J. Math. Anal. Appl. 475 (2019) 1304–1316

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On existence and structure of semiattractors for dynamical systems represented by cocycles

Grzegorz Guzik

AGH University of Science and Technology, Faculty of Applied Mathematics, Al. Mickiewicza 30, 30-059 Krakow, Poland

A R T I C L E   I N F O

Article history:
Received 27 April 2018
Available online 12 March 2019
Submitted by M. Quincampoix

Dedicated to Professor Marek C. Zdun on the occasion of his 70th birthday

Keywords:
Cocycle
Semiattractor
Weak contractivity
Iterated function system
Topological limit
Fibers of semiattractor

A B S T R A C T

We present a new sufficient condition for existence of semiattractors for set-valued semiflows of state multifunctions associated with general cocycle mappings. We assume that a given cocycle has at least one weakly pullback contracting fiber. In particular, one can apply the result to iterated function systems having no contracting mappings. We also introduce and analyze a fiber structure of semiattractors.

1. Introduction

The present paper contains a continuation of our developments in [13], where we discuss properties of cocycle mappings admitting some kind of contractivity along every fiber from the topological point of view. This leads (under some weak assumptions) to the conclusion that the limit set of a pullback trajectory of any bounded set is a singleton uniquely determined for any fiber. A natural question arises: what happens when we have at least one fiber having such a property (we call it here weak pullback contractivity along a fiber, see Section 4 below, cf. also [24])? We prove that in this case we obtain a semiattractor for a set-valued semiflow of so-called state multifunctions. This set can be 'produced' from the union of limit sets along weakly contractive fibers. Semiattractors were first introduced by A. Lasota and J. Myjak (see [21–23]) in the context of iterated function systems, single multifunctions as well as supports of ergodic measures for some transition Markov operators acting on measures. It is remarkable

E-mail address: guzik@agh.edu.pl.

https://doi.org/10.1016/j.jmaa.2019.03.014
0022-247X/

that semiattractors, called also semifractals, being unique closed minimal positively invariant sets, are natural generalization of global strict attractors of iterated function systems and fractal sets deeply studied during last three decades (see for example a survey paper [3]). During the last few years the present author developed new results on semiattractors and other minimal invariant closed sets in the context of general cocycle mappings and semiflows of lower semicontinuous multifunctions [9–12]. In particular, it was shown therein that such sets could be used in studying of the asymptotic behavior of random and nonautonomous dynamical systems, where cocycle mappings arise naturally (for the basic theory see, for example, monographs [1], [6] and [19]). Notice that the theory of such systems is still growing area of nonlinear dynamics (see among others some recent papers [2], [15] [25] and [26]). It is interesting that if there exists a unique probability measure attractive with respect to a family of associated Markov operators, then the semiattractor of a considered system always exists and contains the support of this measure (see [23], [10] and [14]).

As in our previous works we use topological (Kuratowski's) limits instead of commonly used Hausdorff–Pompeiu metric. This approach lets us to consider non-compact attracting sets. Semiattractors being closed sets are often even unbounded.

The paper is organized as follows. The second section contains some useful preliminary results on topological limits, lower semicontinuous multifunctions and semiattractors of set-valued semiflows. In Section 3 we bring together basic facts on cocycle mappings as well as iterated function systems. In Section 4 we introduce a notion of weak pullback contractivity along a fiber, we present some important facts on long time behavior of weak contractive trajectories and the main result (Theorem 4.6) on the existence and a form of semiattractors of such systems. There is an important and new corollary on the behavior of iterated function systems (see Remark 4.7 below): an IFS needs not to contain any contracting mapping, but only at least one weakly contracting word (fiber) to posses the semiattractor. The last section is devoted to fiber structure of semiattractors. In fact, this is the very first trial to look into an internal dynamics of semiattractors. We show that the family of fibers is positively invariant with respect to the cocycle mapping. We prove also that the semiattractor is equal to the closure of the union of its fibers. It is a generalization of known results on a structure of compact attractors of finite iterated function systems (see [18, Proposition 4.3.2], and also [3, Sect. 6]).

2. Preliminaries

Topological limits. Let (X, ϱ) be a metric space. By $B^o(x, \varepsilon)$ we denote the open ball with a center x and a radius ε, a symbol $\mathrm{cl}A$ stays for the closure of $A \subset X$.

Let us recall some basic definitions and notions concerning nets of sets and topological (Kuratowski's) limits (see [5, Ch. 2]). Let $(\Sigma, \leq)$ be a directed set. Any mapping $S : \Sigma \ni \sigma \mapsto A_\sigma \in 2^X$ is called a *net of subsets* of X and denoted as $S = (A_\sigma)_{\sigma\in\Sigma}$. We say that a set $U \subset X$ intersects *almost all* (or *eventually*) sets A_σ if there is a $\sigma_0 \in \Sigma$ such that $A_\sigma \cap U \neq \emptyset$, for every $\sigma \geq \sigma_0$, and we say that U intersects *infinitely many* (or *frequently*) sets A_σ if for every $\sigma_0 \in \Sigma$ there is a $\sigma \geq \sigma_0$ such that $A_\sigma \cap U \neq \emptyset$ holds.

We define the *lower limit* (or *interior limit*) $\liminf_\sigma A_\sigma$ and the *upper limit* (or *exterior limit*) $\limsup_\sigma A_\sigma$ as follows: $x \in \liminf_\sigma A_\sigma$ if for every $\varepsilon > 0$ the ball $B^o(x, \varepsilon)$ intersects almost all sets A_σ, and $x \in \limsup_\sigma A_\sigma$ if for every $\varepsilon > 0$ the ball $B^o(x, \varepsilon)$ intersects infinitely many sets A_σ. If both limits are equal we say that the net $(A_\sigma)_{\sigma\in\Sigma}$ is *topologically convergent*. We denote this common limit as $\lim_\sigma A_\sigma$ and call it a *topological limit* of this net.

One can see that $\liminf_\sigma A_\sigma = \liminf_\sigma\ \mathrm{cl}A_\sigma$ (the same is valid for the upper limit) and $\liminf_\sigma A_\sigma$, $\limsup_\sigma A_\sigma$ are closed sets. Moreover, if $A = \mathrm{cl}\ A$ and $A_\sigma \subset A$ for every $\sigma \in \Sigma$, then $\liminf_\sigma A_\sigma \subset A$ and $\limsup_\sigma A_\sigma \subset A$.

It is clear that the following inclusions hold

$$\bigcap_{\sigma\in\Sigma} \mathrm{cl} A_\sigma \subset \liminf_\sigma A_\sigma \subset \limsup_\sigma A_\sigma \subset \mathrm{cl} \bigcup_{\sigma\in\Sigma} A_\sigma.$$

It can be also verified that the following lemma is true.

Lemma 2.1. *Let $(A_\sigma)_{\sigma\in\Sigma}$ be a net of nonempty subsets of a metric space X. If it is increasing, i.e. $A_{\sigma_1} \subset A_{\sigma_2}$ for every $\sigma_1 < \sigma_2$, then it is topologically convergent, moreover*

$$\lim_\sigma A_\sigma = \mathrm{cl} \bigcup_{\sigma\in\Sigma} A_\sigma.$$

Similarly, if it is decreasing, i.e. $A_{\sigma_1} \subset A_{\sigma_2}$ for every $\sigma_1 > \sigma_2$, then it is topologically convergent, moreover

$$\lim_\sigma A_\sigma = \bigcap_{\sigma\in\Sigma} \mathrm{cl}\, A_\sigma.$$

It is obvious that if a net $(A_\sigma)_{\sigma\in\Sigma}$ is topologically convergent to a nonempty set A, then

$$\lim_\sigma dist(A_\sigma, A) = 0.$$

Here $dist(A, B) = \sup_{x\in A} \varrho(x, B)$.

Other properties of topological limits can be found in [5] and also, in the case of countable sequences, in [20].

Lower semicontinuous multifunctions. Assume that X and Y are nonempty sets. By a *multifunction* $F : X \to \mathcal{P}(Y)$ we mean the mapping from X with values in the family $\mathcal{P}(Y)$ of all nonempty subsets of Y. Given multifunction $F : X \to \mathcal{P}(Y)$ and subsets $A \in \mathcal{P}(X)$, $B \in \mathcal{P}(Y)$ we define sets

$$F(A) := \bigcup_{x\in A} F(x)$$

and

$$F^-(B) := \{x \in X : \ F(x) \cap B \neq \emptyset\}.$$

If in addition Z is a nonempty set and $F : X \to \mathcal{P}(Y)$, $G : Y \to \mathcal{P}(Z)$ are given multifunctions we define the *composition* $G \circ F$ of F and G as a multifunction $G \circ F : X \to \mathcal{P}(Z)$ given by $G \circ F(x) = G(F(x))$.

In all that follows we mostly deal with lower semicontinuous multifunctions, so let now X and Y be topological spaces. A multifunction $F : X \to \mathcal{P}(Y)$ is said to be *lower semicontinuous* (we will write *l.s.c.* for short) if for every open set $V \subset Y$ the set $F^-(V)$ is open in X.

It is known that the below equivalences hold (see, for example, [5, Proposition 2.5.12] and [23, Proposition 2.1]).

Proposition 2.2. *Assume that X and Y are topological spaces. The following conditions are equivalent:*

(i) *a multifunction $F : X \to \mathcal{P}(Y)$ is l.s.c.;*
(ii) *$F(\mathrm{cl}\, B) \subset \mathrm{cl}\, F(B)$ for every $B \subset X$;*
(iii) *if additionally X and Y are metric spaces, then for every sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X and an $x \in X$ the condition $\lim_{n\to\infty} x_n = x$ implies $F(x) \subset \liminf_n \ F(x_n)$.*

Set-valued semiflows and their semiattractors. Let $\mathbb{T}$ be a non-trivial subgroup of additive group $(\mathbb{R}, +)$ of all reals and let $\mathbb{T}^+ := \mathbb{T} \cap (0, \infty)$ and $\mathbb{T}_0^+ := \mathbb{T} \cap [0, \infty)$. We will consider the sets $\mathbb{T}$, $\mathbb{T}^+$ and $\mathbb{T}_0^+$ as directed sets with natural order induced from the real line. These sets can be interpreted as sets of all possible 'times'. Through the whole paper we do not assume any regularity of considered systems with respect to 'time coefficient', so we present a unified approach to all types of 'times' ('discrete', 'continuous' or 'anything between').

Let again (X, ϱ) be a metric space. A family $\{F_t : X \to \mathcal{P}(X) : \ t \in \mathbb{T}^+\}$ is called a *set-valued semiflow* or a *multivalued semidynamical system* (MSDS for short) if the following inclusion holds

$$F_{s+t}(x) \subset F_t \circ F_s(x) \quad \text{for} \ \ s, t \in \mathbb{T}^+ \ \text{ and } \ x \in X.$$

If the equality (the translation equation) holds instead of the inclusion above, i.e.

$$F_{s+t}(x) = F_t \circ F_s(x) \quad \text{for} \ \ s, t \in \mathbb{T}^+ \ \text{ and } \ x \in X,$$

a MSDS is called *strict*. Usually in the definition of a MSDS the standard initial condition

$$F_0(x) = \{x\} \quad \text{for} \ \ x \in X$$

is also added, so any MSDS can be extended on $\mathbb{T}_0^+$. But from some point of view it is reasonable to consider semiflows without that initial condition.

Given a MSDS we define a set

$$C := \bigcap_{x \in X} \liminf_t F_t(x).$$

If C is a nonempty set it is called the *semiattractor* of the MSDS. Obviously, a semiattractor is a closed set. It is also unique, so we are right saying *the* semiattractor.

Let us consider the following condition:

(H) the MSDS $\{F_t : X \to \mathcal{P}(X) : \ t \in \mathbb{T}^+\}$ is strict and the multifunction F_t is l.s.c. for every $t \in \mathbb{T}^+$.

In [11, Proposition 5.6 and Theorem 5.7] we proved what follows.

Proposition 2.3. *Assume that a MSDS $\{F_t : X \to \mathcal{P}(X) : \ t \in \mathbb{T}^+\}$ satisfies condition* (H). *If it admits the semiattractor C, then the following conditions hold:*

(i) *if a non-void set A is such that $F_t(A) \subset A$ for every $t \in \mathbb{T}^+$, then $C \subset A$;*
(ii) *$cl\, F_t(C) = C$;*
(iii) $\lim_t \ F_t(A) = C$ *for every non-void $A \subset C$, in particular* $\lim_t \ F_t(x) = C$ *for every $x \in C$.*

Sets satisfying the condition

$$F_t(A) \subset A \quad \text{for} \ \ t \in \mathbb{T}^+ \tag{2.1}$$

are said to be *positively invariant* (with respect to $\{F_t : \ t \in \mathbb{T}^+\}$). In particular, Proposition 2.3 says that the semiattractor C is a unique minimal closed set positively invariant with respect to the MSDS (cf. [12]).

3. Cocycles and state multifunctions

We are going to define a cocycle mapping with some base map, fiber maps, and also an induced skew product semiflow. Let Ω be a nonempty set and (X, ϱ) be an arbitrary metric space. Usually Ω is called the *base space* and X the *fiber space* or the *phase space*. Let $\theta = \{\theta_t : \Omega \to \Omega : \ t \in \mathbb{T}\}$ be a group of bijective transformations, i.e.

$$\theta_{s+t} = \theta_t \circ \theta_s \quad \text{for} \quad s, t \in \mathbb{T} \quad \text{and} \quad \theta_0 = id_\Omega.$$

The group θ is called the *base flow*. Consider the mapping $\varphi : \mathbb{T}^+ \times \Omega \to X^X$ satisfying the following equation

$$\varphi(s+t, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega) \quad \text{for} \quad s, t \in \mathbb{T}^+ \quad \text{and} \quad \omega \in \Omega. \tag{3.1}$$

Through the paper we will assume that every function $\varphi(t, \omega) : X \to X$ is continuous. This assumption will not be repeated. A pair (θ, φ) is called a *cocycle* (over θ).

Observe that a cocycle (θ, φ) induces a *skew product semigroup* of self-mappings of $\Omega \times X$ given by

$$\Theta_t(\omega, x) = (\theta_t \omega, \varphi(t, \omega)(x)) \quad \text{for} \quad t \in \mathbb{T}^+, \ \ \omega \in \Omega \ \text{ and } \ x \in X,$$

i.e. for every $s, t \in \mathbb{T}^+$,

$$\Theta_{s+t} = \Theta_t \circ \Theta_s.$$

Given a cocycle (θ, φ) we define the family of *state multifunctions* $F_t : X \to \mathcal{P}(X)$, $t \in \mathbb{T}^+$, by

$$F_t(x) := \{\varphi(t, \omega)(x) : \ \omega \in \Omega\}, \tag{3.2}$$

or equivalently [10, Remark 4.2]

$$F_t(x) := \{\varphi(t, \theta_{-t}\omega)(x) : \ \omega \in \Omega\} \tag{3.3}$$

for every $x \in X$. Since all functions $\varphi(t, \omega)$, $t \in \mathbb{T}^+$, $\omega \in \Omega$, are continuous, the state multifunction F_t is l.s.c. for every $t \in \mathbb{T}^+$. It can be verified that state multifunctions form a semiflow which is not necessarily strict (see [11, Example 5.2]).

Suppose that $A \subset X$ is such that

$$\varphi(t, \omega)(A) \subset A \quad \text{for} \quad t \in \mathbb{T}^+ \quad \text{and} \quad \omega \in \Omega. \tag{3.4}$$

Observe that condition (3.4) implies the positive invariance (2.1) of A with respect to the semiflow $\{F_t : t \in \mathbb{T}^+\}$ of state multifunctions.

Lemma 3.1. *Assume that the cocycle (θ, φ) admits a nonempty set $A \subset X$ satisfying condition (3.4). Then for every $\omega \in \Omega$ the net $(\varphi(t, \theta_{-t}\omega)(A))_{t \in \mathbb{T}^+}$ is decreasing, so it is topologically convergent and*

$$\lim_t \varphi(t, \theta_{-t}\omega)(A) = \bigcap_{t \in \mathbb{T}^+} \operatorname{cl} \varphi(t, \theta_{-t}\omega)(A).$$

Proof. Indeed, if $t_1 < t_2$ then there is $\tau \in \mathbb{T}^+$ such that $t_2 = t_1 + \tau$ and hence, using condition (3.4) and the cocycle equation (3.1), we get

$$\begin{aligned}\varphi(t_2, \theta_{-t_2}\omega)(A) &= \varphi(t_1 + \tau, \theta_{-(t_1+\tau)}\omega)(A)\\ &= \varphi(t_1, \theta_{-t_1}\omega) \circ \varphi(\tau, \theta_{-(t_1+\tau)}\omega)(A)\\ &\subset \varphi(t_1, \theta_{-t_1}\omega)(A).\end{aligned}$$

Therefore the assertion is a straightforward consequence of Lemma 2.1. □

Given a cocycle (θ, φ) for $\omega \in \Omega$ and a subset D of X we define, in the standard way, the following *limit set*

$$\mathcal{L}(\omega, D) := \bigcap_{t\in\mathbb{T}^+} \mathrm{cl}\Big(\bigcup_{s\geq t} \varphi(s, \theta_{-s}\omega)(D)\Big) = \limsup_t \varphi(t, \theta_{-t}\omega)(D).$$

Finally define a family $\{A_\omega : \omega \in \Omega\}$ with

$$A_\omega := \mathrm{cl} \bigcup_D \mathcal{L}(\omega, D) \quad \text{for} \quad \omega \in \Omega, \tag{3.5}$$

where the sum on the right-hand side is taken over all bounded subsets D of X.

We say that a cocycle (θ, φ) has the semiattractor C if it is the semiattractor of the semiflow $\{F_t : t \in \mathbb{T}^+\}$ of state multifunctions.

Example 3.2. One of the most important examples of systems which can be represented as cocycles are iterated function systems (IFSs, for short; see [9, Example 3.1], [13, Section 6], and also [19, Example 2.10]). Namely, consider an arbitrary nonempty set Σ and a family of continuous mappings $\{S_\sigma : X \to X : \sigma \in \Sigma\}$. Such a family is called an *iterated function system*. Let now $\Omega = \Sigma^{\mathbb{N}}$ be the set of all sequences on Σ and $\theta : \Omega \to \Omega$ be a left shift operator, i.e. for $\omega = (\sigma_1, \sigma_2, ...)$, $(\theta\omega)(n) = \omega(n+1)$, where $\omega(k)$ denotes the k-th term of the sequence ω. If now for $\omega \in \Omega$,

$$\varphi(1, \omega) := S_{\sigma_1},$$

and for every $n \geq 2$,

$$\varphi(n, \omega) := S_{\sigma_n} \circ ... \circ S_{\sigma_1},$$

the pair (θ, φ) is a discrete cocycle (over the shift θ).

To obtain so-called inverse iterations or inverse process considered by many authors (see, for example [8], [17] and the references therein) let us extend the symbol space in the following way: as $\bar{\Omega}$ consider a set of all two-sided and symmetric sequences $\bar{\omega}$ indexed by $\mathbb{Z} \setminus \{0\}$, namely $\bar{\omega}(n) = \bar{\omega}(-n) = \sigma_n$ for $n \in \mathbb{N}$. Then extending naturally on $\bar{\Omega}$ the left shift operator we obtain

$$\varphi(n, \theta^{-n}\bar{\omega}) = S_{\sigma_1} \circ ... \circ S_{\sigma_n} \quad \text{for} \quad n \in \mathbb{N}.$$

Another effective way of introducing an inverse process is proposed in [17, Remark 1. (iii)].

Consider an IFS $\{S_\sigma(x) : X \to X : \sigma \in \Sigma\}$ and the induced cocycle (θ, φ) as above. Denote by F the *Barnsley–Hutchinson multifunction* given by $F(x) = \{S_\sigma(x) : \sigma \in \Sigma\}$. Denote moreover by F^n the n-th iterate of F. One can see that iterates of the Barnsley–Hutchinson multifunction form a strict MSDS of l.s.c. multifunctions, i.e. $F_n = F^n$ for $n \in \mathbb{N}$ in this case.

4. Weak contractivity

Let (θ, φ) be a cocycle and let $\omega \in \Omega$. We say that the cocycle is *weakly pullback contractive along the fiber* ω if for every nonempty bounded subset D of X and every $\varepsilon > 0$ there is a $t_0 = t_0(\omega, \varepsilon, D) \in \mathbb{T}^+$ such that

$$diam(\varphi(t, \theta_{-t}\omega)(D)) < \varepsilon \tag{4.1}$$

for every $t \geq t_0$ or, equivalently, if for every nonempty bounded set $D \subset X$

$$diam(\varphi(t, \theta_{-t}\omega)(D)) \to 0 \quad \text{as} \quad t \to \infty. \tag{4.2}$$

The set of all $\omega \in \Omega$ such that (4.2) is satisfied for we denote as Ω_T. If $\Omega_T \neq \emptyset$, we denote

$$\mathcal{A}_T = \operatorname{cl} \bigcup_{\omega \in \Omega_T} A_\omega \tag{4.3}$$

and we refer this set as a *target set*.

If $\Omega_T = \Omega$ then the cocycle (θ, φ) is said to be *weakly pullback contractive on fibers*. This condition was extensively studied in our previous paper [13].

The proof of the next result is the same as that of [13, Proposition 4.1], but we present it for the convenience of the reader.

Proposition 4.1. *Assume that the cocycle (θ, φ) is such that $\Omega_T \neq \emptyset$. Then for every nonempty and bounded subsets A, B of X*

$$\liminf_t \ \varphi(t, \theta_{-t}\omega)(A) = \liminf_t \ \varphi(t, \theta_{-t}\omega)(B) \tag{4.4}$$

and

$$\limsup_t \ \varphi(t, \theta_{-t}\omega)(A) = \limsup_t \ \varphi(t, \theta_{-t}\omega)(B) \tag{4.5}$$

for every $\omega \in \Omega_T$.

Proof. Let $A, B \subset X$ be nonempty and bounded and let $\omega \in \Omega_T$. Owing to the symmetry of the condition (4.4) and (4.5) it is sufficient to show that

$$\liminf_t \ \varphi(t, \theta_{-t}\omega)(A) \subset \liminf_t \ \varphi(t, \theta_{-t}\omega)(B)$$

and

$$\limsup_t \ \varphi(t, \theta_{-t}\omega)(A) \subset \limsup_t \ \varphi(t, \theta_{-t}\omega)(B).$$

We prove the first inclusion. The proof of the second one is similar.

Fix $u \in \liminf_t \varphi(t, \theta_{-t}\omega)(A)$ and $\varepsilon > 0$. By the definition of the lower limit, there exists $s_0 \in \mathbb{T}^+$ such that for every $t \geq s_0$,

$$\varphi(t, \theta_{-t}\omega)(A) \cap B^o(u, \varepsilon/2) \neq \emptyset. \tag{4.6}$$

Let now $t_0 = t_0(\varepsilon/2, A \cup B)$ be a number from $\mathbb{T}^+$ corresponding to the sum $A \cup B$ and to $\varepsilon/2$ according to the condition (4.1). Put $\tau_0 := \max\{s_0, t_0\}$ and fix $t \geq \tau_0$. By (4.6) there is a point $w \in \varphi(t, \theta_{-t}\omega)(A)$ such that $\varrho(w, u) < \varepsilon/2$. Therefore there is $x \in A$ such that

$$w = \varphi(t, \theta_{-t}\omega)(x).$$

Take an arbitrary $y \in B$ and set

$$v = \varphi(t, \theta_{-t}\omega)(y).$$

Since $x, y \in A \cup B$ and $t \geq \tau_0 \geq t_0$ the condition (4.1) implies that $\varrho(w, v) < \varepsilon/2$. Consequently, $\varrho(u, v) < \varepsilon$. Since $v \in \varphi(t, \theta_{-t}\omega)(B)$, it follows that

$$\varphi(t, \theta_{-t}\omega)(B) \cap B^o(u, \varepsilon/2) \neq \emptyset.$$

It holds for every $t \geq \tau_0$, therefore from the fact that $\varepsilon > 0$ was arbitrary we infer that $u \in \liminf_t \varphi(t, \theta_{-t}\omega)(B)$. $\square$

Proposition 4.1 implies immediately the corollary.

Corollary 4.2. *If the cocycle (θ, φ) is such that $\Omega_T \neq \emptyset$, then*

$$A_\omega = \limsup_t \varphi(t, \theta_{-t}\omega)(D) \quad \textit{for } \omega \in \Omega_T$$

for every nonempty bounded subset D of X, where A_ω is defined by (3.5).

The next result shows that under some quite weak assumptions sets A_ω for $\omega \in \Omega_T$ are singletons.

Proposition 4.3. *Let (X, ϱ) be a complete metric space. Assume that the cocycle (θ, φ) is such that $\Omega_T \neq \emptyset$. If it admits a nonempty bounded set $A \subset X$ satisfying condition (3.4), then for every $\omega \in \Omega_T$ there is a unique point $x_\omega \in X$ such that $A_\omega = \{x_\omega\}$. Moreover,*

$$A_\omega = \lim_t \varphi(t, \theta_{-t}\omega)(D) \quad \textit{for } \omega \in \Omega_T \tag{4.7}$$

for every nonempty bounded subset D of X. In particular, the target set $\mathcal{A}_T$ given by (4.3) is nonempty.

Proof. Putting $D = A$ we conclude that the assertion is a consequence of Proposition 4.1, Corollary 4.2, Lemma 3.1 and Cantor's characterization of complete metric spaces. The last statement also follows from Proposition 4.1. $\square$

Remark 4.4. It is clear that if X is compact (so it is complete), then it is enough to put $A = X$ in Proposition 4.3.

For any singleton set the formula (4.7) gives us immediately the convergence of pullback trajectories.

Corollary 4.5. *Under assumptions of Proposition 4.3 for every $x \in X$ the pullback trajectory $(\varphi(t, \theta_{-t}\omega)(x))_{t \in \mathbb{T}^+}$ along any fiber $\omega \in \Omega_T$ is convergent. More precisely, for every $x \in X$ and $\omega \in \Omega_T$ we have*

$$\lim_{t \to \infty} \varphi(t, \theta_{-t}\omega)(x) = x_\omega.$$

Now we are in position to formulate the central result of the paper.

Theorem 4.6. *Let (X, ϱ) be a complete metric space. Assume that: the cocycle (θ, φ) is such that $\Omega_T \neq \emptyset$, there exists a nonempty bounded set $A \subset X$ satisfying condition (3.4) and the semiflow $\{F_t : t \in \mathbb{T}^+\}$ of state multifunctions is strict. Then (θ, φ) admits the semiattractor C, moreover*

$$C = \lim_t F_t(\mathcal{A}_T) = \mathrm{cl} \bigcup_{t \in \mathbb{T}^+} F_t(\mathcal{A}_T). \tag{4.8}$$

In this case C is a bounded set.

Proof. Let $x \in X$. Taking into account Proposition 4.3, Corollary 4.2 and Proposition 4.1 for $D = \{x\}$ and $\omega \in \Omega_T$ we infer that

$$\{x_\omega\} = A_\omega = \liminf_t \varphi(t, \theta_{-t}\omega)(A) = \liminf_t \varphi(t, \theta_{-t}\omega)(x) \subset \liminf_t F_t(x).$$

Since x is arbitrary, this implies that $C \neq \emptyset$. Moreover, by the closedness of C we get

$$\mathcal{A}_T \subset C. \tag{4.9}$$

By assumptions, the MSDS $\{F_t : t \in \mathbb{T}^+\}$ of state multifunctions satisfies condition (H), so using (iii) of Proposition 2.3 we obtain from (4.9) that

$$C = \lim_t F_t(\mathcal{A}_T).$$

Finally, by positive invariance of C ((ii) of Proposition 2.3) and inclusion (4.9) we have

$$F_t(\mathcal{A}_T) \subset F_t(C) \subset C$$

for every $t \in \mathbb{T}^+$ and, consequently,

$$\mathrm{cl} \bigcup_{t \in \mathbb{T}^+} F_t(\mathcal{A}_T) \subset C.$$

On the other hand, by properties of topological limits,

$$C = \lim_t F_t(\mathcal{A}_T) \subset \mathrm{cl} \bigcup_{t \in \mathbb{T}^+} F_t(\mathcal{A}_T).$$

Since A is assumed to be a bounded set, using (i) of Proposition 2.3, we infer that C is also bounded. This completes the proof. $\square$

Remark 4.7. (i) In fact, one can observe that by (ii) of Proposition 2.3 we can use any A_ω for $\omega \in \Omega_T$ instead of the whole target set $\mathcal{A}_T$ in the formula (4.8).

(ii) Theorem 4.6 says that the existence of bounded positively invariant subset and at least one weakly pullback contractive fiber guarantee the existence of the semiattractor. So we obtain a new criterion on the existence of the semiattractor for general cocycles. Notice, that in our previously obtained parallel criterion [11, Corollary 5.11] we use selections having globally attractive fixed points.

(iii) One can observe that in particular case of iterated function systems the criterion above gives us a new power. To possess a semiattractor an IFS needs not to contain a contractive mapping in any sense, but only at least one contractive word (fiber) constructed by a composition of some (even infinitely many) transformations.

5. Fibers of semiattractors – the internal dynamics

Let (θ, φ) be a cocycle and let C be its semiattractor. In this section we consider the fibers of semiattractor. Namely, for every $\omega \in \Omega$ define the set

$$C_\omega := \bigcap_{t\in\mathbb{T}^+} \mathrm{cl}\, \varphi(t, \theta_{-t}\omega)(C) = \lim_t \varphi(t, \theta_{-t}\omega)(C) = \mathcal{L}(\omega, C).$$

Due to invariance properties of semiattractors and Lemma 3.1 the sets C_ω are well defined. We refer the sets C_ω as *fibers* of the semiattractor C.

Theorem 5.1. *If C is the semiattractor of a cocycle (θ, φ), then*

$$C = \mathrm{cl} \bigcup_{\omega\in\Omega} C_\omega.$$

Proof. It is easy to observe that

$$\mathrm{cl} \bigcup_{\omega\in\Omega} C_\omega \subset C.$$

To prove the opposite inclusion assume that $y \in C$. Let $\varepsilon > 0$. Since $y \in \liminf_t F_t(y)$, therefore there is $t_0 \in \mathbb{T}^+$ such that

$$B^o(y, \varepsilon/2) \cap F_t(x) \neq \emptyset$$

for every $t \geq t_0$. Given arbitrary $t \geq t_0$, by the definition of F_t, we can find $\omega_0 \in \Omega$ such that

$$\varrho(y, \varphi(t, \theta_{-t}\omega_0)(y)) < \varepsilon/2. \tag{5.1}$$

By the definition of C_{ω_0} and properties of topological limits to $\varepsilon/4$ there exists $s_0 \in \mathbb{T}^+$ such that

$$dist(\varphi(t, \theta_{-t}\omega_0)(C), C_{\omega_0}) < \varepsilon/4$$

for $t \geq s_0$, and, since $y \in C$,

$$\varrho(\varphi(t, \theta_{-t}\omega_0)(y), C_{\omega_0}) < \varepsilon/4$$

for every $t \geq s_0$. Consequently, for an arbitrary $t \geq s_0$ there is $z \in C_{\omega_0}$ such that

$$\varrho(\varphi(t, \theta_{-t}\omega_0)(y), z) < \varepsilon/2. \tag{5.2}$$

Now if $t \geq \max\{t_0, s_0\}$, conditions (5.1) and (5.2) give us

$$\varrho(y, z) < \varepsilon.$$

It means that for every $y \in C$ there is a point $z \in \bigcup_{\omega\in\Omega} C_\omega$ arbitrarily close to y, so

$$y \in \mathrm{cl} \bigcup_{\omega\in\Omega} C_\omega.$$

This ends the proof of the desired equality. $\square$

Proposition 5.2. *If (θ, φ) is a cocycle with semiattractor C, then the family $\{C_\omega : \omega \in \Omega\}$ of its fibers is positively invariant with respect to the cocycle mapping φ, i.e. for every $T \in \mathbb{T}^+$ and $\omega \in \Omega$ the following inclusion holds*

$$\varphi(T, \omega)(C_\omega) \subset C_{\theta_T \omega}.$$

Proof. Fix $T \in \mathbb{T}^+$ and $\omega \in \Omega$. By the continuity of $\varphi(T, \omega)$ and the cocycle property, since $\varphi(T, \theta_{-t}\omega)(C) \subset C$ for every $t \in \mathbb{T}^+$, we get

$$\begin{aligned}\varphi(T, \omega)(C_\omega) &= \varphi(T, \omega)\Big(\bigcap_{t \in \mathbb{T}^+} \mathrm{cl}\, \varphi(t, \theta_{-t}\omega)(C)\Big) \subset \bigcap_{t \in \mathbb{T}^+} \mathrm{cl}\, \varphi(T, \omega) \circ \varphi(t, \theta_{-t}\omega)(C) \\ &= \bigcap_{t \in \mathbb{T}^+} \mathrm{cl}\, \varphi(T + t, \theta_{-t}\omega)(C) = \bigcap_{t \in \mathbb{T}^+} \mathrm{cl}\, \varphi(t, \theta_T(\theta_{-t}\omega)) \circ \varphi(T, \theta_{-t}\omega)(C) \\ &\subset \bigcap_{t \in \mathbb{T}^+} \mathrm{cl}\, \varphi(t, \theta_T(\theta_{-t}\omega))(C) = C_{\theta_T \omega}. \quad \square\end{aligned}$$

Remark 5.3. The results above show that the structure and properties of semiattractors are similar as considered in the theory of random dynamical systems global point attractors (see [7]). Nevertheless, semiattractors not necessarily attract all singletons but only their internal points. In our recent paper [14] we showed that the semiattractor is always contained in the closure of the union of fibers of any point attractor.

Now we are going to generalize some definitions proposed in [4, Section 1.4.2] in the context of iterated function systems on compact spaces and minimal compact invariant sets.

If C is the semiattractor of a cocycle (θ, φ), then it is called *point-fibred* if for every $\omega \in \Omega$ the fiber C_ω is a singleton.

C is called *strongly-fibred* if for every open set $U \subset X$ such that $U \cap C \neq \emptyset$ there is $\omega \in \Omega$ such that $C_\omega \subset U$.

It is called *well-fibred* if for every open set $U \subset X$ such that $U \cap C \neq \emptyset$ and every nonempty bounded subset B of C there is $\omega \in \Omega$ such that $\mathcal{L}(\omega, B) \subset U$.

The following result is obvious.

Proposition 5.4. *Let C be the semiattractor of a cocycle (θ, φ). Then*

(i) *if C is point-fibred, then it is strongly-fibred,*
(ii) *if C is strongly-fibred, then it is well-fibred.*

Remark 5.5. As we see in some examples below, in general the opposite implications do not hold.

Theorem 4.6 implies the following corollary extending [13, Theorem 6.3].

Corollary 5.6. *Let (X, ϱ) be a complete metric space. Assume that the cocycle (θ, φ) is such that $\Omega_T = \Omega$, there exists a nonempty bounded set $A \subset X$ satisfying condition (3.4) and the semiflow $\{F_t : t \in \mathbb{T}^+\}$ of state multifunctions is strict. Then (θ, φ) admits the semiattractor C which is bounded and point-fibred.*

Proof. In fact, since C exists and is bounded, we obtain, by Proposition 4.3,

$$C_\omega = A_\omega = \{x_\omega\}$$

for every $\omega \in \Omega$. $\square$

The following example shows that in order to obtain a point-fibred semiattractor the condition $\Omega_T = \Omega$ is a sufficient only.

Example 5.7. Consider an IFS consisting of four transformations of the plane $\mathbb{R}^2$ into itself given by

$$S_1(x,y) = \Big(\frac{1}{3}x, \frac{1}{2}y\Big), \quad S_2(x,y) = \Big(\frac{1}{3}x + \frac{2}{3}, \frac{1}{2}y\Big),$$
$$S_3(x,y) = \Big(\frac{1}{3}x, y\Big), \quad S_4(x,y) = \Big(\frac{1}{3}x + \frac{2}{3}, y\Big).$$

Since S_1 is a strict contraction having the unique fixed point $(0,0)$, using formula (4.8), by (i) of Remark 4.7 we can get the semiattractor of considered IFS as

$$C = \mathrm{cl} \bigcup_{n\in\mathbb{N}} F^n\big((0,0)\big) = \mathcal{C} \times \{0\},$$

where $\mathcal{C}$ is the standard Cantor set on an interval $[0,1]$. This semiattractor is point-fibred, but for any $R > 0$ we have $diam\ S_3^n\big(B^o((0,0),R)\big) = 2R,\ n \in \mathbb{N}$, so $\Omega_T \neq \Omega$.

The semiattractor of the system satisfying assumptions of Theorem 4.6 need not to be point-fibred.

Example 5.8. Now take into a consideration an IFS consisting of transformations of the plane $\mathbb{R}^2$ into itself given by

$$S_{ij}(x,y) = \Big(\frac{1}{3}x + \frac{i}{3}, \frac{1}{3}y + \frac{j}{3}\Big) \quad \text{for} \quad (i,j) \in \{0,1,2\}^2 \setminus \{(1,1)\}.$$

It is well known that this system has the attractor (fractal in the sense of Hutchinson, see [16]) which is the Sierpinski's carpet contained in the square $[0,1] \times [0,1]$. Let us add to this system the rotation around the origin with the angle $\pi/2$. It is given by $S_{11}(x,y) = (-y,x)$. One can easy observe that such a system has the semiattractor C consisting of four copies of Sierpinski's carpet filling the square $\Delta = [-1,1] \times [-1,1]$. It satisfies assumptions of Theorem 4.6 with $A = \Delta$ and $\Omega_T \neq \Omega$. Observe further that for $\omega = (..., 11, 11, 11, ...)$ we have

$$C_\omega = \bigcap_{n\in\mathbb{N}} S_{11}^n(C) = C,$$

so C is not point-fibred. On the other hand it is not hard to see that for every open set U such that $U \cap C \neq \emptyset$ there is C_ω (being in fact a singleton) with some $\omega \in \Omega$ such that $C_\omega \subset U$, so C is strongly-fibred.

In [4, Example 3.12] it is shown that the system consisting of two diffeomorphism of the circle: the first one being a rotation with irrational rotation number and the second one having attracting fixed point p with derivative in this point equal to one, has the attractor (so the semiattractor, see [23] and [11]) which is well fibred, but it is not strongly-fibred.

Acknowledgment

The research of the author was supported by the Polish Ministry of Science and Higher Education.

References

[1] L. Arnold, Random Dynamical Systems, Springer, Berlin, 1998.
[2] T. Banakh, N. Novosad, Micro and macro fractals generated by multi-valued dynamical system, Fractals 22 (4) (2014), 13 pp.
[3] M.F. Barnsley, A. Vince, Development in fractal geometry, Bull. Math. Sci. 3 (2013) 299–348.
[4] P.G. Barrientos, F.H. Ghane, D. Malicet, A. Sarizadeh, On the chaos game of iterated function systems, Topol. Methods Nonlinear Anal. 49 (1) (2017) 105–132.
[5] R.S. Burachik, A.N. Iusem, Set-Valued Mappings and Enlargements of Monotone Operators, Springer, New York, 2008.
[6] I. Chueshov, Monotone Random Systems Theory and Applications, Springer-Verlag, Berlin, Heidelberg, New York, 2002.
[7] H. Crauel, Random point attractors versus random set attractors, J. Lond. Math. Soc. (2) 63 (2001) 413–427.
[8] P. Diaconis, D. Freedman, Iterated random functions, SIAM Rev. 4 (1) (1999) 45–76.
[9] G. Guzik, Asymptotic stability of discrete cocycles, J. Difference Equ. Appl. 21 (11) (2015) 1044–1057.
[10] G. Guzik, Asymptotic properties of multifunctions, families of measures and Markov operators associated with cocycles, Nonlinear Anal. 130 (2016) 59–75.
[11] G. Guzik, Semiattractors of set-valued semiflows, J. Math. Anal. Appl. 435 (2016) 1321–1334.
[12] G. Guzik, Minimal invariant closed sets of set-valued semiflows, J. Math. Anal. Appl. 449 (2017) 382–396.
[13] G. Guzik, On a class of cocycles having attractors which consists of singletons, Topol. Methods Nonlinear Anal. 50 (2) (2017) 727–739.
[14] G. Guzik, A note on asymptotic behavior of nonautonomous and random dynamical systems, submitted for publication, 2018.
[15] S.C. Hille, T. Szarek, M.A. Ziemlańska, Equicontinuous families of Markov operators in view of asymptotic stability, C. R. Math. Acad. Sci. Paris 355 (12) (2017) 1247–1251.
[16] J.E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J. 30 (1981) 713–747.
[17] R. Kapica, Random iteration and Markov operators, J. Difference Equ. Appl. 22 (2) (2016) 295–305.
[18] B. Kieninger, Iterated Function Systems on Compact Hausdorff Spaces, Ph.D. Thesis, Augsburg University, Berichte aus der Mathematik, Shaker-Verlag, Aachen, 2002.
[19] P.E. Kloeden, M. Rasmussen, Nonautonomous Dynamical Systems, AMS Mathematical Surveys and Monographs, vol. 176, American Mathematical Society, Providence, Rhode Island, 2011.
[20] K. Kuratowski, Topology, vol. I, Academic Press, New York, 1966.
[21] A. Lasota, J. Myjak, Semifractals, Bull. Pol. Acad. Sci. Math. 44 (1) (1996) 5–21.
[22] A. Lasota, J. Myjak, Semifractals on Polish spaces, Bull. Pol. Acad. Sci. Math. 46 (2) (1998) 179–196.
[23] A. Lasota, J. Myjak, Attractors of multifunctions, Bull. Pol. Acad. Sci. Math. 48 (3) (2000) 319–334.
[24] E. Matias, L.J. Díaz, Non-hyperbolic iterated function systems: attractors and stationary measures, available at arXiv:1605.02752, 2016.
[25] T. Szarek, A. Zdunik, Stability of iterated function systems on the circle, Bull. Lond. Math. Soc. 48 (2016) 365–378.
[26] T. Szarek, A. Zdunik, Attractors and invariant measures for random interval homeomorphisms, manuscript.