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Variations on a theme of the Marden and Smirnov operators, differential inequalities for polynomials ☆

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ABSTRACT

In 1949 Marden in his book introduced and studied the differential operator

$$B[f](z) = \lambda_0 f(z) + \lambda_1 \frac{nz}{2} f'(z) + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{f''(z)}{2!},$$

defined on polynomials of degree at most n. Here λ_0, λ_1, and λ_2 are constants such that the polynomial

$$u(z) = \lambda_0 + C_n^1 \lambda_1 z + C_n^2 \lambda_2 z^2$$

has all roots in the half-plane $\operatorname{Re} z \le \frac{n}{4}$. In the book of 2002 Rahman and Schmeisser proved that under simple additional conditions on a couple of polynomials f and F the differential equality $|B[f](z)| \le |B[F](z)|$ holds for $|z| \ge 1$. In this article, in particular, we describe the set of parameters λ_k, for which the inequality of Rahman and Schmeisser takes place for $|z| \ge R$, $R > 1$. As it turns out, in this case the roots of $u(z)$ should belong to the complement of the open disc with diameter $\left(\frac{nR}{2(R+1)}, \frac{nR}{2(R-1)}\right)$. We show that this set can not be extended. This fact does the inequality of Rahman and Schmeisser more applicable for $|z| > 1$. Also for a polynomial f we obtain a sharp estimation of $|f''(z)|$ in terms of $|f(z)|$ and $M_r = \max_{|z|=r} |f(z)|$.

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1. Introduction

By $\mathcal{P}_n$ denote the set of all polynomials in $\mathbb{C}$ of degree at most $n \in \mathbb{N}$. Let $\mathbb{D}$ be the unit disc $\{z \in \mathbb{C} : |z| < 1\}$.

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There are a lot of books and articles devoted to polynomials and differential inequalities for polynomials (see, for example, [1–4], [6–10], [11, ch.V, §1]).

For $z \in \mathbb{C} \setminus \mathbb{D}$ by $\Omega_{|z|}$ denote the image of the disc $\{t \in \mathbb{C} : |t| < |z|\}$ under the mapping $\psi(t) = \dfrac{t}{1+t}$. For $f \in \mathcal{P}_n$ put $S_\alpha[f](z) = zf'(z) - n\alpha f(z)$, α is a constant.

In the article we consider polynomials $f, F \in \mathcal{P}_n$ such that

$$(*)\begin{cases} 1)\ \deg f \le \deg F = n, \\ 2)\ F \text{ has all its zeros in } \overline{\mathbb{D}}, \\ 3)\ |f(z)| \le |F(z)| \text{ on } \partial\mathbb{D}. \end{cases}$$

In [11], in particular, the following theorem was proved:

Theorem A. *[11, ch.V, §1] Let f and F be polynomials possessing conditions $(*)$. Then for $z \in \mathbb{C} \setminus \mathbb{D}$*

$$|S_\alpha[f](z)| \le |S_\alpha[F](z)| \tag{1}$$

for all $\alpha \in \overline{\Omega_{|z|}}$. For $\alpha \in \Omega_{|z|}$ in (1) equality holds at a point $z_0 \in \mathbb{C} \setminus \overline{\mathbb{D}}$ iff $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$.

So, for $\alpha \in \overline{\Omega_{|z|}}$ the Smirnov operator S_α preserves inequality between polynomials.

In [7, p. 65] Marden introduced a differential operator B of second order, $B : \mathcal{P}_n \to \mathcal{P}_n$. This operator B carries a polynomial $f \in \mathcal{P}_n$ into

$$B[f](z) = \lambda_0 f(z) + \lambda_1 \frac{nz}{2} f'(z) + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{f''(z)}{2!},$$

where λ_0, λ_1, and λ_2 are constants such that

$$u(z) = \lambda_0 + C_n^1 \lambda_1 z + C_n^2 \lambda_2 z^2 \neq 0 \quad \text{for} \quad \operatorname{Re} z > \frac{n}{4}. \tag{2}$$

In [10], Rahman and Schmeisser proved that the Marden operator B also preserves inequality between polynomials.

Theorem B. *[10, p. 538–540] Let f and F be polynomials possessing conditions $(*)$. Then*

$$|B[f](z)| \le |B[F](z)|, \quad z \in \mathbb{C} \setminus \mathbb{D}, \tag{3}$$

here the constants $\lambda_0, \lambda_1, \lambda_2$ possess condition (2). For $z_0 \in \mathbb{C} \setminus \overline{\mathbb{D}}$ in (3) equality holds iff $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$.

In the particular case $F(z) = z^n$ Theorem B was earlier proved in [9] by Rahman.

This theorem was generalized in different ways (see, for example [12] and references herewith).

It was proved in [5], that for $|z| \ge R > 1$ condition (2) on constants $\lambda_0, \lambda_1, \lambda_2$ can be weakened in Theorem B. In particular, the following result was presented:

Theorem C. *[5] For $R > 1$ let L_R be the curve*

$$L_R = \left\{ \frac{n}{2}(1 + R\cos\varphi) k(Re^{i\varphi}) : \quad \varphi \in [0, 2\pi] \right\},$$

(see Fig. 1) here $k(z) = \dfrac{z}{(1+z)^2}$ is the Koebe function. Let f and F be polynomials possessing conditions $(*)$, z_1 and z_2 be the roots of $u(z)$. If either a) z_1 and z_2 lie in the half-plane $\operatorname{Re} z < \dfrac{n}{2}\dfrac{R}{1+R}$ or b) the

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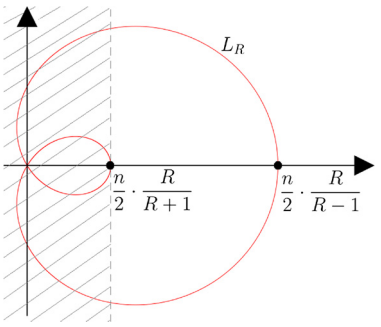


Fig. 1. To Theorem C.

straight line, passing through z_1 and z_2, does not intersect L_R, then

$$|B[f](z)| \le |B[F](z)|, \quad |z| \ge R.$$

In the presented paper we construct new differential operators of second order preserving inequality between polynomials. In section 4 we compare these operators with the Marden operator and prove that for $|z| \ge R > 1$ the new operators essentially generalize the Marden operator. Namely, the set of parameters in our inequalities is much more wider then in Theorem B (cf. Theorem 3). This fact does inequality (3) more applicable for $|z| > 1$. Also we show that, generally speaking, the set of parameters obtained in Theorem 3 can not be extended (cf. Example 1). In section 2 we consider arbitrary linear operators (not necessary differential), defined on polynomials. We investigate the connection between two properties of such operators: preservation of zeros and preservation of inequalities between polynomials.

2. Operators preserving zeros

Let $T : \mathcal{P}_n \to \mathcal{P}_n$ be a linear operator. We shall say that T *preserves zeros* if, for *every* polynomial F of degree n having all its zeros in $\overline{\mathbb{D}}$, the polynomial $T[F]$ also has all zeros in $\overline{\mathbb{D}}$.

Rahman and Schmeisser proved the following

Theorem D. *[10, p. 538] Let f and F be polynomials possessing conditions $(*)$, T be an operator preserving zeros. Then*

$$|T[f](z)| \le |T[F](z)|, \quad z \in \mathbb{C} \setminus \mathbb{D}, \tag{4}$$

for $z_0 \in \mathbb{C} \setminus \overline{\mathbb{D}}$ in (4) equality holds iff $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$. (5)

As it turns out, the converse statement is also true.

Theorem 1. *Let $T : \mathcal{P}_n \to \mathcal{P}_n$ be a linear operator. For all f and F possessing conditions $(*)$ (4) and (5) holds iff T preserves zeros.*

Proof. *Sufficiency* is Theorem D.

Necessity. Assume the converse, let T do not preserve zeros. Then there exist $z_0 \in \mathbb{C}\setminus\overline{\mathbb{D}}$ and a polynomial F with zeros in $\overline{\mathbb{D}}$, $\deg F = n$, such that $T[F](z_0) = 0$.

Consider the polynomial

$$f_1(z) = cF(z), \quad |c| < 1. \tag{6}$$

Then f_1 and F are polynomials possessing conditions $(*)$. Therefore, by (4)

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$$|T[f_1](z_0)| \le |T[F](z_0)| = 0$$

and we have $|T[f_1](z_0)| = |T[F](z_0)|$ for $z_0 \in \mathbb{C}\setminus\overline{\mathbb{D}}$. By (5), $f_1 = e^{i\gamma}F$. This contradicts (6). Hence, T preserves zeros. □

Corollary 1. *Let f and F be polynomials possessing conditions $(*)$. Let $T_1 : \mathcal{P}_n \to \mathcal{P}_m$, $T_2 : \mathcal{P}_m \to \mathcal{P}_k$, $k \le m \le n$, be operators preserving zeros and such that $\deg T_1[F] = m$. Then*

$$|T_2T_1[f](z)| \le |T_2T_1[F](z)|, \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

For $z_0 \in \mathbb{C} \setminus \overline{\mathbb{D}}$ equality holds iff $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$.

Proof. Consider the polynomials $T_1[f]$ and $T_1[F]$. Note that $\deg T_1[f] \le \deg T_1[F] = m$. Since T_1 preserves zeros, all zeros of $T_1[F]$ belong to $\overline{\mathbb{D}}$. By Theorem D,

$$|T_1[f](z)| \le |T_1[F](z)|, \quad z \in \partial\mathbb{D}.$$

Therefore we can apply Theorem D to the polynomials $T_1[f]$, $T_1[F]$, and to the operator T_2. We have

$$|T_2T_1[f](z)| \le |T_2T_1[F](z)|, \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

In addition, in the last inequality for $z_0 \in \mathbb{C} \setminus \overline{\mathbb{D}}$ equality holds iff $T_1[f](z) = e^{i\gamma}T_1[F](z)$. This implies $|T_1[f](z_0)| = |T_1[F](z_0)|$ for $z_0 \in \mathbb{C} \setminus \overline{\mathbb{D}}$. Again, by Theorem D, we have $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$. □

Note that for fixed $z \in \mathbb{C} \setminus \mathbb{D}$ inequality (1) can be written in the form

$$\left|zf'(z) - n\frac{az}{1+az}f(z)\right| \le \left|zF'(z) - n\frac{az}{1+az}F(z)\right|, \quad z \in \mathbb{C} \setminus \mathbb{D},$$

where a is an arbitrary number from $\overline{\mathbb{D}}$, or, equivalently,

$$|\widetilde{S}_a[f](z)| \le |\widetilde{S}_a[F](z)|, \quad z \in \mathbb{C} \setminus \mathbb{D},$$

where

$$\widetilde{S}_a[f](z) = (1+az)f'(z) - naf(z).$$

For the modificated Smirnov operator $\widetilde{S}_a$ the following property holds:

Proposition 1. *Let F be a polynomial, $\deg F = n$, F has all zeros in $\overline{\mathbb{D}}$. Then $\deg \widetilde{S}_a[F] \le n-1$. Moreover, $\deg \widetilde{S}_a[F] = n-1$ for all $a \in \overline{\mathbb{D}}$, except may be one value $a' = \dfrac{F^{(n)}(0)}{F^{(n-1)}(0)}$ if $a' \in \partial\mathbb{D}$. If $a = a'$ then $F(z) = C(1+a'z)^n, C \in \mathbb{C}$, in this case $\widetilde{S}_{a'}[F] \equiv 0$.*

Proof. Denote $F(z) = b_nz^n + ... + b_0$, $b_n \ne 0$. Consider $\widetilde{S}_a[F](z) = c_nz^n + ... + c_0 = (1+az)(nb_nz^{n-1} + (n-1)b_{n-1}z^{n-2} + ... + b_1) - na(b_nz^n + ... + b_0)$. Find

$$c_n = 0, \quad c_{n-1} = nb_n - ab_{n-1}.$$

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Therefore, $\deg \widetilde{S}_a[F] \le n-1$ and $\deg \widetilde{S}_a[F] = n-1$ if $a \ne a' = \dfrac{nb_n}{b_{n-1}} \in \overline{\mathbb{D}}$. Consider the case $a = a' = \dfrac{nb_n}{b_{n-1}} \in \overline{\mathbb{D}}$. Write F in the form $F(z) = K(z-z_1)...(z-z_n), |z_k| \le 1$, $K = const \ne 0$. Then

$$a' = \frac{nb_n}{b_{n-1}} = -\frac{n}{z_1 + ... + z_n} \in \overline{\mathbb{D}}$$

only if $z_1 = ... = z_n$, $z_k \in \partial\mathbb{D}$. Consequently, $a' = -\dfrac{1}{z_1} \in \partial\mathbb{D}$ and

$$F(z) = K(z-z_1)^n = C(1+a'z)^n, \quad C = const \ne 0.$$

Let us compute $\widetilde{S}_{a'}[F]$ for $F(z) = C(1+a'z)^n$:

$$\widetilde{S}_{a'}[F](z) = (1+a'z)Cna'(1+a'z)^{n-1} - Cna'(1+a'z)^n \equiv 0. \quad \square$$

If for a polynomial F with zeros in $\overline{\mathbb{D}}, \deg F = n$, the value $a' = \dfrac{F^{(n)}(0)}{F^{(n-1)}(0)}$ belongs to $\partial\mathbb{D}$, then we shall say that a' is an *exceptional value* for F. By Proposition 1, in this case $F(z) = C(1+a'z)^n$.

From Theorem A (for $\alpha = \frac{az}{1+az}$) and Theorem 1 it follows that for $a \in \mathbb{D}$ the operator $\widetilde{S}_a$ preserves zeros. However, this is not true if $a \in \partial\mathbb{D}$. Indeed, for the polynomial $F(z) = (1+az)^n$, $a \in \partial\mathbb{D}$, we have $\widetilde{S}_a[F](z) \equiv 0$. Nevertheless, the following statement takes place.

Proposition 2. *Let F be a polynomial with zeros in $\overline{\mathbb{D}}$, $\deg F = n$. Let $a \in \partial\mathbb{D}$ be not the exceptional value for F. Then all zeros of $\widetilde{S}_a[F]$ lie in $\overline{\mathbb{D}}$.*

Proof. Assume the converse. Then there exists a point $z_0 \in \mathbb{C} \setminus \overline{\mathbb{D}}$ such that $\widetilde{S}_a[F](z_0) = 0$. This implies $\widetilde{S}_a[F](z) \equiv 0$. Indeed, if $\widetilde{S}_a[F](z) \not\equiv 0$ we can find a neighborhood $U(z_0) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ of z_0 such that $\widetilde{S}_a[F](z) \ne 0$ in $U(z_0) \setminus \{z_0\}$. For $b \in \mathbb{D}$ the operator $\widetilde{S}_b$ preserves zeros. Consequently, $\widetilde{S}_b[F](z) \ne 0$ in $U(z_0)$. Consider the circle $\Gamma = \{z \in \mathbb{C} : |z - z_0| = r\} \subset U(z_0)$. We find a point $z(b) \in \Gamma$ such that

$$\min_{z\in\Gamma} |\widetilde{S}_b[F](z)| = |\widetilde{S}_b[F](z(b))|.$$

Then, by the maximum modulus principle,

$$|\widetilde{S}_b[F](z(b))| \le |\widetilde{S}_b[F](z_0)|.$$

There exists a sequence $b_n \in \mathbb{D}$, $b_n \to a$, such that $z(b_n) \to z^* \in \Gamma$. Passing to the limit in the last inequality we get

$$|\widetilde{S}_a[F](z^*)| \le |\widetilde{S}_a[F](z_0)| = 0.$$

This contradicts the condition $\widetilde{S}_a[F](z) \ne 0$ in $U(z_0) \setminus \{z_0\}$. So, $\widetilde{S}_a[F](z) \equiv 0$. Equivalently,

$$(1+az)F'(z) - naF(z) = 0.$$

Solving this equality we find $F(z) = C(1+az)^n$, $C = const \ne 0$, because $\deg F = n$. Then $\dfrac{F^{(n)}(0)}{F^{(n-1)}(0)} = a \in \partial\mathbb{D}$ and a is the exceptional value for F. This contradiction concludes the proof. $\square$

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3. Differential operators of second order

For $f \in \mathcal{P}_n$, $a_1, a_2 \in \overline{\mathbb{D}}$ apply the operator $\widetilde{S}_{a_2} : \mathcal{P}_{n-1} \to \mathcal{P}_{n-2}$ to the polynomial $\widetilde{S}_{a_1}[f](z)$. We obtain new differential operator of second order

$$\widetilde{S}^2[f](z) \stackrel{def}{=} \widetilde{S}_{a_2}\widetilde{S}_{a_1}[f](z) =$$
$$= (1+a_2z)(a_1f'(z)+(1+a_1z)f''(z)-na_1f'(z))-(n-1)a_2((1+a_1)f'(z)-na_1f(z)) =$$
$$= (1+a_1z)(1+a_2z)f''(z)-(n-1)(a_1(1+a_2z)+a_2(1+a_1z))f'(z)+n(n-1)a_1a_2f(z).$$

From section 2 it follows that the operator $\widetilde{S}^2$ preserves zeros not for all values of parameters a_1, a_2 in contradiction to the Marden operator B.

For the operator $\widetilde{S}^2$ the following theorem holds:

Theorem 2. *Let f and F be polynomials possessing conditions $(*)$. Then*

$$|\widetilde{S}^2[f](z)| \le |\widetilde{S}^2[F](z)|, \quad z \in \mathbb{C}\setminus\mathbb{D}. \tag{7}$$

For $z_0 \in \mathbb{C}\setminus\overline{\mathbb{D}}$ in (7) equality holds only in the following cases:

1) *If one of a_1 and a_2 is exceptional value $a' = \dfrac{F^{(n)}(0)}{F^{(n-1)}(0)} \in \partial\mathbb{D}$ for F then equality holds iff $F = C(1+a'z)^n$, $f = c(1+a'z)^n$, $|c| \le |C|$.*

2) *If $a_1 \ne a' \ne a_2$ then equality holds in one of the cases:*

- **a)** *if a_1, $a_2 \in \mathbb{D}$ then $f = e^{i\gamma}F$, $\forall\gamma \in \mathbb{R}$.*
- **b)** *if $a_1 \in \partial\mathbb{D}$, $a_2 \in \mathbb{D}$ then*
  - **b.1)** *$f = e^{i\gamma}F$, $\forall\gamma \in \mathbb{R}$;*
  - **b.2)** *$F(z) = K(1+a_1z)^n(z-z_1)$, $f(z) = e^{i\gamma}F(z)+C(1+a_1z)^n$, $\forall\gamma \in \mathbb{R}$, $\forall z_1 \in \mathbb{D}$, $\forall K, C \in \mathbb{C}\setminus\{0\}$ such that the circle $\left\{w(\partial\mathbb{D})+1 : w(z) = \dfrac{C}{e^{i\gamma}K}\dfrac{1+a_1z}{z-z_1}\right\}$ is contained in $\overline{\mathbb{D}}$.*

  The case $a_1 \in \mathbb{D}$, $a_2 \in \partial\mathbb{D}$ is symmetric to b).
- **c)** *if a_1, $a_2 \in \partial\mathbb{D}$ then equality holds in one of the following cases:*
  - **c.1)** *$f = e^{i\gamma}F$, $\forall\gamma \in \mathbb{R}$;*
  - **c.2)** *$F(z) = K(1+a_jz)^{n-1}(z-z_1)$, $f(z) = e^{i\gamma}F(z)+C(1+a_jz)^n$, $\forall\gamma \in \mathbb{R}$, $\forall z_1 \in \mathbb{D}$, $\forall K$, $C \in \mathbb{C}\setminus\{0\}$ such that the circle $\left\{w_j(\partial\mathbb{D})+1 : w_j(z) = \dfrac{C}{e^{i\gamma}K}\dfrac{1+a_jz}{z-z_1}\right\}$ is contained in $\overline{\mathbb{D}}$, $j = 1, 2$;*
  - **c.3)** *for $n = 2$, $a_1 \ne a_2$, $F(z) = K(z-z_1)(z-z_2)$, $f(z) = e^{i\gamma}F(z)+C_1(1+a_1z)^2+C(1+a_2z)^2$, $\forall\gamma \in \mathbb{R}$, $\forall z_1 \in \mathbb{D}$, $\forall z_2 \in \partial\mathbb{D}\setminus\left\{-\dfrac{1}{a_1}, -\dfrac{1}{a_2}\right\}$, $\forall C \in \mathbb{C}\setminus\{0\}$, $C_1 = -\left(\dfrac{1+a_2z_2}{1+a_1z_2}\right)^2 C$, $\dfrac{C_1}{C} \ne -\left(\dfrac{a_2}{a_1}\right)^2$, and $K \in \mathbb{C}\setminus\{0\}$ such that the circle $\left\{w_3(\partial\mathbb{D})+1 : w_3(z) = \dfrac{C}{Ke^{i\gamma}}\dfrac{z-\zeta_2}{z-z_1}\right\}$ is contained in $\overline{\mathbb{D}}$, where $\zeta_2 \in \partial\mathbb{D}$ is defined by the equation $\dfrac{1+a_2\zeta_2}{1+a_1\zeta_2} = -\dfrac{1+a_2z_2}{1+a_1z_2}$;*
  - **c.4)** *for $n = 2$, $a_1 \ne a_2$, $F(z) = K(z-z_1)(z-z_2)$, $f(z) = e^{i\gamma}F(z)+C(1+a_1z)^2+C_1(1+a_2z)^2$, $\forall\gamma \in \mathbb{R}$, $\forall z_1, z_2 \in \mathbb{D}$, $\forall C, C_1 \in \mathbb{C}\setminus\{0\}$, $\dfrac{C}{C_1} \ne -\left(\dfrac{a_2}{a_1}\right)^2$, for $K \in \mathbb{C}\setminus\{0\}$ such that*

$$\left|1+\frac{C(1+a_1z)^2+C_1(1+a_2z)^2}{e^{i\gamma}K(z-z_1)(z-z_2)}\right| \le 1 \quad \text{on } \partial\mathbb{D}$$

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and total change of $\arg\dfrac{(e^{it}-\zeta_1)(e^{it}-\zeta_2)}{(e^{it}-z_1)(e^{it}-z_2)}$ *equals* $-\pi$ *when* t *runs the interval* $(\arg\zeta_1, \arg\zeta_2)$, *here* $\zeta_1, \zeta_2 \in \partial\mathbb{D}$ *are the solutions of the equation* $\left(\dfrac{1+a_2\zeta}{1+a_1\zeta}\right)^2 = -\dfrac{C}{C_1}$.

c.5) *for* $n=2$, $a_1 \neq a_2$, $F(z) = A(z-z_1)(z-z_2)$, $f(z) = B(z-z_1)(z-z_2)$, $|B| < |A|$, *and* z_1, $z_2 \in \partial\mathbb{D}$ *such that* $\arg\dfrac{z_2}{z_1} = 2\beta \in (0,\pi]$ *can be arbitrary, the number* $\arg z_1 = \theta_1 \in [0, 2\pi)$ *is doubly defined from the equation*

$$\cos\alpha\cos\beta + \cos(\varphi_1 + \alpha - \theta_1 - \beta) = 0,$$

where $\varphi_1 = \arg a_1 \in [0; 2\pi)$, $\alpha = \dfrac{1}{2}\arg\dfrac{a_2}{a_1} \in (0; \pi/2]$;

c.6) *for* $a_1 = a_2$, $F(z) = K(1+a_1z)^{n-1}(a_1C_1z + C + C_1)$, $f(z) = F(z)(1 + e^{i\gamma}/K)$ $\forall\gamma \in \mathbb{R}$, $\forall K \in \mathbb{C}\setminus\{0\}$, C, $C_1 \in \mathbb{C}\setminus\{0\}$ *such that* $\left|1 + \dfrac{C_1a_1}{e^{i\gamma}K}\right| \le 1$, $\left|\dfrac{C}{C_1} + 1\right| = 1$;

c.7) *for* $a_1 = a_2$, $F(z) = K(1+a_1z)^{n-2}(z-z_1)(a_1C_1z+C+C_1)$, $f(z) = e^{i\gamma}F(z) + C(1+a_1z)^{n-1} + C_1(1+a_1z)^n$, $\forall\gamma \in \mathbb{R}$, $\forall z_1 \in \mathbb{D}$, $\forall K \in \mathbb{C}\setminus\{0\}$, C, $C_1 \in \mathbb{C}\setminus\{0\}$ *such that* $\left|\dfrac{C}{C_1} + 1\right| = 1$ *and the circle* $\left\{w_4(\partial\mathbb{D}) + 1 : w_4(z) = \dfrac{C_1a_1}{e^{i\gamma}K}\dfrac{1+a_1z}{z-z_1}\right\}$ *is contained in* $\overline{\mathbb{D}}$;

c.8) *for* $a_1 = a_2$, $F(z) = K(1+a_1z)^{n-1}(z-z_1)$, $f(z) = e^{i\gamma}F(z) + C(1+a_1z)^{n-1} + C_1(1+a_1z)^n$, $\forall\gamma \in \mathbb{R}$, $\forall z_1 \in \mathbb{D}$, $\forall K \in \mathbb{C}\setminus\{0\}$, C, $C_1 \in \mathbb{C}\setminus\{0\}$ *such that* $\left|\dfrac{C}{C_1} + 1\right| \le 1$ *and the circle* $\left\{w_5(\partial\mathbb{D}) + 1 : w_5(z) = \dfrac{1}{e^{i\gamma}K}\dfrac{a_1C_1z + C + 1}{z - z_1}\right\}$ *is contained in* $\overline{\mathbb{D}}$;

c.9) *for* $a_1 = a_2$, $F(z) = K(1+a_1z)^{n-2}(z-z_1)(z-z_2)$, $f(z) = e^{i\gamma}F(z) + C(1+a_1z)^{n-1} + C_1(1+a_1z)^n$, $\forall\gamma \in \mathbb{R}$, $\forall K \in \mathbb{C}\setminus\{0\}$, *for* z_1, $z_2 \in \mathbb{D}$ *and* C, $C_1 \in \mathbb{C}\setminus\{0\}$ *such that* $\left|\dfrac{C}{C_1} + 1\right| \le 1$ *and*

$$\left|1 + \frac{1}{e^{i\gamma}K}\frac{(1+a_1z)(a_1C_1z + C + 1)}{(z-z_1)(z-z_2)}\right| \le 1 \quad \text{for all } z \in \partial\mathbb{D}.$$

Proof. **1)** Suppose at least one of the values a_1, a_2 is exceptional for F. From commutativity of the Smirnov operator ($\widetilde{S}_{a_2}\widetilde{S}_{a_1} = \widetilde{S}_{a_1}\widetilde{S}_{a_2}$) we can assume that $a_1 = a'$. Then, by Proposition 1, $F(z) = C(1+a'z)^n$ and $\widetilde{S}_{a_1}[F] \equiv 0$. This yields $\widetilde{S}^2[F](z) \equiv 0$.

Since $|f(z)| \le |F(z)|$ on $\partial\mathbb{D}$ we obtain that the point $-\dfrac{1}{a'}$ is a zero of n-th order for f. Consequently, $f(z) = c(1+a'z)^n$, $|c| \le |C|$, and $\widetilde{S}^2[f] \equiv 0$; (7) is fulfilled.

2) Now let a_1 and a_2 be not exceptional values for F.

Consider the case $a_1, a_2 \in \mathbb{D}$. As it was noted in section 2, for $a \in \mathbb{D}$ the operator $\widetilde{S}_a$ preserves zeros. Then, by Corollary 1 applying to $\widetilde{S}_{a_1} : \mathcal{P}_n \to \mathcal{P}_{n-1}$ and $\widetilde{S}_{a_2} : \mathcal{P}_{n-1} \to \mathcal{P}_{n-2}$, we obtain (7). If at least one a_1, a_2 belong to $\partial\mathbb{D}$ then we obtain (7) by passage to the limit.

Now we investigate the case of equality in (7) for $z_0 \in \mathbb{C}\setminus\overline{\mathbb{D}}$. This is the most difficult part of the proof.

a) If a_1, $a_2 \in \mathbb{D}$ then, by Corollary 1, for $z_0 \in \mathbb{C}\setminus\overline{\mathbb{D}}$ equality in (7) holds iff $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$.

b) Suppose one of the parameters (say a_1) belong to $\partial\mathbb{D}$, $a_1 \neq a'$, $a_2 \in \mathbb{D}$. Assume $|\widetilde{S}^2[f](z_0)| = |\widetilde{S}^2[F](z_0)|$ for $z_0 \in \mathbb{C}\setminus\overline{\mathbb{D}}$, i.e. $|\widetilde{S}_{a_2}\widetilde{S}_{a_1}[f](z_0)| = |\widetilde{S}_{a_2}\widetilde{S}_{a_1}[F](z_0)|$. By Proposition 2, $\widetilde{S}_{a_1}[F]$ has all its zeros in $\overline{\mathbb{D}}$. By Theorem A, $|\widetilde{S}_{a_1}[f](z)| \le |\widetilde{S}_{a_1}[F](z)|$ on $\partial\mathbb{D}$. By Proposition 1, $\deg\widetilde{S}_{a_1}[f] \le \deg\widetilde{S}_{a_1}[F] = n-1$. Therefore the polynomials $\widetilde{S}_{a_1}[f]$ and $\widetilde{S}_{a_1}[F]$ possess conditions (∗). The operator $\widetilde{S}_{a_2}$ preserves zeros, because $a_2 \in \mathbb{D}$. Therefore, applying Theorem A to the polynomials $\widetilde{S}_{a_1}[f]$, $\widetilde{S}_{a_1}[F]$ and to the operator $\widetilde{S}_{a_2}$, we get $\widetilde{S}_{a_1}[f](z) = e^{i\gamma}\widetilde{S}_{a_1}[F](z)$, $\gamma \in \mathbb{R}$. Since the operator $\widetilde{S}_{a_1}$ is linear, the last equality can be rewritten as $\widetilde{S}_{a_1}[\psi](z) \equiv 0$, where $\psi(z) = f(z) - e^{i\gamma}F(z)$. Equivalently,

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$$(1+a_1z)\psi'(z)-na_1\psi(z)=0.$$

We find $\psi(z)=C(1+a_1z)^n$. Here and further C means any constant. Consequently,

$$f(z)=e^{i\gamma}F(z)+C(1+a_1z)^n. \tag{8}$$

b.1) If $C=0$ then $f=e^{i\gamma}F$. We have proved b.1).

Consider the case when $C\neq 0$. By $z_1,...,z_k$ denote roots of F that are not equal to $-\dfrac{1}{a_1}$. Note that such a root always exists, because otherwise $F(z)=K(1+a_1z)^n$ (here and further K means any nonzero constant), and we obtain that a_1 is the exceptional value for F. Contradiction. Write F in the form

$$F(z)=K(1+a_1z)^{n-k}(z-z_1)...(z-z_k),\quad 1\le k\le n.$$

Since $|f(z)|\le|F(z)|$ on $\partial\mathbb{D}$ we have

$$\left|1+\frac{C}{e^{i\gamma}}\frac{(1+a_1z)^n}{F(z)}\right|\le 1. \tag{9}$$

By the maximum modulus principle, (9) takes place for all $z\in\mathbb{C}\setminus\mathbb{D}$. It follows from (9) that $z_j\in\mathbb{D}$, $j=1,...,k$. Therefore the total change of $\arg\left\{\dfrac{(1+a_1e^{it})^k}{(e^{it}-z_1)...(e^{it}-z_k)}\right\}$ equals $-k\pi$ when t runs the interval $(\sigma;\sigma+2\pi)$, $\sigma=\arg(-1/a_1)$. Hence, for $k\ge 2$ there exists a point $z\in\partial\mathbb{D}$ such that

$$\operatorname{Re}\left\{\frac{C}{e^{i\gamma}}\frac{(1+a_1z)^n}{F(z)}\right\}>0,$$

i.e. (9) is not true. We obtain a contradiction. So, $C=0$ and $f=e^{i\gamma}F$, $\gamma\in\mathbb{R}$.

b.2) It remains to consider the case $k=1$, i.e. $F(z)=K(1+a_1z)^{n-1}(z-z_1)$, $z_1\in\mathbb{D}$. Then inequality (9) takes the form

$$\left|1+M\frac{1+a_1z}{z-z_1}\right|\le 1,\quad z\in\partial\mathbb{D}, \tag{10}$$

where $M=\dfrac{C}{e^{i\gamma}K}$. For all $z_1\in\mathbb{D}$ the function $w(z)=M\dfrac{1+a_1z}{z-z_1}$ maps $\partial\mathbb{D}$ onto a circle passing through the origin. It is possible to choose $\arg M$ and sufficiently small $|M|$ such that the circle $\{w(\partial\mathbb{D})+1\}$ is contained in $\overline{\mathbb{D}}$ and tangents $\partial\mathbb{D}$ at the point $z=1$. For such M inequality (10) takes place. For f we get

$$\widetilde{S}^2[f](z)=e^{i\gamma}\widetilde{S}^2[F](z)+C\widetilde{S}^2[(1+a_1z)^n](z)=e^{i\gamma}\widetilde{S}^2[F](z),$$

i.e. $\left|\widetilde{S}^2[f](z)\right|\equiv\left|\widetilde{S}^2[F](z)\right|$.

c) Now let $a_1,\ a_2\in\partial\mathbb{D}$. Suppose for $z_0\in\mathbb{C}\setminus\overline{\mathbb{D}}$

$$\left|\widetilde{S}^2[f](z_0)\right|=\left|\widetilde{S}^2[F](z_0)\right|.$$

If $\widetilde{S}^2[F](z)\neq 0$ for all $z\in\mathbb{C}\setminus\overline{\mathbb{D}}$ then by (7)

$$\left|\frac{\widetilde{S}^2[f](z)}{\widetilde{S}^2[F](z)}\right|\le 1,\quad z\in\mathbb{C}\setminus\mathbb{D}.$$

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In addition,

$$\left|\frac{\widetilde{S}^2[f](z_0)}{\widetilde{S}^2[F](z_0)}\right| = 1.$$

Then, as above,

$$\widetilde{S}^2[f](z) = e^{i\gamma}\widetilde{S}^2[F](z), \quad \gamma \in \mathbb{R}.$$

This means $\widetilde{S}^2[\psi](z) \equiv 0$ where $\psi = f - e^{i\gamma}F$, i.e.

$$(1 + a_2 z)H'(z) - (n-1)a_2 H(z) = 0, \tag{11}$$

where $H(z) = \widetilde{S}_{a_1}[\psi](z) = (1+a_1 z)\psi'(z) - n a_1 \psi(z)$. Solving (11), we find $H(z) = C(1+a_2 z)^{n-1}$. Therefore,

$$(1 + a_1 z)\psi'(z) - n a_1 \psi(z) = C(1 + a_2 z)^{n-1}. \tag{12}$$

The homogeneous differential equation

$$(1 + a_1 z)\Psi'(z) - n a_1 \Psi(z) = 0$$

has the solution $\Psi(z) = U(1 + a_1 z)^n$, $U = \text{const}$. Therefore the solution of (12) has the form $\psi(z) = U(z)(1 + a_1 z)^n$. From (12) we get

$$(1 + a_1 z)^{n+1} U'(z) = C(1 + a_2 z)^{n-1}$$

Then

$$U(z) = C \int \frac{(1 + a_2 z)^{n-1}}{(1 + a_1 z)^{n+1}} dz + C_1, \tag{13}$$

here and further C_1 is any constant.

If $\underline{a_1 \neq a_2}$ then

$$U(z) = C \int \frac{(a_1 - a_2 + a_2 t)^{n-1}}{t^{n+1}} dt + C_1 = C \int \sum_{k=0}^{n-1} C_{n-1}^k a_2^k t^{k-n-1} (a_1 - a_2)^{n-1-k} dt + C_1 =$$

$$= C \sum_{k=0}^{n-1} C_{n-1}^k a_2^k (a_1 - a_2)^{n-1-k} \frac{t^{k-n}}{k-n} + C_1 \quad \text{for} \quad t = 1 + a_1 z.$$

Therefore,

$$f(z) = e^{i\gamma}F(z) + U(z)(1 + a_1 z)^n = e^{i\gamma}F(z) + Ct^n + C_1 \sum_{k=0}^{n-1} C_{n-1}^k a_2^k (a_1 - a_2)^{n-1-k} \frac{t^k}{k-n}. \tag{14}$$

Taking into account that

$$\frac{C_{n-1}^k}{k-n} = -\frac{1}{n} C_n^k, \quad k = 0, \ldots, n-1,$$

we rewrite (14) in the form

$$f(z) = e^{i\gamma}F(z) + C_1 t^n + C(a_2 t + a_1 - a_2)^n = \\ = e^{i\gamma}F(z) + C_1(1 + a_1 z)^n + C(1 + a_2 z)^n. \tag{15}$$

c.1) If $C = C_1 = 0$ then $f = e^{i\gamma}F$, $\gamma \in \mathbb{R}$.

c.2) If $C_1 \neq 0,\ C = 0$ then as in b.2) we obtain that for every $\gamma \in \mathbb{R}$, $z_1 \in \mathbb{D}$, $K \in \mathbb{C} \setminus \{0\}$, and all $C_1 \in \mathbb{C}$ such that the circle $\left\{ w_1(\partial\mathbb{D}) + 1 : w_1(z) = \dfrac{C_1}{e^{i\gamma}} \dfrac{1 + a_1 z}{z - z_1} \right\}$ is contained in $\overline{\mathbb{D}}$, for the polynomials $F(z) = K(1 + a_1 z)^{n-1}(z - z_1)$ and $f(z) = e^{i\gamma}F(z) + C_1(1 + a_1 z)^n$ in (7) equality holds. This concludes the proof of c.2).

The case $C_1 = 0,\ C \neq 0$ is analogous to the case $C_1 \neq 0$, $C = 0$.

Now consider the case $C \neq 0,\ C_1 \neq 0,\ a_1 \neq a_2$. Since $|f(z)| \leq |F(z)|$ on $\partial\mathbb{D}$, by (15) and the maximum modulus principle, we have

$$\left| 1 + \frac{C}{e^{i\gamma}} v(z) \right| \leq 1, \quad z \in \mathbb{C} \setminus \mathbb{D}, \tag{16}$$

here

$$v(z) = \frac{q(1 + a_1 z)^n + (1 + a_2 z)^n}{F(z)}, \quad q = \frac{C_1}{C}.$$

If $q = -\left(\dfrac{a_2}{a_1}\right)^n$ then $\deg(q(1 + a_1 z)^n + (1 + a_2 z)^n) < n$. In this case

$$v(z) = \frac{1}{z^p} \sum_{k=0}^{\infty} b_k z^{-k}, \quad p \geq 1,$$

in a neighborhood of the infinity. Therefore there exists z with sufficiently large $|z|$ such that $\dfrac{C}{e^{i\gamma}} v(z) > 0$. Hence, (16) does not take place for some $z \in \mathbb{C} \setminus \mathbb{D}$. Further let $q \neq -\left(\dfrac{a_2}{a_1}\right)^n$.

By $\zeta_1, ..., \zeta_n$ denote the roots of the numerator of $v(z)$, i.e.

$$\frac{1 + a_2\zeta_j}{1 + a_1\zeta_j} = \left(\sqrt[n]{-q}\right)_j = w_j, \quad j = 1, ..., n, \tag{17}$$

here $(\sqrt[n]{-q})_j$ is the j-th branch of $\sqrt[n]{-q}$. Consequently the numerator of $v(z)$ has n simple roots

$$\zeta_j = \frac{1 - w_j}{a_1 w_j - a_2}.$$

Prove that for $n \geq 3$ there exists at least one $\zeta_j \in \mathbb{C} \setminus \overline{\mathbb{D}}$. In other words there exists at least one w_j such that $|1 - w_j| > |a_1 w_j - a_2| = |w_j - a|$, $a = \dfrac{a_2}{a_1} \in \partial\mathbb{D}$. Let P be the perpendicular bisector for the segment $[a, 1]$. Then P divides $\mathbb{C}$ into open half-planes H_1 and H_2, $1 \in H_1$ $a \in H_2$. If $w \in H_2$ then $|w - 1| > |w - a|$. All w_j belong to the circle $\Gamma = \left\{ w \in \mathbb{C} : |w| = |q|^{\frac{1}{n}} \right\}$ and uniformly distributed on Γ. Also P divides the circle Γ into two subarcs of the same length. Consequently if $n \geq 3$ there exists $w_j \in H_2$ and $|w_j - 1| > |w_j - a|$. So, there exists a root ζ_j of the numerator of $v(z)$ lying in $\mathbb{C} \setminus \overline{\mathbb{D}}$. Since all the roots ζ_k of $v(z)$ are simple then in a neighborhood $U(\zeta_j)$ of ζ_j we get

$$q(1 + a_1 z)^n + (1 + a_2 z)^n = b_1(z - \zeta_j) + b_2(z - \zeta_j)^2 + ...,$$

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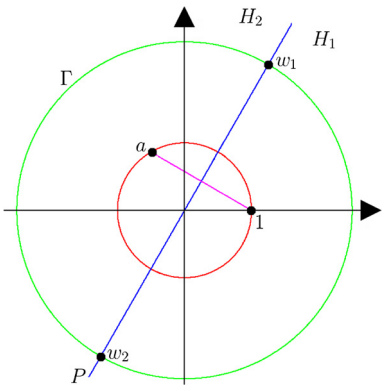


Fig. 2. Case $n = 2$, $C \neq 0$, $C_1 \neq 0$, $a_1 \neq a_2$.

$b_1 \neq 0$. Moreover, $F(\zeta_j) \neq 0$ by condition $(*)$. Therefore there exists $z \in U(\zeta_j) \subset \mathbb{C} \setminus \mathbb{D}$ such that $\dfrac{C}{e^{i\gamma}}\, v(z) > 0$, i.e. (16) does not take place in the case $C \neq 0$ $C_1 \neq 0$, $a_1 \neq a_2$, $n \geq 3$.

It remains to consider the case $n = 2$, $C \neq 0$, $C_1 \neq 0$, $a_1 \neq a_2$.

By (17)

$$w_j = \frac{1 + a_2\zeta_j}{1 + a_1\zeta_j} = \pm\sqrt{-q}, \quad j = 1, 2,$$

where $\sqrt{-q}$ is the main branch of the root. If one of ζ_j belong to $\mathbb{C} \setminus \overline{\mathbb{D}}$ then as in the case $C \neq 0$, $C_1 \neq 0$, $a_1 \neq a_2$, $n \geq 3$ we obtain a contradiction with (16). Consequently, both w_1 and w_2 lie on $P \cap \Gamma$ (see Fig. 2). This yields

$$|1 - w_j| = |w_j - a|, \quad j = 1, 2,$$

and $|\zeta_1| = |\zeta_2| = 1$. In other words, the roots of the numerator of $v(z)$ belong to $\partial\mathbb{D}$.

Let $F(z) = K(z - z_1)(z - z_2)$, z_1, $z_2 \in \overline{\mathbb{D}}$.

If at least one z_1, z_2 belongs to $\partial\mathbb{D}$ and does not equal ζ_1 and ζ_2 or $z_1 = z_2 = \zeta_j$ with the same j then we obtain a contradiction with (16).

c.3) Let $z_1 \in \mathbb{D}$, z_2 equals one of ζ_j (say ζ_1). Then we rewrite (16) in the form

$$\left|1 + \frac{C}{e^{i\gamma}} \frac{q(1 + a_1 z)^2 + (1 + a_2 z)^2}{F(z)}\right| = \left|1 + \frac{C}{Ke^{i\gamma}} \frac{z - \zeta_2}{z - z_1}\right| \leq 1, \quad z \in \partial\mathbb{D}. \tag{18}$$

Arguing as in the case c.2) we can choose K such that (18) takes place. Therefore $\forall\, \gamma \in \mathbb{R}$, $\forall\, z_1 \in \mathbb{D}$, $\forall\, z_2 \in \partial\mathbb{D}$, $-\dfrac{1}{a_2} \neq z_2 \neq -\dfrac{1}{a_1}$, $\forall\, C \in \mathbb{C}$, for $C_1 = -\left(\dfrac{1 + a_2 z_2}{1 + a_1 z_2}\right)^2 C$, and $K \in \mathbb{C} \setminus \{0\}$ such that the circle

$$\left\{w_3(\partial\mathbb{D}) + 1 : w_3 = \frac{C}{Ke^{i\gamma}} \frac{z - \zeta_2}{z - z_1}\right\}$$

($\zeta_2 \in \partial\mathbb{D}$ is the solution of the equation $\left(\dfrac{1 + a_2\zeta}{1 + a_1\zeta}\right)^2 = -\dfrac{C_1}{C}$, $\zeta_2 \neq z_2$, i.e. $\dfrac{1 + a_2\zeta_2}{1 + a_1\zeta_2} = -\dfrac{1 + a_2 z_2}{1 + a_1 z_2}$) is contained in $\mathbb{D}$, the polynomials

$$F(z) = K(z - z_1)(z - z_2), \quad f(z) = e^{i\gamma}F(z) + C_1(1 + a_1 z)^2 + C(1 + a_2 z)^2$$

possess conditions $(*)$. For such f and F we have

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$$\widetilde{S}^2[f](z) = e^{i\gamma}\widetilde{S}^2[F](z) + C_1\widetilde{S}^2[(1+a_1z)^2](z) + C\widetilde{S}^2[(1+a_2z)^2](z) = e^{i\gamma}\widetilde{S}^2[F],$$

i.e. $|\widetilde{S}^2[f](z)| \equiv |\widetilde{S}^2[F](z)|$. This concludes the proof of c.3).

Now assume $z_1,\ z_2 \in \mathbb{D}$. Rewrite (16) in the form

$$\left|1 + C'\frac{(z-\zeta_1)(z-\zeta_2)}{(z-z_1)(z-z_2)}\right| \le 1, \quad z \in \partial\mathbb{D}, \tag{19}$$

here $C' = \dfrac{C_1a_1^2 + Ca_2^2}{e^{i\gamma}K}$. Denote $\zeta_1 = e^{ix}$, $\zeta_2 = e^{iy}$, $x < y < x + 2\pi$, $t = \arg z$. The function

$$\varphi(t) = \arg\frac{(z-\zeta_1)(z-\zeta_2)}{(z-z_1)(z-z_2)}$$

is continuous on $(x, x+2\pi)$ except for the point y. At the point y the function $\varphi(t)$ has a jump that equals π. Denote

$$\xi(t) = \frac{e^{it}-z_1}{e^{it}-\zeta_1}, \quad \eta(t) = \frac{e^{it}-z_2}{e^{it}-\zeta_2}.$$

Then $\varphi(t) = -\arg\xi(t) - \arg\eta(t), t \in (x, x+2\pi)$, further we assume $\arg\xi(x) := \arg\xi(x+0)$, $\arg\eta(y) := \arg\eta(y+0)$. When t runs the half-open interval $[x, x+2\pi)$ the points $\xi(t)$ and $\eta(t)$ runs the straight lines $l_1 = \{\xi(t) : t \in [x, x+2\pi)\}$ and $l_2 = \{\eta(t) : t \in [y, y+2\pi)\}$ on the extended complex plane $\overline{\mathbb{C}}$ in this direction in which the origin lies to the left of the lines l_1 and l_2, because $z_1, z_2 \in \mathbb{D}$. Therefore $\arg\xi(t)$ increases on $(x, x+2\pi)$, $\arg\eta(t)$ increases on $(y, y+2\pi)$. Hence $\varphi(t)$ decreases on the intervals (x, y) and $(y, x+2\pi)$. Moreover the sum of total changes of $\varphi(t)$ on (x, y) and on $(y, x+2\pi)$ equals -2π. If total change of φ is less than $-\pi$ either on (x, y) or on $(y, x+2\pi)$ then as in b) there exists a point $z \in \partial\mathbb{D}$ such that

$$\operatorname{Re}\left\{C'\frac{(z-\zeta_1)(z-\zeta_2)}{(z-z_1)(z-z_2)}\right\} > 0.$$

We obtain a contradiction with (19).

c.4) If total change of φ equals $-\pi$ on both intervals (x, y) and $(y, x+2\pi)$ then (19) takes place for sufficiently small $|C'|$ (large $|K|$) and appropriate $\arg C'(\arg K)$. Namely, if total change of φ equals $-\pi$ on (x, y) then on (x, y) the function φ decreases and takes all values from an interval $(\varphi_0, \varphi_0 - \pi)$. In this case one should take $\arg C' = 3\pi/2 - \varphi_0$ in order to inequality (19) be satisfied.

c.5) Suppose $z_1 = \zeta_1,\ z_2 = \zeta_2$. Then $v(z) \equiv \text{const}$. In this case we get

$$F(x) = A(z-z_1)(z-z_2), \quad f(z) = B(z-z_1)(z-z_2),$$

$z_1, z_2 \in \partial\mathbb{D}$, $|B| \le |A|$ by conditions $(*)$. If $B = e^{i\gamma}A$, $\gamma \in \mathbb{R}$, then $f = e^{i\gamma}F$. Further we shall consider the case $|B| < |A|$. Then f and F possess conditions $(*)$ and for $z_0 \in \mathbb{C} \setminus \overline{\mathbb{D}}$ equality holds in (7) if and only if

$$\frac{1}{A}\widetilde{S}^2[F] = 2(1+a_1z)(1+a_2z) - (a_1+a_2+2a_1a_2z)(2z-z_1-z_2) + 2a_1a_2(z-z_1)(z-z_2) =$$
$$= 2 + (a_1+a_2)(z_1+z_2) + 2a_1a_2z_1z_2 = 0.$$

The last equality is equivalent to

$$2\bar{a}_1\bar{a}_2 + (\bar{a}_1+\bar{a}_2)(z_1+z_2) + 2z_1z_2 = 0. \tag{20}$$

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Denote $\bar{a}_1 = e^{i\varphi_1}$, $\bar{a}_2 = e^{i\varphi_2}$, $z_1 = e^{i\theta_1}$, $z_2 = e^{i\theta_2}$, $\varphi_2 - \varphi_1 = 2\alpha$, $\theta_2 - \theta_1 = 2\beta$. Since $z_1 \neq z_2$, $\zeta_1 \neq \zeta_2$ we can assume that $\alpha \in \left(0; \frac{\pi}{2}\right]$, $\beta \in \left(0; \frac{\pi}{2}\right]$. Rewrite (20):

$$2e^{2i(\varphi_1+\alpha)} + \left|e^{i\varphi_1} + e^{i\varphi_2}\right| e^{i(\varphi_1+\alpha)} \left|e^{i\theta_1} + e^{i\theta_2}\right| e^{i(\theta_1+\beta)} + 2e^{2i(\theta_1+\beta)} = 0. \tag{21}$$

Since

$$\left|e^{i\varphi_1} + e^{i\varphi_2}\right| = \left|e^{i\alpha} + e^{-i\alpha}\right| = 2\cos\alpha, \quad \left|e^{i\theta_1} + e^{i\theta_2}\right| = \left|e^{i\beta} + e^{-i\beta}\right| = 2\cos\beta$$

(21) takes the form:

$$\begin{gathered} e^{2i(\varphi_1+\alpha)} + 2e^{i(\varphi_1+\alpha)}e^{i(\theta_1+\beta)}\cos\alpha\cos\beta + e^{2i(\theta_1+\beta)} = 0 \Longleftrightarrow \\ \cos\alpha\cos\beta + \frac{1}{2}\left(e^{i(\varphi_1+\alpha-\theta_1-\beta)} + e^{i(\theta_1+\beta-\varphi_1-\alpha)}\right) = 0 \Longleftrightarrow \\ \cos\alpha\cos\beta + \cos(\varphi_1+\alpha-\theta_1-\beta) = 0. \end{gathered} \tag{22}$$

On the interval of length 2π for fixed φ_1, α, β equation (22) has two solutions with respect to θ_1. In terms of a_1, a_2, z_1, z_2 this means that for $|B| < |A|$ in (7) equality holds if $F(z) = A(z-z_1)(z-z_2)$, $f(z) = B(z-z_1)(z-z_2)$, here $\arg\dfrac{z_2}{z_1} = 2\beta \in (0,\pi]$ can be arbitrary, $\arg z_1 = \theta_1 \in [0, 2\pi)$ can be doubly defined from equation (22). This completes the proof of c.5).

If $\underline{a_1 = a_2 \in \partial\mathbb{D}}$ then in terms of the above notations (13) takes the form

$$U(z) = C\int \frac{dz}{(1+a_1 z)^2} + C_1 = \frac{C}{1+a_1 z} + C_1$$

and $\psi(z) = C(1+a_1 z)^{n-1} + C_1(1+a_1 z)^n$. From this we get

$$f(z) = e^{i\gamma}F(z) + \psi(z) = e^{i\gamma}F(z) + C(1+a_1 z)^{n-1} + C_1(1+a_1 z)^n \tag{23}$$

As in b) we write $F(z)$ in the form $F(z) = K(1+a_1 z)^{n-k}(z-z_1)...(z-z_k)$, $K = const \neq 0$.

If $\underline{C = C_1 = 0}$ then we obtain the case c.1).

The case $\underline{C = 0,\ C_1 \neq 0}$ is c.2).

Let $\underline{C_1 = 0,\ C \neq 0}$, then $f(z) = e^{i\gamma}F(z) + C(1+a_1 z)^{n-1}$. By $(*)$

$$\left|1 + \frac{C(1+a_1 z)^{k-1}}{e^{i\gamma}K(z-z_1)...(z-z_k)}\right| \leq 1 \quad \text{on} \quad \partial\mathbb{D}. \tag{24}$$

From (24) it follows that $z_j \in \mathbb{D}$. Then total change of $\arg\left\{\dfrac{(1+a_1 e^{it})^{k-1}}{(e^{it}-z_1)...(e^{it}-z_k)}\right\}$ equals $-(k+1)\pi$ as t runs the interval $(\sigma;\sigma+2\pi)$, here $\sigma = \arg(-1/a_1)$. Therefore, as in b.2), (24) does not hold for some $z \in \partial\mathbb{D}$.

Now we consider the case $\underline{C_1 \neq 0 \neq C}$. From $(*)$ we get

$$\left|1 + \frac{(1+a_1 z)^{n-1}(C + C_1 + C_1 a_1 z)}{e^{i\gamma}F(z)}\right| \leq 1, \quad z \in \partial\mathbb{D}. \tag{25}$$

Denote $s = -\dfrac{C/C_1 + 1}{a_1}$,

$$\nu(z) = \frac{C_1 a_1}{e^{i\gamma}K}\frac{(1+a_1 z)^{k-1}(z-s)}{(z-z_1)...(z-z_k)}, \tag{26}$$

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then (25) takes the form

$$|1+\nu(z)|\le 1,\quad z\in\partial\mathbb{D}. \tag{27}$$

Consider all possible cases of location of the point s.

In the case $s\in\mathbb{C}\setminus\overline{\mathbb{D}}$ all $z_j\in\mathbb{D}$ by (27). Arguing as in the case $C_1=0$, $C\neq 0$ we obtain that (27) does not hold for some $z\in\partial\mathbb{D}$.

Consider $s\in\partial\mathbb{D}$. Let one of the roots of $F(z)$ (say z_k) equal s. Then for $k\ge 2$

$$F(z)=K(1+a_1z)^{n-k}(z-z_1)...(z-z_{k-1})(z-s).$$

For $k=1$ we have $F(z)=(1+a_1z)^{n-1}(z-s)$.

c.6) For $k=1$ we rewrite (27) in the form

$$\left|1+\frac{C_1a_1}{e^{i\gamma}K}\right|\le 1. \tag{28}$$

For fixed $a_1\in\partial\mathbb{D}$, $\gamma\in\mathbb{R}$, $K\in\mathbb{C}\setminus\{0\}$, (28) defines a disc in the plane of the variable C_1. So, for all $a_1\in\partial\mathbb{D}$, $\gamma\in\mathbb{R}$, $K\in\mathbb{C}\setminus\{0\}$, and C_1 such that (28) holds, the polynomials $F(z)=K(1+a_1z)^{n-1}(z-s)$, $f(z)=e^{i\gamma}F(z)+C(1+a_1z)^{n-1}+C_1(1+a_1z)^n$ possess conditions $(*)$ for $|s|=\left|\dfrac{C}{C_1}+1\right|=1$. Clearly, for such f and F in (7) equality holds.

c.7) For $k=2$ we have

$$\nu(z)=\frac{C_1a_1}{e^{i\gamma}K}\frac{1+a_1z}{z-z_1}.$$

Arguing as in the case b.2) we see that for fixed $\gamma\in\mathbb{R}$, $z_1\in\mathbb{D}$, $K\in\mathbb{C}\setminus\{0\}$ and proper C_1 (27) holds. Consequently, for described $F(z)=K(1+a_1z)^{n-2}(z-z_1)(z-s)$, $|s|=1$, and $f(z)$, defined by (23), conditions $(*)$ are satisfied and in (7) equality holds. This completes the proof of c.7).

Now let $k\ge 3$. Then

$$\nu(z)=\frac{C_1a_1}{e^{i\gamma}K}\frac{(1+a_1z)^{k-1}}{(z-z_1)...(z-z_{k-1})},$$

by (27) $z_1,...,z_{k-1}\in\mathbb{D}$. Therefore, the total change of $\arg\nu(z)$ equals $-(k-1)\pi\le -2\pi$ and (27) does not take place.

Now we assume that $s\in\partial\mathbb{D}$, but $F(s)\neq 0$, i.e. $F(z)=K(1+a_1z)^{n-k}(z-z_1)...(z-z_k)$. By (27) $z_j\in\mathbb{D}$. Note that $s\neq -1/a_1$. Denote $s=e^{ix}$. For $\nu(z)$, defined by (26), the function $\arg\nu(e^{it})$ is continuous on an interval $(\sigma,\sigma+2\pi)\ni x$ except for the one point x, here $\sigma=\arg(-1/a_1)$. At the point x the function $\arg\nu(e^{it})$ has a jump that equals π. Taking into account this fact we have

$$\lim_{t\to\sigma+2\pi-0}\arg\nu(e^{it})-\lim_{t\to\sigma+0}\arg\nu(e^{it})=(1-k)\pi.$$

Consequently, if $k\ge 3$ then (27) does not hold.

c.8) (case $\left|\frac{C}{C_1}+1\right|=1$). For $k=1$, i.e. $F(z)=K(1+a_1z)^{n-1}(z-z_1)$, $z_1\in\mathbb{D}$, we have

$$\nu(z)=\frac{C_1a_1}{e^{i\gamma}K}\frac{(z-s)}{(z-z_1)}.$$

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Arguing as in the case b.2) we obtain that for fixed $\gamma \in \mathbb{R}$, $z_1 \in \mathbb{D}$, $K \in \mathbb{C} \setminus \{0\}$, $|s| = 1$, there exists C_1 such that (27) takes place. So the equality in (7) holds for the polynomials $F(z) = K(1 + a_1 z)^{n-1}(z - z_1)$, $f(z) = e^{i\gamma}F(z) + C(1 + a_1 z)^{n-1} + C_1(1 + a_1 z)^n$, $|s| = \left|\dfrac{C}{C_1} + 1\right| = 1$, with described parameters. We have proved c.8) for $|s| < 1$.

c.9)(case $\left|\frac{C}{C_1} + 1\right| = 1$). For $\underline{k = 2}$, $s \in \partial\mathbb{D}$, $F(s) \neq 0$, the equality in (7) holds for

$F(z) = K(1 + a_1 z)^{n-2}(z - z_1)(z - z_2)$, and f, described by (23), for those parameters, for which $|f(z)| \leq |F(z)|$ on $\partial\mathbb{D}$.

c.8, c.9)(case $\left|\frac{C}{C_1} + 1\right| < 1$). Consider the case $\underline{s \in \mathbb{D}}$. In this case the total change of $\arg \nu(e^{it})$ equals $-(k-1)\pi$ as t runs the interval $(\sigma; \sigma + 2\pi)$, $\sigma = \arg(-1/a_1)$, $\nu(z)$ is defined by (26). So, for $k \geq 3$ (27) does not take place. For $k = 1$ and $k = 2$ we obtain c.8) and c.9) correspondingly with $|s| = \left|\dfrac{C}{C_1} + 1\right| < 1$.

Above we have considered the case $\widetilde{S}^2[F](z) \neq 0$ for all $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$. We conclude the proof by considering the case $\underline{\widetilde{S}^2[F](z) = 0 \text{ for some } z \in \mathbb{C} \setminus \overline{\mathbb{D}}}$. Arguing as in the proof of Proposition 2 we obtain $\widetilde{S}^2[F](z) \equiv 0$. As in c) we get

$$F(z) = C(1 + a_1 z)^n + C_1(1 + a_2 z)^n.$$

The case $C = C_1 = 0$ is impossible, because $\deg F = n$. The cases $C = 0$, $C_1 \neq 0$ and $C \neq 0$, $C_1 = 0$ are also impossible, because a_1, a_2 are not exceptional values for F. Let $C \neq 0$, $C_1 \neq 0$. Arguing as in c), case $C \neq 0$, $C_1 \neq 0$, we see that for $n \geq 3$ F has a zero in $\mathbb{C} \setminus \overline{\mathbb{D}}$ that is impossible by $(*)$.

For $n = 2$ as in c), case $C \neq 0$, $C_1 \neq 0$, we see that the roots of F must belong to $\partial\mathbb{D}$, i.e.

$F(z) = A(z - z_1)(z - z_2)$, $z_1, z_2 \in \partial\mathbb{D}$. By $(*)$ $f(z) = B(z - z_1)(z - z_2)$, $|B| \leq |A|$. Arguing as in the case c.5) we obtain the same conditions on f and F. In other words, we do not obtain new functions for which in (7) equality holds. □

For fixed $|z| = R \geq 1$ we rewrite inequality (7) in the following form

$$\left|z^2 f''(z) - (n-1)\left(\frac{a_1 z}{1 + a_1 z} + \frac{a_2 z}{1 + a_2 z}\right) z f'(z) + n(n-1)\frac{a_1 z}{1 + a_1 z}\frac{a_2 z}{1 + a_2 z} f(z)\right| \leq$$

$$\leq \left|z^2 F''(z) - (n-1)\left(\frac{a_1 z}{1 + a_1 z} + \frac{a_2 z}{1 + a_2 z}\right) z F'(z) + n(n-1)\frac{a_1 z}{1 + a_1 z}\frac{a_2 z}{1 + a_2 z} F(z)\right|.$$

Denoting $\alpha_1 = \dfrac{a_1 z}{1 + a_1 z}$, $\alpha_2 = \dfrac{a_2 z}{1 + a_2 z}$, we define the new operator

$$S^2[f](z) = z^2 f''(z) - (n-1)(\alpha_1 + \alpha_2) z f'(z) + n(n-1)\alpha_1 \alpha_2 f(z).$$

Since $a_1, a_2 \in \overline{\mathbb{D}}$ then α_1, α_2 belong to the image of the disk $\{t \in \mathbb{C} : |t| \leq |z|\}$ under the mapping $\dfrac{t}{1 + t}$, i.e. $\alpha_1, \alpha_2 \in \overline{\Omega_R}$. Taking into account that $\overline{\Omega_{R_1}} \subset \overline{\Omega_{R_2}}$ for $R_1 < R_2$ we obtain the following

Corollary 2. *Let f and F be polynomials possessing conditions $(*)$, $R \geq 1$. Then*

$$|S^2[f](z)| \leq |S^2[F](z)|, \quad |z| \geq R, \tag{29}$$

here parameters α_1, α_2 from the definition of the operator S^2 belong to $\overline{\Omega_R}$.

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Corollary 3. *Let f be a polynomial, $\deg f = n$. For $r > 0$ denote $M_r = \max\limits_{|z|=r} |f(z)|$. Then for $|z| \geq r$*

$$|f''(z)| \leq \frac{n(n-1)}{(|z|+r)^2}\left(|f(z)| + \frac{|z|^{n-2}}{r^{n-2}}\left(2\frac{|z|}{r}+1\right)M_r\right). \tag{30}$$

For $|z| = r$

$$|f''(z)| \leq \frac{n(n-1)}{4r^2}\left(|f(z)| + 3M_r\right). \tag{31}$$

In (30) and (31) equality holds for $f(z) = z^n$.

Proof. Let $f_r(z) = f(rz)$, $r > 0$. Construct the polynomial $F_r(z) = M_r z^n$. Then f_r and F_r possess conditions $(*)$. Applying Corollary 2 to f_r and F_r we get

$$|S^2[f_r](z)| \leq |S^2[F_r](z)|, \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

Equivalently,

$$\begin{aligned} |z^2r^2f''(rz) - (n-1)(\alpha_1+\alpha_2)rzf'(rz) + n(n-1)\alpha_1\alpha_2 f(rz)| \leq \\ \leq n(n-1)M_r|z|^n|1-\alpha_1-\alpha_2+\alpha_1\alpha_2|, \quad z \in \mathbb{C}\setminus\mathbb{D}, \end{aligned} \tag{32}$$

here α_1, α_2 are any constants from $\overline{\Omega_{|z|}}$. Put $\alpha_1 = -\alpha_2 = \dfrac{|z|}{|z|+1}$. Then we rewrite (32) in the form

$$|z^2r^2f''(rz) - n(n-1)\alpha_1^2 f(rz)| \leq n(n-1)M_r|z|^n|1-\alpha_1^2|.$$

This implies

$$|f''(rz)| \leq \frac{n(n-1)}{(|z|+1)^2r^2}(|f(rz)| + M_r|z|^{n-2}(2|z|+1)), \quad z \in \mathbb{C}\setminus\mathbb{D}.$$

Put $w = rz$. For $|w| \geq r$ we obtain

$$|f''(w)| \leq \frac{n(n-1)}{(|w|+r)^2}\left(|f(w)| + M_r\frac{|w|^{n-2}}{r^{n-2}}\left(\frac{2|w|}{r}+1\right)\right). \tag{33}$$

In (33) equality holds for all $|w| \geq r$ for the function $f(z) = z^n$. Note that on the circle $|w| = r$ (33) give us

$$|f''(w)| \leq \frac{n(n-1)}{4r^2}(|f(w)| + 3M_r).$$

Note that this inequality can be also obtained by double application of inequality (1). □

Remark 1. For $r = 1$ the inequality (31) improves the Bernstein inequality [3, p.169]

$$|f''(z)| \leq n(n-1)M_1, \quad z \in \partial\mathbb{D}.$$

Remark 2. From Theorem 1 and Theorem 2 we obtain that the operator $\widetilde{S}^2$ preserves zeros if $a_1, a_2 \in \mathbb{D}$. If at least one a_1, a_2 belong to $\partial\mathbb{D}$ (say a_1) then $\widetilde{S}^2$ does not preserve zeros, because we can find a polynomial F with zeros in $\overline{\mathbb{D}}$, $\deg F = n$, such that $\widetilde{S}_2[F](z) \equiv 0$. This is, for example, the polynomial $F(z) = C(1+a_1z)^n$, $C = const \neq 0$. That is, not all zeros of $\widetilde{S}^2[F]$ belong to $\overline{\mathbb{D}}$.

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4. Comparison of the operators S_α and S^2 with the Marden operator B

First we compare differential operators of first order: the Smirnov operator $S_\alpha[f](z) = zf'(z) - n\alpha f(z)$ and the Marden operator B of the first order (with $\lambda_2 = 0$), i.e.

$$B[f](z) = \lambda_0 f(z) + \lambda_1 \frac{nz}{2} f'(z).$$

We assume $\lambda_1 \neq 0$, otherwise we obtain the trivial case. In order to inequalities (1) from Theorem A and (3) be fulfilled we should require $\alpha \in \overline{\Omega_{|z|}}$ and the root of the polynomial $u(z) = \lambda_0 + n\lambda_1 z$ should lie in the half-plane $\operatorname{Re} z \leq \dfrac{n}{4}$ by (2). In other words,

$$\operatorname{Re}\left(-\frac{\lambda_0}{\lambda_1 n}\right) \leq \frac{n}{4}.$$

Compare the sets of parameters in Theorem A and Theorem B. In Theorem A this set (coefficient near $-f(z)$) is $\mathcal{A} = \left\{n\alpha : \alpha \in \overline{\Omega_{|z|}}\right\}$. In Theorem B the set of such coefficient near $-f(z)$ is

$$\mathcal{B} = \left\{-\frac{2\lambda_0}{\lambda_1 n} : \operatorname{Re}\left(-\frac{\lambda_0}{\lambda_1 n}\right) \leq \frac{n}{4}\right\} = \left\{t : \operatorname{Re} t \leq \frac{n}{2}\right\}.$$

Consider the differential inequalities from Theorem A and Theorem B for $z \in \partial\mathbb{D}$. In this case $\mathcal{A} = \mathcal{B}$. But for $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ we have $\mathcal{B} \subsetneq \mathcal{A}$. In other words in Theorem A and Theorem B formally the same inequality was obtained

$$|B[f](z)| = \left|\lambda_1 \frac{n}{2} S_\alpha[f](z)\right| \leq |B[F](z)| = \left|\lambda_1 \frac{n}{2} S_\alpha[F](z)\right|,$$

but for different sets of parameters. Moreover, the set of parameters in Theorem A is essentially wider than in Theorem B.

We shall see analogous situation, comparing differential operators of second order. Consider the operator S^2 and the Marden operator B with $\lambda_2 \neq 0$.

Denote by z_1 and z_2 the roots of the polynomial $u(z)$ from condition (2). Denote $z_1 = \dfrac{n}{2}\alpha_1, z_2 = \dfrac{n}{2}\alpha_2$. Then, by (2), $\operatorname{Re}\alpha_k \leq \dfrac{1}{2}, k = 1, 2$. Herewith

$$u(z) = \lambda_0 + \lambda_1 nz + \lambda_2 \frac{n(n-1)}{2} z^2 = \lambda_2 \frac{n(n-1)}{2} \left(z - \frac{n}{2}\alpha_1\right)\left(z - \frac{n}{2}\alpha_2\right).$$

We find

$$\lambda_0 = \lambda_2 \frac{n^3(n-1)}{8}\alpha_1\alpha_2, \quad \lambda_1 = -\lambda_2 \frac{n(n-1)}{4}(\alpha_1 + \alpha_2).$$

Consequently,

$$B[f](z) = \lambda_2 \frac{n^2}{8}\left(n(n-1)\alpha_1\alpha_2 f(z) - (n-1)(\alpha_1 + \alpha_2) z f'(z) + z^2 f''(z)\right) = \lambda_2 \frac{n^2}{8} S^2[f](z).$$

Therefore, for $z \in \partial\mathbb{D}$ the inequalities $|B[f](z)| \leq |B[F](z)|$ and $|S^2[f](z)| \leq |S^2[F](z)|$ are equivalent. But for $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ the set of parameters $\left(\alpha_k \in \Omega_{|z|}\right)$ in the last inequality is much more wider than in Theorem B.

The new condition on the roots $z_k \in \dfrac{n}{2}\overline{\Omega_{|z|}}$ give us the following result.

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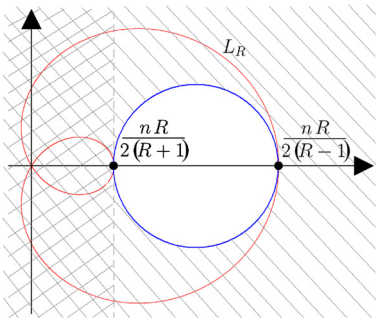


Fig. 3. Comparison of Theorem B, Theorem C, and Theorem 3.

Theorem 3. *Let f and F be polynomials possessing conditions $(*)$, $R \geq 1$. If the roots of the polynomial*

$$u(z) = \lambda_0 + \lambda_1 C_n^1 z + \lambda_2 C_n^2 z^2$$

lie in $\frac{n}{2}\overline{\Omega_R}$ then

$$|B[f](z)| \leq |B[F](z)|, \quad |z| \geq R.$$

If $R = 1$ in Theorem 3 we don't obtain new result, because $\frac{n}{2}\overline{\Omega_R} = \left\{z : \operatorname{Re} z \leq \frac{n}{4}\right\}$. In other words, we obtain the set of parameters from Theorem B.

If $R > 1$ we have much more weaker condition on the roots of $u(z)$ than in Theorem B and Theorem C. Moreover, comparing Theorem 3 with Theorem C, it is interesting to note that in Theorem C both roots of $u(z)$ should lie either in the half-plane $\operatorname{Re} z < \frac{n}{2}\frac{R}{1+R}$ or in a half-plane that boundary is tangent to the curve L_R. In Theorem 3 the roots of $u(z)$ are any points from $\frac{n}{2}\overline{\Omega_R}$ (see Fig. 3).

The following example shows that in the cases when $|z| = 1$ or $n = 2$ the set of parameters in (29) can not be extended.

Example 1. Consider $f(z) = 1, F(z) = z^n$. Suppose $\alpha_1 = \alpha_2 = \alpha \in \mathbb{C} \setminus \overline{\Omega_{|z|}}$. If (29) takes place for such f, F, α_1, and α_2, then

$$|\alpha|^2 \leq |z^n - 2\alpha z^n + \alpha^2 z^n| = |z|^n |1-\alpha|^2, \quad z \in \mathbb{C} \setminus \mathbb{D}.$$

In this case we have $\left|\frac{\alpha}{1-\alpha}\right| \leq |z|^{\frac{n}{2}}$. On the other hand, since $\alpha \notin \Omega_{|z|}$ we obtain $\left|\frac{\alpha}{1-\alpha}\right| > |z|$. Therefore if $|z| = 1$ or $n = 2$ we obtain a contradiction and (29) does not take place.

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