_{i=1}^n v_{x_ix_i}\Big)^2 \leq n\sum_{i=1}^n |v_{x_ix_i}|^2 \leq n|D^2v|^2 \tag{2.5}$$

to estimate the last integral as follows:

$$\begin{aligned}
2\alpha\int_\Omega \nabla u\nabla v|\nabla v|^{2(q-1)} &= -2(q-1)\alpha\int_\Omega u|\nabla v|^{2(q-2)}\nabla v\cdot\nabla|\nabla v|^2 - 2\alpha\int_\Omega u|\nabla v|^{2(q-1)}\Delta v\\
&\leq (q-1)\eta\int_\Omega |\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2 + \frac{(q-1)\alpha^2}{\eta}\int_\Omega u^2|\nabla v|^{2(q-1)}\\
&\quad + 2d_2\int_\Omega |\nabla v|^{2(q-1)}|D^2v|^2 + \frac{n\alpha^2}{2d_2}\int_\Omega u^2|\nabla v|^{2(q-1)}.
\end{aligned}$$

Combining these two results, we obtain the desired inequality (2.3). □

Lemma 2.6. *Let f satisfy the logistic condition (1.4). Then, for any $p \geq 1, q \geq 1$, the solution (u, v) of the KS system (1.3) verifies*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u^p|\nabla v|^{2q} + (p-1)(d_1-\epsilon)p\int_\Omega u^{p-2}|\nabla u|^2|\nabla v|^{2q} + 2\beta q\int_\Omega u^p|\nabla v|^{2q}\\
&+ (q-1)(d_2-\eta)q\int_\Omega u^p|\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2 + \mu p\int_\Omega u^{p+1}|\nabla v|^{2q}\\
&\leq \frac{\chi^2(p-1)p}{4\epsilon}\int_\Omega u^p|\nabla v|^{2(q+1)} + \chi pq\int_\Omega u^p|\nabla v|^{2(q-1)}\nabla v\nabla|\nabla v|^2
\end{aligned}$$

$$
\begin{aligned}
&- (d_1+d_2)pq\int_\Omega u^{p-1}|\nabla v|^{2(q-1)}\nabla u\nabla|\nabla v|^2\\
&+\frac{q\alpha^2}{(p+1)^2}\Big[\frac{(q-1)}{\eta}+\frac{n}{2d_2}\Big]\int_\Omega u^{p+2}|\nabla v|^{2(q-1)}\\
&+ap\int_\Omega u^{p-1}|\nabla v|^{2q}+d_2q\int_{\partial\Omega}u^p|\nabla v|^{2(q-1)}\frac{\partial|\nabla v|^2}{\partial\nu}
\end{aligned}\tag{2.6}
$$

for all $t\in(0,T_{max})$ and for any $\epsilon\in(0,d_1)$ and $\eta\in(0,d_2)$.

Proof. We use the u and v-equations in (1.3), (2.4), no flux boundary conditions for u and v and integration by parts to compute honestly that

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega u^p|\nabla v|^{2q} &= p\int_\Omega u^{p-1}|\nabla v|^{2q}[\nabla\cdot(d_1\nabla u-\chi u\nabla v)+f(u)]\\
&\quad+2q\int_\Omega u^p|\nabla v|^{2(q-1)}\nabla v\cdot(d_2\nabla\Delta v-\beta\nabla v+\alpha\nabla u)\\
&=-p\int_\Omega(d_1\nabla u-\chi u\nabla v)\Big[(p-1)u^{p-2}\nabla u|\nabla v|^{2q}+qu^{p-1}|\nabla v|^{2(q-1)}\nabla|\nabla v|^2\Big]\\
&\quad+p\int_\Omega u^{p-1}|\nabla v|^{2q}f(u)+d_2q\int_\Omega u^p|\nabla v|^{2(q-1)}\Delta|\nabla v|^2-2\beta q\int_\Omega u^p|\nabla v|^{2q}\\
&\quad-2d_2q\int_\Omega u^p|\nabla v|^{2(q-1)}|D^2v|^2+2\alpha q\int_\Omega u^p|\nabla v|^{2(q-1)}\nabla u\nabla v\\
&=-(d_1+d_2)pq\int_\Omega u^{p-1}|\nabla v|^{2(q-1)}\nabla u\nabla|\nabla v|^2-d_1(p-1)p\int_\Omega u^{p-2}|\nabla u|^2|\nabla v|^{2q}\\
&\quad+p\int_\Omega u^{p-1}|\nabla v|^{2q}f(u)+\chi(p-1)p\int_\Omega u^{p-1}|\nabla v|^{2q}\nabla u\nabla v\\
&\quad+\chi pq\int_\Omega u^p|\nabla v|^{2(q-1)}\nabla v\nabla|\nabla v|^2-d_2q(q-1)\int_\Omega u^p|\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2\\
&\quad-2d_2q\int_\Omega u^p|\nabla v|^{2(q-1)}|D^2v|^2-2\beta q\int_\Omega u^p|\nabla v|^{2q}\\
&\quad+2\alpha q\int_\Omega u^p|\nabla v|^{2(q-1)}\nabla u\nabla v+d_2q\int_{\partial\Omega}u^p|\nabla v|^{2(q-1)}\frac{\partial|\nabla v|^2}{\partial\nu}.
\end{aligned}\tag{2.7}
$$

The logistic condition $f(u)\le a-\mu u^2$ gives rise to

$$
p\int_\Omega u^{p-1}|\nabla v|^{2q}f(u)\le ap\int_\Omega u^{p-1}|\nabla v|^{2q}-\mu p\int_\Omega u^{p+1}|\nabla v|^{2q}.\tag{2.8}
$$

A simple use of Young's inequality with ϵ shows

$$
\chi(p-1)p\int_\Omega u^{p-1}|\nabla v|^{2q}\nabla u\nabla v\le\epsilon(p-1)p\int_\Omega u^{p-2}|\nabla u|^2|\nabla v|^{2q}+\frac{\chi^2(p-1)p}{4\epsilon}\int_\Omega u^p|\nabla v|^{2(q+1)}.\tag{2.9}
$$

Upon integration by parts, applications of Young's inequality with ϵ and (2.5), we find that

$$\begin{aligned}
&2\alpha q \int_\Omega u^p|\nabla v|^{2(q-1)}\nabla u \nabla v \\
&= -\frac{2(q-1)q\alpha}{p+1}\int_\Omega u^{p+1}|\nabla v|^{2(q-2)}\nabla v\cdot\nabla|\nabla v|^2 - \frac{2q\alpha}{p+1}\int_\Omega u^{p+1}|\nabla v|^{2(q-1)}\Delta v \\
&\le q(q-1)\eta\int_\Omega u^p|\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2 + \frac{(q-1)q\alpha^2}{(p+1)^2\eta}\int_\Omega u^{p+2}|\nabla v|^{2(q-1)} \\
&\quad + 2d_2 q\int_\Omega u^p|\nabla v|^{2(q-1)}|D^2 v|^2 + \frac{qn\alpha^2}{2d_2(p+1)^2}\int_\Omega u^{p+2}|\nabla v|^{2(q-1)}.
\end{aligned} \tag{2.10}$$

Substituting (2.8), (2.9) and (2.10) into (2.7), after suitable rearrangements, we obtain the key inequality (2.6). □

3. Logistic damping prevents blow-up in 3-D setting

To get a clear and better understanding about how strong a logistic damping can prevent blowup phenomenon in chemotaxis systems with logistic sources, we first explore the issue in the physically relevant case of $n = 3$. In this section, we will provide details to the algorithm leading to the main result (i) of Theorem 1.1.

For the full-parameter chemotaxis-growth system (1.3) in 3-D, we observe that, to show the L^∞-boundedness of u, it is enough to show the $L^{\frac{3}{2}+\epsilon}$-boundedness of u for some $\epsilon > 0$, thanks to the boundedness criterion obtained in [50] via Moser iteration and in [2] via semigroup theory. This enables us to find out how large should μ be so that blow-up is impossible. To achieve our goal, we need to carefully collect the appearing constants in each derived inequalities. We shall indeed prove that $\|u(t)\|_{L^2}$ is bounded. For this purpose, inspired by [42], our analysis consists of deriving a delicate Gronwall inequality for the coupled functional

$$z(t) := \delta_1\int_\Omega u^2(\cdot,t) + \delta_2\int_\Omega u(\cdot,t)|\nabla v(\cdot,t)|^2 + \delta_3\int_\Omega |\nabla v(\cdot,t)|^4 \tag{3.1}$$

of the form

$$z'(t) + \epsilon z(t) \le C_\epsilon, \quad t\in(0,T_{\max})$$

for some carefully chosen positive constants $\delta_1,\delta_2,\delta_3$, $\epsilon > 0$ and perhaps large C_ϵ independent of t. Once this is done, the L^2-and hence the L^∞-boundedness of u will be obtained.

Upon easy applications of Lemma 2.4 with $p = 2$ and Lemma 2.5 with $q = 2$, we end up with the following two lemmas.

Lemma 3.1. *Let f satisfy the logistic condition (1.4). Then the solution (u, v) of the KS system (1.3) satisfies*

$$\frac{d}{dt}\int_\Omega u^2 + 2(d_1-\epsilon_1)\int_\Omega |\nabla u|^2 + 2\mu\int_\Omega u^3 \le \frac{\chi^2}{2\epsilon_1}\int_\Omega u^2|\nabla v|^2 + 2a\int_\Omega u \tag{3.2}$$

for all $t \in (0, T_{max})$ and for any $\epsilon_1 \in (0, d_1)$.

Lemma 3.2. *The solution (u,v) of the KS system (1.3) satisfies*

$$\begin{aligned} &\tfrac{d}{dt}\int_\Omega |\nabla v|^4 + 2(d_2-\epsilon_2)\int_\Omega |\nabla|\nabla v|^2|^2 + 4\beta\int_\Omega |\nabla v|^4 \\ &\le 2(\tfrac{n}{2d_2}+\tfrac{1}{\epsilon_2})\alpha^2\int_\Omega u^2|\nabla v|^2 + 2d_2\int_{\partial\Omega}|\nabla v|^2\tfrac{\partial}{\partial\nu}|\nabla v|^2 \end{aligned} \tag{3.3}$$

for all $t\in(0,T_{max})$ and for any $\epsilon_2\in(0,d_2)$.

Lemma 3.3. *Under the logistic condition (1.4), the solution (u,v) of the KS system (1.3) satisfies*

$$\begin{aligned} &\tfrac{d}{dt}\int_\Omega u|\nabla v|^2 + (\mu-\tfrac{\chi^2}{4\epsilon_3})\int_\Omega u^2|\nabla v|^2 + 2\beta\int_\Omega u|\nabla v|^2 \\ &\le \tfrac{(d_1+d_2)^2}{4\epsilon_4}\int_\Omega |\nabla u|^2 + (\epsilon_3+\epsilon_4)\int_\Omega |\nabla|\nabla v|^2|^2 + \tfrac{n\alpha^2}{8d_2}\int_\Omega u^3 + a\int_\Omega |\nabla v|^2 + d_2\int_{\partial\Omega} u\tfrac{\partial}{\partial\nu}|\nabla v|^2 \end{aligned} \tag{3.4}$$

for all $t\in(0,T_{max})$ and for any $\epsilon_3,\epsilon_4>0$.

Proof. Lemma 2.6 with $p=1=q$ reads as

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u|\nabla v|^2 + 2\beta\int_\Omega u|\nabla v|^2 + \mu\int_\Omega u^2|\nabla v|^2 \\ &\quad\le \chi\int_\Omega u\nabla v\nabla|\nabla v|^2 - (d_1+d_2)\int_\Omega \nabla u\nabla|\nabla v|^2 + \frac{n\alpha^2}{8d_2}\int_\Omega u^3 + a\int_\Omega |\nabla v|^2 + d_2\int_{\partial\Omega} u\frac{\partial|\nabla v|^2}{\partial\nu} \end{aligned} \tag{3.5}$$

for all $t\in(0,T_{\max})$. Now, applying repeatedly Young's inequality with ϵ and taking into account (3.2) and (3.3), we infer that, for any $\epsilon_3,\epsilon_4>0$,

$$\chi\int_\Omega u\nabla v\nabla|\nabla v|^2 \le \frac{\chi^2}{4\epsilon_3}\int_\Omega u^2|\nabla v|^2 + \epsilon_3\int_\Omega |\nabla|\nabla v|^2|^2 \tag{3.6}$$

as well as

$$-(d_1+d_2)\int_\Omega \nabla u\nabla|\nabla v|^2 \le \frac{(d_1+d_2)^2}{4\epsilon_4}\int_\Omega |\nabla u|^2 + \epsilon_4\int_\Omega |\nabla|\nabla v|^2|^2. \tag{3.7}$$

Then we substitute (3.6) and (3.7) into (3.5) to conclude (3.4). □

With these preparations, we are now ready to study the coupled functional (3.1) by means of Lemmas 3.1, 3.2 and 3.3.

Lemma 3.4. *Under the logistic condition (1.4), the solution (u,v) of the KS system (1.3) satisfies the inequality*

$$\begin{aligned} &\frac{d}{dt}\Big\{\delta_1\int_\Omega u^2 + \delta_2\int_\Omega u|\nabla v|^2 + \delta_3\int_\Omega |\nabla v|^4\Big\} + (2\mu\delta_1 - \frac{n\alpha^2}{8d_2}\delta_2)\int_\Omega u^3 \\ &+ [2(d_1-\epsilon_1)\delta_1 - \frac{(d_1+d_2)^2}{4\epsilon_4}\delta_2]\int_\Omega |\nabla u|^2 + 4\beta\delta_3\int_\Omega |\nabla v|^4 \\ &\quad+ [2(d_2-\epsilon_2)\delta_3 - (\epsilon_3+\epsilon_4)\delta_2]\int_\Omega |\nabla|\nabla v|^2|^2 + 2\beta\delta_2\int_\Omega u|\nabla v|^2 \end{aligned} \tag{3.8}$$

$$+ [(\mu - \frac{\chi^2}{4\epsilon_3})\delta_2 - 2(\frac{n}{2d_2} + \frac{1}{\epsilon_2})\alpha^2\delta_3 - \frac{\chi^2}{2\epsilon_1}\delta_1] \int_\Omega u^2|\nabla v|^2$$

$$\leq 2a\delta_1 \int_\Omega u + a\delta_2 \int_\Omega |\nabla v|^2 + 2d_2\delta_3 \int_{\partial\Omega} |\bigtriangledown v|^2 \frac{\partial}{\partial\nu}|\bigtriangledown v|^2 + d_2\delta_2 \int_{\partial\Omega} u\frac{\partial}{\partial\nu}|\bigtriangledown v|^2$$

for any $t \in (0, T_{max})$ *and any positive constants* δ_1, δ_2 *and* δ_3 *and* $\epsilon_1 \in (0, d_1)$, $\epsilon_2 \in (0, d_2)$ *and* $\epsilon_3, \epsilon_4 > 0$.

Proof. By evident multiplications and additions from Lemmas 3.1–3.3, one can readily derive the inequality (3.8). □

Motivated by (3.8), to find the possibly smallest lower bound μ_0 for the damping rate μ that could be obtained using such method, we wish to choose ϵ_i and δ_i to satisfy

$$\begin{cases} 2\mu\delta_1 - \frac{n\alpha^2}{8d_2}\delta_2 > 0, \\ 2(d_1 - \epsilon_1)\delta_1 - \frac{(d_1+d_2)^2}{4\epsilon_4}\delta_2 > 0, \\ 2(d_2 - \epsilon_2)\delta_3 - (\epsilon_3 + \epsilon_4)\delta_2 > 0, \\ (\mu - \frac{\chi^2}{4\epsilon_3})\delta_2 - 2(\frac{n}{2d_2} + \frac{1}{\epsilon_2})\alpha^2\delta_3 - \frac{\chi^2}{2\epsilon_1}\delta_1 \geq 0. \end{cases} \tag{3.9}$$

The minimizer of the minimization problem (3.9) in μ will give us the smallest damping rate μ_0 we are seeking. Our goal is then to choose ϵ_i and δ_i so that μ is minimized. By eliminations from (3.9), we end up with

$$\begin{cases} \mu > \frac{n\alpha^2}{16d_2}\frac{\delta_2}{\delta_1}, \quad \frac{\delta_2}{\delta_1} < \frac{8(d_1-\epsilon_1)\epsilon_4}{(d_1+d_2)^2}, \\ \mu > \frac{\chi^2}{4\epsilon_3} + 2(\frac{n}{2d_2} + \frac{1}{\epsilon_2})\alpha^2 \cdot \frac{\epsilon_3+\epsilon_4}{2(d_2-\epsilon_2)} + \frac{\chi^2}{2\epsilon_1} \cdot \frac{\frac{(d_1+d_2)^2}{4\epsilon_4}}{2(d_1-\epsilon_1)} \end{cases} \tag{3.10}$$

for any $\epsilon_1 \in (0, d_1), \epsilon_2 \in (0, d_2)$ and $\epsilon_3, \epsilon_4 > 0$. Next, we shall first minimize the expression in second line on the right-hand side of (3.10). Notice that

$$\frac{\chi^2}{2\epsilon_1} \cdot \frac{\frac{(d_1+d_2)^2}{4\epsilon_4}}{2(d_1-\epsilon_1)} \geq \frac{\chi^2}{d_1^2}\frac{(d_1+d_2)^2}{4\epsilon_4} \tag{3.11}$$

with equality if and only $\epsilon_1 = d_1/2$, and

$$2(\frac{n}{2d_2} + \frac{1}{\epsilon_2})\alpha^2 \cdot \frac{\epsilon_3+\epsilon_4}{2(d_2-\epsilon_2)} \geq \Big[\frac{n\alpha}{(\sqrt{2n+4}-2)d_2}\Big]^2(\epsilon_3+\epsilon_4) \tag{3.12}$$

with equality if and only if

$$\epsilon_2 = \frac{(\sqrt{2n+4}-2)}{n}d_2.$$

Now, by algebraic calculations from (3.10), (3.11) and (3.12), we deduce that the second expression on the right-hand side of (3.10) achieves its minimum

$$\mu_0 = \mu_0(n, d_1, d_2, \alpha, \chi) := \frac{n}{\sqrt{2n+4}-2}(\frac{1}{d_1} + \frac{2}{d_2})\alpha|\chi| \tag{3.13}$$

if and only if

$$\begin{cases} \epsilon_1 = \frac{d_1}{2}, & \epsilon_2 = \frac{(\sqrt{2n+4}-2)}{n} d_2, \\ \epsilon_3 = \frac{(\sqrt{2n+4}-2)d_2}{2n\alpha}|\chi|, & \epsilon_4 = \frac{(\sqrt{2n+4}-2)(d_1+d_2)d_2}{2nd_1\alpha}|\chi|. \end{cases} \tag{3.14}$$

For such well-chosen ϵ_i according to (3.14), for any $\mu > \mu_0$ as given in (3.13), we can choose $\delta_i (i = 1, 2, 3)$ in a way

$$\delta_1 > \frac{(d_1 + d_2)^2}{8\epsilon_4(d_1 - \epsilon_1)}, \qquad \delta_2 = 1, \qquad \delta_3 > \frac{\epsilon_3 + \epsilon_4}{2(d_2 - \epsilon_2)} \tag{3.15}$$

so that the second to the fourth inequality in (3.9) are satisfied. Next, for the first constraint in (3.9), we observe

$$\frac{(d_1 + d_2)^2}{8\epsilon_4(d_1 - \epsilon_1)} \ge \frac{n\alpha^2}{16d_2}\frac{1}{\mu} \Longleftrightarrow \mu \ge \frac{n\alpha^2}{16d_2}\frac{8(d_1 - \epsilon_1)\epsilon_4}{(d_1 + d_2)^2} = \frac{(\sqrt{2n+4} - 2)}{8(d_1 + d_2)}\alpha|\chi|$$

and

$$\frac{(\sqrt{2n+4} - 2)}{8(d_1 + d_2)}\alpha|\chi| < \frac{n}{\sqrt{2n+4} - 2}(\frac{1}{d_1} + \frac{2}{d_2})\alpha|\chi| = \mu_0.$$

Hence, the first and thus all inequalities in (3.9) are satisfied.

Finally, for any $\mu > \mu_0$ as given in (3.13), we can first fix $\epsilon_i (i = 1, 2, 3, 4)$ complying with (3.14) and then, based on (3.15), we choose $\delta_i (i = 1, 2, 3)$ to further satisfy

$$\begin{cases} \frac{(d_1+d_2)^2}{8\epsilon_4(d_1-\epsilon_1)} < \delta_1 < \frac{2\epsilon_1}{\chi^2}\left[\mu - \frac{\chi^2}{4\epsilon_3} - \frac{(\epsilon_3+\epsilon_4)}{(d_2-\epsilon_2)}(\frac{n}{2d_2} + \frac{1}{\epsilon_2})\alpha^2\right], \\ \delta_2 = 1, \\ \frac{\epsilon_3+\epsilon_4}{2(d_2-\epsilon_2)} < \delta_3 < \frac{(\mu - \frac{\chi^2}{4\epsilon_3} - \frac{\chi^2}{2\epsilon_1}\delta_1)}{2(\frac{n}{2d_2} + \frac{1}{\epsilon_2})\alpha^2} \end{cases} \tag{3.16}$$

so that all inequalities in (3.9) are satisfied.

For any $\mu > \mu_0$ as defined in (3.13), we shall illustrate that all integrals on the right-hand side of (3.8) can be controlled by the dissipative terms on its left and, as a result, yielding the assertion that $\|u(t)\|_{L^2}$ is uniformly bounded. Therefore, we conclude the statement (i) of Theorem 1.1 in physically relevant setting $n = 3$ by the $L^{\frac{n}{2}+\epsilon}$-criterion in [2,50].

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, f satisfy the logistic condition (1.4) and $d_1, d_2, \alpha, \beta > 0$, $a \ge 0$ and $\chi \in \mathbb{R}$. Then, for any $\mu > \mu_0(n, d_1, d_2, \alpha, \chi)$ as given in (3.13), there exists a constant $C(u_0, v_0)$ such that the unique nonnegative solution (u, v) of the KS system (1.3) satisfies*

$$\int_\Omega u^2(\cdot, t) + \int_\Omega u(\cdot, t)|\nabla v(\cdot, t)|^2 + \int_\Omega |\nabla v(\cdot, t)|^4 \le C(u_0, v_0), \qquad t \in (0, T_{max}).$$

If $n \le 3$, then the solution (u, v) exists globally in time, i.e., $T_{max} = \infty$ and $(u(\cdot, t), v(\cdot, t))$ is uniformly bounded in $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ for all $t \in (0, \infty)$.

Proof. It follows from Lemma 3.4 that the quantity

$$z(t) := \delta_1 \int_\Omega u^2 + \delta_2 \int_\Omega u|\nabla v|^2 + \delta_3 \int_\Omega |\nabla v|^4, \qquad t \in (0, T_{\max}),$$

fulfills

$$
\begin{aligned}
&z'(t) + \beta z(t) + [2(d_1 - \epsilon_1)\delta_1 - \tfrac{(d_1+d_2)^2}{4\epsilon_4}\delta_2] \textstyle\int_\Omega |\nabla u|^2 \\
&\quad + [2(d_2 - \epsilon_2)\delta_3 - (\epsilon_3 + \epsilon_4)\delta_2] \textstyle\int_\Omega |\nabla|\nabla v|^2|^2 + 3\beta\delta_3 \int_\Omega |\nabla v|^4 \\
&\quad + \beta\delta_2 \textstyle\int_\Omega u|\nabla v|^2 + (2\mu\delta_1 - \tfrac{n\alpha^2}{8d_2}\delta_2) \int_\Omega u^3 \\
&\quad \le 2a\delta_1 \textstyle\int_\Omega u + a\delta_2 \int_\Omega |\nabla v|^2 + \beta\delta_1 \int_\Omega u^2 \\
&\qquad + 2d_2\delta_3 \textstyle\int_{\partial\Omega} |\nabla v|^2 \tfrac{\partial}{\partial\nu} |\nabla v|^2 + d_2\delta_2 \int_{\partial\Omega} u \tfrac{\partial}{\partial\nu} |\nabla v|^2.
\end{aligned}
\tag{3.17}
$$

In the sequel, we bound the integrals on the right-hand side of (3.17) in terms of the dissipative terms on its left-hand side.

Let us first focus on controlling the boundary integrals in (3.17). So far, there are a couple of existing ways to handle these boundary integrals, see [17,35,53], for instance. Here we would like to provide an alternative and transparent way to remove the technical assumption that the domain Ω be convex. The starting point is based on the pointwise geometric inequality

$$\frac{\partial|\nabla w|^2}{\partial\nu} \le K_1(\Omega)|\nabla w|^2 \quad \text{on } \partial\Omega, \tag{3.18}$$

which holds for any bounded smooth domain $\Omega \subset \mathbb{R}^n$ and any w satisfying $\frac{\partial w}{\partial\nu} = 0$ on $\partial\Omega$, cf. [28, Lemma 4.2]. Here and below, K_i will denote some inessential constants. Notice also a user-friendly version of trace inequality with ϵ (cf. [32, Remark 52.9]): for any $\epsilon > 0$, one has

$$\|w\|_{L^2(\partial\Omega)} \le \epsilon\|\nabla w\|_{L^2(\Omega)} + C_\epsilon\|w\|_{L^2(\Omega)}, \quad \forall w \in H^1(\Omega). \tag{3.19}$$

Indeed, this is immediately implied, upon a use of Young's inequality with ϵ, by the following version of trace inequality:

$$\|w\|_{L^2(\partial\Omega)} \le K_2\|w\|_{L^2(\Omega)}^{\frac{1}{2}}\|w\|_{H^1(\Omega)}^{\frac{1}{2}}, \quad \forall w \in H^1(\Omega). \tag{3.20}$$

As a matter of fact, by the property of trace inequality (the trace operator T maps $H^{\frac{1}{2}}(\Omega)$ continuously onto $L^2(\partial\Omega)$), one has

$$\|w\|_{L^2(\partial\Omega)} \le K_3\|w\|_{H^{\frac{1}{2}}(\Omega)}, \quad \forall w \in H^{\frac{1}{2}}(\Omega).$$

On the other hand, it follows from the fact that $H^{\frac{1}{2}}$ interpolates the spaces $H^0 = L^2$ and H^1 that

$$\|w\|_{H^{\frac{1}{2}}(\Omega)} \le K_4\|w\|_{L^2(\Omega)}^{\frac{1}{2}}\|w\|_{H^1(\Omega)}^{\frac{1}{2}}, \quad \forall w \in H^1(\Omega).$$

A collection of these two estimates directly leads to (3.20).

Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ by Kondrachov and that $L^2(\Omega)$ is continuously embedded in $L^1(\Omega)$, Lion's lemma says, for any $\eta > 0$,

$$\|w\|_{L^2(\Omega)} \le \eta\|\nabla w\|_{L^2(\Omega)} + \eta\|w\|_{L^2(\Omega)} + C_\eta\|w\|_{L^1(\Omega)}, \quad \forall w \in H^1(\Omega).$$

Combing this with (3.19), we conclude, for any $\epsilon > 0$,

$$\|w\|_{L^2(\partial\Omega)} \le \epsilon\|\nabla w\|_{L^2(\Omega)} + C_\epsilon\|w\|_{L^1(\Omega)}, \quad \forall w \in H^1(\Omega). \tag{3.21}$$

This gives another elementary proof for the equivalent trace inequality stated in [35, P. 13, line -4].

Now, with (3.18) and (3.21) at hand, we can cope with the boundary integrals in (3.17) as follows: for any $\epsilon > 0$,

$$\begin{aligned}
&2d_2\delta_2 \int_{\partial\Omega} |\bigtriangledown v|^2 \tfrac{\partial}{\partial\nu}|\bigtriangledown v|^2 + d_2\delta_3 \int_{\partial\Omega} u\tfrac{\partial}{\partial\nu}|\bigtriangledown v|^2 \\
&\leq 2d_2\delta_2 K_1 \int_{\partial\Omega} |\bigtriangledown v|^4 + d_2\delta_3 K_1 \int_{\partial\Omega} u|\bigtriangledown v|^2 \\
&\leq K_5 \int_{\partial\Omega} |\bigtriangledown v|^4 + K_6 \int_{\partial\Omega} u^2 = K_5\||\nabla v|^2\|^2_{L^2(\partial\Omega)} + K_6\|u\|^2_{L^2(\partial\Omega)} \\
&\leq \epsilon \int_\Omega |\nabla|\nabla v|^2|^2 + C_\epsilon\Big(\int_\Omega |\bigtriangledown v|^2\Big)^2 + \epsilon\int_\Omega |\nabla u|^2 + C_\epsilon\Big(\int_\Omega u\Big)^2.
\end{aligned} \tag{3.22}$$

For any $\mu > \mu_0(n, d_1, d_2, \alpha, \chi)$ as given in (3.13), we first fix $\epsilon_i, i = 1, 2, 3, 4$ according to (3.14), and then fix δ_1, δ_2 and δ_3 complying with (3.16). In this way, the inequality (3.9) is satisfied. Finally, we fix ϵ according to

$$\epsilon = \frac{1}{2}\min\Big\{2(d_1 - \epsilon_1)\delta_1 - \frac{(d_1 + d_2)^2}{4\epsilon_4}\delta_3, \quad 2(d_2 - \epsilon_2)\delta_3 - (\epsilon_3 + \epsilon_4)\delta_2\Big\}.$$

Then the boundary integrals in (3.22) will be absorbed by the terms on the left-hand side of (3.17) and Lemma 2.3.

Next, notice that

$$\beta\delta_1 \int_\Omega u^2 \leq (2\mu\delta_1 - \frac{n\alpha^2}{8d_2}\delta_2) \int_\Omega u^3 + K_7|\Omega|, \tag{3.23}$$

where

$$K_7 = \max\{\beta\delta_1 u^2 - (2\mu\delta_1 - \frac{n\alpha^2}{8d_2}\delta_2)u^3 | u \geq 0\} < \infty.$$

Finally, we substitute (3.22) and (3.23) into (3.17) and use the boundedness of $\|u\|_{L^1}$ and $\|\nabla v\|_{L^2}$ as in Lemma 2.3 to conclude

$$z'(t) + \beta z(t) \leq C(u_0, v_0), \qquad t \in [0, T_{\max}),$$

which together the definition of z simply leads to

$$\begin{aligned}
z(t) &= \delta_1 \int_\Omega u^2 + \delta_2 \int_\Omega u|\nabla v|^2 + \delta_3 \int_\Omega |\nabla v|^4 \\
&\leq \delta_1 \int_\Omega u_0^2 + \delta_2 \int_\Omega u_0|\nabla v_0|^2 + \delta_3 \int_\Omega |\nabla v_0|^4 + \tfrac{C(u_0,v_0)}{\beta}, \quad t \in [0, T_{\max}).
\end{aligned}$$

This shows the uniform L^2-boundedness of $u(\cdot, t)$ for any $t \in [0, T_{\max})$. Consequently, by Moser iteration, cf. the $L^{\frac{n}{2}+\epsilon}$-criterion in [50, Theorem 1.1] or [2, Lemma 3.2] with $n = 3$, we infer that $T_{\max} = \infty$ and that $(u(\cdot, t), v(\cdot, t))$ is uniformly bounded in $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ for all $t \in (0, \infty)$. □

4. How strong a logistic damping can prevent blowups in n-D

4.1. Blow-up prevention by logistic source in nonconvex domains

In this subsection, based on the detailed algorithm in 3-D, we wish to provide a clue on how to compute the explicit logistic damping rate μ_0 that suppresses blow-up whenever $\mu > \mu_0$ in n-D ($n \geq 4$). We will mainly do it for $n = 4, 5$. Our procedure suggests that an explicit logistic damping rate μ_0 suppressing blow-up whenever $\mu > \mu_0$, which enjoys the properties as described in Remark 1.2, is also available in more higher dimensions.

When $n \le 5$, it amounts to ensuring the uniform boundedness of $\|u\|_{L^3(\Omega)}$. In such setup, the core analysis then lies in deriving a subtle estimate for

$$z(t) := \delta_1 \int_\Omega u^3 + \delta_2 \int_\Omega u^2|\nabla v|^2 + \delta_3 \int_\Omega u|\nabla v|^4 + \delta_4 \int_\Omega |\nabla v|^6, \quad t \in (0, T_{\max}) \tag{4.1}$$

of the form

$$z'(t) + \zeta z(t) \le C_\zeta, \quad t \in (0, T_{\max}) \tag{4.2}$$

for some well chosen positive constants δ_i and $\zeta > 0$ and perhaps large C_ζ independent of t. Once this is done, the L^3-and hence the L^∞-boundedness of u will be obtained, by the boundedness principles [2,50].

Upon trivial applications of Lemma 2.4 with $p = 3$ and Lemma 2.5 with $q = 3$, we achieve the following two lemmas.

Lemma 4.1. *Let f satisfy the logistic condition (1.4). Then the solution (u, v) of the KS system (1.3) satisfies*

$$\frac{d}{dt}\int_\Omega u^3 + 6(d_1 - \epsilon)\int_\Omega u|\nabla u|^2 + 3\mu \int_\Omega u^4 \le \frac{3\chi^2}{2\epsilon}\int_\Omega u^3|\nabla v|^2 + 3a\int_\Omega u^2 \tag{4.3}$$

for all $t \in (0, T_{max})$ and for any $\epsilon \in (0, d_1)$.

Lemma 4.2. *The solution (u, v) of the KS system (1.3) satisfies*

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla v|^6 &+ 6(d_2 - \eta)\int_\Omega |\nabla v|^2|\nabla|\nabla v|^2|^2 + 6\beta\int_\Omega |\nabla v|^6 \\ &\le 3\Big(\frac{2\alpha^2}{\eta} + \frac{n\alpha^2}{2d_2}\Big)\int_\Omega u^2|\nabla v|^4 + 3d_2\int_{\partial\Omega} |\nabla v|^4 \frac{\partial}{\partial\nu}|\nabla v|^2\end{aligned} \tag{4.4}$$

for all $t \in (0, T_{max})$ and for any $\epsilon_2 \in (0, d_2)$.

Lemma 4.3. *Let f satisfy the logistic condition (1.4). Then the solution (u, v) of the KS system (1.3) verifies*

$$\begin{aligned}\frac{d}{dt}\int_\Omega u|\nabla v|^4 &+ 4\beta\int_\Omega u|\nabla v|^4 + \Big(\mu - \frac{\chi^2}{2\epsilon_1}\Big)\int_\Omega u^2|\nabla v|^4 + 2(d_2 - \eta)\int_\Omega u|\nabla|\nabla v|^2|^2 \\ &\le 2(\epsilon_1 + \epsilon_2)\int_\Omega |\nabla v|^2|\nabla|\nabla v|^2|^2 + \frac{(d_1 + d_2)^2}{2\epsilon_2}\int_\Omega |\nabla u|^2|\nabla v|^2 \\ &\quad + \frac{\alpha^2}{2}\Big(\frac{1}{\eta} + \frac{n}{2d_2}\Big)\int_\Omega u^3|\nabla v|^2 + a\int_\Omega |\nabla v|^4 + 2d_2\int_{\partial\Omega} u|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}\end{aligned} \tag{4.5}$$

for all $t \in (0, T_{max})$ and for any $\epsilon \in (0, d_1)$, $\eta \in (0, d_2)$ and $\epsilon_1, \epsilon_2 > 0$.

Proof. Lemma 2.6 with $p = 1$ and $q = 2$ reads as

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u|\nabla v|^4 + 4\beta\int_\Omega u|\nabla v|^4 + 2(d_2-\eta)\int_\Omega u|\nabla|\nabla v|^2|^2 + \mu\int_\Omega u^2|\nabla v|^4\\&\le 2\chi\int_\Omega u|\nabla v|^2\nabla v\nabla|\nabla v|^2 - 2(d_1+d_2)\int_\Omega |\nabla v|^2\nabla u\nabla|\nabla v|^2 + \frac{\alpha^2}{2}(\frac{1}{\eta}+\frac{n}{2d_2})\int_\Omega u^3|\nabla v|^2\\&\quad + a\int_\Omega |\nabla v|^4 + 2d_2\int_{\partial\Omega} u|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}.\end{aligned}\tag{4.6}$$

Taking into account (4.4) of Lemma 4.2, we estimate

$$2\chi\int_\Omega u|\nabla v|^2\nabla v\nabla|\nabla v|^2 \le 2\epsilon_1\int_\Omega |\nabla v|^2|\nabla|\nabla v|^2|^2 + \frac{\chi^2}{2\epsilon_1}\int_\Omega u^2|\nabla v|^4\tag{4.7}$$

and

$$-2(d_1+d_2)\int_\Omega |\nabla v|^2\nabla u\nabla|\nabla v|^2 \le 2\epsilon_2\int_\Omega |\nabla v|^2|\nabla|\nabla v|^2|^2 + \frac{(d_1+d_2)^2}{2\epsilon_2}\int_\Omega |\nabla u|^2|\nabla v|^2.\tag{4.8}$$

Then combining (4.6), (4.7) and (4.8), we get the desired inequality (4.5). □

Lemma 4.4. *Let f satisfy the logistic condition (1.4). Then the solution (u, v) of the KS system (1.3) verifies*

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^2|\nabla v|^2 + 2(d_1-\epsilon)\int_\Omega |\nabla u|^2|\nabla v|^2 + 2\beta\int_\Omega u^2|\nabla v|^2 + (2\mu - \frac{\chi^2}{2\epsilon_3})\int_\Omega u^3|\nabla v|^2\\&\le \frac{\chi^2}{2\epsilon}\int_\Omega u^2|\nabla v|^4 + 2\epsilon_4\int_\Omega u|\nabla u|^2 + \Big[2\epsilon_3 + \frac{(d_1+d_2)^2}{2\epsilon_4}\Big]\int_\Omega u|\nabla|\nabla v|^2|^2\\&\quad + \frac{n\alpha^2}{18d_2}\int_\Omega u^4 + 2a\int_\Omega u|\nabla v|^2 + d_2\int_{\partial\Omega} u^2\frac{\partial|\nabla v|^2}{\partial\nu}\end{aligned}\tag{4.9}$$

for all $t \in (0, T_{max})$ and for any $\epsilon \in (0, d_1)$, $\eta \in (0, d_2)$ and $\epsilon_3, \epsilon_4 > 0$.

Proof. It follows from (2.6) with $p = 2$ and $q = 1$ in Lemma 2.6 that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^2|\nabla v|^2 + 2(d_1-\epsilon)\int_\Omega |\nabla u|^2|\nabla v|^2 + 2\beta\int_\Omega u^2|\nabla v|^2 + 2\mu\int_\Omega u^3|\nabla v|^2\\&\le \frac{\chi^2}{2\epsilon}\int_\Omega u^2|\nabla v|^4 + 2\chi\int_\Omega u^2\nabla v\nabla|\nabla v|^2 - 2(d_1+d_2)\int_\Omega u\nabla u\nabla|\nabla v|^2 + \frac{\alpha^2}{9}\frac{n}{2d_2}\int_\Omega u^4\\&\quad + 2a\int_\Omega u|\nabla v|^2 + d_2\int_{\partial\Omega} u^2\frac{\partial|\nabla v|^2}{\partial\nu}.\end{aligned}\tag{4.10}$$

Taking into consideration (4.5) and (4.3), we bound

$$2\chi\int_\Omega u^2\nabla v\nabla|\nabla v|^2 \le 2\epsilon_3\int_\Omega u|\nabla|\nabla v|^2|^2 + \frac{\chi^2}{2\epsilon_3}\int_\Omega u^3|\nabla v|^2\tag{4.11}$$

and

$$-2(d_1+d_2)\int_\Omega u\nabla u\nabla|\nabla v|^2 \le 2\epsilon_4\int_\Omega u|\nabla u|^2 + \frac{(d_1+d_2)^2}{2\epsilon_4}\int_\Omega u|\nabla|\nabla v|^2|^2. \tag{4.12}$$

A substitution of (4.10) and (4.11) into (4.12) yields (4.9). □

Now, we are well-prepared to estimate the time evolution of the coupled functional (4.1) by means of Lemmas 4.1, 4.2, 4.3 and 4.4.

Lemma 4.5. *Under the logistic condition (1.4), the solution (u, v) of the KS system (1.3) satisfies the key inequality*

$$\begin{aligned}
&\frac{d}{dt}\Big\{\delta_1\int_\Omega u^3 + \delta_2\int_\Omega u^2|\nabla v|^2 + \delta_3\int_\Omega u|\nabla v|^4 + \delta_4\int_\Omega |\nabla v|^6\Big\} + 6\beta\delta_4\int_\Omega |\nabla v|^6\\
&\quad + 2[3(d_2-\eta)\delta_4 - (\epsilon_1+\epsilon_2)\delta_3]\int_\Omega |\nabla v|^2|\nabla|\nabla v|^2|^2 + (3\mu\delta_1 - \frac{n\alpha^2}{18d_2}\delta_2)\int_\Omega u^4\\
&\quad + [2(d_1-\epsilon)\delta_2 - \frac{(d_1+d_2)^2}{2\epsilon_2}\delta_3]\int_\Omega |\nabla u|^2|\nabla v|^2 + 2\beta\delta_2\int_\Omega u^2|\nabla v|^2\\
&\quad + \Big[(2(d_2-\eta)\delta_3 - (2\epsilon_3 + \frac{(d_1+d_2)^2}{2\epsilon_4})\delta_2\Big]\int_\Omega u|\nabla|\nabla v|^2|^2\\
&\quad + 4\beta\delta_3\int_\Omega u|\nabla v|^4 + 2[3(d_1-\epsilon)\delta_1 - \epsilon_4\delta_2]\int_\Omega u|\nabla u|^2\\
&\quad + \Big[(2\mu - \frac{\chi^2}{2\epsilon_3})\delta_2 - \frac{3\chi^2}{2\epsilon}\delta_1 - \frac{\alpha^2}{2}(\frac{1}{\eta} + \frac{n}{2d_2})\delta_3\Big]\int_\Omega u^3|\nabla v|^2\\
&\quad + \Big[(\mu - \frac{\chi^2}{2\epsilon_1})\delta_3 - \frac{\chi^2}{2\epsilon}\delta_2 - 3(\frac{2\alpha^2}{\eta} + \frac{n\alpha^2}{2d_2})\delta_4\Big]\int_\Omega u^2|\nabla v|^4\\
&\le 3a\delta_1\int_\Omega u^2 + 2a\delta_2\int_\Omega u|\nabla v|^2 + a\delta_3\int_\Omega |\nabla v|^4 + 3d_2\delta_4\int_{\partial\Omega} |\nabla v|^4\frac{\partial}{\partial\nu}|\nabla v|^2\\
&\quad + 2d_2\delta_3\int_{\partial\Omega} u|\bigtriangledown v|^2\frac{\partial}{\partial\nu}|\bigtriangledown v|^2 + d_2\delta_2\int_{\partial\Omega} u^2\frac{\partial}{\partial\nu}|\bigtriangledown v|^2
\end{aligned} \tag{4.13}$$

for all $t\in(0,T_{max})$ and for all $\epsilon_i,\delta_i>0$ and $\epsilon\in(0,d_1)$ and $\eta\in(0,d_2)$.

Proof. By honest computations from Lemmas 4.1–4.4 and by evident multiplications and additions, one can readily derive the lemma. □

As to the boundary integrals on the right-hand side of (4.13), we deduce from (3.18), (3.21) and Young's inequality with epsilon that

$$\begin{aligned}
&3d_2\delta_4\int_{\partial\Omega}|\nabla v|^4\frac{\partial}{\partial\nu}|\nabla v|^2+2d_2\delta_3\int_{\partial\Omega}u|\bigtriangledown v|^2\frac{\partial}{\partial\nu}|\bigtriangledown v|^2+d_2\delta_2\int_{\partial\Omega}u^2\frac{\partial}{\partial\nu}|\bigtriangledown v|^2\\
&\le 3d_2\delta_4K_1\int_{\partial\Omega}|\nabla v|^6+2d_2\delta_3K_1\int_{\partial\Omega}u|\bigtriangledown v|^4+d_2\delta_2K_1\int_{\partial\Omega}u^2|\bigtriangledown v|^2\\
&\le K_8\int_{\partial\Omega}u^3+K_8\int_{\partial\Omega}|\bigtriangledown v|^6=K_8\|u^{\frac{3}{2}}\|^2_{L^2(\partial\Omega)}++K_8\||\bigtriangledown v|^3\|^2_{L^2(\partial\Omega)}\\
&\le \xi\int_\Omega u|\nabla u|^2+C_\xi\Big(\int_\Omega u^{\frac{3}{2}}\Big)^2+\sigma\int_\Omega|\nabla v|^2|\nabla|\nabla v|^2|^2+C_\sigma\Big(\int_\Omega|\nabla v|^3\Big)^2
\end{aligned}\tag{4.14}$$

for any $\xi > 0$ and $\sigma > 0$.

Based on this boundary integral estimate and (4.13), we wish to select $\epsilon, \eta, \epsilon_i, \delta_i$ and μ such that

$$\begin{cases}
2[3(d_1-\epsilon)\delta_1-\epsilon_4\delta_2]>0,\\
2[3(d_2-\eta)\delta_4-(\epsilon_1+\epsilon_2)\delta_3]>0,\\
2(d_1-\epsilon)\delta_2-\frac{(d_1+d_2)^2}{2\epsilon_2}\delta_3\ge 0,\\
2(d_2-\eta)\delta_3-(2\epsilon_3+\frac{(d_1+d_2)^2}{2\epsilon_4})\delta_2\ge 0,\\
3\mu\delta_1-\frac{n\alpha^2}{18d_2}\delta_2\ge 0,\\
(2\mu-\frac{\chi^2}{2\epsilon_3})\delta_2-\frac{3\chi^2}{2\epsilon}\delta_1-\frac{\alpha^2}{2}(\frac{1}{\eta}+\frac{n}{2d_2})\delta_3\ge 0,\\
(\mu-\frac{\chi^2}{2\epsilon_1})\delta_3-\frac{\chi^2}{2\epsilon}\delta_2-3(\frac{2\alpha^2}{\eta}+\frac{n\alpha^2}{2d_2})\delta_4\ge 0.
\end{cases}\tag{4.15}$$

Only the third constraint intertwines with the fourth constraint, while this can be fulfilled since we can choose $\epsilon, \eta, \epsilon_i$ such that

$$\frac{(d_1+d_2)^2}{4\epsilon_2(d_1-\epsilon)}\le\frac{2(d_2-\eta)}{2\epsilon_3+\frac{(d_1+d_2)^2}{2\epsilon_4}}\iff\frac{(d_1+d_2)^2}{4(d_1-\epsilon)(d_2-\eta)}\le\frac{\epsilon_2}{\epsilon_3+\frac{(d_1+d_2)^2}{4\epsilon_4}}<\frac{\epsilon_2}{\epsilon_3}.\tag{4.16}$$

Then algebraic manipulations from (4.15) and (4.16) show that

$$\begin{aligned}
3\mu&\ge\tfrac{n\alpha^2}{54d_2}\tfrac{\delta_2}{\delta_1}+\tfrac{\chi^2}{4\epsilon_3}+\tfrac{3\chi^2}{4\epsilon}\tfrac{\delta_1}{\delta_2}+\tfrac{\alpha^2}{4}(\tfrac{1}{\eta}+\tfrac{n}{2d_2})\tfrac{\delta_3}{\delta_2}+\tfrac{\chi^2}{2\epsilon_1}+\tfrac{\chi^2}{2\epsilon}\tfrac{\delta_2}{\delta_3}+3(\tfrac{2\alpha^2}{\eta}+\tfrac{n\alpha^2}{2d_2})\tfrac{\delta_4}{\delta_3}\\
&>\sqrt{\tfrac{n}{18d_2\epsilon}}\alpha|\chi|+\tfrac{\chi^2}{4\epsilon_3}+\sqrt{\tfrac{1}{2\epsilon}(\tfrac{1}{\eta}+\tfrac{n}{2d_2})}\alpha|\chi|+\tfrac{\chi^2}{2\epsilon_1}+(\tfrac{2\alpha^2}{\eta}+\tfrac{n\alpha^2}{2d_2})\tfrac{\epsilon_1+\epsilon_2}{d_2-\eta}\\
&>\Big\{\sqrt{\tfrac{n}{18d_2\epsilon}}+\sqrt{\tfrac{1}{2\epsilon}(\tfrac{1}{\eta}+\tfrac{n}{2d_2})}+\sqrt{\tfrac{1}{(d_2-\eta)}(\tfrac{2}{\eta}+\tfrac{n}{2d_2})}\Big[\sqrt{2}+\tfrac{(d_1+d_2)}{2\sqrt{(d_1-\epsilon)(d_2-\eta)}}\Big]\Big\}\alpha|\chi|
\end{aligned}\tag{4.17}$$

With these preparations, we obtain the second assertion (ii) in Theorem 1.1.

Lemma 4.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, f satisfy the logistic condition (1.4) and $d_1, d_2, \alpha, \beta > 0$, $a \ge 0$ and $\chi \in \mathbb{R}$. Then, for any*

$$\mu>\mu_0(n,d_1,d_2,\alpha,\chi):=\max\Big\{\frac{1}{3}h(n,d_1,d_2),\frac{n}{\sqrt{2n+4}-2}(\frac{1}{d_1}+\frac{2}{d_2})\Big\}\alpha|\chi|\tag{4.18}$$

with

$$\begin{aligned}
h(n,d_1,d_2)=\inf_{0<\epsilon<d_1,0<\eta<d_2}&\Big\{\sqrt{\tfrac{n}{18d_2\epsilon}}+\sqrt{\tfrac{1}{2\epsilon}(\tfrac{1}{\eta}+\tfrac{n}{2d_2})}\\
&+\sqrt{\tfrac{1}{(d_2-\eta)}(\tfrac{2}{\eta}+\tfrac{n}{2d_2})}\Big[\sqrt{2}+\tfrac{(d_1+d_2)}{2\sqrt{(d_1-\epsilon)(d_2-\eta)}}\Big]\Big\},
\end{aligned}$$

there exists a constant $C(u_0, v_0)$ such that the unique nonnegative solution (u, v) of the chemotaxis-growth system (1.3) satisfies

$$\int_\Omega u^3(\cdot,t) + \int_\Omega u^2(\cdot,t)|\nabla v(\cdot,t)|^2 + \int_\Omega u|\nabla v|^4 + \int_\Omega |\nabla v(\cdot,t)|^6 \le C(u_0,v_0), \forall t \in (0, T_{max}).$$

If $n \le 5$, the solution (u, v) exists globally in time, i.e., $T_{max} = \infty$ and $(u(\cdot,t), v(\cdot,t))$ is uniformly bounded in $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ for all $t \in (0,\infty)$.

Proof. Due to $\mu > \mu_0$ given in (4.18), we first know from Lemma 3.5 that

$$3a\delta_1 \int_\Omega u^2 + 2a\delta_2 \int_\Omega u|\nabla v|^2 + a\delta_3 \int_\Omega |\nabla v|^4 \le K_9(u_0,v_0) \tag{4.19}$$

for all $t \in (0, T_{\max})$. Moreover, due to (4.17), the fact that $\mu > \mu_0$ allows us to fix $\epsilon, \eta, \epsilon_i, \delta_i$ satisfying (4.16) and (4.15). Indeed, we first choose $(\epsilon, \eta) = (\epsilon_0, \eta_0)$ so that h is minimized, then we choose ϵ_1 to be the minimizer of

$$\frac{\chi^2}{2\epsilon_1} + (\frac{2\alpha^2}{\eta} + \frac{n\alpha^2}{2d_2})\frac{\epsilon_1}{d_2-\eta},$$

take $\epsilon_3 = 1$ and then fix ϵ_2 and ϵ_4 so that (4.16) is satisfied. Then, based on (4.15), (4.16) and (4.17), all δ_i can be chosen readily.

Upon such well chosen $\epsilon, \eta, \epsilon_i$ and δ_i, using (4.13), (4.14) and arguing as Lemma 3.5, we can easily deduce a Gronwall inequality for the coupled quantity z as defined by (4.1) of the form (4.2). As a matter of fact, by (4.13) and (4.14), we have

$$\begin{aligned} z'(t) + \beta z(t) &+ 2[3(d_1-\epsilon)\delta_1 - \epsilon_4\delta_2 - \xi] \textstyle\int_\Omega u|\nabla u|^2 + (3\mu\delta_1 - \frac{n\alpha^2}{18d_2}\delta_2)\int_\Omega u^4 \\ &+ 2[3(d_2-\eta)\delta_4 - (\epsilon_1+\epsilon_2)\delta_3 - \sigma] \textstyle\int_\Omega |\nabla v|^2|\nabla|\nabla v|^2|^2 \\ &\le 3a\delta_1 \textstyle\int_\Omega u^2 + 2a\delta_2\int_\Omega u|\nabla v|^2 + a\delta_3 \int_\Omega |\nabla v|^4 \\ &\quad + \beta\delta_1 \textstyle\int_\Omega u^3 + C_\xi\Big(\int_\Omega u^{\frac{3}{2}}\Big)^2 + C_\sigma\Big(\int_\Omega |\nabla v|^3\Big)^2. \end{aligned} \tag{4.20}$$

Now, by fixing ξ and σ sufficiently small, we deduce from (4.19), (4.20) and Hölder inequality that

$$z'(t) + \beta z(t) \le C(u_0, v_0),$$

directly yielding the desired boundedness stated in the lemma. In particular, it guarantees that $\|u(\cdot,t)\|_{L^3(\Omega)}$ is uniformly bounded. Thus, in the case of $n \le 5$, the $L^{\frac{n}{2}+\epsilon}$-criterion [2,50] shows, that $(u(\cdot,t), v(\cdot,t))$ is uniformly bounded in $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ for all $t \in (0,\infty)$. $\square$

The algorithm for proving L^2 and L^3-boundedness of u may be in principle carried over to obtain L^r-boundedness of u for $r = 4, 5, \cdots$ inductively by establishing a Gronwall inequality for the coupled quantity (motivated by [42] again)

$$z(t) := \sum_{k=0}^{r} \delta_i \int_\Omega u^k |\nabla v|^{2(r-k)}, \qquad t \in (0, T_{\max}) \tag{4.21}$$

of the form

$$z'(t) + \epsilon z(t) \le C_\epsilon, \quad t \in (0, T_{\max})$$

for some carefully chosen positive constants δ_i and small $\epsilon > 0$ and perhaps large C_ϵ independent of t. In this process, as r becomes large, we will have more terms to handle and more boundary integrals will appear. While, simplification of the process is possible. In fact, for $r = 2, 3, \cdots$, suppose that $\mu > \mu_0^{(r)}(n, d_1, d_2, \alpha, \chi)$ is the condition under which L^r-boundedness of u (indeed, uniform boundedness of $z(t)$ as in (4.21)) is guaranteed. Then, based on the procedures for $r = 2$ and $r = 3$, when $\mu > \mu_0^{(r-1)}(n, d_1, d_2, \alpha, \chi)$, the process for the next step r may be continued with the assumption that $a = 0$ and Ω be convex.

Once this is done, the L^r-boundedness of u will be obtained. Further, by choosing $r = \lfloor \frac{n}{2} \rfloor + 1$, the $L^{\frac{n}{2}+\epsilon}$-criterion in [2,50] ensures the desired L^∞-boundedness of u. Here, we state the following expected general result that offers a quantitative description on when logistic damping dominates over chemotactic aggregation for (1.3) in $\Omega \subset \mathbb{R}^n$. While, we have to leave a rigorous examination for future study.

Proposition 4.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, f satisfy the logistic condition (1.4) and $d_1, d_2, \alpha, \beta > 0$, $a \ge 0$ and $\chi \in \mathbb{R}$. Then, for any natural number $r \ge 2$, there exist a function $\theta_0^{(r)}(n, d_1, d_2)$ which tends to infinity as $d_1 \to 0$ or $d_2 \to 0$ (and is decreasing in d_1, d_2) and constant $C(u_0, v_0)$ such that, for any*

$$\mu > \theta_0^{(r)}(n, d_1, d_2)\alpha|\chi|,$$

the nonnegative solution (u, v) of the KS system (1.3) satisfies

$$\sum_{k=0}^{r} \int_\Omega u^k |\nabla v|^{2(r-k)} \le C(u_0, v_0), \qquad t \in (0, T_{max}).$$

As a result, if $n \le 2r - 1$, then (u, v) exists globally in time and $(u(\cdot, t), v(\cdot, t))$ is uniformly bounded in $L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ for all $t \in (0, \infty)$.

4.2. The special case that $d_1 = d_2$ and $\Omega \subset \mathbb{R}^n$ is convex

In the special case that $\tau = 1$ (equal diffusion rates) in (1.1), $\chi > 0$ (positive chemotaxis) and $\Omega \subset \mathbb{R}^n$ is convex, the explicit lower bound

$$\mu > \frac{n}{4}\chi$$

ensuring global boundedness to the solution of (1.1) has been elucidated in [42] by establishing a parabolic inequality for a combination of u and $|\nabla v|^2$. Here, in this subsection, for the full-parameter chemotaxis model (1.3), we will write down all the details for convenience and completeness.

Lemma 4.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth convex domain, f satisfy the logistic condition (1.4) and $d_1, d_2, \alpha, \beta > 0$, $a \ge 0$. Assume that*

$$d_1 = d_2, \qquad \chi > 0, \qquad \mu > \frac{n}{4d_1}\alpha\chi,$$

the unique nonnegative solution (u, v) of the chemotaxis-growth system (1.3) exists globally in time, i.e., $T_{max} = \infty$ and is bounded in the following way:

$$u \le \max\Big\{ \max_{\bar\Omega\times[0,1]} u, \frac{1}{2\alpha} \max\Big\{ \max_{\bar\Omega}(2\alpha u(\cdot,1) + \chi|\nabla v(\cdot,1)|^2), \frac{a\alpha}{\beta} + \frac{\alpha\beta}{\mu - \frac{n\alpha}{4d_1}\chi} \Big\}\Big\}$$

and

$$v \le \max\Big\{ \max_{\bar\Omega\times[0,1]} v, \frac{1}{2\beta} \max\Big\{ \max_{\bar\Omega}(2\alpha u(\cdot,1) + \chi|\nabla v(\cdot,1)|^2), \frac{a\alpha}{\beta} + \frac{\alpha\beta}{\mu - \frac{n\alpha}{4d_1}\chi} \Big\}\Big\}$$

on $\bar\Omega \times [0,\infty)$. *In addition, for any* $\epsilon > 0$,

$$|\nabla v| \le \frac{1}{\chi} \max\Big\{ \max_{\bar\Omega}(2\alpha u(\cdot,\epsilon) + \chi|\nabla v(\cdot,\epsilon)|^2), \frac{a\alpha}{\beta} + \frac{\alpha\beta}{\mu - \frac{n\alpha}{4d_1}\chi} \Big\} \text{ on } \bar\Omega \times [\epsilon,\infty).$$

Moreover, if $\|\nabla v_0\|_{L^\infty(\Omega)} < \infty$, *then* ϵ *can be chosen to be zero.*

Proof. Taking the gradient first and then multiplying it by ∇v in the v-equation of (1.3) and then using (2.4) and the fact that $d_1 = d_2$, one derives

$$(|\nabla v|^2)_t = d_1 \triangle |\nabla v|^2 - 2d_1|D^2 v|^2 - 2\beta|\nabla v|^2 + 2\alpha \nabla u \nabla v. \tag{4.22}$$

Multiplying the u-equation of (1.3) by 2α and the equation (4.22) by χ yields

$$(2\alpha u + \chi|\nabla v|^2)_t = d_1\Delta(2\alpha u + \chi|\nabla v|^2) - 2d_1\chi|D^2v|^2 - 2\beta\chi|\nabla v|^2 - 2\alpha\chi u\Delta v + 2\alpha f(u).$$

Notice from Cauchy–Schwarz inequality and (2.5) that

$$-2\alpha\chi u\Delta v \le \frac{n\chi\alpha^2}{2d_1}u^2 + \frac{2d_1}{n}\chi|\Delta v|^2 \le \frac{n\chi\alpha^2}{2d_1}u^2 + 2d_1\chi|D^2v|^2,$$

we then deduce from the logistic condition $f(u) \le a - \mu u^2$ that

$$(2\alpha u + \chi|\nabla v|^2)_t \le d_1\Delta(2\alpha u + \chi|\nabla v|^2) - 2\beta\chi|\nabla v|^2 + 2a\alpha - 2\alpha(\mu - \frac{n\alpha}{4d_1}\chi)u^2. \tag{4.23}$$

Recall that, for a convex domain Ω, we have the well-known fact that $\frac{\partial}{\partial\nu}(|\nabla v|^2) \le 0$ for any function satisfying $\frac{\partial v}{\partial\nu} = 0$ on $\partial\Omega$; see Matano [27, Lemma 5.3]. Consequently, from $\frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0$ on $\partial\Omega$, we get

$$\frac{\partial}{\partial\nu}w := \frac{\partial}{\partial\nu}(2\alpha u + \chi|\nabla v|^2) = 2\alpha\frac{\partial u}{\partial\nu} + 2\chi\frac{\partial}{\partial\nu}|\nabla v|^2 = 2\chi\frac{\partial}{\partial\nu}|\nabla v|^2 \le 0 \text{ on } \partial\Omega. \tag{4.24}$$

Now, since $\mu > \frac{n\alpha}{4d_1}\alpha\chi$, using elementary calculations we find that

$$w_t - d_1\Delta w + 2\beta w \le M, \tag{4.25}$$

where

$$M = \max\{2a\alpha - 2\alpha(\mu - \frac{n\alpha}{4d_1}\chi)u^2 + 4\alpha\beta u | u \ge 0\} = 2a\alpha + \frac{2\alpha\beta^2}{\mu - \frac{n\alpha}{4d_1}\chi} < \infty.$$

Recall that from Lemma 2.2 that $(u,v) \in C^{2,1}(\bar\Omega \times (0,T_{\max}))$. So, we can perform a small time shift and treat any positive time as the "initial time". Then, by (4.25) and (4.24), we conclude from the maximum principle and the Hopf boundary point lemma, for any $\epsilon \in (0,T_{\max})$, that

$$w = 2\alpha u + \chi|\nabla v|^2 \le \max\Big\{\max_{\bar\Omega}(2\alpha u(\cdot,\epsilon) + \chi|\nabla v(\cdot,\epsilon)|^2), \frac{M}{2\beta}\Big\} \text{ on } \bar\Omega\times[\epsilon, T_{\max}).$$

This gives the uniform boundedness for the coupled quantity $2\alpha u + \chi|\nabla v|^2$, and then the positiveness of χ shows the L^∞-boundedness of u and ∇v:

$$u \le \max\Big\{\max_{\bar\Omega\times[0,\epsilon]} u, \frac{1}{2\alpha}, \max\Big\{\max_{\bar\Omega}(2\alpha u(\cdot,\epsilon) + \chi|\nabla v(\cdot,\epsilon)|^2), \frac{M}{2\beta}\Big\}\Big\}$$

on $\bar\Omega\times[\epsilon, T_{\max})$ and

$$|\nabla v| \le \frac{1}{\chi}\max\Big\{\max_{\bar\Omega}(2\alpha u(\cdot,\epsilon) + \chi|\nabla v(\cdot,\epsilon)|^2), \frac{M}{2\beta}\Big\} \text{ on } \bar\Omega\times[\epsilon, T_{\max}).$$

These combined with the blow-up criterion (2.1) of Lemma 2.2 show that (u,v) exists globally in time, i.e., $T_{\max} = \infty$.

Finally, an application of maximum principle to the v-equation in (1.3) on $\bar\Omega\times[1,\infty)$ yields the bound for v. Indeed, $|\nabla v|$ is bounded if $|\nabla v_0|$ is. $\square$

5. Proof of the large time behavior for the KS model (1.8)

In this section, we show the proof of Theorem 1.3, which relies on finding so-called Lyapounov functionals. Here, we will present all the necessary details for the clarity of obtaining the explicit convergence rates.

Proof. We modified the functional in [9] as

$$H(t) = \int_\Omega\Big(u - \frac{\kappa}{\mu} - \frac{\kappa}{\mu}\ln(\frac{\mu}{\kappa}u)\Big) + \delta\int_\Omega(v - \frac{\alpha\kappa}{\beta\mu})^2, \quad \delta = \frac{\kappa\chi^2}{8d_1d_2\mu}. \tag{5.1}$$

Differentiating H, using the chemotaxis-logistic system (1.8) and integrating by parts, we deduce from Cauchy–Schwarz inequality that

$$\begin{aligned}
\frac{d}{dt}H(t) &= \int_\Omega \frac{u-\frac{\kappa}{\mu}}{u}u_t + 2\delta\int_\Omega(v - \frac{\alpha\kappa}{\beta\mu})v_t + 2\delta\int_\Omega(v-\frac{\alpha\kappa}{\beta\mu})(d_2\Delta v - \beta v + \alpha u)\\
&= \int_\Omega \frac{u-\frac{\kappa}{\mu}}{u}\Big(\nabla\cdot(d_1\nabla u - \chi u\nabla v) + u(\kappa-\mu u)\Big) - 2d_2\delta\int_\Omega|\nabla v|^2 + 2\delta\int_\Omega(v-\frac{\alpha\kappa}{\beta\mu})(-\beta v + \alpha u)\\
&= -\frac{\kappa}{\mu}d_1\int_\Omega\frac{|\nabla u|^2}{u^2} + \frac{\kappa}{\mu}\chi\int_\Omega\frac{\nabla u}{u}\cdot\nabla v - \mu\int_\Omega(u-\frac{\kappa}{\mu})^2\\
&\quad - 2d_2\delta\int_\Omega|\nabla v|^2 - 2\beta\delta\int_\Omega(v-\frac{\alpha\kappa}{\beta\mu})^2 + 2\alpha\delta\int_\Omega(u-\frac{\alpha\kappa}{\beta\mu})(v-\frac{\alpha\kappa}{\beta\mu})\\
&\le -2d_2(\delta - \frac{\kappa\chi^2}{8d_1d_2\mu})\int_\Omega|\nabla v|^2 - (\mu - 2\alpha\delta\epsilon)\int_\Omega(u-\frac{\kappa}{\mu})^2 - 2\delta(\beta - \frac{\alpha}{4\epsilon})\int_\Omega(v-\frac{\alpha\kappa}{\beta\mu})^2\\
&= -(\mu-2\alpha\delta\epsilon)\int_\Omega(u-\frac{\kappa}{\mu})^2 - 2\delta(\beta-\frac{\alpha}{4\epsilon})\int_\Omega(v-\frac{\alpha\kappa}{\beta\mu})^2
\end{aligned} \tag{5.2}$$

for any $\epsilon > 0$. Now, we wish to minimize μ by choosing ϵ and δ such that

$$\begin{cases} \mu - 2\alpha\delta\epsilon > 0, \\ \beta - \frac{\alpha}{4\epsilon} > 0. \end{cases} \iff \begin{cases} \mu > \frac{\kappa\alpha\chi^2}{4d_1d_2\mu}\epsilon, \\ \frac{\alpha}{4\beta} < \epsilon < \frac{4d_1d_2\mu^2}{\alpha\kappa\chi^2}. \end{cases} \iff \mu^2 > \frac{\kappa\alpha^2\chi^2}{16d_1d_2\beta}. \tag{5.3}$$

The last constraint is guaranteed by our assumption (1.9). Next, for fixed ϵ obeying (5.3), we set

$$\eta = \min\Big\{(\mu - 2\alpha\delta\epsilon), 2\delta(\beta - \frac{\alpha}{4\epsilon})\Big\} = \min\Big\{(\mu - \frac{\alpha\kappa\chi^2}{4d_1d_2\mu}\epsilon), \frac{\kappa\chi^2}{4d_1d_2\mu}(\beta - \frac{\alpha}{4\epsilon})\Big\}, \tag{5.4}$$

and then we infer from (5.2), (5.3) and (5.4) that

$$\frac{d}{dt}H(t) \le -\eta\Big(\int_\Omega (u - \frac{\kappa}{\mu})^2 + \int_\Omega (v - \frac{\alpha\kappa}{\beta\mu})^2\Big). \tag{5.5}$$

Since $H(t) \ge 0$, an integration of (5.5) from any $t_0 \ge 0$ to t yields

$$\eta\int_{t_0}^{t}\Big(\int_\Omega (u - \frac{\kappa}{\mu})^2 + \int_\Omega (v - \frac{\alpha\kappa}{\beta\mu})^2\Big) \le H(t_0),$$

giving trivially

$$\int_{t_0}^{\infty}\Big(\int_\Omega (u - \frac{\kappa}{\mu})^2 + \int_\Omega (v - \frac{\alpha\kappa}{\beta\mu})^2\Big) \le \frac{H(t_0)}{\eta}. \tag{5.6}$$

Thanks to Theorem 1.1, the condition $\mu > \mu_0$ ensures that (u, v) is globally bounded and classical. Then from the parabolic regularity, we see that $\int_\Omega (u - \frac{\kappa}{\mu})^2 + \int_\Omega (v - \frac{\alpha\kappa}{\beta\mu})^2$ is uniformly bounded and uniformly continuous in t. This allows one to deduce from (5.6) that

$$\int_\Omega (u - \frac{\kappa}{\mu})^2 + \int_\Omega (v - \frac{\alpha\kappa}{\beta\mu})^2 \to 0, \quad \text{as } t \to \infty. \tag{5.7}$$

Since u, v are smooth and bounded, the standard parabolic regularity for parabolic equations (cf. [21]) shows there are $\sigma \in (0, 1)$ and C such that

$$\|u\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\bar\Omega\times[t,t+1])} + \|v\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\bar\Omega\times[t,t+1])} \le C, \quad \forall t \ge 1. \tag{5.8}$$

Hence, by the Gagliardo–Nirenberg inequality, (5.7) and (5.8), we obtain

$$\begin{aligned} \|u(\cdot,t) - \tfrac{\kappa}{\mu}\|_{L^\infty(\Omega)} &\le C_{GN}\|u(\cdot,t) - \tfrac{\kappa}{\mu}\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}}\|u - \tfrac{\kappa}{\mu}\|_{L^2(\Omega)}^{\frac{2}{n+2}} \\ &\le C\|u(\cdot,t) - \tfrac{\kappa}{\mu}\|_{L^2(\Omega)}^{\frac{2}{n+2}} \to 0, \text{ as } t \to \infty. \end{aligned} \tag{5.9}$$

In the same way, we get

$$\|v(\cdot,t) - \frac{\alpha\kappa}{\beta\mu}\|_{L^\infty(\Omega)} \le C\|v(\cdot,t) - \frac{\alpha\kappa}{\beta\mu}\|_{L^2(\Omega)}^{\frac{2}{n+2}} \to 0, \text{ as } t \to \infty. \tag{5.10}$$

Based on the definition of H in (5.1), (5.9) and (5.5), we calculate via the L'Hôpital's rule that

$$\lim_{u\to\frac{\kappa}{\mu}} \frac{u - \frac{\kappa}{\mu} - \frac{\kappa}{\mu}\ln(\frac{\mu}{\kappa}u)}{(u - \frac{\kappa}{\mu})^2} = \frac{\mu}{2\kappa}.$$

This together with (5.9) allows one to find $t_1 \geq 0$ such that

$$\frac{\mu}{4\kappa}(u-\frac{\kappa}{\mu})^2 \leq u-\frac{\kappa}{\mu}-\frac{\kappa}{\mu}\ln(\frac{\mu}{\kappa}u) \leq \frac{\mu}{\kappa}(u-\frac{\kappa}{\mu})^2, \qquad t \geq t_1,$$

and then the definition of H in (5.1) entails

$$\min\{\frac{\mu}{4\kappa},\delta\}\Big(\int_\Omega (u-\frac{\kappa}{\mu})^2+\int_\Omega (v-\frac{\alpha\kappa}{\beta\mu})^2\Big) \leq H(t), \qquad t \geq t_1 \tag{5.11}$$

and

$$H(t) \leq \max\{\frac{\mu}{\kappa},\delta\}\Big(\int_\Omega (u-\frac{\kappa}{\mu})^2+\int_\Omega (v-\frac{\alpha\kappa}{\beta\mu})^2\Big), \qquad t \geq t_1. \tag{5.12}$$

Combining (5.5) and (5.12), we obtain an ordinary differential inequality:

$$\frac{d}{dt}H(t) \leq -\frac{\eta}{\max\{\frac{\mu}{\kappa},\delta\}}H(t), \quad t \geq t_1,$$

directly yielding

$$H(t) \leq H(t_1)e^{-\frac{\eta}{\max\{\frac{\mu}{\kappa},\delta\}}(t-t_1)}, \quad t \geq t_1.$$

This in conjunction with (5.11) tells us that

$$\int_\Omega (u-\frac{\kappa}{\mu})^2+\int_\Omega (v-\frac{\alpha\kappa}{\beta\mu})^2 \leq \frac{H(t_1)}{\min\{\frac{\mu}{4\kappa},\delta\}}e^{-\frac{\eta}{\max\{\frac{\mu}{\kappa},\delta\}}(t-t_1)}, \quad t \geq t_1.$$

With this decay estimate at hand, we then derive from (5.9) and (5.10) that there exists a large constant $C>0$ such that

$$\|u(\cdot,t)-\frac{\kappa}{\mu}\|_{L^\infty(\Omega)}+\|v(\cdot,t)-\frac{\alpha\kappa}{\beta\mu}\ \|_{L^\infty(\Omega)} \leq Ce^{-\frac{\eta}{(n+2)\max\{\frac{\mu}{\kappa},\delta\}}(t-t_1)}, \quad t \geq t_1.$$

Then plugging in the definitions of δ in (5.1), η in (5.4) and taking $\epsilon=\epsilon_0$ in (5.3), we obtain the desired exponential decay estimate (1.10).

In the case of $\kappa=0$, the uniform boundedness and global existence of solution does not affect as long as $\mu>\mu_0$. We integrate the first equation in the KS system (1.8) and use Hölder inequality to obtain

$$\frac{d}{dt}\int_\Omega u=-\mu\int_\Omega u^2 \leq -\mu|\Omega|^{-1}\Big(\int_\Omega u\Big)^2, \quad t>0,$$

which enables us to deduce

$$\int_\Omega u \leq \Big[(\int_\Omega u_0)^{-1}+\mu|\Omega|^{-1}t\Big]^{-1} \leq \frac{c_1}{t+1}, \quad t>0. \tag{5.13}$$

As before, the boundedness of u and Gagliardo–Nirenberg inequality entails

$$\begin{aligned}\|u(\cdot,t)\|_{L^\infty(\Omega)} &\le C_{GN}\|u(\cdot,t)\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+1}}\|u(\cdot,t)\|_{L^1(\Omega)}^{\frac{1}{n+1}} \\ &\le C\Big[(\textstyle\int_\Omega u_0)^{-1}+\mu|\Omega|^{-1}t\Big]^{-\frac{1}{n+1}}, \quad t>0.\end{aligned} \tag{5.14}$$

An integration of the second equation in (1.8) shows

$$\frac{d}{dt}\int_\Omega v=-\beta\int_\Omega v+\alpha\int_\Omega u\le-\beta\int_\Omega v+\frac{c_2}{t+1}. \tag{5.15}$$

Solving this Gronwall inequality, we obtain

$$\int_\Omega v\le\|v_0\|_{L^1}e^{-\beta t}+c_2\frac{\int_0^t\frac{e^{\beta s}}{s+1}}{e^{\beta t}}\le\|v_0\|_{L^1}e^{-\beta t}+\frac{c_3}{t+1}\le\frac{c_4}{t+1},$$

where we used the fact that

$$\lim_{t\to\infty}\frac{(t+1)\int_0^t\frac{e^{\beta s}}{s+1}}{e^{\beta t}}=\frac{1}{\beta}<\infty,\quad \lim_{t\to\infty}(t+1)e^{-\beta t}=0.$$

Then we conclude from (5.14) with u replaced by v that

$$\|v(\cdot,t)\|_{L^\infty(\Omega)}\le\frac{c_4}{(t+1)^{n+1}},\quad t>0. \tag{5.16}$$

In the case of $\kappa<0$, we integrate the first equation in (1.8) to get

$$\frac{d}{dt}\int_\Omega u=\kappa\int_\Omega u-\mu\int_\Omega u^2\le\kappa\int_\Omega u,\quad t>0,$$

and thus

$$\int_\Omega u\le e^{\kappa t}\int_\Omega u_0,\quad t>0. \tag{5.17}$$

Then the GN inequality (5.14) implies that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}\le c_5e^{\frac{\kappa}{n+1}t},\quad t>0. \tag{5.18}$$

Combining (5.17) and (5.15), we derive

$$\|v(\cdot,t)\|_{L^1}\le\begin{cases}\|v_0\|_{L^1}e^{-\beta t}+c_6te^{-\beta t}, & \text{if } \beta=-\kappa,\\ \|v_0\|_{L^1}e^{-\beta t}+c_7\frac{e^{\kappa t}-e^{-\beta t}}{\beta+\kappa}, & \text{if } \beta\ne-\kappa.\end{cases}\le c_9e^{-\frac{1}{2}\min\{\beta,-\kappa\}t}.$$

With this decay estimate at hand, the GN inequality (5.14) gives rise to

$$\|v(\cdot,t)\|_{L^\infty(\Omega)}\le c_{10}e^{-\frac{1}{2(n+1)}\min\{\beta,-\kappa\}t},\forall t\ge0. \tag{5.19}$$

Extracting the essential ingredients of the estimates (5.14), (5.16), (5.18) and (5.19), we readily conclude the decay estimates (1.10), (1.11) and (1.12). □

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References

[1] X. Bai, M. Winkler, Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics, Indiana Univ. Math. J. 65 (2016) 553–583.
[2] N. Bellomo, A. Bellouquid, Y. Tao, M. Winkler, Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues, Math. Models Methods Appl. Sci. 25 (9) (2015) 1663–1763.
[3] A. Blanchet, J. Carrillo, P. Laurencot, Critical mass for a Patlak–Keller–Segel model with degenerate diffusion in higher dimensions, Calc. Var. Partial Differential Equations 35 (2009) 133–168.
[4] X. Cao, Boundedness in a quasilinear parabolic–parabolic Keller–Segel system with logistic source, J. Math. Anal. Appl. 412 (2014) 181–188.
[5] A. Chertock, A. Kurganov, X. Wang, Y. Wu, On a chemotaxis model with saturated chemotactic flux, Kinet. Relat. Models 5 (2012) 51–95.
[6] T. Cieślak, P. Laurencot, Finite time blow-up for a one-dimensional quasilinear parabolic–parabolic chemotaxis system, Ann. Inst. H. Poincaré Anal. Non Linéaire 27 (2010) 437–446.
[7] T. Cieślak, C. Stinner, New critical exponents in a fully parabolic quasilinear Keller–Segel system and applications to volume filling models, J. Differential Equations 258 (2015) 2080–2113.
[8] K. Fujie, M. Winkler, T. Yokota, Blow-up prevention by logistic sources in a parabolic–elliptic Keller–Segel system with singular sensitivity, Nonlinear Anal. 109 (2014) 56–71.
[9] X. He, S. Zheng, Convergence rate estimates of solutions in a higher dimensional chemotaxis system with logistic source, J. Math. Anal. Appl. 436 (2016) 970–982.
[10] B. Hu, Y. Tao, Boundedness in a parabolic–elliptic chemotaxis-growth system under a critical parameter condition, Appl. Math. Lett. 64 (2017) 1–7.
[11] T. Hillen, A. Potapov, The one-dimensional chemotaxis model: global existence and asymptotic profile, Math. Methods Appl. Sci. 27 (2004) 1783–1801.
[12] T. Hillen, K. Painter, A user's guide to PDE models for chemotaxis, J. Math. Biol. 58 (2009) 183–217.
[13] T. Hillen, K. Painter, Spatio-temporal chaos in a chemotaxis model, Phys. D 240 (2011) 363–375.
[14] D. Horstmann, G. Wang, Blow-up in a chemotaxis model without symmetry assumptions, European J. Appl. Math. 12 (2001) 159–177.
[15] D. Horstmann, From 1970 until now: the Keller–Segel model in chemotaxis and its consequence I, Jahresber. Dtsch. Math.-Ver. 105 (2003) 103–165.
[16] D. Horstmann, M. Winkler, Boundedness vs. blow-up in a chemotaxis system, J. Differential Equations 215 (2005) 52–107.
[17] S. Ishida, K. Seki, T. Yokota, Boundedness in quasilinear Keller–Segel systems of parabolic–parabolic type on non-convex bounded domains, J. Differential Equations 256 (8) (2014) 2993–3010.
[18] K. Kang, A. Stevens, Blowup and global solutions in a chemotaxis-growth system, Nonlinear Anal. 135 (2016) 57–72.
[19] E. Keller, L. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret. Biol. 26 (1970) 399–415.
[20] E. Keller, L. Segel, Model for chemotaxis, J. Theoret. Biol. 30 (1971) 225–234.
[21] O. Ladyzhenskaya, V. Solonnikov, N. Uralceva, Linear and Quasilinear Equations of Parabolic Type, AMS, Providence, RI, 1968.
[22] J. Lankeit, Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source, J. Differential Equations 258 (2015) 1158–1191.
[23] J. Lankeit, Chemotaxis can prevent thresholds on population density, Discrete Contin. Dyn. Syst. Ser. B 20 (2015) 1499–1527.
[24] P. Laurencota, N. Mizoguchib, Finite time blowup for the parabolic–parabolic Keller–Segel system with critical diffusion, Ann. Inst. H. Poincaré Anal. Non Linéaire 34 (1) (2017) 197–220.
[25] K. Lin, C. Mu, Global dynamics in a fully parabolic chemotaxis system with logistic source, Discrete Contin. Dyn. Syst. 36 (2016) 5025–5046.
[26] M. Ma, Z. Wang, Global bifurcation and stability of steady states for a reaction–diffusion-chemotaxis model with volume-filling effect, Nonlinearity 28 (2015) 2639–2660.
[27] H. Matano, Asymptotic behavior and stability of solutions of semilinear diffusion equations, Publ. Res. Inst. Math. Sci. 15 (1979) 401–454.
[28] N. Mizoguchi, P. Souplet, Nondegeneracy of blow-up points for the parabolic Keller–Segel system, Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (2014) 851–875.
[29] T. Nagai, T. Senba, K. Yoshida, Application of the Trudinger–Moser inequality to a parabolic system of chemotaxis, Funkcial. Ekvac. 40 (1997) 411–433.
[30] K. Osaki, A. Yagi, Finite dimensional attractor for one-dimensional Keller–Segel equations, Funkcial. Ekvac. 44 (2001) 441–469.

[31] K. Osaki, T. Tsujikawa, A. Yagi, M. Mimura, Exponential attractor for a chemotaxis-growth system of equations, Nonlinear Anal. 51 (2002) 119–144.
[32] P. Quittner, P. Souplet, Superlinear Parabolic Problems. Blow-up, Global Existence and Steady States, Birkhäuser Advanced Texts, Birkhäuser Verlag, Basel, 2007.
[33] T. Senba, T. Suzuki, Parabolic system of chemotaxis: blowup in a finite and the infinite time, Methods Appl. Anal. 8 (2001) 349–367.
[34] Y. Tao, M. Winkler, Boundedness in a quasilinear parabolic–parabolic Keller–Segel system with subcritical sensitivity, J. Differential Equations 252 (2012) 692–715.
[35] Y. Tao, M. Winkler, Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system, Z. Angew. Math. Phys. 66 (2015) 2555–2573.
[36] Y. Tao, M. Winkler, Persistence of mass in a chemotaxis system with logistic source, J. Differential Equations 259 (2015) 6142–6161.
[37] J. Tello, M. Winkler, A chemotaxis system with logistic source, Comm. Partial Differential Equations 32 (2007) 849–877.
[38] L. Wang, Y. Li, C. Mu, Boundedness in a parabolic–parabolic quasilinear chemotaxis system with logistic source, Discrete Contin. Dyn. Syst. 34 (2014) 789–802.
[39] Z. Wang, T. Xiang, A class of chemotaxis systems with growth source and nonlinear secretion, arXiv:1510.07204, 2015.
[40] M. Winkler, Chemotaxis with logistic source: very weak global solutions and their boundedness properties, J. Math. Anal. Appl. 348 (2008) 708–729.
[41] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller–Segel model, J. Differential Equations 248 (2010) 2889–2905.
[42] M. Winkler, Boundedness in the higher-dimensional parabolic–parabolic chemotaxis system with logistic source, Comm. Partial Differential Equations 35 (2010) 1516–1537.
[43] M. Winkler, Does a 'volume-filling effect' always prevent chemotactic collapse?, Math. Methods Appl. Sci. 33 (2010) 12–24.
[44] M. Winkler, K. Djie, Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect, Nonlinear Anal. 72 (2010) 1044–1064.
[45] M. Winkler, Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction, J. Math. Anal. Appl. 384 (2011) 261–272.
[46] M. Winkler, Finite-time blow-up in the higher-dimensional parabolic–parabolic Keller–Segel system, J. Math. Pures Appl. 100 (2013) 748–767.
[47] M. Winkler, Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening, J. Differential Equations 257 (2014) 1056–1077.
[48] M. Winkler, How far can chemotactic cross-diffusion enforce exceeding carrying capacities?, J. Nonlinear Sci. 24 (2014) 809–855.
[49] M. Winkler, Emergence of large population densities despite logistic growth restrictions in fully parabolic chemotaxis systems, Discrete Contin. Dyn. Syst. Ser. B 22 (2017) 2777–2793.
[50] T. Xiang, Boundedness and global existence in the higher-dimensional parabolic–parabolic chemotaxis system with/without growth source, J. Differential Equations 258 (2015) 4275–4323.
[51] T. Xiang, On a class of Keller–Segel chemotaxis systems with cross-diffusion, J. Differential Equations 259 (2015) 4273–4326.
[52] C. Yang, X. Cao, Z. Jiang, S. Zheng, Boundedness in a quasilinear fully parabolic Keller–Segel system of higher dimension with logistic source, J. Math. Anal. Appl. 430 (2015) 585–591.
[53] Q. Zhang, Y. Li, Boundedness in a quasilinear fully parabolic Keller–Segel system with logistic source, Z. Angew. Math. Phys. 66 (2015) 2473–2484.
[54] J. Zheng, Boundedness of solutions to a quasilinear parabolic–parabolic Keller–Segel system with a logistic source, J. Math. Anal. Appl. 431 (2015) 867–888.
[55] J. Zheng, Boundedness and global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with nonlinear logistic source, J. Math. Anal. Appl. 450 (2017) 1047–1061.