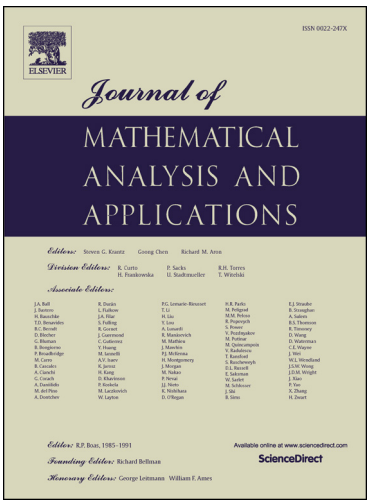
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PII: S0022-247X(18)30301-9
DOI: https://doi.org/10.1016/j.jmaa.2018.04.011
Reference: YJMAA 22165

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 19 July 2017

Please cite this article in press as: J. Hao, X. Chen, Exponential decay of a thermoelastic system for a thin plate under periodic sunlight, *J. Math. Anal. Appl.* (2018), https://doi.org/10.1016/j.jmaa.2018.04.011

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Exponential Decay of A Thermoelastic System for A Thin Plate under Periodic Sunlight *

Jianghao Hao[a†], Xinfu Chen[a,b]

[a]School of Mathematical Science, Shanxi University, Taiyuan, Shanxi, 030006, China
[b]Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA

Abstract

In this paper we investigate a thermoelastic system with Dirichlet or Neumann or natural boundary conditions. For completeness we derive the thermoelastic system model and show the stability results for different boundary conditions which are based on the analysis of the corresponding eigenvalue problems. Further we derive the equilibrium ambient temperature equations that admits a periodic solution modeling the daily heat source from the sun.

Keywords: Thermoelastic plate; Model; Stability

1 Introduction

We consider long time behavior of the dimensionless thermoelastic model

$$\begin{cases} w_{tt} + \Delta^2 w + \alpha\Delta\theta = 0 & \text{in } \Omega\times[0,\infty), \\ \theta_t - k\Delta\theta - \alpha\Delta w_t = -\gamma(t)\theta & \text{in } \Omega\times[0,\infty), \\ w(\cdot,0) = w_0(\cdot),\ w_t(\cdot,0) = w_1(\cdot), \theta(\cdot,0) = \theta_0(\cdot) & \text{in } \Omega, \end{cases} \tag{1.1}$$

subject to different sets of boundary conditions; here w is vertical deflection from the "middle surface" of a thin plate, θ is vertical temperature gradient of thin plate and $\gamma(t) = 4\sigma T^3(t)$, where $T(t)$ is the ambient temperature of the plate. Our model is slightly different from that in [1, 2] so we provide our detailed derivation in the next section. We would like to point out that θ here is not temperance but the temperature gradient in vertical direction, so the coefficient $\gamma(t) = 4\sigma T(t)^3$ arises from the Stefan-Boltzmann radiation Law [3, 4]

$$T_t = S(t) - \sigma T^4,$$

where $S(t)$ is the radiation received from the Sun. We assume that S is periodic and T reaches its dynamical equilibrium (see derivation in Appendix A). Hence, we assume that $\gamma(t)$ is a fixed function satisfying

$$\gamma(t) = \gamma(t+1) > 0 \quad \text{for every } t \geq 0. \tag{1.2}$$

We shall consider the following boundary conditions:

(1) Dirichlet Boundary Condition

$$\theta = w = \Delta w = 0 \quad \text{on } \partial\Omega\times[0,\infty). \tag{1.3}$$

Physically, this means that the boundary of the place is kept to the temperature of the environment and its edge is fixed.

*This work was partially supported by NSF grant DMS-1008905, NNSF of China (61374089) and NSF of Shanxi Province (2014011005-2), TYAL of Shanxi, and 100 talent program of Shanxi.
†Corresponding author.
E-mail: hjhao@sxu.edu.cn, xinfu@pitt.edu

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(2) Neumann Boundary Condition

$$\frac{\partial\theta}{\partial n} = \frac{\partial w}{\partial n} = \frac{\partial \Delta w}{\partial n} = 0 \quad \text{on } \partial\Omega \times [0,\infty). \tag{1.4}$$

Here $\partial w/\partial n$ is the directional derivative normal to $\partial\Omega$. This set of boundary conditions means that the plate is thermally insulated at the boundary and the edge of the plate is free.

Note that if the initial value $[\theta(\cdot,0), w(\cdot,0), w_t(\cdot,0)] = [\theta_0, w_0, w_1]$ is a constant vector, then the solution is given by

$$[\theta(x,t),\ w(x,t)] = [\theta_0 e^{-\int_0^t \gamma(s)ds},\ \ w_0 + w_1 t], \quad \forall\, t \geqslant 0.$$

Hence, by linearity, we assume that under the Neumann boundary condition, the initial values satisfy

$$\int_\Omega \theta_0(x)dx = 0, \quad \int_\Omega w_0(x)dx = 0, \quad \int_\Omega w_1(x)dx = 0. \tag{1.5}$$

(3) Natural Boundary Condition

$$\frac{\partial\theta}{\partial n} = w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega \times [0,\infty). \tag{1.6}$$

We call it natural since it is the physical application in our mind. This corresponds to the thermal insulated boundary with edge clamped.

In recent years the asymptotic properties of thermoelastic systems have been widely investigated. The stability of the associated solution is influenced highly by the structural of the problem. For bounded domain, there are many results in particular on exponential stability; see, [5–15]. For works arising in control theory we refer to [2, 16, 17]. For results for the Cauchy problem or in general exterior domains, see, [13–15, 18–20].

Coleman and Noll [21] developed a rigorous theory of the thermostatics for continuous bodies in arbitrary states of strain, considered stability in minimum-energy sense, in which the energy function involves either the Helmholtz free energy at a given prescribed temperature (isothermal stability) or the internal energy, with the total entropy in the body prescribed (adiabatic stability).

Dafermos [22] considered boundary-initial value problems with rather general homogeneous boundary conditions on bounded domains for classical linear anisotropic thermoelasticity. He proved that solutions to such problems are asymptotically stable as the time t tends to infinity.

Kim in [6] studied the problem (1.1) without the source term $-\gamma(t)\theta$. He showed the exponential decay of the solution. This result was improved in [23] by showing that the problem of the thermoelastic plate are associated with an analytic semigroup. Denk et al. [15] also considered this problem in a bounded or exterior C^4-domain Ω of R^n $(n \geq 2)$. By proving a resolvent estimate they showed that the C_0-semigroup is analytic. In case Ω is bounded they also obtained an exponential stability result.

Fabrizio et al. [24] investigated a thermoelastic plate with inhomgeneous external sources of force and energy, and obtained the existence, uniqueness and asymptotic stability of the solutions as consequence of the dissipation properties of the material.

In monograph [25], Chueshov and Lasiecka gave an extensive study on the von-kármán plate system both in pure elastic and thermoelastic cases. They analyzed the wellposedness and its long-time behavior and the existence of attracting sets.

Recently, Lasiecka et al. [26] studied a quasilinear thermoelastic plate of Kirchhoff-Love type and gave the short-time existence and uniqueness of classical solutions. Furthermore, they used barrier techniques to prove the global existence and exponential stability of solutions under a small condition on the initial data.

Quintanilla and Racke [27] investigated the well-posedness, exponential stability or the lack thereof, of thermoelastic systems in materials where two temperatures are involved related by an elliptic equation. They obtained exponential and nonexponential stabilty, thus providing another examples where the change from Fourier's to Cattaneo's law leads to a loss of exponential stability.

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Note that

$$0 = \int_\Omega \Big\{ 2w_t[w_{tt} + \Delta^2 w - \Delta\theta] + \theta[\theta_t - k\Delta\theta + \Delta w_t + \gamma(t)\theta] \Big\} dx$$

$$= \frac{d}{dt}\int_\Omega \left[w_t^2 + |\Delta w|^2 + \theta^2 \right] dx + 2\int_\Omega \left(k|\nabla\theta|^2 + \gamma(t)\theta^2 \right) dx + J_B,$$

where

$$J_B = 2\int_{\partial\Omega} \left[w_t \frac{\partial \Delta w}{\partial n} - \left(\frac{\partial w}{\partial n}\right)_t \Delta w - w_t \frac{\partial \theta}{\partial n} + \left(\frac{\partial w}{\partial n}\right)_t \theta - k\theta \frac{\partial \theta}{\partial n} \right] ds.$$

Under any set the above boundary conditions (1.3), (1.4), or (1.6), we have $J_B = 0$. Hence, we have energy dissipation

$$\frac{d}{dt}E(t) = -2\int_\Omega \left[k|\nabla\theta|^2 + \gamma(t)\theta^2 \right] dx, \tag{1.7}$$

where

$$E(t) = \int_\Omega \left[w_t^2 + |\Delta w|^2 + \theta^2 \right] dx. \tag{1.8}$$

In theory of linear elasticity, one is concerned about the rate of convergence of the system to its equilibrium state. In this paper we want to show that $(\Delta w, \theta)$ approaches zero exponential fast.

This paper is organized as follows. In the next section, we derive the model for completeness. In Section 3, we consider the system (1.1) under the Dirichlet and Neumann boundary condition respectively. In Section 4 we consider the natural boundary condition. In Appendix A, we derive the equilibrium ambient temperature equation which defines the periodic function $\gamma(t)$.

2 Derivation of the Model

In this section, we follow [1, 2], provide a derivation of our thermoelastic model for thin plate.

2.1 The Equation for Elasticity of Plate

Denote by $\bar{\Omega} \times [-h/2, h/2] \subset \mathbb{R}^3$ the reference domain occupied by the thin plate at equilibrium with $h > 0$ small enough. We denote by $\mathbf{U} = [U^1, U^2, U^3]'$ (prime for transpose) the displacement, i.e., the material particle referenced by (x, z) is located at $(x, z)' + \mathbf{U}(x, z, t)$ at time t. The *strain energy* $\mathbf{E}_s$ and the *kinetic energy* $\mathbf{E}_k$ at time t are defined by

$$\mathbf{E}_s(t) = \frac{1}{2}\int_{-h/2}^{h/2}\int_\Omega \varepsilon_{ij}\sigma_{ij} dx dz, \qquad \mathbf{E}_k(t) = \frac{1}{2}\int_{h/2}^{h/2}\int_\Omega \rho \mathbf{U}_t \cdot \mathbf{U}_t dx dz,$$

where $\varepsilon_{ij}, \sigma_{ij}$ and ρ are *strain tensor, stress tensor, and density*, respectively. We assume the following:

1. *Linear strain-displacement relation:*

$$\varepsilon_{ij} = \frac{1}{2}\Big(U^i_{x_j} + U^j_{x_i} \Big).$$

2. *Isotropic linear stress-strain relation (Hookie's Law):*

$$\sigma_{ij} = \frac{E}{1+\mu}\Big(\varepsilon_{ij} + \frac{\mu\varepsilon_{kk} - \nu p}{1-\mu}\delta_{ij} \Big), \tag{2.1}$$

where E is the Young's modulus (force/area, pulling pressure needed to double the length) and μ the Poisson's ratio (of elongation and transverse shrinkage when stretched), $\frac{\nu p}{1+\mu}$ denotes the thermal strain and p is the thermal pressure.

We remark that the denominator $1 - \mu$ in (2.1) can be viewed as the 2D version of the relation $\varepsilon_{ij} = E^{-1}[(1+\mu)\sigma_{ij} - \mu\sigma_{kk}\delta_{ij}]$; in the classical 3D Hookie's law, the denominator $1 - \mu$ in (2.1) should be $1 - 2\mu$ with physical range $0 < \mu < 1/2$. Here we use $1 - \mu$ instead of $1 - 2\mu$ is for the purpose to match our final equation to that derived in [1, 2].

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3. *No transverse shear:*

$$\mathbf{U}(x,z,t) = \mathbf{u}(x,t) - z\nabla w(x,t), \qquad \mathbf{u} = [u^1, u^2, w]',$$

here $w(x,t)$ is the *vertical deflection* of the "middle surface", referenced by the plane $z = 0$.

4. *Linear thermal expansion:*

$$p = \alpha\, T - \beta_0 \tag{2.2}$$

where $\beta_0 \geqslant 0$ and $\alpha > 0$ are constants and T is the absolute temperature.

Since we are interested in the vertical deflection function w, without loss of generality we assume that $u^1 = u^2 \equiv 0$. Substituting the above relations into the integral for $\mathbf{E}_s$ one finds that

$$\mathbf{E}_s = \frac{D}{2}\int_\Omega \Big\{(\Delta w)^2 + 2(w_{x_1x_2}w_{x_1x_2} - w_{x_1x_1}w_{x_2x_2}) + \nu\alpha\theta\Delta w\Big\}dx,$$

where $D = Eh^3/[12(1-\mu^2)]$ is the *modulus of flexural rigidity* (calorie or joules, energy per unit length per unit curvature of, for example, a bow) and θ is the *average vertical temperature gradient* defined by

$$\theta(x,t) := \frac{12}{h^3}\int_{-h/2}^{h/2} zT(x,z,t)dz = \frac{12}{h^3}\int_{-h/2}^{h/2}\Big[\frac{h^2}{8} - \frac{z^2}{2}\Big]T_z(x,y,t)dz = T_z(x,\hat{z},t).$$

We assume the law of *"Principle of Virtual Work"*, i.e., the first variation of the Lagrangian $\int_0^t[\mathbf{E}_s(\tau)+\mathbf{E}_k(\tau)]d\tau$ with respect to w is zero. This leads to the elastic equation

$$h\,[\rho w_t]_t(x,t) - \tfrac{1}{12}h^3\,\mathrm{div}(\rho\nabla w_t)_t + \Delta[D\Delta w + \alpha\nu\theta] = 0 \qquad \text{in } \ \Omega\times(0,\infty).$$

Neglecting the second term and change of density and scaling appropriately, we then obtain our dimensionless *thermoelastic equation*

$$w_{tt} + \Delta[\Delta w + \alpha\theta] = 0. \tag{2.3}$$

2.2 The Equation for Temperature Gradient

We start from the following *conservation of energy* equation

$$(\rho cT + \eta\,\mathrm{div}\,\mathbf{U})_t - \mathrm{div}(\kappa\nabla T) = S(t)e^{\beta z} - \sigma T^4,$$

where $\mathrm{div} = [\partial_{x_1}, \partial_{x_2}, \partial_z]$, $\nabla T = [T_{x_1}, T_{x_2}, T_z]'$, and

$T(x,z,t)$:	*thermodynamic temperature* at time t and reference point (x,z),
$(\rho cT)_t$:	rate of change of thermal energy, where c is the *specific heat*,
$\mathrm{div}(\kappa\nabla T)$:	thermal diffusion,
$S(t)$:	energy received from the Sun's radiation (at the middle of the plate),
β :	penetration coefficient of sunlight into the depth of plate,
σT^4 :	energy loss due to *Stefan-Boltzmann radiation* of black body,
$\mathrm{div}\mathbf{U}$:	volume expansion,
$(\eta\,\mathrm{div}\mathbf{U})_t$:	energy needed for volume expansion.

Next, similar to that in [2], we assume that the temperature has the expansion

$$T(x,z,t) = T_0(x,t) + z\tilde{\theta}(x,t) + \text{higher order terms},$$

where $T_0(x,t)$ is the temperature at the middle surface and $\tilde{\theta}(x,t)$ is the average vertical temperature gradient. Substituting the above approximation into the energy conservation equation we obtain

$$\begin{aligned}
&(\rho cT_0 + \eta\,\mathrm{div}\,\mathbf{u})_t - \kappa\Delta T_0 = S(t) - \sigma T_0^4, \qquad \text{in } \ \Omega\times(0,\infty),\\
&(\rho c\tilde{\theta} - \eta\Delta w)_t - \kappa\Delta\tilde{\theta} = -4\sigma T_0^3\tilde{\theta} + \beta S(t), \quad \text{in } \ \Omega\times(0,\infty).
\end{aligned}$$

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Neglecting the density change and cross-sectional displacement (i.e., $\rho_t = 0$ and $u^1 = u^2 \equiv 0$), we obtain

$$\rho c T_{0t} - \kappa \Delta T_0 = S(t) - \sigma T_0^4, \qquad \text{in } \Omega \times (t_0, \infty). \tag{2.4}$$

Here $t_0 \ll -1$ is the initial time when the physical apparatus is set. We complete the system by supplying no-flux boundary condition and initial temperature distribution

$$\partial_n T_0 = 0 \ \text{ on } \ \partial\Omega \times [t_0, \infty), \qquad T_0(\cdot, t_0) = T^0(\cdot) \geqslant 0. \tag{2.5}$$

We shall assume that S is a non-negative non-trivial periodic function with period 1:

$$S(t) = S(t+1) \geqslant (\not\equiv) 0, \qquad \forall\, t \in (-\infty, \infty). \tag{2.6}$$

Then once can show (see Appendix A) that

$$\lim_{t_0 \to -\infty} T_0(x,t) = T_*(t),$$

where T_* is the unique periodic positive solution of the ODE

$$\rho c \frac{dT_*(t)}{dt} = S(t) - \sigma T_*^4(t), \qquad T_*(t) = T_*(t+1) > 0, \quad \forall\, t \in (-\infty, \infty).$$

Hence, we replace T_0 by $T_*(t)$. Finally, we set $\theta(x,t) = \tilde{\theta}(x,t) - \theta_0(t)$ where $\theta_0(t)$ is the periodic solution of

$$\rho c \theta_0'(t) = -4\sigma T_*^3(t)\theta_0(t) + \beta S(t), \qquad \forall\, t \in (-\infty, \infty).$$

Then we obtain, after suitable scaling, the dimensionless *thermal gradient equation*

$$\theta_t - k\Delta\theta = \alpha \Delta w_t - \gamma(t)\theta, \qquad \text{in } \Omega \times (0, \infty), \tag{2.7}$$

where $\gamma(t) = 4\sigma T_*^3(t)$ is a known positive periodic function:

$$\gamma(t) = \gamma(t+1) > 0, \quad \forall\, t \in (-\infty, \infty). \tag{2.8}$$

Here we can make the dimensionless parameter α the same as that in (2.3) since we can scale either θ or w; we use the same letter α as that in (2.2) (in which α has unit pascal/Kevin=Newton/meter^2/Kevin) is for the purpose to indicate that the *thermal effect to elasticity is realized by means of thermal expansion.*

We remark that in our intension of application, the apparatus is in space orbit, regarded as a near vacuum environment. If the apparatus is in atmosphere, then by Newton's law of cooling, a positive constant should be added to $\gamma(t)$, to account for the energy loss/gain due to contact of apparatus' surface to the cold/warm environment.

Remark 2.1. *For the Dirichlet boundary conditions* (1.3) *and the Neumann boundary conditions* (1.4) *we shall investigate our thermoelastic system in a two-dimensional setting, i.e., Ω is a two dimensional bounded domain.*

For the natural boundary conditions (1.6) *we need to solve an eigenvalue problem. In order to solve the eigenvalue problem, we focus our attention to a one-dimensional setting $\Omega = (-\ell, \ell)$; this can be deduced by assuming that the cross section of the thin plate is a two dimensional infinite long slab, $\{(x,y) \mid x \in (-\ell, \ell), y \in (-\infty, \infty)\}$ or some rectangular shape with special initial and boundary condition so that the solution does not depend on y. Then we can treat Ω as if it were a one dimensional interval.*

3 The Dirichlet and Neumman Boundary Conditions

In this section, we consider (1.1) under the Dirichlet Boundary Condition (1.3) or the Neumann Boundary Condition (1.4) [together with (1.5)]. We shall show the following:

Theorem 3.1. *Assume that k and α are positive constants and $\gamma(t)$ is a continuous and non-negative period function. Then there exists positive constants M and ν that depends only on α, k, Ω, and $\|\gamma\|_{L^\infty((0,\infty))}$ such that any solution of* (1.1) *subject to either the Dirichlet boundary condition* (1.3) *or the Neumann boundary condition* (1.4) *and the normalization condition* (1.5) *satisfies*

$$E(t) \leq ME(0)e^{-\nu t}, \qquad \forall t > 0,$$

where $E(t)$ is the energy defined in (1.8).

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3.1 Conversion to Parabolic System

We consider the vector valued function

$$\mathbf{U} = \begin{pmatrix} \Delta w \\ w_t \\ \theta \end{pmatrix}.$$

Then (1.1) is equivalent to

$$\mathbf{U}_t = A\Delta\mathbf{U} - \gamma(t)B\mathbf{U}, \qquad \text{in } \Omega \times (0,\infty), \tag{3.1}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -\alpha \\ 0 & \alpha & k \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The Dirichlet boundary condition (1.3) is equivalent to

$$U = \mathbf{0}, \qquad \text{on } \partial\Omega \times [0,\infty). \tag{3.2}$$

The Neumman boundary condition (1.4) is equivalent to

$$\frac{\partial \mathbf{U}}{\partial n} = \mathbf{0}, \qquad \text{on } \partial\Omega \times [0,\infty). \tag{3.3}$$

The energy estimate (1.7) can be derived equivalently by, denoting by $\mathbf{U}^T$ the transpose of $\mathbf{U}$,

$$\begin{aligned} \frac{d}{dt}\int_\Omega |\mathbf{U}|^2 dx &= \int_\Omega (\mathbf{U}_t^T\mathbf{U} + \mathbf{U}^T\mathbf{U}_t)dx \\ &= \int_\Omega (\Delta\mathbf{U}^T A^T\mathbf{U} + \mathbf{U}^T A\Delta\mathbf{U} - \gamma\mathbf{U}^T(B+B^T)\mathbf{U})dx \\ &= \int_\Omega \left[-\nabla\mathbf{U}^T(A+A^T)\nabla\mathbf{U} - \gamma(t)\mathbf{U}^T(B^T+B)\mathbf{U}\right]dx \\ &= -2\int_\Omega (k|\nabla\theta|^2 + \gamma(t)\theta^2)dx. \end{aligned} \tag{3.4}$$

In the sequel, we shall derive

$$\frac{d}{dt}\widetilde{E}(t) \le -\nu\widetilde{E}(t),$$

for some constant $\nu > 0$ and appropriately defined $\widetilde{E}(t)$ that is equivalent to $E(t)$.

3.2 Eigenvalues of the Matrix A

We investigate the eigenvalues of the matrix A.

(i) Since $\alpha > 0$, for each eigenvalue λ of A, its eigenspace is one dimensional, expanded by

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \lambda\alpha \\ -1-\lambda^2 \end{bmatrix}.$$

(ii) The characteristic polynomial of A is

$$p(\lambda) = \det(\lambda\mathbf{I} - A) = \lambda^3 - k\lambda^2 + (1+\alpha^2)\lambda - k.$$

Note that $\lambda = \pm\mathbf{i}$ is not an eigenvalue, so $|v_3| = |1+\lambda^2| > 0$. Consequently, if $(\lambda, \mathbf{v})$ is an eigenpair, denoting by $(\bar\lambda, \bar{\mathbf{v}})$ the complex conjugate of $(\lambda, \mathbf{v})$, we have

$$(\lambda + \bar\lambda)|\mathbf{v}|^2 = (A\mathbf{v})^T\bar{\mathbf{v}} + \mathbf{v}^T(A\bar{\mathbf{v}}) = \mathbf{v}^T(A+A^T)\bar{\mathbf{v}} = 2k|v_3|^2 > 0.$$

Thus, $\mathrm{Re}(\lambda) > 0$. Namely, the real part of each eigenvalue of A is positive.

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(iii) The characteristic equation $p(\lambda) = 0$ can be written as

$$k = f(\alpha, \lambda) := \lambda + \frac{\alpha^2 \lambda}{1 + \lambda^2}.$$

The roots of $f_\lambda(\alpha, \lambda) = 0$ are $\lambda = \lambda_\pm(\alpha) := [\alpha^2 - 2 \pm \alpha\sqrt{\alpha^2 - 8}]/2$. Hence, the following holds:

(a) If $0 < \alpha < \sqrt{8}$ and $k > 0$ or $a \geqslant \sqrt{8}$ and $k \in (0, f(\alpha, \lambda_+(\alpha))) \cup (f(\lambda_-(\alpha), \alpha), \infty)$, then there are two complex eigenvalues;

(b) If $\alpha = \sqrt{8}$ and $k = f(\lambda_\pm(\alpha), \alpha) = 3\sqrt{3}$, then $\lambda = \sqrt{3}$ is an eigenvalue of multiplicity 3;

(c) If $\alpha > \sqrt{8}$ and $k = f(\alpha, \lambda_+(\alpha))$ or $k = f(\alpha, \lambda_-(\alpha))$, then there are exactly two real eigenvalues;

(d) If $\alpha > \sqrt{8}$ and $f(\alpha, \lambda_+(\alpha)) < k < f(\alpha, \lambda_-(\alpha))$, then there are three distinct real eigenvalues.

We write $A = P^{-1}\Lambda P$ where P is a non-singular real matrix and Λ is the Jordan canonical form of A. Depending on α and k, we derive from the above observation that there are the following situations:

1. A has three distinct positive eigenvalues:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \Big(\lambda_3 > \lambda_2 > \lambda_1 > 0\Big).$$

2. A has only two real positive eigenvalues:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & \lambda_2 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \quad \Big(\lambda_3 = \lambda_2 > 0, \lambda_1 > 0\Big).$$

3. A has one positive eigenvalue of multiplicity three:

$$\Lambda = \begin{pmatrix} \nu_1 & \frac{1}{2}\nu_1 & 0 \\ 0 & \nu_1 & \frac{1}{2}\nu_1 \\ 0 & 0 & \nu_1 \end{pmatrix}, \quad \Big(\nu_3 = \nu_2 = \nu_1 = \sqrt{3}\Big).$$

4. A has one real eigenvalue and two complex eigenvalues:

$$\Lambda = \begin{pmatrix} \nu_1 & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{pmatrix}, \quad \Big(\nu_1 > 0, a = \mathrm{Re}(\lambda_2) > 0, b = \mathrm{Im}\,(\lambda_2) > 0\Big).$$

In all these cases,

$$\rho := \min\{\nu_1, \mathrm{Re}(\nu_2), \mathrm{Re}(\nu_3)\} > 0,$$

$$\xi^T \Lambda \xi \geqslant \frac{1}{2}\rho|\xi|^2 \qquad \forall\, \xi \in \mathbb{R}^3.$$

Now we work on the function

$$\mathbf{V} := P\mathbf{U}.$$

Then multiplying $\mathbf{U}_t = A\Delta\mathbf{U} - \gamma(t)B\mathbf{U}$ by P we obtain

$$\mathbf{V}_t = \Lambda\Delta\mathbf{V} - \gamma(t)PB\mathbf{U}. \tag{3.5}$$

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3.3 The Dirichlet Boundary Condition

First we consider the Dirichlet boundary condition (1.3).

Note that

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\mathbf{V}|^2dx &= \int_\Omega 2\mathbf{V}^T\mathbf{V}_t dx = 2\int_\Omega \Big[\mathbf{V}^T\Lambda\Delta\mathbf{V} - \gamma(t)\mathbf{V}^TPB\mathbf{U}\Big]dx\\ &= -2\int_\Omega \Big[\nabla\mathbf{V}^T\Lambda\nabla\mathbf{V} + \gamma(t)\mathbf{V}^TPB\mathbf{U}\Big]dx\\ &\le -\rho\int_\Omega |\nabla\mathbf{V}|^2dx + 2\gamma(t)\Big(\int_\Omega |\mathbf{V}|^2dx\Big)^{1/2}\|P\|\Big(\int_\Omega\theta^2dx\Big)^{1/2}.\end{aligned}$$

Since $\mathbf{V}|_{\partial\Omega} = 0$, we have Poincaré inequality, for some positive constant $C_0 = C_0(\Omega)$,

$$\int_\Omega |\nabla\mathbf{V}|^2dx \geqslant C_0\int_\Omega |\mathbf{V}|^2dx. \tag{3.6}$$

Hence,

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\mathbf{V}|^2dx &\le -\rho C_0\int_\Omega |\mathbf{V}|^2dx + 2\gamma(t)\Big(\int_\Omega |\mathbf{V}|^2dx\Big)^{1/2}\|P\|\Big(\int_\Omega\theta^2dx\Big)^{1/2}\\ &\le -\frac{\rho C_0}{2}\int_\Omega |\mathbf{V}|^2dx + \frac{2\gamma^2(t)\|P\|^2}{C_0\rho}\int_\Omega\theta^2dx.\end{aligned}$$

We introduce positive constants

$$\delta := \frac{C_0\rho}{\|P\|^2\|\gamma\|_{L^\infty}}, \qquad\qquad \nu := \frac{\rho C_0\delta}{2[\|P^{-1}\|^2+\delta]}.$$

Then,

$$\begin{aligned}\frac{d}{dt}\int_\Omega \big(|\mathbf{U}|^2+\delta|\mathbf{V}|^2\big)\,dx &\le -\int_\Omega\Big(2k|\nabla\theta|^2 + \frac{\rho C_0\delta}{2}|\mathbf{V}|^2\Big)dx - 2\gamma\Big[1-\frac{\delta\|P\|^2\gamma}{C_0\rho}\Big]\int_\Omega\theta^2dx\\ &\leqslant -\frac{\rho C_0\delta}{2}\int_\Omega|\mathbf{V}|^2dx\\ &\le -\frac{\rho C_0\delta}{2}\int_\Omega\frac{|\mathbf{U}|^2+\delta|\mathbf{V}|^2}{\|P^{-1}\|^2+\delta}dx = -\nu\int_\Omega\Big(|\mathbf{U}|^2+\delta|\mathbf{V}|^2\Big)dx,\end{aligned}$$

where we have used $|\mathbf{U}| = |P^{-1}\mathbf{V}| \leqslant \|P^{-1}\|\,|\mathbf{V}|$. It then follows from Gronwall's inequality that

$$\begin{aligned}E(t) = \int_\Omega|\mathbf{U}|^2dx &\leqslant \int_\Omega\big(|\mathbf{U}|^2+\delta|\mathbf{V}|^2\big)\,dx\\ &\leqslant e^{-\nu t}\int_\Omega\big(|\mathbf{U}(\cdot,0)|^2+\delta|\mathbf{V}(\cdot,0)|^2\big)\,dx \leqslant [1+\delta\|P\|^2]E(0)e^{-\nu t}.\end{aligned}$$

The assertion of Theorem 3.1 for Dirichlet boundary condition thus follows.

3.4 The Neumann Boundary Condition

Next we consider the Neumann boundary condition (1.4). Then

$$\begin{aligned}\int_\Omega\Delta w dx &= \int_{\partial\Omega}\frac{\partial w}{\partial n}ds = 0,\\ \frac{d}{dt}\int_\Omega w_t(x,t)dx &= -\int_\Omega\Delta(\Delta w+\alpha\theta)dx = -\int_{\partial\Omega}\Big\{\frac{\partial\Delta w}{\partial n}+\alpha\frac{\partial\theta}{\partial n}\Big\}ds = 0,\\ \frac{d}{dt}\int_\Omega\theta(x,t)dx &= -\gamma(t)\int_\Omega\theta dx + \int_{\partial\Omega}\Big(k\frac{\partial\theta}{\partial n}+\alpha\frac{\partial w_t}{\partial n}\Big)ds = -\gamma(t)\int_\Omega\theta(x,t)dx.\end{aligned}$$

In view of (1.5), we find that

$$\int_\Omega w_t(x,t)dx = 0, \quad \int_\Omega\theta(x,t)dx = 0, \quad t\geqslant 0.$$

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Thus, we have

$$\int_\Omega \mathbf{U}(x,t)dx = 0, \qquad \int_\Omega \mathbf{V}(x,t)dx = P\int_\Omega \mathbf{U}(x,t)dx = 0, \quad t \geqslant 0.$$

Using Poincaré inequality again, we know that (3.6) still holds. Same proof as above we obtain the exponential decay of the energy E. This completes the proof of Theorem 3.1.

3.5 The Regularizing Property

The system (3.5) is a parabolic system, so it should have the regularizing property; i.e., the solution immediately becomes smooth. Here we prove such a result.

We denote by $\{\lambda_i, \varphi_i(x)\}_{i=1}^\infty$ the complete set of eigenpairs of $-\Delta$ subject to the Dirichlet (for (1.3)) or Neumann (for 1.4) boundary condition e.g.

$$\begin{cases} -\Delta\varphi_i = \lambda_i\varphi_i & \text{in } \Omega, \\ \varphi_i = 0 \text{ or } \dfrac{\partial\varphi_i}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$
$$\int_\Omega \varphi_i^2(x)dx = 1, \quad 0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots.$$

By separation of variables the solution can be written as

$$\mathbf{U}(x,t) = \sum_{i=1}^\infty \mathbf{U}_i(t)\varphi_i(x),$$

where $\mathbf{U}_i(t) = [u_{i1}\ u_{i2}\ u_{i3}]^T$ is the solution of

$$\begin{cases} \mathbf{U}_i'(t) = -\lambda_i A\mathbf{U}_i - \gamma(t)B\mathbf{U}_i, \\ \mathbf{U}_i(0) = \mathbf{C}_i := \int_\Omega \varphi_i \mathbf{U}(\cdot,0)dx. \end{cases}$$

We shall study the decay rate of $\mathbf{U}_i$. To study the evolution of $\mathbf{U}_i$, it suffices to study the solution $\mathbf{u}_i(z,\cdot)$ of the initial value problem

$$\begin{cases} \mathbf{u}_i{}' = -\lambda_i A\mathbf{u}_i - \gamma(t)B\mathbf{u}_i, \quad t > 0, \\ \mathbf{u}_i(z,0) = z \in \mathbb{R}^3. \end{cases}$$

We use the change of variable $\mathbf{v}_i = P\mathbf{u}_i$. Then

$$\begin{cases} \mathbf{v}_i' = \lambda_i\Lambda\mathbf{v}_i - \gamma(t)PB\mathbf{u}_i, \\ \mathbf{v}(z,0) = Pz. \end{cases}$$

Let

$$\delta := \frac{\lambda_1\rho}{\|\gamma\|_{L^\infty}\|P\|^2}, \qquad \nu := \frac{\lambda_1\rho\delta}{2[\|P^{-1}\| + \delta]}.$$

Then using $B = B^T = B^2$ we have

$$\begin{aligned} \frac{d}{dt}\Big(|\mathbf{u}_i|^2 + \delta|\mathbf{v}_i|^2\Big) &= -\lambda_i\mathbf{u}_i^T(A + A^T)\mathbf{u}_i - 2\gamma\mathbf{u}_i^T B^T B\mathbf{u}_i + \delta\lambda_i\mathbf{v}_i^T(\Lambda + \Lambda^T)\mathbf{v}_i - 2\delta\gamma\mathbf{v}_i^T PB\mathbf{u}_i \\ &\leqslant -2\gamma(t)|B\mathbf{u}_i|^2 - \delta\rho\lambda_i|\mathbf{v}_i|^2 + 2\delta\gamma(t)|\mathbf{v}_i|\ \|P\|\ |B\mathbf{u}_i| \\ &\leqslant -2\gamma(t)|B\mathbf{u}_i|^2 - \delta\rho\lambda_i|\mathbf{v}_i|^2 + \delta\Big(\frac{\rho\lambda_i}{2}|\mathbf{v}_i|^2 + \frac{2\gamma^2(t)\|P\|^2}{\rho\lambda_i}|B\mathbf{u}_i|^2\Big) \\ &\leqslant -\frac{\delta\rho\lambda_i}{2}|\mathbf{v}|_i^2 \\ &\leqslant -\frac{\delta\rho\lambda_i}{2}\frac{|\mathbf{u}_i|^2 + \delta|\mathbf{v}_i|^2}{\|P^{-1}\|^2 + \delta} \\ &\leqslant -\nu\lambda_i(|\mathbf{u}_i|^2 + \delta|\mathbf{v}_i|^2). \end{aligned}$$

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Thus,

$$\begin{aligned} |\mathbf{u}_i(z,t)|^2 &\leqslant |\mathbf{u}_i|^2 + \delta|\mathbf{v}_i|^2 \\ &\leqslant (|\mathbf{u}_i(z,0)|^2 + \delta|\mathbf{v}(z,0)|^2)e^{-\nu\lambda t} \\ &\leqslant |z|^2[1+\|P^{-1}\|^2]\, e^{-\nu\lambda_i t}. \end{aligned}$$

Note that $\mathbf{U}_i(t) = \mathbf{u}_i(\mathbf{U}_i(0), t)$ we find that

$$|\mathbf{U}_i(t)|^2 \leqslant |\mathbf{U}_i(0)|^2[1+\|P^{-1}\|^2]\, e^{-\nu\lambda_i t}. \tag{3.7}$$

Now for every integer $m \geqslant 0$,

$$\begin{aligned} \Big\|(-\Delta)^m \mathbf{U}(\cdot,t)\Big\|^2_{L^2(\Omega)} &= \Big\|\sum_{i=1}^{\infty} \lambda_i^m \mathbf{U}_i(t)\varphi_i(x)\Big\|^2_{L^2(\Omega)} \\ &= \sum_{i=1}^{\infty} \lambda_i^{2m}|\mathbf{U}_i(t)|^2 \\ &\leqslant (1+\|P^{-1}\|^2)\sum_{i=1}^{\infty}\lambda_i^{2m}e^{-\lambda_i\nu t}\|\mathbf{U}_i(0)\|^2 \\ &\leqslant \frac{1+\|P^{-1}\|^2}{(\nu t)^{2m}}\Big(\sup_{z>0} z^{2m}e^{-\nu z}\Big)\sum_{i=1}^{\infty}\|\mathbf{U}_i(0)\|^2 \\ &= \frac{1+\|P^{-1}\|^2}{(\nu t)^{2m}}\Big(\sup_{z>0} z^{2m}e^{-\nu z}\Big)\|\mathbf{U}(\cdot,0)\|^2_{L^2(\Omega)} < \infty. \end{aligned}$$

Now we consider the boundary condition of $(-\Delta)^m\mathbf{U}(\cdot,t)$ on $\partial\Omega$.

(i) Suppose we have the Dirichlet boundary condition $\mathbf{U} = 0$ on $\partial\Omega \times (0,\infty)$. Differentiating the equation

$$(-\Delta)\mathbf{U} = -A^{-1}\mathbf{U}_t - \gamma(t)A^{-1}B\mathbf{U},$$

we obtain

$$(-\Delta)^{m+1}\mathbf{U} = -A^{-1}[(-\Delta)^m\mathbf{U}]_t - \gamma(t)A^{-1}B(-\Delta)^m\mathbf{U}.$$

By induction, if we have $(-\Delta)^m\mathbf{U} = 0$ on $\partial\Omega\times(0,\infty)$, then we have $(-\Delta)^{m+1}\mathbf{U} = 0$ on $\partial\Omega\times(0,\infty)$. Thus, by elliptic estimates, assuming $\partial\Omega \in C^\infty$, we have $\mathbf{U}(\cdot,t) \in H^{2m}(\Omega)$ for every integer m. That implies that $\mathbf{U}(\cdot,t) \in C^\infty(\bar\Omega)$ for every $t > 0$.

(ii) The Neumann boundary condition can be similarly considered.

In summary, we have the following:

Theorem 3.2. *Let (w,θ) be the solution of (1.1) subject to either Dirichlet boundary condition (1.3) or Neumann boundary condition (1.4). Assume that the initial data $(\theta_0, w_0, w_1) \in L^2(\Omega)$ and $\partial\Omega \in C^\infty$. Then for every $t > 0$, $[\theta(\cdot,t), w(\cdot,t), w_t(\cdot,t)] \in C^\infty(\bar\Omega)$ (assuming $\partial\Omega \in C^\infty$).*

Example. In the one space dimension with $\Omega = (0,\ell)$, under the Neumann boundary condition, the solution is given by

$$\begin{bmatrix} w_{xx}(x,t) \\ w_t(x,t) \\ \theta(x,t) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{|\Omega|}\int w_1(y)dy \\ \frac{e^{-\int_0^t\gamma(s)ds}}{|\Omega|}\int_\Omega \theta_0(y)dy \end{bmatrix} + \sum_{i=1}^{\infty}\mathbf{U}_i(t)\cos\Big(\frac{i\pi x}{\ell}\Big),$$

$$(-1)^m\frac{\partial^{2m}\mathbf{U}(x,t)}{\partial x^{2m}} = \sum_{i=1}^{\infty}\mathbf{U}_i(t)\left[\frac{i\pi}{\ell}\right]^{2m}\cos\left(\frac{i\pi x}{l}\right) \in L^2(\Omega) \quad \forall\, m \in \mathbb{N}.$$

Hence, in view of (3.7), we find that $\mathbf{U}(\cdot,t) \in C^\infty(\overline{\Omega})$ for every $t > 0$.

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4 Exponential Decay under Natural Boundary Condition

In this section, we consider the problem (1.1) with the Natural boundary condition (1.6).

As we shall see below, we need to solve an eigenvalue problem. For simplicity, we shall assume that the cross section of our thin place is an infinitely long slab $(-\ell, \ell) \times (-\infty, \infty)$, or some other rectangular shape with special boundary condition so that the solution does not depend on y. Thus, we assume that

$$\Omega = (-\ell, \ell) \in \mathbb{R} \qquad (\ell > 0).$$

Set

$$\Phi(x,t) := \begin{bmatrix} w \\ w_t \\ \theta \end{bmatrix}, \qquad \mathcal{B}\Phi := \begin{bmatrix} w \\ \partial_n w_t \\ \partial_n \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \partial_n & 0 \\ 0 & 0 & \partial_n \end{bmatrix} \Phi.$$

Then the system (1.1), (1.6) can be written, in vector form, as

$$\begin{cases} \Phi_t = \begin{bmatrix} 0 & 1 & 0 \\ -\Delta^2 & 0 & -\alpha\Delta \\ 0 & \alpha\Delta & k\Delta - \gamma \end{bmatrix} \Phi & \text{in } \Omega, \\ \mathcal{B}\Phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Without loss of generality, we assume that

$$\int_\Omega \theta(x,t)dx = 0, \quad t \geqslant 0.$$

4.1 The Case $\gamma \equiv 0$

We first consider the case $\gamma \equiv 0$. Using separation of variables, we search for solutions of the form

$$\Phi(x,t) = \Psi(x)e^{\lambda t}.$$

This leads to the eigenproblem, for (λ, Ψ):

$$\begin{cases} \left(\lambda I - \begin{bmatrix} 0 & 1 & 0 \\ -\Delta^2 & 0 & -\alpha\Delta \\ 0 & \alpha\Delta & k\Delta \end{bmatrix}\right) \Psi = 0 & \text{in } \Omega, \\ B\Psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Componentwise, with $\Psi = [\psi_1, \psi_2, \psi_3]$,

$$\psi_2 = \lambda\psi_1, \qquad \Delta^2\psi_1 + \lambda\psi_2 + \alpha\Delta\psi_3 = 0, \quad \lambda\psi_3 = \Delta(\alpha\psi_2 + k\psi_3). \tag{4.2}$$

If $\lambda = 0$, the solution is given by $\psi_1 \equiv 0, \psi_2 \equiv 0$, and ψ_3 =constant. For simplicity, we focus our attention to the case $\int_\Omega \theta(x,t)dx = 0$ for all t. Then $\lambda = 0$ is not an eigenvalue.

In the sequel, we consider $\lambda \neq 0$. Then we know that the original system is stable, so $\mathrm{Re}(\lambda) < 0$. To solve the eigenvalue system, we introduce an auxiliary function

$$S := \frac{\alpha\psi_2 + k\psi_3}{\alpha\lambda^2}.$$

Then the third equation in (4.2) gives $\psi_3 = \alpha\lambda\Delta S$ and $\psi_2 = (\alpha\lambda^2 S - k\psi_3)/\alpha = \lambda^2 S - k\lambda\Delta S$; together with the first two equations in (4.2) we find

$$\begin{cases} \psi_1 = \lambda S - k\Delta S, \\ \psi_2 = \lambda[\lambda S - k\Delta S], \\ \psi_3 = \alpha\lambda\Delta S, \\ \Big\{[\Delta^2 + \lambda^2][\lambda \mathbf{I} - k\Delta] + \alpha^2\lambda\Delta^2\Big\} S = 0. \end{cases} \tag{4.3}$$

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It is then easy to show that (4.2) and (4.3) are equivalent. Together with the boundary condition for Ψ, the system (4.1) is equivalent to the sixth-order elliptic problem for the scalar function S:

$$\begin{cases} \Big(k\Delta^3 - [\alpha^2+1]\lambda\Delta^2 + k\lambda^2\Delta - \lambda^3\mathbf{I}\Big)S = 0 & \text{in } \Omega, \\ \frac{\partial\Delta S}{\partial n} = 0, \quad \frac{\partial S}{\partial n} = 0, \quad (\lambda\mathbf{I} - k\Delta)S = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

We decompose the characteristic polynomial of A by

$$p(\lambda) := \det(\lambda\mathbf{I} - A) = \lambda^3 - k\lambda^2 + (1+\alpha^2)\lambda - k = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3),$$

where $(\lambda_1, \lambda_2, \lambda_3)$ are eigenvalues of A. Note that

$$\lambda_1 > 0, \quad \mathrm{Re}(\lambda_i) > 0, \quad \lambda_1\lambda_2\lambda_3 = k > 0, \qquad \lambda_1 + \lambda_2 + \lambda_3 = k.$$

Then the differential equation for S in Ω can be written as

$$\big(\lambda\mathbf{I} - \lambda_1\Delta\big)\big(\lambda\mathbf{I} - \lambda_2\Delta\big)\big(\lambda\mathbf{I} - \lambda_3\Delta\big)S = 0 \qquad \text{in } \Omega. \tag{4.5}$$

For the moment, we do not know how to deal with the eigenproblem (4.4) for general domain Ω. Hence, in the sequel, we consider a special one dimensional case

$$\Omega = (-\ell, \ell).$$

Suppose $S(x)$ is a solution of (4.4). Define

$$S_{\mathrm{even}}(x) = \frac{1}{2}\Big(S(x) + S(-x)\Big), \qquad S_{\mathrm{odd}}(x) = \frac{1}{2}\Big(S(x) - S(-x)\Big).$$

One can verify that both S_{even} and $S_{\mathrm{odd}}(x)$ are solutions of (4.4). On the other hand, since

$$S(x) = S_{\mathrm{even}}(x) + S_{\mathrm{odd}}(x),$$

we conclude that every solution of (4.4) can be decomposed as an even solution and old solution. Thus, we study the problem in function class of even and old functions. This reduction allows us to study only the problem on the half interval $(0, \ell)$ in the class of even and odd functions separately.

First we study even solutions of (4.4). The equation (4.5) can be written as

$$\left(\frac{d^2}{dx^2} - \frac{\lambda}{\lambda_1}\right)\left(\frac{d^2}{dx^2} - \frac{\lambda}{\lambda_2}\right)\left(\frac{d^2}{dx^2} - \frac{\lambda}{\lambda_3}\right)S = 0. \tag{4.6}$$

For simplicity of presentation, we assume that A has three distinct eigenvalues. The general even solution of (4.6) is

$$S = \sum_{i=1}^{3} C_i \cos\left(\sqrt{-\frac{\lambda}{\lambda_i}}\, x\right).$$

To continue, we use the convention for non-zero complex value and its square root by

$$\zeta = Re^{\mathbf{i}\theta}, \quad R > 0, \quad -\pi < \theta \leqslant \pi, \quad \mathrm{Arg}(\zeta) := \theta, \quad \sqrt{\zeta} = \sqrt{R}e^{\frac{\theta}{2}i}.$$

Set

$$z = \sqrt{-\lambda}\,\ell, \qquad \mathrm{Re}(z) > 0, \qquad \mu_i = \sqrt{\frac{1}{\lambda_i}}.$$

Then

$$\begin{aligned} S(x) &= \sum_{i=1}^{3} C_i \cos\Big(\frac{z\mu_i x}{\ell}\Big), \\ S'(\ell) &= -\frac{z}{\ell}\sum_{i=1}^{3} C_i\mu_i \sin(\mu_i z), \\ S'''(\ell) &= \frac{z^3}{\ell^3}\sum_{i=1}^{3} C_i\mu_i^3 \sin(\mu_i z), \\ \lambda S(\ell) - kS''(\ell) &= -\frac{z^2}{\ell^2}\sum_{i=1}^{3} C_i[1 - k\mu_i^2]\cos(\mu_i z). \end{aligned}$$

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Hence λ is an eigenvalue if and only if $\lambda = -z^2/\ell^2$ and z with $\mathrm{Re}(z) > 0$ is the root of

$$Q(z) = 0,$$

where $Q(\cdot)$ is an entire even function defined by

$$Q(z) = \det \begin{bmatrix} \mu_1 \sin(\mu_1 z) & \mu_2 \sinh(\mu_2 z) & \mu_3 \sin(\mu_3 z) \\ \mu_1^3 \sin(\mu_1 z) & \mu_2^3 \sin(\mu_2 z) & \mu_3^3 \sin(\mu_3 z) \\ (1 - k\mu_1^2)\cos(\mu_1 z) & (1 - k\mu_2^2)\cos(\mu_2 z) & (1 - k\mu_3^2)\cos(\mu_3 z) \end{bmatrix}.$$

We consider two cases.

Case 1. $\sin(\mu_1 z)\sin(\mu_2 z)\sin(\mu_3 z) = 0$. Then, for some $j = 1, 2$, or 3 and integer $n \geqslant 1$,

$$\mu_j z = n\pi \quad \Rightarrow -\lambda = \frac{z^2}{\ell^2} = \frac{(n\pi)^2}{\mu_j^2 \ell^2} = \lambda_j \frac{(n\pi)^2}{\ell^2}.$$

Thus, except $\lambda = 0$, we have

$$\mathrm{Arg}(-\lambda) \in [-\theta^*, \theta^*], \qquad \theta^* := \mathrm{Arg}(\lambda_2) = -\mathrm{Arg}(\lambda_3).$$

Case2. $\sin(\mu_2 z)\sin(\mu_2 z)\sin(\mu_3 z) \neq 0$. Then the roots of $Q(z)$ is the same as the roots of $\tilde{Q}$ defined by

$$\begin{aligned} \tilde{Q}(z) &= \frac{Q(z)}{\sin(\mu_2 z)\sin(\mu_2 z)\sin(\mu_3 z)} \\ &= \det \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 \\ \mu_1^3 & \mu_2^3 & \mu_3^3 \\ (1 - k\mu_1^2)\cot(\mu_1 z) & (1 - k\mu_2^2)\cot(\mu_2 z) & (1 - k\mu_3^2)\cot(\mu_3 z) \end{bmatrix} \end{aligned}$$

Since we know the dynamical system is stable, so $\mathrm{Re}(-\lambda) > 0$ and $|\mathrm{Arg}(z)| < \frac{\pi}{4}$. Thus, we denote

$$-\lambda = Re^{\theta \mathbf{i}}, \quad \lambda_j = R_j e^{\mathbf{i}\theta_j}, \quad z = \sqrt{-\lambda}\ell = \sqrt{R}\ell e^{\mathbf{i}\theta/2}, \qquad \mu_j = \sqrt{\frac{1}{\lambda}_j} = \frac{e^{-\mathbf{i}\theta_j/2}}{\sqrt{R_j}}.$$

Then

$$\mu_i z = \sqrt{\frac{R\ell^2}{R_j}}\, e^{\mathbf{i}(\theta - \theta_i)/2} =: x + \mathbf{i}\, y$$

where $x > 0$ since $|\theta| < \pi/2$ and $|\theta_i| < \pi/2$. When $y > 0$, we have

$$\begin{aligned} \cot(x + \mathbf{i}y) + \mathbf{i} &= \frac{\mathbf{i}[e^{\mathbf{i}(x+\mathbf{i}y)} + e^{-\mathbf{i}(x+\mathbf{i}y)}]}{e^{\mathbf{i}(x+\mathbf{i}y)} - e^{-\mathbf{i}(x+\mathbf{i}y)}} + \mathbf{i} \\ &= \frac{2\mathbf{i}e^{-y+\mathbf{i}x}}{e^{-y+\mathbf{i}x} - e^{y-\mathbf{i}x}} = \frac{2e^{-2y+2\mathbf{i}x}}{\mathbf{i}[1 - e^{-2y+2\mathbf{i}x}]}. \end{aligned}$$

Thus, when $y > 0$,

$$\left|\cot(x + \mathbf{i}y) + \mathbf{i}\right| \leqslant \sqrt{\frac{2e^{-2y}}{(1 - e^{-2y})^2 + e^{-2y}\sin^2 x}} \leqslant \frac{\sqrt{2}e^{-y}}{1 - e^{-2y}}.$$

Similarly, when $y < 0$,

$$\left|\cot(x + \mathbf{i}y) - \mathbf{i}\right| \leqslant \sqrt{\frac{2e^{2y}}{(1 - e^{2y})^2 + e^{-2y}\sin^2 x}} \leqslant \frac{\sqrt{2}e^{-|y|}}{1 - e^{-2|y|}}.$$

Hence, by the definition of $P(z)$ we have the following:

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Lemma 4.1. *Let $-\lambda = Re^{\mathbf{i}\theta}$ and $z = \sqrt{-\lambda}\ell$.*
(i) If $\theta_2 < \theta < \pi$, then

$$P(z) = -\mathbf{i}A_0 + O(1)\frac{e^{-y}}{1-e^{-2y}}$$

where $O(1)$ is bounded by a constant depending only on α and k, and

$$\begin{aligned} A_0 &= \det\begin{bmatrix} \mu_1 & \mu_2 & \mu_3 \\ \mu_1^3 & \mu_2^3 & \mu_3^3 \\ 1-k\mu_1^2 & 1-k\mu_2^2 & 1-k\mu_3^2 \end{bmatrix}. \\ &= (\mu_1-\mu_2)(\mu_2-\mu_3)(\mu_3-\mu_1)(\mu_1+\mu_2+\mu_3+k\mu_1\mu_2\mu_3) \\ &= (\mu_1-\mu_2)(\mu_2-\mu_3)(\mu_3-\mu_1)(\mu_1+\mu_2+\mu_3+\sqrt{k}), \\ y &= \min\left\{\sqrt{\frac{R\ell^2}{R_1}}\sin\frac{|\theta|}{2}, \sqrt{\frac{R\ell^2}{R_2}}\sin\frac{|\theta-\theta_2|}{2}, \sqrt{\frac{R\ell^2}{R_3}}\sin\frac{|\theta-\theta_3|}{2}\right\}. \end{aligned}$$

(ii) Similarly, if $-\pi < \theta < \theta_3(=-\theta_2)$ or if $0 < |\theta| < \theta_2$, then with the same y as above,

$$P(z) = (\pm)\mathbf{i}A_0 + O(1)\frac{e^{-y}}{1-e^{-2y}}.$$

Sine $A_0 \neq 0$, we see that there exists a positive constant Y such that $P(z) \neq 0$ if $|y| > Y\ell$. Consequently, if $\lambda = -Re^{\mathbf{i}\theta}$ is an eigenvalue, then $y \leqslant Y\ell$, which implies that

$$\min\left\{\sin^2\frac{\theta-\theta_1}{2},\ \sin^2\frac{\theta-\theta_2}{2},\ \sin^2\frac{\theta-\theta_3}{2}\right\} \leqslant Y^2\frac{\max\{\lambda_1, |\lambda_2|, |\lambda_3|\}}{|\lambda|}.$$

After similar considerations for odd function and case when A has eigenvalues of multiplicity more than one, we have the following:

Corollary 4.2. *Let σ be the spectrum of the problem* (4.4)*. Then*

$$\overline{\lim_{\lambda\in\sigma, |\lambda|\to\infty}} \min\left\{|\mathrm{Arg}(-\lambda)|,\ \Big||\mathrm{Arg}(-\lambda)| - \theta_2\Big|\right\} = 0.$$

Hence, by the general theory of sectorial operator, we have the following

Proposition 4.3. *Let (θ, w) be solution of* (1.1)*, where $\gamma \equiv 0$, subject to the natural boundary condition* (1.6) *and normalization $\int_\Omega \theta_0(x) = 0$. Then there exist positive constants M and ν such that*

$$E(t) \leqslant ME(0)e^{-\nu t}, \quad \forall\, t \geqslant 0. \tag{4.7}$$

4.2 The General Case

Now we consider the general case, i.e., we remove the assumption $\gamma \equiv 0$ imposed in the previous subsection. We want to show that the energy E defined in (1.8) decays exponentially fast as $t \to \infty$. By linearity, we assume, without loss of generality, that

$$\int_\Omega \theta(x,0)dx = 0.$$

Then for any $t \geqslant 0$, $\int_\Omega \theta(x,t)\,dx = 0$, so by Poincar'e inequality,

$$\int_\Omega |\nabla\theta(x,t)|^2 dx \geqslant C_0\int_\Omega \theta(x,t)^2 dx. \quad \forall\, t > 0.$$

Let $\varepsilon > 0$ and T be positive integer to be determined.

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First of all, using the energy identity

$$\frac{dE(t)}{dt} = -k\int_\Omega |\nabla\ \theta|^2 dx - \gamma(t)\int_\Omega \theta^2 dx \leq -kC_0\int_\Omega \theta^2 dx,$$

we obtain

$$E(T) \leqslant E(0) - kC_0\int_0^T \theta^2(x,\tau)d\tau. \tag{4.8}$$

(1) Suppose $\int_0^T\int_\Omega \theta^2(x,t)dxt \geq \varepsilon E(0)$, then

$$E(T) \leq [1-\varepsilon]E(0). \tag{4.9}$$

(2)Suppose $\int_0^T\int_\Omega \theta^2(x,t)dxdt \leq \varepsilon E(0)$. Denote by $(\tilde{w},\tilde{\theta})$ the solution of

$$\begin{cases} \tilde{w}_{tt} + \Delta^2\tilde{w} + \alpha\Delta\tilde{\theta} = 0 \quad \tilde{\theta}_t - k\Delta\tilde{\theta} - \alpha\Delta\tilde{w}_t = 0 & \text{in } \Omega\times(0,\infty), \\ \tilde{\theta}(\cdot,0) = \theta(\cdot,0), \quad \tilde{w}(\cdot,0) = w(\cdot,0), \quad \tilde{w}_t(\cdot,0) = w_t(\cdot,0) & \text{on } \Omega\times\{0\}, \\ \dfrac{\partial\tilde{\theta}}{\partial n} = w = \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty). \end{cases}$$

Set

$$W = w - \tilde{w}, \quad \Theta = \theta - \tilde{\theta}.$$

Then

$$\begin{cases} W_{tt} + \Delta^2 W + \alpha\Delta\Theta = 0, \quad \Theta_t - k\Delta\Theta - \alpha\Delta W_t = -\gamma(t)\theta & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial\Theta}{\partial n} = W = \dfrac{\partial W}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ W(\cdot,0) = \theta(\cdot,0) = W_t(\cdot,0) = 0 & \text{on } \Omega\times\{0\}. \end{cases}$$

Note that $\int_\Omega \Theta(x,t)dx = 0$, so $\int_\Omega |\nabla\Theta(x,t)|^2 dx \geqslant C_0\int_\Omega \theta^2 dx$. Thus

$$\begin{aligned} \frac{d}{dt}\int_\Omega \left(W_t^2 + |\Delta W|^2 + \Theta^2\right)dx &= -2k\int_\Omega |\nabla\Theta|^2 dx - 2\gamma(t)\int_\Omega \Theta\theta dx \\ &\leq -2kC_0\int_\Omega \Theta^2 dx + 2\|\gamma\|_{L^\infty}\sqrt{\int_\Omega \Theta^2 dx}\sqrt{\int_\Omega \theta^2 dx} \\ &\leq \frac{\|\gamma\|_{L^\infty}^2}{2kC_0}\int_\Omega \theta^2 dx. \end{aligned}$$

After integrating we obtain

$$R(T) := \int_\Omega \left(W_t^2 + |\Delta W|^2 + \Theta^2\right)dx\Big|_{t=T} \quad \leqslant \quad \frac{\|\gamma\|_{L^\infty}^2}{2kC_0}\int_0^T\int_\Omega \theta^2(x,t)dxdt \leqslant C_1\varepsilon E(0),$$

where

$$C_1 := \frac{\|\gamma\|_{L^\infty}^2}{2kC_0}.$$

Next define

$$\tilde{E}(t) = \int_\Omega \left\{\tilde{w}_t^2 + |\Delta\tilde{w}|^2 + \tilde{\theta}^2\right\}dx.$$

Then

$$\begin{aligned} E(T) &= \tilde{E}(T) + 2\int_\Omega \left(W_t w_t + 2\Delta W\Delta w + \Theta\theta\right)dx + R(T) \\ &\leqslant \tilde{E}(T) + 2\sqrt{E(T)}\sqrt{R(T)} + R(T) \\ &\leqslant \tilde{E}(T) + 2\sqrt{E(0)}\sqrt{R(T)} + R(T). \end{aligned}$$

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By Proposition 4.3, we have $\tilde{E}(T) \leqslant ME(0)e^{-\nu T}$. Hence, using the estimate for $R(T)$ we have

$$\begin{aligned} E(T) &\leq ME(0)e^{-\nu T} + E(0)\Big\{2\sqrt{C_1\varepsilon} + C_1\varepsilon\Big\} \\ &= E(0)\Big\{Me^{-\nu T} + 2\sqrt{C_1\varepsilon} + C_1\varepsilon\Big\}. \end{aligned}$$

Now combine with the first case we then obtain, in any case

$$\frac{E(T)}{E(0)} \leqslant \max\Big\{1-\varepsilon, \quad Me^{-\nu T} + 2\sqrt{C_1\varepsilon} + C_1\varepsilon\Big\}.$$

Now we fix the parameters ε and T as follows. We take

$$\varepsilon = \frac{1}{16(C_1+1)}, \qquad T := \frac{1}{\nu}\ln(4M), \quad \hat{\nu} = -\frac{1}{T}\ln\Big[1 - \frac{1}{16(1+C_1)}\Big].$$

Then

$$Me^{-\nu T} + 2\sqrt{C_1\varepsilon} + C_1\varepsilon \leqslant \frac{1}{4} + 2\sqrt{\frac{1}{16}} + \frac{1}{16} = \frac{13}{16}.$$

Hence,

$$E(T) \leqslant \Big[1 - \frac{1}{16(1+C_1)}\Big]E(0) = E(0)e^{-\tilde{\nu}T}.$$

This implies that E decays exponentially fast; indeed, for any $t > 0$, set $n = [t/T]$ the integer part of t/T, then by induction

$$E(t) \leqslant e^{-n\tilde{\nu}T}E(t-nT) \leqslant e^{-\tilde{\nu}t+\tilde{\nu}(t-nT)}E(0) \leqslant e^{-\tilde{\nu}t}e^{\tilde{\nu}T}E(0).$$

Hence, we have the following:

Theorem 4.4. *Assume that Ω is an one dimensional interval. There exists positive constants M and ν such that if (θ, w) is a solution of* (1.1) *subject to the natural boundary condition* (1.6)*, then*

$$E(t) \leqslant ME(0)e^{-\nu t}, \quad \forall\, t \geqslant 0.$$

Appendix

A Periodic Solution

In this appendix we show that the following problem admits a globally stable periodic solution:

$$T' = S(t) - \sigma T^4, \tag{A.1}$$

where σ is a positive constant, and

$$S(\cdot) \geqslant 0, \quad S(\cdot) \not\equiv 0, \quad S(t+1) = S(t), \quad t \in \mathbb{R}.$$

More precisely, we shall prove the following:

Theorem A.1. *Assume that $S(\cdot)$ is a continuous, non-negative, and non-trivial continuous and periodic function with period one. Then* (A.1) *admits a unique positive periodic solution $T_*(\cdot)$ that is globally stable, i.e., there exists a positive constant $\nu > 0$ such that for any $a \geqslant 0$, if denote by $T(a,t)$ the solution of* (A.1) *with initial condition $T(0) = a$, then*

$$|T(a,t) - T_*(t)| \leqslant M(a)e^{-\nu t}.$$

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Proof. Note that the set $[0,\infty)$ is invariant under the flow (A.1) since $S \geqslant 0$ and $\sigma > 0$. In physics T is absolute temperature so it is required that $T \geqslant 0$. For possible extension to general case, we write (A.1) in to a general form

$$T' = f(t,T). \tag{A.2}$$

We assume the following:

(1) $f, f_T \in C(\mathbb{R}\times[0,\infty))$, $f(t+1,T) = f(t,T)$ for every $T \geqslant 0$ and $t \in \mathbb{R}$.

(2) $f(\cdot,0) \geqslant 0$, $f(\cdot,0) \not\equiv 0$; there exists $M > 0$ such that $f(t,T) < 0$ for every $T > M$.

(3) $f_T(t,T) < 0$ for every $t \in \mathbb{R}$ and $T > 0$.

For each $a \geqslant 0$, we denote by $T(a;t)$ the solution of

$$\begin{cases} T' = f(t,T), & t > 0, \\ T|_{t=0} = a. \end{cases}$$

We consider the Poincaré map $\mathbf{T}(a) := T(a,1)$. One can check the following,

(i) Since $f(\cdot,0) \geqslant 0$, 0 is a sub-solution, so $T(a,t) \geqslant 0$ for each $a \geqslant 0$ and $t \geqslant 0$. In addition, since $f(\cdot,0) \not\equiv 0$, we have $\mathbf{T}(0) := T(0,1) > 0$.

(ii) Since $f(\cdot,T) < 0$ for every $M > 0$, we find that $\mathbf{T}(a) < a$ for every $a \geqslant M$.

(iii) By variation of parameters, we find, for every $a > 0$,

$$\mathbf{T}'(a) = \frac{\partial T(a,1)}{\partial a} = \exp\left(\int_0^1 f_T(t,T(a,t))dt\right) < 1.$$

Thus, $\mathbf{T}$ is a contraction so there exists a unique fixed point, which we denote by a_*. This implies that $T(a_*,\cdot)$ is the unique periodic solution of (A.2).

Finally, for any $a \geqslant 0, a \neq a_*$, we have, using fundamental theorem of calculus,

$$\frac{T(a,1)-a_*}{a-a_*} = \frac{\mathbf{T}(a)-\mathbf{T}(a_*)}{a-a_*} = \int_0^1 \frac{\partial \mathbf{T}(a_*+\theta(a-a_*))}{\partial a} d\theta < 1.$$

Hence, as $n \to \infty$, $\lim_{n\to\infty} T(a,n) = a_*$. Consequently, fix $\varepsilon > 0$, there exists a large integer N depending on ε and $a \geqslant 0$ such that $|T(a,t) - T(a_*,t)| \leqslant \varepsilon$ for every $t \geqslant N$. Consequently, setting

$$\rho = \max_{b\in[a_*-\varepsilon,a_*+\varepsilon]} \mathbf{T}'(b),$$

we have, when $t = N + n + s$ with $s \in [0,1)$,

$$\begin{aligned} |T(a,t) - T(a_*,t)| &\leqslant |T(a,N+n) - T(a_*,N+n)| \\ &\leqslant \rho^n |T(a,N) - T(a_*,N)| \\ &\leqslant \varepsilon\rho^n \leqslant \frac{\varepsilon}{\rho^{N+1}}\rho^t = M(a)e^{-\nu t}, \end{aligned}$$

where $M(a) = \varepsilon\rho^{-N-1}$ and $\nu = -\ln\rho$. This completes the proof of the Theorem. □

Acknowledgements. Thanks to Professor Zhuangyi Liu for many valuable discussions.

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