J. Math. Anal. Appl. 475 (2019) 1385–1403

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Multiplicity and concentration results for fractional Schrödinger system with steep potential wells

Liejun Shen

School of Mathematics and Statistics, Central China Normal University, China

A R T I C L E   I N F O

Article history:
Received 11 May 2018
Available online 16 March 2019
Submitted by J. Shi

Keywords:
Nehari manifold
Fibering map
Multiplicity
Concentration
Steep potential wells
Concave-convex

A B S T R A C T

This paper is concerned with the fractional coupled Schrödinger system. By using the Nehari manifold and fibering map, we obtain the multiplicity and concentration of solutions for the given problem with steep potential wells, where some new estimates will be established. In particular, although there exist concave-convex nonlinearities in the coupled system, it is not necessary to assume that the corresponding Lebesgue norms of the weight functions of the convex terms need to be small enough.

1. Introduction and main results

In this paper, we study the multiplicity and concentration behavior of nontrivial solutions for the following coupled elliptic system:

$$\begin{cases} (-\Delta)^s u + \lambda V(x)u = f(x)|u|^{q-2}u + \frac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, & x \in \mathbb{R}^N, \\ (-\Delta)^s v + \lambda W(x)v = g(x)|v|^{q-2}v + \frac{\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $(-\Delta)^s$ is the fractional Laplacian operator with $s \in (0,1)$, the parameter $\lambda > 0$, $1 < q < 2$, $\alpha,\ \beta > 1$ with $\alpha + \beta < 2_s^* = 2N/(N-2s)$ and $N \geq 3$. We first assume that $V(x)$ and $W(x)$ satisfy the following conditions:

(VW_1) *$V(x), W(x) \in C(\mathbb{R}^N, \mathbb{R})$ with $V(x), W(x) \geq 0$ on $\mathbb{R}^N$;*

(VW_2) *there exists $c > 0$ such that the set $\Sigma \triangleq \left\{x \in \mathbb{R}^N : V(x)W(x) < c^2\right\}$ has positive finite Lebesgue measure;*

E-mail address: liejunshen@163.com.

https://doi.org/10.1016/j.jmaa.2019.03.021

(VW_3) $\Omega_1 \triangleq intV^{-1}(0)$ *and* $\Omega_2 \triangleq intW^{-1}(0)$ *are nonempty and have smooth boundaries with* $\overline{\Omega_1} = V^{-1}(0)$, $\overline{\Omega_2} = W^{-1}(0)$, *and* $\Omega_1 \cap \Omega_2 \neq \emptyset$.

For the weight functions $f(x)$ and $g(x)$, we assume

(FG_1) $f,\ g \in L^\infty(\mathbb{R}^N)$ *satisfy* $\Theta_f \subset \Sigma$ *and* $\Theta_g \subset \Sigma$, *where* Σ *is given by* $(VW)_2$ *and*

$$\Theta_f \triangleq \{x \in \mathbb{R}^N : f(x) > 0\}, \quad \Theta_g \triangleq \{x \in \mathbb{R}^N : g(x) > 0\};$$

(FG_2) $f,\ g \in L^{2_s^*/(2_s^*-q)}(\mathbb{R}^N)$.

The aforementioned assumptions $(VW_1)-(VW_3)$ are firstly proposed by Bartsch-Wang in their cerebrated paper [3] to study a scalar Schrödinger equation. The potential $\lambda V(x)$ and $\lambda W(x)$ with the above hypotheses are usually called by the steep potential wells.

The fractional Laplacian, $(-\Delta)^s u$, in this paper can be represented [32, Lemma 3.2] as

$$(-\Delta)^s u(x) = -\frac{1}{2} C_N(s) \int\limits_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \quad \forall\ x \in \mathbb{R}^N,$$

where

$$C_N(s) = \left(\int\limits_{\mathbb{R}^N} \frac{1-\cos\xi_1}{|y|^{N+2s}} d\xi \right)^{-1}, \quad \xi = (\xi_1, \xi_2, \cdots, \xi_N).$$

As a consequence of [32, Proposition 3.4 and Proposition 3.6], the natural inner product and norm of $H^s(\mathbb{R}^N)$ can be defined as

$$(u,\varphi)_{H^s(\mathbb{R}^N)} = \int\limits_{\mathbb{R}^N} \left[(-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi + u\varphi\right] dx, \quad \text{and} \quad \|u\|_{H^s(\mathbb{R}^N)} = \big(u,u\big)^{\frac{1}{2}}_{H^s(\mathbb{R}^N)}.$$

Also the homogeneous fractional Sobolev space $D^{s,2}(\mathbb{R}^N)$ is defined by

$$D^{s,2}(\mathbb{R}^N) = \left\{ u \in L^{2_s^*}(\mathbb{R}^N) : |(-\Delta)^{\frac{s}{2}} u| \in L^2(\mathbb{R}^N) \right\}$$

which is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{D^{s,2}(\mathbb{R}^N)} = \left(\int\limits_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right)^{\frac{1}{2}}.$$

According to [12], there exists a best constant $S_s > 0$ such that

$$S_s = \inf\left\{ \int\limits_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx : u \in D^{s,2}(\mathbb{R}^N) \ \text{ and } \ |u|_{2_s^*} = 1 \right\} > 0, \tag{1.2}$$

where $|\cdot|_r$ denotes the standard norm of the usual Lebesgue space with $1 \le r \le \infty$.

To deal with (1.1), we introduce the following spaces

$$E_1 \triangleq \left\{ u \in D^{s,2}(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} V(x)u^2 dx < +\infty \right\}$$

and

$$E_2 \triangleq \left\{ u \in D^{s,2}(\mathbb{R}^N) : \int\limits_{\mathbb{R}^N} W(x)u^2 dx < +\infty \right\}.$$

Thus the natural space in this paper is the space $E \triangleq E_1 \times E_2$, which is a Hilbert space equipped with the inner product and norm

$$(z, \zeta) = \int\limits_{\mathbb{R}^N} \left[(-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi + V(x) u \varphi + (-\Delta)^{\frac{s}{2}} v (-\Delta)^{\frac{s}{2}} \psi + W(x) v \psi \right] dx$$

and

$$\|z\| = (z, z)^{1/2}$$

for any $z = (u, v) \in E$ and $\zeta = (\varphi, \psi) \in E$. Given $\lambda > 0$, we let $E_\lambda \triangleq (E, \|\cdot\|_\lambda)$ be endowed with the inner product and norm

$$(z, \zeta)_\lambda = \int\limits_{\mathbb{R}^N} \left[(-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi + \lambda V(x) u \varphi + (-\Delta)^{\frac{s}{2}} v (-\Delta)^{\frac{s}{2}} \psi + \lambda W(x) v \psi \right] dx$$

and

$$\|z\|_\lambda = (z, z)_\lambda^{1/2}.$$

Obviously, $\|z\| \leq \|z\|_\lambda$ if $\lambda \geq 1$. We will show the multiplicity and concentration results of nontrivial solutions of (1.1) by looking for critical points of the associated functional

$$J_\lambda(z) = \frac{1}{2}\|z\|_\lambda^2 - \frac{1}{q} I_{f,g}(z) - \frac{1}{\alpha + \beta} \int\limits_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx,$$

where

$$I_{f,g}(z) \triangleq \int\limits_{\mathbb{R}^N} \left[f(x)|u|^q + g(x)|v|^q \right] dx.$$

In view of [42], the critical points of $J_\lambda(z)$ are in fact the (weak) solutions of (1.1). We say that $z = (u, v)$ is a (weak) solution of (1.1) if for any $\zeta = (\varphi, \psi) \in E$ there holds

$$\int\limits_{\mathbb{R}^N} \left[(-\Delta)^{\frac{s}{2}} u (-\Delta)^{\frac{s}{2}} \varphi + \lambda V(x) u \varphi + (-\Delta)^{\frac{s}{2}} v (-\Delta)^{\frac{s}{2}} \psi + \lambda W(x) v \psi \right] dx$$
$$= \int\limits_{\mathbb{R}^N} \left[f(x)|u|^{q-2} u \varphi + g(x)|v|^{q-2} v \psi \right] dx$$

$$+\frac{\alpha}{\alpha+\beta}\int\limits_{\mathbb{R}^N}|u|^{\alpha-2}u\varphi|v|^{\beta}dx+\frac{\beta}{\alpha+\beta}\int\limits_{\mathbb{R}^N}|u|^{\alpha}|v|^{\beta-2}v\psi dx. \tag{1.3}$$

The scalar case of (1.1), that is,

$$(-\Delta)^s u+\lambda V(x)=\xi(x)|u|^{q-2}+\eta(x)|u|^{r-2}u,\quad x\in\mathbb{R}^N, \tag{1.4}$$

where $s\in(0,1]$, $1<q<2<r<2_s^*$ with $N\geq 3$ and ξ,η are two weight functions, has been paid attention by many scholars in the last serval decades, see [11,27] for example. Equations like (1.4) come from the following fractional Schrödinger equation

$$(-\Delta)^s u+V(x)u=h(x,u),\quad x\in\mathbb{R}^N$$

used to study the standing wave solutions $\psi(t,x)=u(x)e^{-i\omega t}$ for the equation

$$i\hbar\frac{\partial\psi}{\partial t}=\hbar^2(-\Delta)^{\alpha}\psi+W(x)\psi-h(x,\psi),\quad x\in\mathbb{R}^N,$$

where $\hbar$ is the Planck's constant, $W:\mathbb{R}^N\to\mathbb{R}$ is an external potential and h is a suitable nonlinearity. Since the fractional Schrödinger equation appears in problems involving nonlinear optics, plasma physics and condensed matter physics, it is one of the main objects of the fractional quantum mechanic. To know more about the study of fractional Schrödinger equations, the reader can refer to [22,23,8,15,35,4,19,27,28,21,36,37] and the references therein for example.

There are extensive bibliographies in the study of the coupled elliptic systems on bounded domain see [16,20,10,33,34] and their references therein for example. As to the whole space, we refer the reader to [2,17,29,31,30]. In [17], under the assumptions $(VW_1)-(VW_3)$, the authors consider a similar problem

$$\begin{cases}-\Delta u+\lambda V(x)u=\frac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, & x\in\mathbb{R}^N,\\ -\Delta v+\lambda W(x)v=\frac{\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & x\in\mathbb{R}^N,\end{cases}$$

where $\alpha,\ \beta>1$ with $\alpha+\beta<2^*=2N/(N-2)$ and $N\geq 3$. They establish a positive least energy solution for the above problem by Mountain-Pass theorem and explore the phenomenon of concentration of solutions. Meanwhile, for any $k\in\mathbb{N}^+$, they show that the above problem admits at least k nontrivial solutions as well as the concentration result for large $\lambda>0$ by using the well-known Symmetric Mountain-Pass theorem [1]. Subsequently, for $\alpha,\ \beta>1$ with $\alpha+\beta<6$, Lv-Xiao [29] study the following coupled system of Kirchhoff type

$$\begin{cases}-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+\lambda V(x)u=\frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, & x\in\mathbb{R}^3,\\ -\left(a+b\int_{\mathbb{R}^3}|\nabla v|^2dx\right)\Delta v+\lambda W(x)v=\frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & x\in\mathbb{R}^3,\end{cases}$$

by assuming

(H_1) *$V(x),W(x)\in C(\mathbb{R}^3,[0,\infty))$ and $\Omega\triangleq intV^{-1}(0)=intW^{-1}(0)$ is nonempty with smooth boundary and $\overline{\Omega}=V^{-1}(0)=W^{-1}(0)$;*

(H_2) *there exist $M_1,M_2>0$ such that the sets $\{x\in\mathbb{R}^3:V(x)<M_1\}$ and $\{x\in\mathbb{R}^3:W(x)<M_2\}$ have positive finite Lebesgue measures.*

They obtain the existence and multiplicity of solutions for large $\lambda > 0$, but don't consider the concentration of nontrivial solutions.

However, to the best knowledge of us, it seems that there are few works on the multiplicity and concentration of solutions to the coupled elliptic system involving concave-convex nonlinearities with steep potential wells. In the present paper, we mainly follow the idea of [17]. Let us point out that although the idea was used before for other different problems, the adaptation of the procedure to our problem is not trivial at all. We establish some new estimation such as the inequality (2.5) below which will play an important role in our proof. Moreover, the key inequalities, such as (2.17) and (2.18) below which are used to prove the (PS) condition, don't seem to have appeared in previous literature. What we want to emphasize here is that there are two usual ways to study the concentration results, one is by the aid of the vanishing lemma [25,26], see [44,39,24,40,43,11,27,38] for example, the other is via the Nehari method on the limit system, see [17] for example. But the above two methods can not be applied directly to our case. On one hand, the work space $E = E_1 \times E_2$ which is a subspace of $D^{s,2}(\mathbb{R}^N) \times D^{s,2}(\mathbb{R}^N)$, not a subspace of $H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$, prevents us using the vanishing lemma. On the other hand, we establish two distinct nontrivial solutions for the limit system (see (1.5) below), so the Nehari argument does not work. Combining (2.5) and (2.18), we obtain (3.2) and (3.3) to overcome this difficult.

Now we give our main results.

Theorem 1.1. *If $1 < q < 2$, $\alpha,\ \beta > 1$ with $\alpha + \beta < 2_s^*$, assume that $(FG_1) - (FG_2)$ and $(VW_1) - (VW_3)$ as well as the following condition*

$$(VW_4) \qquad 0 < |\Sigma|^{\frac{2_s^*-q}{2_s^*(2-q)} + \frac{2s(1-\mu)}{(\alpha+\beta-2)Np}} < \left[\frac{2-q}{\alpha+\beta-q}\left(\frac{1}{2}S_s^{-1}\right)^{(\mu-1)/p} S_s^{(2_s^*/2)^2\mu/p}\right]^{\frac{1}{\alpha+\beta-2}} \times \left[\frac{(\alpha+\beta-2)S_s^{q/2}}{(\alpha+\beta-q)(|f|_\infty+|g|_\infty)}\right]^{\frac{1}{2-q}},$$

where $|\Sigma|$ denotes the Lebesgue measure of Σ given in (VW_2), and

$$p = \frac{2_s^*}{2_s^* - \alpha - \beta + 2} \in (1, 2_s^*/2) \quad \text{and} \quad \mu = \frac{2(p-1)}{2_s^* - 2} = \frac{2(\alpha+\beta-2)}{(2_s^* - \alpha - \beta + 2)(2_s^* - 2)} \in (0,1).$$

Then there exists a constant $\Lambda > 0$ such that (1.1) has two positive solutions $z_\lambda^+ = (u_\lambda^+, v_\lambda^+)$ and $z_\lambda^- = (u_\lambda^-, v_\lambda^-)$.

On the concentration of solutions, we have the following result.

Theorem 1.2. *Let $z_\lambda^\pm$ be the solutions obtained in Theorem 1.1. Then $z_\lambda^\pm \to z^\pm$ in E as $\lambda \to \infty$, and z^+ (z^-) is a (nontrivial) solution of*

$$\begin{cases} (-\Delta)^s u = f(x)|u|^{q-2}u + \frac{\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta, & x \in \Omega_1, \\ (-\Delta)^s v = g(x)|v|^{q-2}v + \frac{\beta}{\alpha+\beta}|u|^\alpha|v|^{\beta-2}v, & x \in \Omega_2, \\ u = 0, & x \in \partial\Omega_1, \\ v = 0, & x \in \partial\Omega_2. \end{cases} \tag{1.5}$$

Furthermore, we have the following conclusions:

(1) if $\Theta_f \cap \Omega_1 = \emptyset$ and $\Theta_g \cap \Omega_2 = \emptyset$, then $z^+ \equiv 0$;
(2) if $|\Theta_f \cap \Omega_1| > 0$ and $|\Theta_g \cap \Omega_2| > 0$, then $z^+ \neq 0$;
(3) $z^+ \neq z^-$.

The paper is organized as follows. In Section 2, we provide several lemmas, which are crucial in proving our main results. In Section 3, the proofs of Theorems 1.1 and 1.2 are established.

Notations. Throughout this paper we shall denote by C and C_i $(i = 1, 2, \cdots)$ for various positive constants whose exact value may change from lines to lines but are not essential to the analysis of the problem. The set A^c is the complement set of A in $\mathbb{R}^N$. We use "$\to$" and "$\rightharpoonup$" to denote the strong and weak convergence in the related function space, respectively. For any $\rho > 0$ and any $x \in \mathbb{R}^N$, $B_\rho(x)$ denotes the ball of radius ρ centered at x, that is, $B_\rho(x) := \{y \in \mathbb{R}^N : |y - x| < \rho\}$.

Let $(X, \|\cdot\|)$ be a Banach space with its dual space $(X^*, \|\cdot\|_*)$, and Φ be its functional on X. The Palais-Smale sequence at level $c \in \mathbb{R}$ ($(PS)_c$ sequence in short) corresponding to Φ means that $\Phi(x_n) \to c$ and $\Phi'(x_n) \to 0$ as $n \to \infty$, where $\{x_n\} \subset X$. If for any $(PS)_c$ sequence $\{x_n\}$ in X, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x_0$ in X for some $x_0 \in X$, then we say that the functional Φ satisfies the so called $(PS)_c$ condition.

2. Preliminaries

In this section, we present some preliminaries for the main results of this paper. It is worth mentioning here that the main idea of the following lemma comes from [17, Lemmas 2.1-2.2], however, we obtain a totally different estimate which enables us to show the (PS) condition of J_λ in another way.

In view of the constants p and μ given by the assumption (VW_4), there hold

$$\frac{\alpha - 1}{2_s^*} + \frac{\beta - 1}{2_s^*} + \frac{1}{p} = 1 \tag{2.1}$$

and

$$p = 2_s^*\mu/2 + (1 - \mu). \tag{2.2}$$

Lemma 2.1. *If $\alpha,\ \beta > 1$ with $\alpha + \beta < 2_s^*$ and assume $(VW_1) - (VW_2)$. Then for any $z = (u, v) \in E_\lambda$, there exists $\Lambda_0 \triangleq \max\{1, S_s|\Sigma|^{-\frac{2s}{N}}c^{-1}\}$ such that*

$$\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx \le \left(\frac{1}{2} S_s^{-1} |\Sigma|^{\frac{2s}{N}}\right)^{(1-\mu)/p} S_s^{-(2_s^*/2)^2\mu/p} \|z\|_\lambda^{\alpha+\beta}.$$

Proof. Recalling (VW_2), by using the Hölder inequality and (1.2) we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |uv| dx &\le \int_{\Sigma} (|u|^{2_s^*})^{\frac{1}{2_s^*}} (|v|^{2_s^*})^{\frac{1}{2_s^*}} 1^{\frac{2s}{N}} dx + \frac{1}{\lambda c} \int_{\Sigma^c} (\lambda V(x) u^2)^{\frac{1}{2}} (\lambda W(x) v^2)^{\frac{1}{2}} dx \\
&\le |u|_{2_s^*} |v|_{2_s^*} |\Sigma|^{\frac{2s}{N}} + \frac{1}{\lambda c} \left(\int_{\Sigma^c} \lambda V(x) u^2 dx \right)^{\frac{1}{2}} \left(\int_{\Sigma^c} \lambda W(x) v^2 dx \right)^{\frac{1}{2}} \\
&\le S_s^{-1} |\Sigma|^{\frac{2s}{N}} |(-\Delta)^{\frac{s}{2}} u|_2 |(-\Delta)^{\frac{s}{2}} v|_2 + \frac{1}{2\lambda c} \int_{\mathbb{R}^N} \lambda V(x) u^2 + \lambda W(x) v^2 dx \\
&\le \frac{1}{2} \max\left\{ S_s^{-1} |\Sigma|^{\frac{2s}{N}}, \frac{1}{\lambda c} \right\} \|z\|_\lambda^2.
\end{aligned} \tag{2.3}$$

It follows from (1.2) and (2.2) that

$$
\begin{aligned}
\int_{\mathbb{R}^N} |uv|^p dx &\le \left(\int_{\mathbb{R}^N} |uv|^{\frac{2_s^*}{2}} dx \right)^{\mu} \left(\int_{\mathbb{R}^N} |uv| dx \right)^{1-\mu} \le \left[\frac{1}{2} \left(|u|_{2_s^*}^{2_s^*} + |v|_{2_s^*}^{2_s^*} \right) \right]^{\mu} \left(\int_{\mathbb{R}^N} |uv| dx \right)^{1-\mu} \\
&\le \left[\frac{S_s^{-2_s^*/2}}{2} \left(|(-\Delta)^{\frac{s}{2}} u|_2^{2_s^*} + |(-\Delta)^{\frac{s}{2}} v|_2^{2_s^*} \right) \right]^{\mu} \left(\int_{\mathbb{R}^N} |uv| dx \right)^{1-\mu} \\
&\le S_s^{-2_s^* \mu/2} \|z\|_\lambda^{2_s^* \mu} \left(\int_{\mathbb{R}^N} |uv| dx \right)^{1-\mu}. \qquad (2.4)
\end{aligned}
$$

Combining (2.1) and (2.3)-(2.4), we derive

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx \\
&\le \left(\int_{\mathbb{R}^N} |u|^{2_s^*} dx \right)^{\frac{\alpha-1}{2_s^*}} \left(\int_{\mathbb{R}^N} |v|^{2_s^*} dx \right)^{\frac{\beta-1}{2_s^*}} \left(\int_{\mathbb{R}^N} |uv|^p dx \right)^{\frac{1}{p}} \\
&\le S_s^{-\frac{\alpha+\beta-2}{2}} \|z\|_\lambda^{\alpha+\beta-2} \left(\int_{\mathbb{R}^N} |uv|^p dx \right)^{\frac{1}{p}} \le S_s^{-(2_s^*/2)^2 \mu/p} \|z\|_\lambda^{\alpha+\beta-2+2_s^*\mu/p} \left(\int_{\mathbb{R}^N} |uv| dx \right)^{\frac{1-\mu}{p}} \\
&\le \left(\frac{1}{2} \max\left\{ S_s^{-1} |\Sigma|^{\frac{2s}{N}}, \frac{1}{\lambda c} \right\} \right)^{(1-\mu)/p} S_s^{-(2_s^*/2)^2 \mu/p} \|z\|_\lambda^{\alpha+\beta-2+2_s^*\mu/p+2(1-\mu)/p} \\
&= \left(\frac{1}{2} S_s^{-1} |\Sigma|^{\frac{2s}{N}} \right)^{(1-\mu)/p} S_s^{-(2_s^*/2)^2 \mu/p} \|z\|_\lambda^{\alpha+\beta} \qquad (2.5)
\end{aligned}
$$

when $\lambda \ge \Lambda_0$. The proof is complete. □

According to (FG_1) and by some direct computations, there holds

$$
\begin{aligned}
\int_{\mathbb{R}^N} f(x)|u|^q dx &= \int_{\Sigma} f(x)|u|^q dx + \int_{\Sigma^c} f(x)|u|^q dx \le \int_{\Sigma} f(x)|u|^q dx \\
&\le |f|_\infty |\Sigma|^{(2_s^*-q)/2_s^*} \left(S_s^{-1} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx \right)^{q/2} \le |f|_\infty |\Sigma|^{(2_s^*-q)/2_s^*} S_s^{-q/2} \|z\|_\lambda^q. \qquad (2.6)
\end{aligned}
$$

Similarly, we have

$$
\int_{\mathbb{R}^N} g(x)|v|^q dx \le |g|_\infty |\Sigma|^{(2_s^*-q)/2_s^*} S_s^{-q/2} \|z\|_\lambda^q. \qquad (2.7)
$$

Next we will study the so-called Nehari manifold because the variational functional $J_\lambda(z)$ is not bounded from below on the whole space E_λ. Let us define

$$
\mathcal{N}_\lambda = \left\{ z \in E_\lambda \backslash \{0\} : \langle J_\lambda'(z), z \rangle = 0 \right\}
$$

and then any nontrivial solution of (1.1) belongs to $\mathcal{N}_\lambda$. Obviously, $z \in \mathcal{N}_\lambda$ if and only if

$$
\|z\|_\lambda^2 = I_{f,g}(z) + \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx \quad \text{and} \quad z = (u, v) \quad \text{with} \quad u \ne 0 \text{ and } v \ne 0.
$$

The following lemma tells us the behavior of $J_\lambda(z)$ on $\mathcal{N}_\lambda$.

Lemma 2.2. *The functional $J_\lambda(z)$ is coercive and bounded from below on $\mathcal{N}_\lambda$.*

Proof. For any $z \in \mathcal{N}_\lambda$, since $1 < q < 2 < \alpha + \beta$, by (2.6) and (2.7)

$$\begin{aligned} J_\lambda(z) = J_\lambda(z) - \frac{1}{\alpha+\beta}\langle J'_\lambda(z), z\rangle &= \left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\|z\|_\lambda^2 - \left(\frac{1}{q} - \frac{1}{\alpha+\beta}\right) I_{f,g}(z) \\ &\geq \left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\|z\|_\lambda^2 - \left(\frac{1}{q} - \frac{1}{\alpha+\beta}\right)(|f|_\infty + |g|_\infty)|\Sigma|^{(2_s^*-q)/2_s^*} S_s^{-q/2}\|z\|_\lambda^q \\ &\geq -\frac{(2-q)(\alpha+\beta-2)}{2q(\alpha+\beta)}\left(\frac{\alpha+\beta-q}{\alpha+\beta-2}(|f|_\infty + |g|_\infty)|\Sigma|^{(2_s^*-q)/2_s^*} S_s^{-q/2}\right)^{\frac{1}{2-q}} \triangleq -M_0, \end{aligned}$$

which yields that $J_\lambda(z)$ is coercive and bounded from below on $\mathcal{N}_\lambda$. $\square$

The Nehari manifold $\mathcal{N}_\lambda$ is closely linked to the function $\varphi_{\lambda,z}(t) = J_\lambda(tz)$ for any $t > 0$. As we all know, the above map was introduced by Drábek-Pohožaev [13] and discussed in Brown-Zhang [6] (or Hsu [20], Chen-Kuo-Wu [9]). For any $z \in E_\lambda$, one has

$$\begin{aligned} \varphi_{\lambda,z}(t) &= \frac{t^2}{2}\|z\|_\lambda^2 - \frac{t^q}{q} I_{f,g}(z) - \frac{t^{\alpha+\beta}}{\alpha+\beta}\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx, \\ \varphi'_{\lambda,z}(t) &= t\|z\|_\lambda^2 - t^{q-1} I_{f,g}(z) - t^{\alpha+\beta-1}\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx, \\ \varphi''_{\lambda,z}(t) &= \|z\|_\lambda^2 - (q-1)t^{q-2} I_{f,g}(z) - (\alpha+\beta-1)t^{\alpha+\beta-1}\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx. \end{aligned}$$

It is easy to see that for any $z \in E_\lambda$ and $t > 0$, there holds

$$t\varphi'_{\lambda,z}(t) = t^2\|z\|_\lambda^2 - t^q I_{f,g}(z) - t^{\alpha+\beta}\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx,$$

which gives that $\varphi'_{\lambda,z}(t) = 0$ if and only if $tz \in \mathcal{N}_\lambda$. In particular, $\varphi'_{\lambda,z}(1) = 0$ if and only if $z \in \mathcal{N}_\lambda$. Arguing as Brown-Zhang [6], we split $\mathcal{N}_\lambda$ into three parts:

$$\begin{aligned} \mathcal{N}_\lambda^+ &= \{z \in \mathcal{N}_\lambda : \varphi''_{\lambda,z}(1) > 0\}, \\ \mathcal{N}_\lambda^0 &= \{z \in \mathcal{N}_\lambda : \varphi''_{\lambda,z}(1) = 0\}, \\ \mathcal{N}_\lambda^- &= \{z \in \mathcal{N}_\lambda : \varphi''_{\lambda,z}(1) < 0\}. \end{aligned}$$

Therefore for any $z \in \mathcal{N}_\lambda$, we have

$$\begin{aligned} \varphi''_{\lambda,z}(1) &= \|z\|_\lambda^2 - (q-1) I_{f,g}(z) - (\alpha+\beta-1)\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx \\ &= (2-q)\|z\|_\lambda^2 - (\alpha+\beta-q)\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx \qquad (2.8) \\ &= (2-\alpha-\beta)\|z\|_\lambda^2 + (\alpha+\beta-q) I_{f,g}(z). \qquad (2.9) \end{aligned}$$

As a direct consequence of (2.9), one has

$$I_{f,g}(z) > 0 \text{ for any } z \in \mathcal{N}_\lambda^0 \cup \mathcal{N}_\lambda^+.$$

It is similar to the argument in Brown-Zhang [6, Theorem 2.3] that we can derive the following result.

Lemma 2.3. *Suppose $z \in E_\lambda$ is a local minimizer for $J_\lambda(z)$ on $\mathcal{N}_\lambda$ and $z \notin \mathcal{N}_\lambda^0$, then $J'_\lambda(z) = 0$ in E_λ^*.*

Inspired by the above lemma, we need to study when the case $\mathcal{N}_\lambda^0 = \emptyset$ happens.

Lemma 2.4. *For $1 < q < 2$ and $\alpha, \beta > 1$ with $\alpha + \beta < 2_s^*$, if $\lambda > \Lambda_0$ and assume $(VW_1) - (VW_2)$, (VW_4) and (FG_1), then $\mathcal{N}_\lambda^0 = \emptyset$.*

Proof. Arguing it indirectly and suppose for any $z \in \mathcal{N}_\lambda^0$, using (2.5) and (2.8) we have

$$(2-q)\|z\|_\lambda^2 \le (\alpha+\beta-q)\left(\frac{1}{2}S_s^{-1}|\Sigma|^{\frac{2s}{N}}\right)^{(1-\mu)/p} S_s^{-(2_s^*/2)^2\mu/p}\|z\|_\lambda^{\alpha+\beta}, \tag{2.10}$$

which implies that

$$\|z\|_\lambda \ge \left[\frac{2-q}{\alpha+\beta-q}\left(\frac{1}{2}S_s^{-1}|\Sigma|^{\frac{2s}{N}}\right)^{(\mu-1)/p} S_s^{(2_s^*/2)^2\mu/p}\right]^{\frac{1}{\alpha+\beta-2}}.$$

Similarly, by using (2.6), (2.7) and (2.9) we obtain

$$\|z\|_\lambda \le \left[\frac{(\alpha+\beta-q)(|f|_\infty+|g|_\infty)|\Sigma|^{(2_s^*-q)/2_s^*}}{(\alpha+\beta-2)S_s^{q/2}}\right]^{\frac{1}{2-q}}.$$

It follows from the above two formulas that

$$|\Sigma|^{\frac{2_s^*-q}{2_s^*(2-q)}+\frac{2s(1-\mu)}{(\alpha+\beta-2)Np}} \ge \left[\frac{2-q}{\alpha+\beta-q}\left(\frac{1}{2}S_s^{-1}\right)^{(\mu-1)/p} S_s^{(2_s^*/2)^2\mu/p}\right]^{\frac{1}{\alpha+\beta-2}} \left[\frac{(\alpha+\beta-2)S_s^{q/2}}{(\alpha+\beta-q)(|f|_\infty+|g|_\infty)}\right]^{\frac{1}{2-q}},$$

which is a contradiction to (VW_4). The proof is complete. □

To find solutions of (1.1), it is necessary to consider whether $\mathcal{N}_\lambda^\pm$ are nonempty.

Lemma 2.5. *For $1 < q < 2$ and $\alpha,\ \beta > 1$ with $\alpha + \beta < 2_s^*$ and assume $(VW_1) - (VW_2)$, (VW_4) and (FG_1). Then for any $\lambda > \Lambda_0$ and $z \in E_\lambda \backslash \{0\}$, there exists*

$$t_{max} = \left(\frac{(2-q)\|z\|_\lambda^2}{(\alpha+\beta-q)\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta dx}\right)^{\frac{1}{\alpha+\beta-2}} > 0,$$

such that

(i) if $I_{f,g}(z) \le 0$, there is a unique $t^- > t_{max}$ such that $t^- z \in \mathcal{N}_\lambda^-$ and

$$J_\lambda(t^- z) = \sup_{t\ge 0} J_\lambda(tz);$$

(ii) if $I_{f,g}(z) > 0$, there are unique t^+ and t^- with $0 < t^+ < t_{max} < t^-$ such that $t^\pm z \in \mathcal{N}_\lambda^\pm$ and

$$J_\lambda(t^+ z) = \inf_{0 \le t \le t_{max}} J_\lambda(tz) \quad \text{and} \quad J_\lambda(t^- z) = \sup_{t \ge t_{max}} J_\lambda(tz).$$

Proof. The proof is standard, we omit it here (see e.g. Brown-Wu [7, Lemma 2.6] and Hsu [20, Lemma 3.5] for example). □

From Lemma 2.4, we know that $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ for any $\lambda > \Lambda_0$. Moreover, by Lemma 2.5, $\mathcal{N}_\lambda^\pm \neq \emptyset$; by Lemma 2.2, we may define

$$m_\lambda = \inf_{z \in \mathcal{N}_\lambda} J_\lambda(z), \quad m_\lambda^+ = \inf_{z \in \mathcal{N}_\lambda^+} J_\lambda(z), \quad m_\lambda^- = \inf_{z \in \mathcal{N}_\lambda^-} J_\lambda(z).$$

Then we have the following result.

Lemma 2.6. *If $1 < q < 2$, $\alpha, \ \beta > 1$ with $\alpha + \beta < 2_s^*$, and assume $(VW_1) - (VW_4)$ and $(FG_1) - (FG_2)$. Then for any $\lambda > \Lambda_0$ there exists $d_0 > 0$ independent of λ such that $m_\lambda^+ < 0 < d_0 < m_\lambda^-$. In particular, we have $m_\lambda^+ = m_\lambda$.*

Proof. For any $z \in \mathcal{N}_\lambda^+$, by (2.8) we have

$$(2-q)\|z\|_\lambda^2 > (\alpha+\beta-q) \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx,$$

which implies that

$$\begin{aligned} J_\lambda(z) = J_\lambda(z) - \frac{1}{q}\langle J_\lambda'(z), z\rangle &= \frac{q-2}{2q}\|z\|_\lambda^2 + \frac{\alpha+\beta-q}{q(\alpha+\beta)} \int_{\mathbb{R}^N} |u|^\alpha |v|^\beta dx \\ &< -\frac{2-q}{q}\left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\|z\|_\lambda^2 < 0. \end{aligned}$$

Thus we obtain that $m_\lambda^+ < 0$.

Similar to (2.10), we can derive

$$\|z\|_\lambda > \left[\frac{2-q}{\alpha+\beta-q}\left(\frac{1}{2}S_s^{-1}|\Sigma|^{\frac{2s}{N}}\right)^{(\mu-1)/p} S_s^{(2_s^*/2)^2\mu/p}\right]^{\frac{1}{\alpha+\beta-2}} \quad \text{for any} \quad z \in \mathcal{N}_\lambda^-. \tag{2.11}$$

Then for any $z \in \mathcal{N}_\lambda^- \subset \mathcal{N}_\lambda$ and by (2.6)-(2.7), we have

$$\begin{aligned} J_\lambda(z) &= J_\lambda(z) - \frac{1}{\alpha+\beta}\langle J_\lambda'(z), z\rangle = \frac{\alpha+\beta-2}{2(\alpha+\beta)}\|z\|_\lambda^2 - \frac{\alpha+\beta-q}{q(\alpha+\beta)} I_{f,g}(z) \\ &\ge \|z\|_\lambda^q\left[\left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\|z\|_\lambda^{2-q} - \left(\frac{1}{\alpha+\beta} - \frac{1}{q}\right)(|f|_\infty + |g|_\infty)|\Sigma|^{(2_s^*-q)/2_s^*} S_s^{-q/2}\right]. \end{aligned} \tag{2.12}$$

Combining (2.11) and (2.12), by (VW_4) there exists $d_0 > 0$ independent of λ such that $m_\lambda^- \ge d_0$. The proof is complete. □

The following lemma is an another version of Brézis-Lieb lemma [5], which is proved by Han [18].

Lemma 2.7. *Let $\{(u_n, v_n)\} \subset E_\lambda$ be such that $(u_n, v_n) \rightharpoonup (u, v)$ in E_λ, then*

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} \left[|u_n|^\alpha |v_n|^\beta - |u_n - u|^\alpha |v_n - v|^\beta - |u|^\alpha |v|^\beta \right] dx = 0. \tag{2.13}$$

To find the critical points of J_λ, we need the following compactness result.

Lemma 2.8. *If $1 < q < 2$, α, $\beta > 1$ with $\alpha + \beta < 2_s^*$, and assume $(VW_1) - (VW_4)$ and $(FG_1) - (FG_2)$, then there exists $D > 0$ and $\Lambda = \Lambda(D) > 0$ such that J_λ satisfies the $(PS)_c$ condition in E_λ for any $c < D$ and $\lambda > \Lambda$.*

Proof. Let $\{z_n\} = \{(u_n, v_n)\}$ be a (PS)-sequence with $c < D$. We know from Lemma 2.2 that $\{z_n\}$ is bounded in E_λ, and then there exists $z = (u, v) \in E_\lambda$ such that $z_n \rightharpoonup z$ in the sense of a subsequence. Furthermore, we may assume that $u_n \rightharpoonup u$ in E_1 and $v_n \rightharpoonup v$ in E_2. As the argument in [17], one has

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} |u_n|^{\alpha-2} u_n \varphi |v_n|^\beta dx = \int\limits_{\mathbb{R}^N} |u|^{\alpha-2} u \varphi |v|^\beta dx, \quad \text{for any} \quad \varphi \in C_0^\infty(\mathbb{R}^N)$$

and

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} |u_n|^\alpha |v_n|^{\beta-2} v_n \psi dx = \int\limits_{\mathbb{R}^N} |u|^\alpha |v|^{\beta-2} v \psi, \quad \text{for any} \quad \psi \in C_0^\infty(\mathbb{R}^N).$$

Using the Lebesgue theorem, by (FG_2) one has

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} f(x) |u_n|^{q-2} u_n \varphi dx = \int\limits_{\mathbb{R}^N} f(x) |u|^{q-2} u \varphi dx$$

and

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} g(x) |v_n|^{q-2} v_n \varphi dx = \int\limits_{\mathbb{R}^N} g(x) |v|^{q-2} v \varphi dx.$$

Hence we have

$$0 = \lim_{n\to\infty} \langle J'_\lambda(z_n), \zeta \rangle = \langle J'_\lambda(z), \zeta \rangle, \quad \text{for any} \quad \zeta = (\varphi, \psi) \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N),$$

which yields that $J'_\lambda(z) = 0$.

Now we show that $z_n \to z$ in E_λ. In fact, let $\overline{z}_n \triangleq z_n - z$ with $\overline{u}_n \triangleq u_n - u$ and $\overline{v}_n \triangleq v_n - v$. By (2.13) and (FG_2), we have

$$J_\lambda(\overline{z}_n) = J_\lambda(z_n) - J_\lambda(z) + o(1) \quad \text{and} \quad J'_\lambda(\overline{z}_n) = J'_\lambda(z_n) + o(1). \tag{2.14}$$

On the other hand, by (FG_2) we obtain

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} f(x) |\overline{u}_n|^q dx = \lim_{n\to\infty} \int\limits_{\mathbb{R}^N} g(x) |\overline{v}_n|^q dx = 0. \tag{2.15}$$

If $z \equiv 0$, $J_\lambda(z) = 0$; if $z \not\equiv 0$, $J_\lambda(z) \geq -M_0$ by Lemma 2.2. In a word, we always have $J_\lambda(z) \geq -M_0$ which together with (2.14)-(2.15) implies that

$$\begin{aligned} D+M_0 \geq c-J_\lambda(z) &= J_\lambda(\overline{z}_n)-\frac{1}{\alpha+\beta}\langle J_\lambda'(\overline{z}_n),\overline{z}_n\rangle+o(1) \\ &= \frac{\alpha+\beta-2}{2(\alpha+\beta)}\|\overline{z}_n\|_\lambda^2+o(1). \end{aligned} \tag{2.16}$$

Recalling (VW_2), one has

$$\begin{aligned} \int\limits_{\mathbb{R}^N}|\overline{u}_n\overline{v}_n|dx &= \int\limits_{\Sigma^c}|\overline{u}_n\overline{v}_n|dx+\int\limits_{\Sigma}|\overline{u}_n\overline{v}_n|dx=\int\limits_{\Sigma^c}|\overline{u}_n\overline{v}_n|dx+o(1) \\ &\leq \frac{1}{\lambda c}\int\limits_{\Sigma^c}\left(\lambda V(x)|\overline{u}_n|^2\right)^{\frac{1}{2}}\left(\lambda W(x)|\overline{v}_n|^2\right)^{\frac{1}{2}}dx+o(1)\leq\frac{1}{2\lambda c}\|\overline{z}_n\|_\lambda^2+o(1). \end{aligned} \tag{2.17}$$

It is similar to (2.1)-(2.5) and using (2.16)-(2.17) we derive

$$\begin{aligned} \int\limits_{\mathbb{R}^N}|\overline{u}_n|^\alpha|\overline{v}_n|^\beta dx &\leq \left(\frac{1}{2\lambda c}\right)^{(1-\mu)/p}S_s^{-(2_s^*/2)^2\mu/p}\|\overline{z}_n\|_\lambda^{\alpha+\beta}+o(1) \\ &\leq \left(\frac{1}{2\lambda c}\right)^{(1-\mu)/p}S_s^{-(2_s^*/2)^2\mu/p}\left(\frac{2(D+M_0)(\alpha+\beta)}{\alpha+\beta-2}\right)^{(\alpha+\beta-2)/2}\|\overline{z}_n\|_\lambda^2+o(1) \\ &\leq \frac{1}{2}\|\overline{z}_n\|_\lambda^2+o(1), \end{aligned} \tag{2.18}$$

whenever

$$\lambda\geq\Lambda=\Lambda(D)\triangleq\max\left\{\Lambda_0,2^{p/(1-\mu)}\left(\frac{1}{2c}\right)S_s^{-(2_s^*/2)^2\mu/(1-\mu)}\left(\frac{2(D+M_0)(\alpha+\beta)}{\alpha+\beta-2}\right)^{\frac{p(\alpha+\beta-2)}{2(1-\mu)}}\right\}.$$

Consequently, by (2.18) we have

$$o(1)=\langle J_\lambda'(\overline{z}_n),\overline{z}_n\rangle=\|\overline{z}_n\|_\lambda^2-\int\limits_{\mathbb{R}^N}|\overline{u}_n|^\alpha|\overline{v}_n|^\beta+o(1)\geq\frac{1}{2}\|\overline{z}_n\|_\lambda^2+o(1),$$

which yields that $\overline{z}_n\to 0$ in E_λ. The proof is complete. □

3. Proof of Theorems 1.1 and 1.2

In this section, we will prove Theorems 1.1 and 1.2. Using the Ekeland's variational principle [14] and the argument in [41] (or see [20, Proposition 4.1] for example), we have the following result.

Lemma 3.1. *Under the assumptions of Lemma 2.6, then for any $\lambda>\Lambda_0$, $J_\lambda(z)$ has two (PS) sequences $\{z_n^-\}\subset\mathcal{N}_\lambda^-$ and $\{z_n^+\}\subset\mathcal{N}_\lambda^+$ at the levels m_λ^- and m_λ^+, respectively.*

Proposition 3.2. *If $1<q<2$, $\alpha,\ \beta>1$ with $\alpha+\beta<2_s^*$, and assume $(VW_1)-(VW_4)$ and $(FG_1)-(FG_2)$. Then for each $\lambda>\Lambda_0$ the functional $J_\lambda(z)$ has a minimizer $z_\lambda^+\in\mathcal{N}_\lambda^+\subset\mathcal{N}_\lambda$ and there hold*

(i) $J_\lambda(z_\lambda^+)=m_\lambda^+=m_\lambda<0$;
(ii) z_λ^+ *is a nontrivial solution of* (1.1).

Proof. By Lemma 3.1, there exists $\{z_n^+\} = \{(u_n^+, v_n^+)\} \subset \mathcal{N}_\lambda^+$ such that it is a (PS) sequence of J_λ at the level m_λ^+. By Lemma 2.2, we know that $\{z_n^+\}$ is bounded in E_λ, passing to a subsequence if necessary, $z_n^+ \rightharpoonup z_\lambda^+ = (u_\lambda^+, v_\lambda^+)$ in E_λ.

We first claim that $z_\lambda^+ \neq 0$. Arguing it indirectly and suppose that $z_\lambda^+ \equiv 0$, hence

$$\lim_{n\to\infty} I_{f,g}(z_n^+) = I_{f,g}(z_\lambda^+) = 0,$$

which implies that

$$\begin{aligned} 0 = \langle J'_\lambda(z_n^+), z_n^+\rangle &= \|z_n^+\|_\lambda^2 - I_{f,g}(z_n^+) - \int_{\mathbb{R}^N} |u_n^+|^\alpha |v_n^+|^\beta dx \\ &= \|z_n^+\|_\lambda^2 - \int_{\mathbb{R}^N} |u_n^+|^\alpha |v_n^+|^\beta dx + o(1). \end{aligned}$$

Therefore we have

$$\begin{aligned} J_\lambda(z_n^+) &= \frac{1}{2}\|z_n^+\|_\lambda^2 - \frac{1}{q} I_{f,g}(z_n^+) - \frac{1}{\alpha+\beta}\int_{\mathbb{R}^N} |u_n^+|^\alpha |v_n^+|^\beta dx \\ &= \left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\|z_n^+\|_\lambda^2 + o(1), \end{aligned}$$

which is a contradiction to $\lim_{n\to\infty} J_\lambda(z_n^+) = m_\lambda < 0$ by Lemma 2.6. Hence $z_\lambda^+ \neq 0$. By the property of weak convergence, it is easy to see that $z_\lambda^+ \in \mathcal{N}_\lambda^+$.

Now we show that $z_n^+ \to z_\lambda^+$ in E_λ. Suppose the contrary, that is, $\|z^+\|_\lambda < \liminf_{n\to\infty} \|z_n^+\|_\lambda$, then

$$\begin{aligned} m_\lambda^+ \le J_\lambda(z_\lambda^+) &= J_\lambda(z_\lambda^+) - \frac{1}{\alpha+\beta}\langle J'_\lambda(z_\lambda^+), z_\lambda^+\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\|z_\lambda^+\|_\lambda^2 - \left(\frac{1}{q} - \frac{1}{\alpha+\beta}\right) I_{f,g}(z_\lambda^+) \\ &< \liminf_{n\to\infty} J_\lambda(z_n^+) = m_\lambda^+, \end{aligned}$$

a contradiction! Consequently, $z_n^+ \to z_\lambda^+$ in E_λ and $J_\lambda(z_\lambda^+) = m_\lambda^+$. In particular, z_λ^+ is a nontrivial solution of (1.1) by Lemma 2.3. The proof is complete. □

In view of Lemma 2.8, it is necessary to estimate m_λ^- carefully. To do it, we choose two nonzero functions, φ_{Ω_1} and ψ_{Ω_2}, to satisfy $\varphi_{\Omega_1} \in C_0^\infty(\Omega_1)$ and $\psi_{\Omega_2} \in C_0^\infty(\Omega_2)$, where Ω_1 and Ω_2 are given by (VW_4). Set $e = (\varphi_{\Omega_1}, \psi_{\Omega_2}) \in E_\lambda$, then by (VW_3) we have

$$J_\lambda(te) = \frac{t^2}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\varphi_{\Omega_1}|^2 + |(-\Delta)^{\frac{s}{2}}\psi_{\Omega_2}|^2 dx - \frac{t^q}{q} I_{f,g}(e) - \frac{t^{\alpha+\beta}}{\alpha+\beta}\int_{\mathbb{R}^N} |\varphi_{\Omega_1}|^\alpha |\psi_{\Omega_2}|^\beta dx.$$

For any $\lambda \ge \Lambda_0$, by Lemma 2.5 there exist positive constants t_0^- and D_0 independent of λ such that $t_0^- e \in \mathcal{N}_\lambda^-$ and

$$\sup_{t\ge 0} J_\lambda(te) = J_\lambda(t_0^- e) = D_0 > 0.$$

Therefore we have

$$m_\lambda^- \leq D_0 < +\infty, \quad \text{for any} \quad \lambda \geq \Lambda_0. \tag{3.1}$$

We now establish another solution of (1.1).

Proposition 3.3. *If $1 < q < 2$, α, $\beta > 1$ with $\alpha + \beta < 2_s^*$ and assume $(VW_1) - (VW_4)$ and $(FG_1) - (FG_2)$, then for each $\lambda \geq \Lambda = \Lambda(D_0)$ the functional $J_\lambda(z)$ has a minimizer $z_\lambda^- \in \mathcal{N}_\lambda^- \subset \mathcal{N}_\lambda$ and there hold*

(i) $J_\lambda(z_\lambda^-) = m_\lambda^- > 0$;
(ii) z_λ^- *is a nontrivial solution of* (1.1).

Proof. By Lemma 3.1, there exists $\{z_n^-\} = \{(u_n^-, v_n^-)\} \subset \mathcal{N}_\lambda^-$ such that it is a (PS) sequence of J_λ at the level m_λ^-. By Lemma 2.2, we know that $\{z_n^-\}$ is bounded in E_λ, passing to a subsequence if necessary, $z_n^- \rightharpoonup z_\lambda^- = (u_\lambda^-, v_\lambda^-)$ in E_λ. Recalling (3.1), we set

$$\Lambda = \Lambda(D_0) \triangleq \max\left\{\Lambda_0, 2^{p/(1-\mu)}\left(\frac{1}{2c}\right)S_s^{-(2_s^*/2)^2\mu/(1-\mu)}\left(\frac{2(D_0 + M_0)(\alpha+\beta)}{\alpha+\beta-2}\right)^{\frac{p(\alpha+\beta-2)}{2(1-\mu)}}\right\},$$

up to a sequence if necessary, we obtain $z_n^- \to z_\lambda^-$ in E_λ by Lemma 2.8. Hence $J_\lambda(z_\lambda^-) = m_\lambda^- > 0$ by Lemma 2.6 and z_λ^- is a nontrivial solution of (1.1) by Lemma 2.3. $\square$

We are in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Combining Propositions 3.2 and 3.3, the system (1.1) has two nontrivial solutions $z_\lambda^\pm$ satisfying

$$J_\lambda(z_\lambda^+) = m_\lambda^+ < 0 < d_0 \leq J_\lambda(z_\lambda^-) = m_\lambda^-.$$

The remainder is to show $z_\lambda^\pm$ are positive. Indeed, we set $|z_\lambda^\pm| \triangleq (|u_\lambda^\pm|, |v_\lambda^\pm|)$ then $J_\lambda(z_\lambda^\pm) = J_\lambda(|z_\lambda^\pm|) = m_\lambda^\pm$ and $|z_\lambda^\pm| \in \mathcal{N}_\lambda^\pm$. By Lemma 2.3, $|z_\lambda^\pm|$ are solutions of (1.1) and hence we may assume that $z_\lambda^\pm = |z_\lambda^\pm|$ are nonnegative. Moreover, if there exists $x_0 \in \mathbb{R}^N$ such that $u_\lambda^\pm(x_0) = v_\lambda^\pm(x_0) = 0$, then $(-\Delta)^s z_\lambda^\pm(x_0) = \big((-\Delta)^s u_\lambda^\pm(x_0), (-\Delta)^s v_\lambda^\pm(x_0)\big) = (0,0)$ and thus

$$(-\Delta)^s z_\lambda^\pm(x_0) = -\frac{1}{2}C_N(s)\int\limits_{\mathbb{R}^N}\frac{z_\lambda^\pm(x_0+y) + z_\lambda^\pm(x_0-y) - 2z_\lambda^\pm(x_0)}{|y|^{N+2s}}dy,$$

which gives that

$$\int\limits_{\mathbb{R}^N}\frac{z_\lambda^\pm(x_0+y) + z_\lambda^\pm(x_0-y)}{|y|^{N+2s}}dy = 0.$$

Therefore we obtain $z_\lambda^\pm(x) \equiv (0,0)$, a contradiction! Hence $z_\lambda^\pm$ are positive. $\square$

Next, we investigate the concentration result for the solutions obtained in Theorem 1.1 and give the proof of Theorem 1.2.

Proof of Theorem 1.2. For any sequence $\lambda_n \to +\infty$, let $z_{\lambda_n}^\pm = z_n^\pm = (u_n^\pm, v_n^\pm)$ be the critical points of J_{λ_n} obtained in Theorem 1.1. In view of Lemma 2.6 and (3.1), we have

$$
\begin{aligned}
D_0 &\geq J_{\lambda_n}(z_n^{\pm}) - \frac{1}{\alpha+\beta}\langle J'_{\lambda_n}(z_n^{\pm}), z_n^{\pm}\rangle = \left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\|z_n^{\pm}\|_{\lambda_n}^2 - \left(\frac{1}{q} - \frac{1}{\alpha+\beta}\right)I_{f,g}(z_n^{\pm}) \\
&\geq \left(\frac{1}{2} - \frac{1}{\alpha+\beta}\right)\|z_n^{\pm}\|_{\lambda_n}^2 - \left(\frac{1}{q} - \frac{1}{\alpha+\beta}\right)(|f|_\infty + |g|_\infty)|\Sigma|^{(2_s^*-q)/2_s^*} S_s^{-q/2}\|z_n^{\pm}\|_{\lambda_n}^q,
\end{aligned}
$$

which yields that $\|z_n^{\pm}\|_{\lambda_n}$ are bounded. In particular, we have $\|z_n^{\pm}\|$ are bounded, passing to a subsequence if necessary, there exist $z^{\pm} = (u^{\pm}, v^{\pm}) \in E$ such that $z_n^{\pm} \rightharpoonup z^{\pm}$ in E. For any $\varphi \in C_0^\infty(\Omega_1)$, then inserting $(\varphi, 0)$ into (1.3) there holds

$$
\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n^{\pm}(-\Delta)^{\frac{s}{2}}\varphi dx = \int_{\mathbb{R}^N} f(x)|u_n^{\pm}|^{q-2}u_n^{\pm}\varphi dx + \frac{\alpha}{\alpha+\beta}\int_{\mathbb{R}^N} |u_n^{\pm}|^{\alpha-2}u_n^{\pm}\varphi|v_n^{\pm}|^{\beta}dx,
$$

which implies that

$$
\int_{\Omega_1\cup\Omega_2} (-\Delta)^{\frac{s}{2}} u^{\pm}(-\Delta)^{\frac{s}{2}}\varphi dx = \int_{\Omega_1\cup\Omega_2} f(x)|u^{\pm}|^{q-2}u^{\pm}\varphi dx + \frac{\alpha}{\alpha+\beta}\int_{\Omega_1\cup\Omega_2} |u^{\pm}|^{\alpha-2}u^{\pm}\varphi|v^{\pm}|^{\beta}dx,
$$

for any $\varphi \in C_0^\infty(\Omega_1)$. Similarly, for any $\psi \in C_0^\infty(\Omega_2)$,

$$
\int_{\Omega_1\cup\Omega_2} (-\Delta)^{\frac{s}{2}} v^{\pm}(-\Delta)^{\frac{s}{2}}\psi dx = \int_{\Omega_1\cup\Omega_2} g(x)|v^{\pm}|^{q-2}v^{\pm}\psi dx + \frac{\beta}{\alpha+\beta}\int_{\Omega_1\cup\Omega_2} |u^{\pm}|^{\alpha}|v^{\pm}|^{\beta-2}v^{\pm}\psi dx.
$$

We now show that $u^{\pm} \equiv 0$ in Ω_1^c. By $u_n^{\pm} \to u^{\pm}$ a.e. in $\mathbb{R}^N$ one has

$$
0 \leq \int_{\mathbb{R}^N\setminus V^{-1}(0)} V(x)|u^{\pm}|^2 dx \leq \liminf_{n\to\infty}\int_{\mathbb{R}^N} V(x)|u_n^{\pm}|^2 dx \leq \frac{1}{\lambda_n}\|z_n^{\pm}\|_{\lambda_n}^2 \to 0 \ \text{ as } \ n \to \infty,
$$

which implies that $u^{\pm} = 0$ a.e. in $V^{-1}(0)$. In view of (VW_3), we know that $u^{\pm} \in H_0^s(\Omega_1)$. Similarly, we have $v^{\pm} \in H_0^s(\Omega_2)$. Thus $z^{\pm} = (u^{\pm}, v^{\pm})$ are solutions of (1.5). We claim that

Claim 1: $z_n^{\pm} \to z^{\pm}$ in E.

Denote $\kappa_n^{\pm} \triangleq u_n^{\pm} - u^{\pm}$ and $\omega_n^{\pm} \triangleq v_n^{\pm} - v^{\pm}$, then $\tau_n^{\pm} \triangleq z_n^{\pm} - z^{\pm} = (\kappa_n^{\pm}, \omega_n^{\pm})$. Since we have demonstrated that $\|z_n^{\pm}\|_{\lambda_n}$ are bounded, without loss of generality, we can assume there exist constants $M^{\pm} > 0$ independent of n such that $\|z_n^{\pm}\|_{\lambda_n} \leq M^{\pm} < +\infty$. Consequently, we have $\|z^{\pm}\|_{\lambda_n} \leq M^{\pm}$ and then $\|\tau_n^{\pm}\|_{\lambda_n} \leq 2M^{\pm}$. In view of (VW_2),

$$
\begin{aligned}
\int_{\mathbb{R}^N} |\kappa_n^{\pm}\omega_n^{\pm}|dx &= \int_{\Sigma^c} |\kappa_n^{\pm}\omega_n^{\pm}|dx + \int_{\Sigma} |\kappa_n^{\pm}\omega_n^{\pm}|dx = \int_{\Sigma^c} |\kappa_n^{\pm}\omega_n^{\pm}|dx + o(1) \\
&\leq \frac{1}{\lambda_n c}\int_{\Sigma^c} \left(\lambda_n V(x)|\kappa_n^{\pm}|^2\right)^{\frac{1}{2}}\left(\lambda_n W(x)|\omega_n^{\pm}|^2\right)^{\frac{1}{2}} dx + o(1) \\
&\leq \frac{1}{2\lambda_n c}\|\tau_n^{\pm}\|_{\lambda_n}^2 + o(1) \leq \frac{2(M^{\pm})^2}{\lambda_n c} + o(1) = o(1).
\end{aligned}
$$

Similar to (2.4), the above formula gives that

$$
\int_{\mathbb{R}^N} |\kappa_n^{\pm}\omega_n^{\pm}|^p dx \leq S_s^{-2_s^*\mu/2}\|\tau_n^{\pm}\|_{\lambda_n}^{2_s^*\mu}\left(\int_{\mathbb{R}^N} |\kappa_n^{\pm}\omega_n^{\pm}|dx\right)^{1-\mu} = o(1).
$$

As (2.5), we have

$$\int_{\mathbb{R}^N} |\kappa_n^{\pm}|^{\alpha}|\omega_n^{\pm}|^{\beta}dx \leq S_s^{-\frac{\alpha+\beta-2}{2}}\|\tau_n^{\pm}\|_{\lambda_n}^{\alpha+\beta-2}\Bigg(\int_{\mathbb{R}^N} |\kappa_n^{\pm}\omega_n^{\pm}|^{p}dx\Bigg)^{\frac{1}{p}} = o(1). \tag{3.2}$$

Combining (2.13) and (3.2), there holds

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n^{\pm}|^{\alpha}|v_n^{\pm}|^{\beta}dx = \int_{\mathbb{R}^N} |u^{\pm}|^{\alpha}|v^{\pm}|^{\beta}dx = \int_{\Omega_1\cup\Omega_2} |u^{\pm}|^{\alpha}|v^{\pm}|^{\beta}dx. \tag{3.3}$$

Therefore by (3.3) we derive

$$\begin{aligned}\|z^{\pm}\|^2 &\leq \lim_{n\to\infty}\|z_n^{\pm}\|^2 \leq \lim_{n\to\infty}\|z_n^{\pm}\|_{\lambda_n}^2 = \lim_{n\to\infty}\Bigg(I_{f,g}(z_n^{\pm}) + \int_{\mathbb{R}^N} |u_n^{\pm}|^{\alpha}|v_n^{\pm}|^{\beta}dx\Bigg)\\ &= \int_{\Omega_1\cup\Omega_2}\Big[f(x)|u^{\pm}|^{q} + g(x)|v^{\pm}|^{q}\Big]dx + \int_{\Omega_1\cup\Omega_2} |u^{\pm}|^{\alpha}|v^{\pm}|^{\beta}dx \leq \|z^{\pm}\|^2,\end{aligned}$$

where we have used the fact that $z^{\pm}\in H_0^s(\Omega_1)\times H_0^s(\Omega_2)$ are solutions of (1.5) in the last inequality. Thus we complete the proof of Claim 1.

Claim 2: $z^- \neq 0$.

Arguing (2.3) for $\lambda\equiv 1$, and repeating the proving process of (2.5) we have

$$\int_{\mathbb{R}^N} |u_n^{-}|^{\alpha}|v_n^{-}|^{\beta}dx \leq \Bigg(\frac{1}{2}\max\Big\{S_s^{-1}|\Sigma|^{\frac{2s}{N}}, \frac{1}{c}\Big\}\Bigg)^{(1-\mu)/p} S_s^{-(2_s^*/2)^2\mu/p}\|z_n^-\|^{\alpha+\beta} \triangleq C_0\|z_n^-\|^{\alpha+\beta}.$$

In view of (2.8), we obtain

$$(2-q)\|z_n^-\|^2 \leq (2-q)\|z_n^-\|_{\lambda_n}^2 < (\alpha+\beta-q)\int_{\mathbb{R}^N} |u_n^{-}|^{\alpha}|v_n^{-}|^{\beta}dx \leq C_0(\alpha+\beta-q)\|z_n^-\|^{\alpha+\beta},$$

which together with $z_n^- \to z^-$ in E implies that $z^-\neq 0$. The proof of Claim 2 is complete.

- $\Theta_f\cap\Omega_1=\emptyset$ and $\Theta_g\cap\Omega_2=\emptyset$.

In view of (2.9), we have

$$\begin{aligned}\|z^+\|^2 &= \lim_{n\to\infty}\|z_n^+\|^2 \leq \lim_{n\to\infty}\|z_n^+\|_{\lambda_n}^2 \leq \frac{\alpha+\beta-q}{\alpha+\beta-2}\lim_{n\to\infty} I_{f,g}(z_n^+) = \frac{\alpha+\beta-q}{\alpha+\beta-2} I_{f,g}(z^+)\\ &= \frac{\alpha+\beta-q}{\alpha+\beta-2}\Bigg(\int_{\Omega_1} f(x)|u^+|^q dx + \int_{\Omega_2} g(x)|v^+|^q dx\Bigg) \leq 0,\end{aligned}$$

which yields that $z^+\equiv 0$.

- $|\Theta_f\cap\Omega_1|>0$ and $|\Theta_g\cap\Omega_2|>0$.

Choosing $\varphi_{\Theta_f\cap\Omega_1}\in C_0^{\infty}(\Theta_f\cap\Omega_1)$ and $\psi_{\Theta_g\cap\Omega_2}\in C_0^{\infty}(\Theta_g\cap\Omega_2)$ to be nontrivial, where Ω_1, Ω_2 and Θ_f, Θ_g are given by (VW_3) and (FG_1). Set $e=(\varphi_{\Theta_f\cap\Omega_1},\psi_{\Theta_f\cap\Omega_2})\in E$, then

$$J_{\lambda_n}(te) = \frac{t^2}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\varphi_{\Theta_f\cap\Omega_1}|^2 + |(-\Delta)^{\frac{s}{2}}\psi_{\Theta_g\cap\Omega_2}|^2 - \frac{t^{\alpha+\beta}}{\alpha+\beta}\int_{\mathbb{R}^N} |\varphi_{\Theta_f\cap\Omega_1}|^\alpha |\psi_{\Theta_g\cap\Omega_2}|^\beta dx$$
$$-\frac{t^q}{q}\Big(\int_{\Theta_f\cap\Omega_1} f(x)|\varphi_{\Theta_f\cap\Omega_1}|^q dx + \int_{\Theta_g\cap\Omega_2} g(x)|\varphi_{\Theta_g\cap\psi_2}|^q dx\Big).$$

For any $\lambda_n \geq \Lambda_0$, by Lemma 2.5 there exist constants $t_0^+ > 0$ and $\varrho_0 < 0$ independent of λ_n such that $t_0^+ e \in \mathcal{N}_{\lambda_n}^+$ and

$$\inf_{0\leq t\leq t_0^+} J_{\lambda_n}(te) = J_{\lambda_n}(t_0^+ e) = \varrho_0 < 0,$$

which implies that $J_{\lambda_n}(z_n^+) = m_{\lambda_n}^+ \leq \varrho_0 < 0$. Set

$$J(z) = \frac{1}{2}\int_{\Omega_1\cup\Omega_2} |(-\Delta)^{\frac{s}{2}}u|^2 + |(-\Delta)^{\frac{s}{2}}v|^2 dx - \frac{1}{q}I_{f,g}|_{\Omega_1\cup\Omega_2}(z) - \frac{1}{\alpha+\beta}\int_{\Omega_1\cup\Omega_2} |u|^\alpha|v|^\beta dx,$$

where

$$I_{f,g}|_{\Omega_1\cup\Omega_2}(z) = \int_{\Omega_1\cup\Omega_2} f(x)|u|^q + g(x)|v|^q dx.$$

Thus by $z^+ \in H_0^s(\Omega_1)\times H_0^s(\Omega_2)$,

$$J(z^+) = \frac{1}{2}\|z^+\|^2 - \frac{1}{q}I_{f,g}(z^+) - \frac{1}{\alpha+\beta}\int_{\mathbb{R}^N} |u^+|^\alpha|v^+|^\beta dx \leq \lim_{n\to\infty} J_{\lambda_n}(z_n^+) \leq \varrho_0 < 0,$$

and then $z^+ \neq 0$. The remainder is to show $z^+ \neq z^-$. In fact,

$$\begin{aligned}
0 < d_0 \leq J_{\lambda_n}(z_n^-) &= J_{\lambda_n}(z_n^-) - \frac{1}{2}\langle J'_{\lambda_n}(z_n^-), z_n^-\rangle \\
&= \Big(\frac{1}{2}-\frac{1}{q}\Big)I_{f,g}(z_n^-) + \Big(\frac{1}{2}-\frac{1}{\alpha+\beta}\Big)\int_{\mathbb{R}^N} |u_n^-|^\alpha|v_n^-|^\beta dx \\
&= \Big(\frac{1}{2}-\frac{1}{q}\Big)I_{f,g}(z^-) + \Big(\frac{1}{2}-\frac{1}{\alpha+\beta}\Big)\int_{\mathbb{R}^N} |u^-|^\alpha|v^-|^\beta dx + o(1) \\
&= J(z^-) - \frac{1}{2}\langle J'(z^-), z^-\rangle + o(1) = J(z^-) + o(1),
\end{aligned}$$

which gives that $J(z^+) \leq \varrho_0 < 0 < d_0 \leq J(z^-)$. Therefore $z^+ \neq z^-$. □

Acknowledgments

The author would like to thank the anonymous referee for carefully reading the manuscript and valuable comments that greatly helped improve this paper. The author was supported by NSFC (Grant No. 11371158, 11771165), the program for Changjiang Scholars and Innovative Research Team in University (No. IRT13066) and the excellent doctoral dissertation cultivation from Central China Normal University (Grant No. 2018YBZZ068).

References

[1] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.
[2] A. Ambrosetti, E. Colorado, D. Ruiz, Multi-bump solitons to linearly coupled systems of nonlinear Schrödinger equations, Calc. Var. Partial Differential Equations 30 (2007) 85–112.
[3] T. Bartsch, Z. Wang, Existence and multiplicity results for superlinear elliptic problems on $\mathbb{R}^N$, Comm. Partial Differential Equations 20 (1995) 1725–1741.
[4] G. Bisci, V. Rădulescu, Ground state solutions of scalar field fractional Schrödinger equations, Calc. Var. Partial Differential Equations 54 (2015) 2985–3008.
[5] H. Brézis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 88 (1983) 486–490.
[6] K. Brown, Y. Zhang, The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function, J. Differential Equations 193 (2003) 481–499.
[7] K. Brown, T. Wu, A semilinear elliptic system involving nonlinear boundary condition and sign-changing weigh function, J. Math. Anal. Appl. 337 (2008) 1326–1336.
[8] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32 (2007) 1245–1260.
[9] C. Chen, Y. Kuo, T. Wu, The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions, J. Differential Equations 250 (2011) 1876–1908.
[10] Z. Chen, W. Zou, Positive least energy solutions and phase separation for coupled Schrödinger equations with critical exponent, Arch. Ration. Mech. Anal. 205 (2012) 515–551.
[11] Y. Cheng, T. Wu, Multiplicity and concentration of positive solutions for semilinear elliptic equations with steep potential, Commun. Pure Appl. Anal. 15 (2016) 2457–2473.
[12] A. Cotsiolis, N. Tavoularis, Best constants for Sobolev inequalities for higher order fractional derivatives, J. Math. Anal. Appl. 295 (2004) 225–236.
[13] P. Drábek, S. Pohozaev, Positive solutions for the p-Laplacian: application of the fibering method, Proc. Roy. Soc. Edinburgh Sect. A 127 (1997) 703–726.
[14] I. Ekeland, Nonconvex minimization problems, Bull. Amer. Math. Soc. 1 (1979) 443–473.
[15] P. Felmer, A. Quaas, J. Tan, Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian, Proc. Roy. Soc. Edinburgh Sect. A 142 (2012) 1237–1262.
[16] D. Figueiredot, E. Mitidieri, A maximum principle for an elliptic system and applications to semilinear problems, SIAM J. Math. Anal. 17 (1986) 836–849.
[17] M. Furtado, E. Silva, M. Xavier, Multiplicity and concentration of solutions for elliptic systems with vanishing potentials, J. Differential Equations 249 (2010) 2377–2396.
[18] P. Han, The effect of the domain topology on the number of positive solutions of an elliptic system involving Sobolev exponents, Houston J. Math. 32 (2006) 1241–1257.
[19] X. He, W. Zou, Existence and concentration result for the fractional Schrödinger equations with critical nonlinearities, Calc. Var. Partial Differential Equations 55 (2016) 1–39.
[20] T. Hsu, Multiple positive solutions for a critical quasilinear elliptic system with concave-convex nonlinearities, Nonlinear Anal. 71 (2009) 2688–2698.
[21] N. Ikoma, Existence of solutions of scalar field equations with fractional operator, J. Fixed Point Theory Appl. 19 (2017) 649–690.
[22] N. Laskin, Fractional quantum mechanics and Lévy path integrals, Phys. Lett. A 268 (2000) 298–305.
[23] N. Laskin, Fractional Schrödinger equation, Phys. Rev. E 66 (2002) 56.
[24] Csésar E. Torres Ledesma, Existence and concentration of solutions for a non-linear fractional Schrödinger equation with steep potential well, Commun. Pure Appl. Anal. 15 (2016) 535–547.
[25] P.L. Lions, The concentration-compactness principle in the calculus of variation. The locally compact case. Part I, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 109–145.
[26] P.L. Lions, The concentration-compactness principle in the calculus of variation. The locally compact case. Part II, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 223–283.
[27] Y. Liu, Z. Liu, Multiplicity and concentration of solutions for fractional Schrödinger equation with sublinear perturbation and steep potential, Comput. Math. Appl. 72 (2016) 1629–1640.
[28] W. Liu, Existence of multi-bump solutions for the fractional Schrödinger-Poisson system, J. Math. Phys. 57 (2016) 091502.
[29] D. Lv, J. Xiao, Existence and multiplicity results for a coupled system of Kirchhoff type equations, Electron. J. Qual. Theory Differ. Equ. 6 (2014) 1–10.
[30] D. Lv, S. Peng, Existence and asymptotic behavior of vector solutions for coupled nonlinear Kirchhoff-type systems, J. Differential Equations 263 (2017) 8947–8978.
[31] R. Mandel, Minimal energy solutions for repulsive nonlinear Schrödinger systems, J. Differential Equations 257 (2014) 450–468.
[32] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012) 521–573.
[33] S. Peng, W. Shuai, Q. Wang, Multiple positive solutions for linearly coupled nonlinear elliptic systems with critical exponent, J. Differential Equations 263 (2017) 709–731.
[34] S. Peng, Q. Wang, Z. Wang, On coupled nonlinear Schrödinger systems with mixed couplings, Trans. Amer. Math. Soc. (2019), https://doi.org/10.1090/tran/7383, in press.
[35] S. Secchi, Ground state solutions for nonlinear fractional Schrödinger equations in $\mathbb{R}^N$, J. Math. Phys. 54 (2013) 031501.

[36] L. Shen, Existence result for fractional Schrödinger-Poisson systems involving a Bessel operator without Ambrosetti-Rabinowitz condition, Comput. Math. Appl. 75 (2018) 296–306.
[37] L. Shen, X. Yao, Least energy solutions for a class of fractional Schrödinger-Poisson systems, J. Math. Phys. 59 (2018) 081501.
[38] L. Shen, Multiplicity and concentration results for fractional Schrödinger-Poisson systems involving a Bessel operator, Math. Methods Appl. Sci. 41 (2018) 7599–7611.
[39] J. Sun, T. Wu, Ground state solutions for an indefinite Kirchhoff type problem with steep potential well, J. Differential Equations 256 (2014) 1771–1792.
[40] J. Sun, T. Wu, On the nonlinear Schrödinger-Poisson systems with sign-changing potential, Z. Angew. Math. Phys. 66 (2015) 1649–1669.
[41] G. Tarantello, On nonhomogeneous elliptic involving critical Sobolev exponent, Ann. Inst. H. Poincaré Anal. Non Linéaire 9 (1992) 281–304.
[42] M. Willem, Minimax Theorems, Birkhäuser, Boston, 1996.
[43] Y. Ye, C. Tang, Existence and multiplicity of solutions for Schrödinger-Poisson equations with sign-changing potential, Calc. Var. Partial Differential Equations 53 (2015) 383–411.
[44] L. Zhao, H. Liu, F. Zhao, Existence and concentration of solutions for Schrödinger-Poisson equations with steep well potential, J. Differential Equations 255 (2013) 1–23.