



Evolution of infinite populations of immigrants: Micro- and mesoscopic description

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ARTICLE INFO

Article history:

Received 12 September 2018

Available online 18 April 2019

Submitted by Y. Du

Keywords:

Complex system

Infinite population

Diversity

Population dynamics

Stochastic semigroup

Kinetic equation

ABSTRACT

A model is proposed of an infinite population of entities immigrating to a noncompact habitat, in which the newcomers are repelled by the already existing population. The evolution of such a population is described at micro- and mesoscopic levels. The microscopic states are probability measures on the corresponding configuration space. States of populations without interactions are Poisson measures, fully characterized by their densities. The evolution of micro-states is Markovian and obtained from the Kolmogorov equation with the use of correlation functions. The mesoscopic description is made by a kinetic equation for the densities. We show that the micro-states are approximated by the Poissonian states characterized by the densities obtained from the kinetic equation. Both micro- and mesoscopic descriptions are performed and their interrelations are analyzed, that includes also discussing the problem of the appearance of a spatial diversity in such populations.

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1. Introduction

It is a common viewpoint that the observed spatial diversity of a complex system appears due to the interplay between environmental heterogeneities and interactions between its constituents. This equally refers to systems studied in biology [11], chemistry [19], physics and other sciences [9]. Despite a substantial progress achieved during the last decades it is still a challenging analytic problem to elaborate adequate models and mathematical means for studying these phenomena, see [5,9,21].

In this work, we introduce and analyze a seemingly simple model in which point entities arrive at random in a noncompact habitat, for convenience chosen to be \mathbb{R}^d , $d \geq 1$. The already existing entities may repel the newcomers. No other actions – like birth, death (departure), jumps – are taken into account. The entities can be immigrants, molecules, ions, micro-organisms, etc. The simplicity of the model allows us to analyze various aspects of its description at micro- and mesoscopic levels and their interconnections. At the same time, the model is rich enough to capture the main peculiarities of the dynamics of such populations, and

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some of the aspects of its theory developed here are quite demanding. The model can also be used as a part of more involved models, for which our present analysis can be used as a starting point in their study.

If the arrival intensity b (probability density per time) depends only on the location $x \in \mathbb{R}^d$, at the microscopic level the dynamics of the population can be described as a spatio-temporal Poisson process, cf. [10], for which the probability $P_{t,\Lambda}(N)$ of having N entities at time t in a compact set $\Lambda \subset \mathbb{R}^d$ is given by the Poisson law¹

$$P_{t,\Lambda}(N) = \frac{1}{N!} \left(t \int_{\Lambda} b(x) dx \right)^N \exp \left(-t \int_{\Lambda} b(x) dx \right). \quad (1.1)$$

In this case, the particle density is $\varrho_t(x) = tb(x)$ and the only source of the spatial diversity is the x -dependence of b . As is typical for infinite systems of this kind, their microscopic description turns into a hard mathematical problem whenever one wants to include interactions. A more practical approach is to describe such systems by using aggregate characteristics, like particle density. In this case, however, one loses the possibility to directly include interactions as the individual entities do not participate in the description. A usual bypass here – borrowed from statistical physics where it is known as the mean-field approach – is to make the model parameters state-dependent. This leads to so called phenomenological (or heuristic) models that often have no microscopic analogies and thus provide rather superficial description of the corresponding phenomena. At this level, the evolution of our model is described by the following equation

$$\frac{d}{dt} \varrho_t(x) = b(x, \varrho_t) = b(x) \exp \left(- \int_{\mathbb{R}^d} K(x, y) \varrho_t(y) dy \right), \quad (1.2)$$

where $K(x, y)$ is a positive kernel, and hence the interaction is repulsive. By setting $K(x, y) = \phi(x - y) = \phi(y - x)$ one obtains a translation invariant version. It is possible to show, see Theorem 2.9 below, that this equation has a unique positive solution $\varrho_t \in L^\infty(\mathbb{R}^d)$ if ϕ is bounded and integrable and b is bounded. In the homogeneous case $b(x) \equiv b_*$, this solution (with the zero initial condition) is also homogeneous and can be obtained explicitly in the form

$$\varrho_t(x) \equiv \frac{1}{\langle \phi \rangle} \log \left(1 + b_* \langle \phi \rangle t \right), \quad \langle \phi \rangle := \int_{\mathbb{R}^d} \phi(x) dx. \quad (1.3)$$

To illustrate that the homogeneity of the solution may be unstable to arbitrarily small perturbations of the homogeneity of b let us consider the following version of (1.2). Assume that there exist two (compact) patches, A and B , such that $b|_A = b_A$, $b|_B = b_B$ and $b(x) = 0$ whenever x is outside of $A \cup B$. Assume also that $K(x, y) = K(y, x) = 1$ for $x \in A$ and $y \in B$, $K(x, y) = \alpha \geq 0$ for $x, y \in A$ and for $x, y \in B$, and $K(x, y) = 0$ otherwise. Then consider the problem in (1.2) with the zero initial condition. Clearly, in this case one gets $\varrho_t(x) = 0$ for all $t \geq 0$ and x outside of $A \cup B$. Thus, one can consider the patches only. The restrictions $\varrho_t|_A =: \varrho_t^A$ and $\varrho_t|_B =: \varrho_t^B$ satisfy the system of equations

$$\frac{d}{dt} \varrho_t^A = b_A \exp(-\alpha \varrho_t^A - \varrho_t^B), \quad \frac{d}{dt} \varrho_t^B = b_B \exp(-\varrho_t^A - \alpha \varrho_t^B). \quad (1.4)$$

The case of $\alpha < 1$ (resp. $\alpha > 1$) corresponds to the situation in which the repulsion from the entities in the other patch is stronger (resp. weaker) than that of the entities in the same patch. The solution of this

¹ The described population gets instantly infinite if $\int_{\mathbb{R}^d} b(x) dx = +\infty$; i.e., if b is not integrable, as it is in the homogeneous case with constant b .

system clearly exists. However, it can explicitly be obtained only in particular cases. In general, one might integrate (1.4) and then analyze the solution. Recalling that (1.4) is subject to the zero initial condition, for $\alpha \neq 1$ we get

$$\exp((\alpha - 1)\varrho_t^A) - 1 = \frac{b_A}{b_B} \left(\exp((\alpha - 1)\varrho_t^B) - 1 \right). \quad (1.5)$$

For $\alpha = 1$, after some calculations we obtain an explicit solution

$$\varrho_t^A = \frac{b_A}{b_A + b_B} \log(1 + (b_A + b_B)t), \quad \varrho_t^B = \frac{b_B}{b_A + b_B} \log(1 + (b_A + b_B)t). \quad (1.6)$$

In the latter case, both densities increase with time ad infinitum logarithmically, and $\varrho_t^A \approx \varrho_t^B$ (at all t) whenever $b_A \approx b_B$ (more precisely, $\varrho_t^A/\varrho_t^B = b_A/b_B$). For $\alpha > 1$ and $b_A \neq b_B$, by (1.5) we conclude that both ϱ_t^A and ϱ_t^B increase ad infinitum and $\varrho_t^A \approx \varrho_t^B$ whenever $b_A \approx b_B$. That is, in both these cases no clear difference between the patches appears if the intensities b_A and b_B are close to each other. In the homogeneous case $b_A = b_B = b_*$, for all α we get

$$\varrho_t^A = \varrho_t^B = \varrho_t = \frac{1}{1 + \alpha} \log(1 + (1 + \alpha)b_*t),$$

that resembles (1.3). For $\alpha < 1$, however, the equality between the patches in the homogeneous case gets unstable. Assuming $b_A < b_B$, by (1.5) we conclude that ϱ_t^B should increase ad infinitum whereas ϱ_t^A tend to $\varrho_\infty^A = [\log b_B - \log(b_B - b_A)]/(1 - \alpha)$. This effect of increasingly diverse patches persists regardless how small is $b_B - b_A$. Having this effect in mind, one might be interested in developing a more or less complete theory of (1.2), in particular, in revealing which properties of b and K are responsible for pattern formation. In part, this is done in Theorem 2.9 below, where we show the existence of a unique classical solution and derive some information on its properties. Clearly, in studying (1.2) numerical methods can be essentially helpful. We plan to realize this in a separate work.

As mentioned above, a theory based on kinetic equations like in (1.2) is pretty rough and can only be considered as the first step in the study of the corresponding phenomena where spatial correlations are taken into account in a mean-field like way. To make the next step one should (in one or another way) obtain a ‘true’ and ‘complete’ system – hierarchy – of equations, that include correlation functions of second, third, and higher orders. Then one might think of a decoupling of this hierarchy, see, e.g., [1,18]. A yet more important issue is whether a solution of this ‘true’ hierarchy gives correlation functions corresponding to a microscopic state. If yes, then the evolution obtained by solving the hierarchy can ‘represent’ – in a certain sense – the evolution of the micro-states and thus yield a microscopic description of the considered phenomenon. In this work, for our model – introduced in Section 2 – we give partial answers to these questions. This includes the following:

- (i) Deriving a hierarchy of evolution equations for correlation functions from the Kolmogorov equation describing the microscopic evolution of observables (subsection 2.4 and Appendix).
- (ii) Proving the existence and uniqueness of solutions of the evolution equation for correlation functions that yields the evolution $k_0 \rightarrow k_t$; then proving that each k_t is the correlation function of a unique micro-state μ_t (Theorem 2.6), that yields the evolution of states $\mu_0 \rightarrow \mu_t$.
- (iii) Under an additional condition on the repulsion, proving (Theorem 2.8) that the expected number of entities contained in a compact Λ increases in time at most logarithmically, cf. (1.3).
- (iv) Proving the existence and uniqueness of solutions of (1.2) and describing some of its properties (Theorem 2.9).

- (v) Deriving (by a scaling procedure) a mesoscopic evolution equation – that coincides with the kinetic equation (1.2) – from the Kolmogorov equation describing the evolution of micro-states of the model. Proving that the mesoscopic limit of the state μ_t is the Poisson state characterized by the density that solves (1.2) (Theorem 2.10).

The paper is organized as follows. In Section 2, we briefly introduce the mathematical framework in which we then build up the theory. Then we introduce the model by defining the generator of its Markov evolution. Thereafter, we discuss in detail how to derive a weak evolution of states of infinite particle systems, including the model considered in this work. The main idea of this is to use correlation functions the evolution of which is described by an evolution equation obtained from the Kolmogorov equation in Appendix. The equation for correlation functions has a hierarchical form and is considered in an ascending scale of Banach space chosen in such a way that they contain the correlation functions of the so-called sub-Poissonian states. To relate the microscopic theory based on the Kolmogorov equation to the one that uses kinetic equations we introduce the notion of Poisson approximability of the micro-states (Definition 2.5) based on the passage from the micro- to the meso-scale. Finally, we formulate our results in Theorems 2.6 – 2.10 and Corollary 2.7. In subsection 2.7 we comment these results and give additional information on their meaning and relevance. In Section 3, we give the proof of Theorems 2.6 and 2.8. The proof of the latter theorem is rather simple and based on the results of Theorem 2.6. Its proof, however, is the most involved part of this work. It is based on a combination of a number of methods of studying evolution equations in scales of Banach spaces, including the theory of stochastic semigroups in AL -spaces. Section 4 contains the proof of Theorems 2.9 and 2.10 describing the mesoscopic evolution and its connection to the microscopic evolution. In the concluding part of the paper we placed Appendix containing technicalities.

2. The model and the results

Here we give the description of the model preceded by a short presentation of the preliminaries (see [12, 13] for more) and then briefly outline the main aspects of the theory. Thereafter, we formulate the results.

2.1. Preliminaries

The microscopic description of the dynamics of the model which we introduce and study in this work is conducted as the Markov evolution of states of an infinite population of point entities dwelling \mathbb{R}^d , cf. [7, 12, 13]. The phase space of the population is the set Γ of all locally finite subsets $\gamma \subset \mathbb{R}^d$ – configurations, equipped with a standard (vague) topology and the corresponding Borel σ -field $\mathcal{B}(\Gamma)$. This makes $(\Gamma, \mathcal{B}(\Gamma))$ a standard Borel space that allows one to consider probability measures on Γ as states of the system. In the sequel, $\mathcal{P}(\Gamma)$ will stand for the set of all such measures. Along with states $\mu \in \mathcal{P}(\Gamma)$ one employs *observables* – appropriate measurable functions $F : \Gamma \rightarrow \mathbb{R}$. The mentioned local finiteness means that the observable $\Gamma \ni \gamma \mapsto |\gamma \cap \Lambda| =: N_\Lambda(\gamma)$ takes finite values only. Here Λ is a compact subset of \mathbb{R}^d and $|\cdot|$ stands for cardinality. Among the states one can distinguish those characterized by the corresponding densities – locally integrable functions $\varrho_\mu : \mathbb{R}^d \rightarrow \mathbb{R}_+ := [0, +\infty)$ such that

$$\mu(N_\Lambda) = \int_{\Lambda} \varrho_\mu(x) dx, \quad (2.1)$$

where dx is Lebesgue’s measure and the notation $\mu(F) = \int F d\mu$ is used for appropriate measures and observables. Note that $\mu(N_\Lambda)$ is the (expected) number of ‘particles’ contained in Λ in state μ . Each Poissonian measure π_ϱ , see [10], is completely characterized by its density ϱ . In the homogeneous case, $\varrho(x) \equiv \kappa > 0$, and thus π_κ is invariant with respect to the translations of \mathbb{R}^d .

The dynamics of a given system can be described as the evolution of observables by solving the Kolmogorov equation

$$\frac{d}{dt}F_t = LF_t, \quad F_t|_{t=0} = F_0. \quad (2.2)$$

The evolution of states $\mathcal{P}(\Gamma) \ni \mu_0 \rightarrow \mu_t \in \mathcal{P}(\Gamma)$ is then usually obtained (in the weak sense) by means of the rule

$$\mu_t(F_0) = \mu_0(F_t), \quad (2.3)$$

where the function $t \mapsto F_t$ is obtained from (2.2) with F_0 running through a sufficiently large (measure-defining) set of observables. As such a set one can take $\mathcal{F} = \{F^\theta : \theta \in \Theta\}$, where

$$F^\theta(\gamma) = \prod_{x \in \gamma} (1 + \theta(x)), \quad \gamma \in \Gamma, \quad (2.4)$$

and Θ stands for the set of compactly supported continuous functions $\theta : \mathbb{R}^d \rightarrow (-1, 0]$.

The model studied in this work is a particular case of a more general one describing a system of point entities arriving in (immigrating) and departing from (emigrating) \mathbb{R}^d . Such systems are described by the Kolmogorov equations with

$$\begin{aligned} (LF)(\gamma) &= \int_{\mathbb{R}^d} c_+(x, \gamma) [F(\gamma \cup x) - F(\gamma)] dx \\ &+ \sum_{x \in \gamma} c_-(x, \gamma \setminus x) [F(\gamma \setminus x) - F(\gamma)]. \end{aligned} \quad (2.5)$$

Here and in the sequel, in the expressions like $\gamma \cup x$ we treat x as a singleton configuration. If the immigration and emigration rates c_\pm are state-independent, the evolution of states $\mu_0 \rightarrow \mu_t$ can be constructed explicitly, see [13, Sect. 2.3]. In this case: (a) $\varrho_{\mu_t}(x) = \varrho_{\mu_0}(x) + tc_+(x)$ if $c_-(x) = 0$; (b) $\varrho_{\mu_t}(x) \leq \varrho_\infty(x)$ for some $\varrho_\infty(x) < \infty$ if $c_-(x) > 0$. By (2.1) (b) implies

$$\mu_t(N_\Lambda) \leq C_\Lambda, \quad (2.6)$$

holding for all $t > 0$ and compact $\Lambda \subset \mathbb{R}^d$ such that $c_-(x) \geq \sigma_\Lambda > 0$ for all $x \in \Lambda$. In particular, if the condition $c_-(x) \geq \sigma > 0$ is satisfied globally (for all x), then (2.6) holds also globally, i.e., for all compact Λ . The main question studied in [13] was whether a local competition alone can yield the global boundedness as in (2.6). In this case, c_- has the form

$$c_-(x, \gamma) = \sum_{y \in \gamma} a_-(x, y),$$

with an appropriate competition kernel $a_-(x, y) \geq 0$. Here, however, the particles in the system interact with each other and the proof of the existence of the evolution of states turns into an essential problem. Nevertheless, in [13] the weak evolution $\mu_0 \rightarrow \mu_t$ was constructed for which it was shown that (2.6) holds for each compact Λ whenever there exists a box $\Delta \subset \mathbb{R}^d$ such that its disjoint translates Δ_l , $l \in \mathbb{Z}^d$, cover \mathbb{R}^d and $a_-(x, y) \geq a_0 > 0$ for x, y running through each of Δ_l . In the present article, we study the particular case of (2.5) in which emigration is absent, i.e., $c_-(x, \gamma) \equiv 0$, and the immigration term is taken in the form, cf. (1.2),

$$c_+(x, \gamma) = b(x) \exp \left(- \sum_{y \in \gamma} \phi(x - y) \right), \quad (2.7)$$

where $\phi : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is the *repulsion* potential that is assumed to be integrable and bounded. According to (2.7) the already existing population decreases the immigration rate as compared to the free case $\phi(x) \equiv 0$. As mentioned above, in this free case one has

$$\mu_t(N_\Lambda) = \mu_0(N_\Lambda) + t \int_\Lambda b(x) dx.$$

Then one of the questions of the theory is how the repulsion can attenuate the growth of $\mu_t(N_\Lambda)$ in time. Note that according to the assumed boundedness of ϕ , which, in particular, excludes a hard-core repulsion, we have that $c_+(x, \gamma) > 0$ for all those x where $b(x) > 0$. Thus, one cannot expect that $\mu_t(N_\Lambda)$ is bounded in time since any kind of emigration is absent in the model.

2.2. The model

In this article, we introduce and study the model described by the Kolmogorov equation with the following operator, cf. (2.5)

$$(LF)(\gamma) = \int_{\mathbb{R}^d} b(x) \exp \left(- \sum_{y \in \gamma} \phi(x - y) \right) [F(\gamma \cup x) - F(\gamma)] dx. \quad (2.8)$$

Assumption 2.1. *In general, we assume that: (a) b and ϕ are nonnegative, measurable and bounded from above by \bar{b} and $\bar{\phi}$, respectively; (b) ϕ is integrable and its L^1 -norm is denoted by $\langle \phi \rangle$, see (1.3). Additionally, we assume: (c) there exist $r > 0$ and $\phi_* > 0$ such that*

$$\phi(x) \geq \phi_*, \quad \text{whenever } |x| \leq r.$$

Notably, we do not assume that b is integrable, which means that we allow the particle system described by (2.8) with such b be instantly infinite, even if it is initially empty. Note also that, for $\theta \in \Theta$, by (2.4) it follows that

$$|(LF^\theta)(\gamma)| \leq \bar{b} \|\theta\|_{L^1(\mathbb{R}^d)}, \quad (2.9)$$

holding for all $\gamma \in \Gamma$.

2.3. The weak evolution of states

For a given $\mu \in \mathcal{P}(\Gamma)$, its *Bogoliubov* (generating) functional is $B_\mu(\theta) = \mu(F^\theta)$ with F^θ defined in (2.4). It thus is considered as a map from Θ to \mathbb{R}_+ . For a Poisson measure π_ϱ characterized by density ϱ , it is

$$B_{\pi_\varrho}(\theta) = \exp \left(\int_{\mathbb{R}^d} \varrho(x) \theta(x) dx \right). \quad (2.10)$$

In our consideration, Poisson measures play the role of reference states. In view of this, we restrict our attention to the subset $\mathcal{P}_{\text{exp}}(\Gamma) \subset \mathcal{P}(\Gamma)$ containing all those $\mu \in \mathcal{P}(\Gamma)$ for each of which B_μ can be continued,

as a function of θ , to an exponential type entire function on $L^1(\mathbb{R}^d)$. It can be shown that a given μ belongs to $\mathcal{P}_{\text{exp}}(\Gamma)$ if and only if its functional B_μ can be written down in the form

$$B_\mu(\theta) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} k_\mu^{(n)}(x_1, \dots, x_n) \theta(x_1) \cdots \theta(x_n) dx_1 \cdots dx_n, \quad (2.11)$$

where $k_\mu^{(n)}$ is the n -th order correlation function of μ . It is a symmetric element of $L^\infty((\mathbb{R}^d)^n)$ for which

$$\|k_\mu^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)} \leq \exp(\vartheta n), \quad n \in \mathbb{N}_0, \quad (2.12)$$

with a certain $\vartheta \in \mathbb{R}$. It is known, see Proposition 2.3 below, that $k_\mu^{(n)}(x_1, \dots, x_n) \geq 0$ for all n and (Lebesgue-) almost all x_1, \dots, x_n . Then (2.12) can be rewritten in the form

$$0 \leq k^{(n)}(x_1, \dots, x_n) \leq \exp(\vartheta n), \quad n \in \mathbb{N}_0. \quad (2.13)$$

Note that $k_{\pi_\varkappa}^{(n)}(x_1, \dots, x_n) = \varkappa^n$. Note also that $k^{(0)} = 1$ and $k_\mu^{(1)}$ is the density of the system in state μ . The set Γ contains a subset, $\Gamma^{(0)}$, consisting of a single element – the empty configuration. If μ is such that $\mu(\Gamma^{(0)}) = 1$, then the system in this state is empty. Its correlation function k_μ satisfies (2.12) with any ϑ . In this case, we will allow ϑ in (2.12) and (2.13) be $-\infty$. For $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$ and a compact Λ , by (2.13) and [13, eq. (4.5)] it follows that

$$\forall m \in \mathbb{N} \quad \mu(N_\Lambda^m) \leq \mu_{\pi_\varkappa}(N_\Lambda^m), \quad \varkappa = e^\vartheta, \quad (2.14)$$

with ϑ satisfying (2.13) for this μ . That is why the elements of $\mathcal{P}_{\text{exp}}(\Gamma)$ are called sub-Poissonian states. By (2.14) one readily gets that $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$ also satisfies

$$\forall \alpha > 0 \quad \mu(F_\Lambda^\alpha) \leq \exp((e^\alpha - 1)e^\vartheta |\Lambda|) := C_\Lambda^\alpha(\mu), \quad F_\Lambda^\alpha(\gamma) := e^{\alpha N_\Lambda(\gamma)}, \quad (2.15)$$

where ϑ is the same as in (2.14) and $|\Lambda|$ stands for Lebesgue's measure of Λ .

Let $\Gamma^{(n)}$, $n \in \mathbb{N}_0$ stand for the set of n -point configurations. It is an element of $\mathcal{B}(\Gamma)$. Then so is $\Gamma_0 = \cup_{n \geq 0} \Gamma^{(n)}$ – the set of all finite configurations. In the sequel, we will use also the measurable space $(\Gamma_0, \mathcal{B}(\Gamma_0))$ where $\mathcal{B}(\Gamma_0)$ is a sub-field of $\mathcal{B}(\Gamma)$ containing all measurable subsets of Γ_0 . It can be shown, see [7], that a function $G : \Gamma_0 \rightarrow \mathbb{R}$ is $\mathcal{B}(\Gamma)/\mathcal{B}(\mathbb{R})$ -measurable if and only if, for each $n \in \mathbb{N}$, there exists a symmetric Borel function $G^{(n)} : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$ such that

$$G(\eta) = G^{(n)}(x_1, \dots, x_n), \quad \text{for } \eta = \{x_1, \dots, x_n\}. \quad (2.16)$$

Definition 2.2. A measurable function $G : \Gamma_0 \rightarrow \mathbb{R}$ is said to have bounded support if: (a) there exists a compact $\Lambda \subset \mathbb{R}^d$ such that $G(\eta) = 0$ whenever $\eta \cap (\mathbb{R}^d \setminus \Lambda) \neq \emptyset$; (b) there exists $N \in \mathbb{N}_0$ such that $G(\eta) = 0$ whenever $|\eta| > N$. By $\Lambda(G)$ and $N(G)$ we denote the smallest Λ and N with the properties just mentioned. By $B_{\text{bs}}(\Gamma_0)$ we denote the set of all bounded functions of bounded support.

In our study, we use the following subset of $B_{\text{bs}}(\Gamma_0)$

$$B_{\text{bs}}^*(\Gamma_0) = \{G \in B_{\text{bs}}(\Gamma_0) : (KG)(\eta) \geq 0\}, \quad (KG)(\eta) := \sum_{\xi \subset \eta} G(\xi). \quad (2.17)$$

Note that the cone $B_{\text{bs}}^+(\Gamma_0) = \{G \in B_{\text{bs}}(\Gamma_0) : G(\eta) \geq 0\}$ is a proper subset of $B_{\text{bs}}^*(\Gamma_0)$.

The Lebesgue-Poisson measure λ on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is defined by the following formula

$$\int_{\Gamma_0} G(\eta) \lambda(d\eta) = G(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G^{(n)}(x_1, \dots, x_n) dx_1 \cdots dx_n, \quad (2.18)$$

holding for all appropriate $G : \Gamma_0 \rightarrow \mathbb{R}$, that obviously includes $G \in B_{\text{bs}}(\Gamma_0)$. Like in (2.16), we introduce $k_\mu : \Gamma_0 \rightarrow \mathbb{R}$ such that $k_\mu(\eta) = k_\mu^{(n)}(x_1, \dots, x_n)$ for $\eta = \{x_1, \dots, x_n\}$, $n \in \mathbb{N}$. We also set $k_\mu(\emptyset) = 1$. With the help of the measure introduced in (2.18), the expressions for B_μ in (2.10) and (2.11) can be combined into the following formulas

$$\begin{aligned} B_\mu(\theta) &= \int_{\Gamma_0} k_\mu(\eta) \prod_{x \in \eta} \theta(x) \lambda(d\eta) =: \int_{\Gamma_0} k_\mu(\eta) e(\theta; \eta) \lambda(d\eta) \\ &= \int_{\Gamma} \prod_{x \in \gamma} (1 + \theta(x)) \mu(d\gamma) = \int_{\Gamma} F^\theta(\gamma) \mu(d\gamma), \end{aligned} \quad (2.19)$$

cf. (2.4). Thereby, one can transform the action of L on F , see (2.5), to the action of L^Δ on k_μ according to the rule

$$\int_{\Gamma} (LF_\theta)(\gamma) \mu(d\gamma) = \int_{\Gamma_0} (L^\Delta k_\mu)(\eta) e(\theta; \eta) \lambda(d\eta). \quad (2.20)$$

Correspondingly, along with the Kolmogorov equation (2.2) one can consider

$$\frac{d}{dt} k_t = L^\Delta k_t, \quad k_t|_{t=0} = k_{\mu_0}, \quad (2.21)$$

where $\mu_0 \in \mathcal{P}_{\text{exp}}(\Gamma)$ is the initial state. Then the construction of the evolution of states $\mu_0 \rightarrow \mu_t$ will be realized in the following steps:

- (i) proving the existence of a unique solution k_t of the Cauchy problem in (2.21);
- (ii) proving that this solution k_t is the correlation function of a unique $\mu_t \in \mathcal{P}_{\text{exp}}(\Gamma)$.

Upon realizing this program one will be able to identify μ_t by its values on the members of \mathcal{F} – a measure-defining class – computed according to the formula

$$\mu_t(F^\theta) = \int_{\Gamma_0} k_t(\eta) e(\theta; \eta) \lambda(d\eta), \quad (2.22)$$

see (2.19). In realizing item (ii), we will use the following statement, see [14, Theorems 6.1 and 6.2 and Remark 6.3].

Proposition 2.3. *A measurable function $k : \Gamma_0 \rightarrow \mathbb{R}$ is the correlation function of a unique $\mu \in \mathcal{P}_{\text{exp}}(\Gamma_0)$ if and only if it satisfies: (a) $k(\emptyset) = 1$; (b) the estimate in (2.12) holds for some $\vartheta \in \mathbb{R}$ and all $n \in \mathbb{N}$; (c) for each $G \in B_{\text{bs}}^*(\Gamma_0)$, see (2.17), the following holds*

$$\langle\langle G, k \rangle\rangle := \int_{\Gamma_0} G(\eta) k(\eta) \lambda(d\eta) \geq 0. \quad (2.23)$$

2.4. Evolution of correlation functions

Now we turn to defining the Cauchy problem (2.21) in appropriate Banach spaces. In view of (2.12), we set

$$\|k\|_{\vartheta} = \sup_{n \in \mathbb{N}_0} \left(\exp(-\vartheta n) \|k^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)} \right) = \operatorname{ess\,sup}_{\eta \in \Gamma_0} (\exp(-\vartheta|\eta|) |k(\eta)|), \quad (2.24)$$

and then

$$\mathcal{K}_{\vartheta} = \{k : \Gamma_0 \rightarrow \mathbb{R} : \|k\|_{\vartheta} < \infty\}, \quad \vartheta \in \mathbb{R}.$$

By (2.24) one clearly gets that

$$|k(\eta)| \leq \exp(\vartheta|\eta|) \|k\|_{\vartheta}, \quad \eta \in \Gamma_0, \quad (2.25)$$

and

$$\mathcal{K}_{\vartheta} \hookrightarrow \mathcal{K}_{\vartheta'}, \quad \text{for } \vartheta' > \vartheta, \quad (2.26)$$

that is, \mathcal{K}_{ϑ} is continuously embedded in $\mathcal{K}_{\vartheta'}$. The latter fact allows one to employ the whole ascending scale of Banach spaces $\{\mathcal{K}_{\vartheta}\}_{\vartheta \in \mathbb{R}}$.

In this paper, we extensively use the Minlos lemma, cf. [7, eq. (2.2)] and/or [14, Appendix A], according to which the following holds

$$\int_{\Gamma_0} \int_{\Gamma_0} G(\eta \cup \xi) H(\eta, \xi) \lambda(d\eta) \lambda(d\xi) = \int_{\Gamma_0} G(\eta) \left(\sum_{\xi \subset \eta} H(\xi, \eta \setminus \xi) \right) \lambda(d\eta) \quad (2.27)$$

for appropriate $G, H : \Gamma_0 \rightarrow \mathbb{R}$. In Appendix, by means of (2.27) and (2.20) we show that

$$(L^{\Delta}k)(\eta) = \sum_{x \in \eta} b(x) e(\tau_x; \eta \setminus x) (W_x k)(\eta \setminus x), \quad (2.28)$$

$$(W_x k)(\eta) := \int_{\Gamma_0} e(t_x; \xi) k(\eta \cup \xi) \lambda(d\xi),$$

$$\tau_x(y) := e^{-\phi(x-y)}, \quad t_x(y) := \tau_x(y) - 1.$$

By (2.25) we then have

$$\begin{aligned} |(W_x k)(\eta)| &\leq \|k\|_{\vartheta} e^{\vartheta|\eta|} \int_{\Gamma_0} e(|t_x|; \xi) e^{\vartheta|\xi|} \lambda(d\xi) \\ &= \|k\|_{\vartheta} e^{\vartheta|\eta|} \exp \left(e^{\vartheta} \int_{\mathbb{R}^d} (1 - e^{-\phi(x-y)}) dy \right) \\ &\leq \|k\|_{\vartheta} e^{\vartheta|\eta|} \exp(\langle \phi \rangle e^{\vartheta}), \end{aligned} \quad (2.29)$$

where we have used the assumed properties of ϕ , see (1.3). Now we apply (2.29) in (2.28) and get

$$|(L^\Delta k)(\eta)| \leq |\eta| e^{|\eta| \bar{b}} \|k\|_\vartheta \exp(\langle \phi \rangle e^\vartheta - \vartheta), \quad (2.30)$$

that holds for each $\vartheta \in \mathbb{R}$. Set

$$\mathcal{D}_\vartheta = \left\{ k \in \mathcal{K}_\vartheta : \exists C_k > 0 \ |k(\eta)| \leq \frac{C_k e^{|\eta|}}{1 + |\eta|} \right\}. \quad (2.31)$$

Similarly as in obtaining (2.30) one shows that $L^\Delta k \in \mathcal{K}_\vartheta$ for each $k \in \mathcal{D}_\vartheta$. Thus, we define in \mathcal{K}_ϑ the (unbounded) linear operator $L_\vartheta^\Delta := (L^\Delta, \mathcal{D}_\vartheta)$. By (2.25) and (2.31) we readily obtain that

$$\forall \vartheta_0 < \vartheta \quad \mathcal{K}_{\vartheta_0} \subset \mathcal{D}_\vartheta. \quad (2.32)$$

Moreover, by employing in (2.30) the estimate $|\eta| e^{-v|\eta|} \leq (ve)^{-1}$, $v > 0$, we conclude that, for each $\vartheta' > \vartheta$, cf. (2.26), L^Δ can be defined as a bounded linear operator – denoted by $L_{\vartheta', \vartheta}^\Delta$ – acting from \mathcal{K}_ϑ to $\mathcal{K}_{\vartheta'}$, the operator norm of which satisfies

$$\|L_{\vartheta', \vartheta}^\Delta\| \leq \frac{\beta(\vartheta) e^{-\vartheta}}{e^{(\vartheta' - \vartheta)}}, \quad \beta(\vartheta) := \bar{b} \exp(\langle \phi \rangle e^\vartheta). \quad (2.33)$$

In the sequel, we will consider two types of linear operators defined by (2.28): (a) unbounded operators $L_\vartheta^\Delta : \mathcal{D}_\vartheta \rightarrow \mathcal{K}_\vartheta$, $\vartheta \in \mathbb{R}$; (b) bounded operators $L_{\vartheta', \vartheta}^\Delta : \mathcal{K}_\vartheta \rightarrow \mathcal{K}_{\vartheta'}$ with $\vartheta' > \vartheta$. These operators satisfy

$$\forall \vartheta' > \vartheta \quad \forall k \in \mathcal{K}_\vartheta \quad L_{\vartheta', \vartheta}^\Delta k = L_{\vartheta'}^\Delta k. \quad (2.34)$$

Now we fix some $\vartheta \in \mathbb{R}$ and consider the following Cauchy problem in \mathcal{K}_ϑ

$$\frac{d}{dt} k_t = L_\vartheta^\Delta k_t, \quad k_t|_{t=0} = k_0. \quad (2.35)$$

Definition 2.4. By a (classical) solution of the problem in (2.35) on a time interval $[0, T)$ we understand a continuous map $[0, T) \ni t \mapsto k_t \in \mathcal{D}_\vartheta \subset \mathcal{K}_\vartheta$, which is continuously differentiable on $(0, T)$ and such that both equalities in (2.35) are satisfied. We say that such a solution is global (in time) if $T = +\infty$.

Then item (i) of the program mentioned above assumes proving the existence of such solutions. Note, however, that a priori k_t that solves (2.35) need not be the correlation function of any state, and hence has no direct relation to the evolution of the considered system. To realize item (ii) that establishes such a relation we show that this solution has the property $k_t \in \mathcal{K}^*$ where

$$\mathcal{K}^* := \bigcup_{\vartheta \in \mathbb{R}} \mathcal{K}_\vartheta^*, \quad \mathcal{K}_\vartheta^* = \{k \in \mathcal{K}_\vartheta : k(\emptyset) = 1 \text{ and } \forall G \in B_{\text{bs}}^*(\Gamma_0) \ \langle\langle G, k \rangle\rangle \geq 0\}, \quad (2.36)$$

see (2.23).

2.5. The mesoscopic description

It is believed that the description of an infinite interacting particle system by means of kinetic equations is in a sense equivalent to considering it at a more coarse-grained (mesoscopic) spatial scale, see [2, Chapter 8] and [20]. Typically, passing from the micro- to the mesoscopic levels is made with the help of a scale parameter $\varepsilon \in (0, 1]$ in such a way that $\varepsilon = 1$ corresponds to the micro-level, whereas the limit $\varepsilon \rightarrow 0$ yields the description in which the corpuscular structure disappears and the system turns into a medium characterized by density ϱ . In this limit, instead of interactions one deals with state-dependent external

forces, that is typical to the mean-field approach of statistical physics. The evolution $\varrho_0 \rightarrow \varrho_t$ is obtained from a kinetic equation. It approximates the evolution of states as it may be seen from the mesoscopic level provided these states exist. As the Poissonian state π_ϱ is completely characterized by its density, cf. (2.10) (note that $k_{\pi_\varrho}(\eta) = \prod_{x \in \eta} \varrho(x)$), considering densities only can be interpreted as approximating the states μ_t by the corresponding Poissonian states π_{ϱ_t} . In view of this, we introduce the following notion, cf. [7, p. 70].

Definition 2.5. A state $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$ is said to be Poisson-approximable if: (i) there exist $\vartheta \in \mathbb{R}$ and a bounded measurable $\varrho : \mathbb{R}^d \rightarrow [0, +\infty)$ such that both k_μ and k_{π_ϱ} lie in the same \mathcal{K}_ϑ ; (ii) there exists a continuous map $(0, 1] \ni \varepsilon \mapsto q^\varepsilon \in \mathcal{K}_\vartheta$ such that $q^1 = k_\mu$ and $\|q^\varepsilon - k_{\pi_\varrho}\|_\vartheta \rightarrow 0$ as $\varepsilon \rightarrow 0^+$.

Our aim is to show that the evolution of states $\mu_0 \rightarrow \mu_t$ discussed above in subsection 2.3 preserves the property just defined relative to the time dependent density ϱ_t obtained as a solution of the kinetic equation in (2.38), understood similarly as in Definition 2.4.

2.6. The results

First we establish the existence of the evolution of states as discussed in subsections 2.3 and 2.4 and describe some of its properties. Then we turn to the mesoscopic scale.

2.6.1. The evolution of states

Theorem 2.6. Let the model satisfy items (a) and (b) of Assumption 2.1 and let the initial state $\mu_0 \in \mathcal{P}_{\text{exp}}(\Gamma)$ and $\vartheta_0 \in \mathbb{R}$ be such that $k_{\mu_0} \in \mathcal{K}_{\vartheta_0}$. Set $\vartheta_t = \log(e^{\vartheta_0} + \bar{b}t)$, $t \geq 0$. Then there exists the unique map $[0, +\infty) \ni t \mapsto k_t \in \mathcal{K}^*$, see (2.36), such that $k_0 = k_{\mu_0}$ and the following holds:

(i) for all $t > 0$,

$$0 \leq k_t(\eta) \leq \exp(\vartheta_t |\eta|), \quad \eta \in \Gamma_0,$$

and hence $k_t \in \mathcal{K}_{\vartheta_t}^*$;

(ii) for each $T > 0$ and $t < T$, the map $t \mapsto k_t \in \mathcal{K}_{\vartheta_t} \subset \mathcal{D}_{\vartheta_T}$, cf. (2.32), is continuous in $\mathcal{K}_{\vartheta_T}$ on $[0, T)$, continuously differentiable on $(0, T)$ and satisfies

$$\frac{d}{dt} k_t = L_{\vartheta_T}^\Delta k_t.$$

The proof of this statement is performed in Section 3 below.

Corollary 2.7. For each $\mu_0 \in \mathcal{P}_{\text{exp}}(\Gamma)$, there exists the unique map $[0, +\infty) \ni t \mapsto \mu_t \in \mathcal{P}_{\text{exp}}(\Gamma)$ such that μ_t is the state of the population at time t and $\mu_t|_{t=0} = \mu_0$. These states μ_t are identified by their values $\mu_t(F^\theta) = \langle e(\theta; \cdot), k_t \rangle$ with k_t as in Theorem 2.6, see (2.22). Moreover, for each $\theta \in \Theta$, the map $(0, +\infty) \ni t \mapsto \mu_t(F^\theta) \in \mathbb{R}$ is differentiable and the following holds, see (2.9),

$$\frac{d}{dt} \mu_t(F^\theta) = \mu_t(LF^\theta) = \langle e(\theta; \cdot), L_{\vartheta_T}^\Delta k_t \rangle,$$

with any T satisfying $T > t$.

Our next result establishes an upper bound for $\mu_t(N_\Lambda)$, cf. (2.1). For r as in item (c) of Assumption 2.1, we set $\Delta_x = \{y \in \mathbb{R}^d : |x - y| \leq r/2\}$ and let v be the volume (Lebesgue's measure) of this ball.

Theorem 2.8. Let μ_t , $t > 0$ be the state of the population as in Corollary 2.7. In addition to items (a) and (b), assume that also item (c) of Assumption 2.1 holds true. Then, for each compact $\Lambda \subset \mathbb{R}^d$ and $t > 0$, the following holds

$$\mu_t(N_\Lambda) \leq \frac{m_\Lambda}{\phi_*} \log \left(C_{\Delta_0}^{\phi_*}(\mu_0) + (e^{\phi_*} - 1) \bar{b} v t \right), \quad (2.37)$$

where m_Λ is the minimum number of the balls Δ_x that cover Λ and $C_{\Delta_0}^{\phi_*}(\mu_0)$ is given by the right-hand side of (2.15) with $\vartheta = \vartheta_0$, the same as in Theorem 2.6. If $\mu_0(\Gamma^{(0)}) = 1$, i.e., if the system is initially empty, then one may take $C_{\Delta_0}^{\phi_*}(\mu_0) = 1$.

2.6.2. The mesoscopic evolution

As mentioned in Introduction, the mesoscopic theory of the evolution of our model is based on the kinetic equation in (1.2) the translation invariant version of which is

$$\frac{d}{dt} \varrho_t(x) = b(x) \exp \left(-(\phi * \varrho_t)(x) \right), \quad (2.38)$$

where

$$(\phi * \varrho_t)(x) := \int_{\mathbb{R}^d} \phi(x-y) \varrho_t(y) dy. \quad (2.39)$$

The definition of its classical solution is pretty similar to that given in Definition 2.4. Let $\varrho_0 \in L^\infty(\mathbb{R}^d)$ be the initial condition in (2.38). Then we set

$$b^+ = \langle \phi \rangle \operatorname{ess\,sup}_{x \in \mathbb{R}^d} b(x) e^{-(\phi * \varrho_0)(x)}, \quad b^- = \langle \phi \rangle \operatorname{ess\,inf}_{x \in \mathbb{R}^d} b(x) e^{-(\phi * \varrho_0)(x)}, \quad (2.40)$$

and also

$$\omega_+(t) = \omega_-(t) + (b^+ - b^-)t, \quad \text{for } b^+ > b^-, \quad (2.41)$$

$$\omega_-(t) = \log \left(\frac{b^+}{b^+ - b^-} - \frac{b^-}{b^+ - b^-} e^{-(b^+ - b^-)t} \right), \quad b^+ > b^-,$$

$$\omega_+(t) = \omega_-(t) = \log(1 + bt), \quad \text{for } b^+ = b^- = b.$$

Theorem 2.9. For each positive $\varrho_0 \in L^\infty(\mathbb{R}^d)$, the kinetic equation in (2.38), (2.39) has a unique global classical solution $\varrho_t \in L^\infty(\mathbb{R}^d)$ that for all $t > 0$ and almost all $x \in \mathbb{R}^d$ satisfies

$$\varrho_0(x) + \frac{\omega_-(t)}{\langle \phi \rangle} \leq \varrho_t(x) \leq \varrho_0(x) + \frac{\omega_+(t)}{\langle \phi \rangle}. \quad (2.42)$$

Theorem 2.10. Let k_t and ϱ_t be the solutions described by Theorems 2.6 and 2.9, respectively. Assume also that the initial state μ_0 in Theorem 2.6 is Poisson approximable. That is, there exist $\vartheta_0 \in \mathbb{R}$ and a continuous map $(0, 1] \ni \varepsilon \mapsto q_{0,\varepsilon} \in \mathcal{K}_{\vartheta_0}$ such that $q_{0,1} = k_{\mu_0}$ and $\|q_{0,\varepsilon} - k_{\pi_{\vartheta_0}}\|_{\vartheta_0} \rightarrow 0$ as $\varepsilon \rightarrow 0^+$. Then there exist $\vartheta > \vartheta_0$, $T > 0$ and continuous maps $(0, 1] \ni \varepsilon \mapsto q_{t,\varepsilon} \in \mathcal{K}_\vartheta$, $t \in [0, T]$ such that

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{t \in [0, T]} \|q_{t,\varepsilon} - k_{\pi_{\vartheta_t}}\|_\vartheta = 0. \quad (2.43)$$

2.7. Comments

2.7.1. The choice of the model

Our choice of the translation invariant repulsion made in (2.8) is motivated only by the convenience and simplicity of the presentation. A more general version could contain $K(x, y)$ instead of $\phi(x - y)$, cf. (1.2). In this case, the theory developed below would also be valid after a proper modification. As mentioned in Introduction, the model can be employed as a part of more involved models. Let us mention some of them. As might be seen from the example in (1.4), (1.5), the homogeneous solutions can be unstable to small perturbations of the homogeneous rate b . With this regard, a generalization of the model can be made by allowing b be dependent on some extra parameters, e.g., be random. A possible choice might be

$$b(x) = b \exp \left(\sum_{y \in \omega} \Phi(x, y) \right),$$

where $\omega \subset \mathbb{R}^d$ is the configuration of some entities (attraction centers), distributed e.g., according to a Poisson law, and $\Phi(x, y)$ is an attraction/repulsion potential. Another possibility, close to just mentioned, is to consider a two-type system of the Widom-Rowlinson kind [3,8,15,16]. In this system, the particles of different types would repel each other whereas those of the same type not interact. Then instead of (1.2) we would have

$$\begin{aligned} \frac{d}{dt} \varrho_{0,t}(x) &= b_0(x) \exp \left(- \int_{\mathbb{R}^d} K(x, y) \varrho_{1,t}(y) dy \right), \\ \frac{d}{dt} \varrho_{1,t}(x) &= b_1(x) \exp \left(- \int_{\mathbb{R}^d} K(x, y) \varrho_{0,t}(y) dy \right), \end{aligned}$$

that strongly resemble the system in (1.4), which manifests the instability to appearing patterns discussed in Introduction. Another possibility to modify the model is to include some motion of the entities, e.g., random jumps as in the Kawasaki model [4].

2.7.2. Theorem 2.6 and Corollary 2.7

These statements establish the existence and uniqueness of the global in time evolution of states through the evolution of the corresponding correlation functions. In general, here we follow the scheme developed and used in [3,4,13,17]. Its main aspects are outlined in subsection 2.3. Technically, the proof of Theorem 2.6 is the most involved part of this work, divided into several steps. The hardest one is to prove that the evolution $k_0 \rightarrow k_t$ is such that k_t is the correlation function of a unique state. Usually, this link between correlation functions and micro-state is not even discussed.

In the course of preparing the very formulation of Theorem 2.6 we have obtained the ‘true’ hierarchy of evolution equations for correlation functions of all orders mentioned in Introduction. It is hidden in (2.21) or (2.35). To see it one has to recall that $k(\{x_1, \dots, x_n\}) = k^{(n)}(x_1, \dots, x_n)$ and $k^{(n)}$ is the n -th order correlation function, cf. (2.16). Then the equation for $k_t^{(1)}$ (the first member of the hierarchy) reads, see (2.18) and (2.28),

$$\begin{aligned} \frac{d}{dt} k_t^{(1)}(x) &= b(x) \\ &+ b(x) \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} \left(\prod_{i=1}^n [e^{-\phi(x-x_i)} - 1] \right) k_t^{(n)}(x_1, \dots, x_n) dx_1 \cdots dx_n. \end{aligned}$$

In contrast to birth-and-death models [1,7,18], this equation contains correlation functions of all orders. Its ‘naive’ decoupling consists in setting $k_t^{(n)}(x_1, \dots, x_n) \approx k_t^{(1)}(x_1) \cdots k_t^{(1)}(x_n)$, which yields

$$\frac{d}{dt} k_t^{(1)}(x) = b(x) \exp \left(- \int_{\mathbb{R}^d} (1 - e^{-\phi(x-y)}) k_t^{(1)}(y) dy \right),$$

that can be considered as a version of (2.38), (2.39). Noteworthy,

$$\hat{\phi}(x-y) := 1 - e^{-\phi(x-y)} \leq \phi(x-y).$$

To realize a more advanced decoupling here, one should write the next equations in the hierarchy.

2.7.3. Theorem 2.8

This statement gives an example of the self-regulation of a complex system due to interactions between its constituent, cf. [12,13]. As mentioned in Introduction and Section 2, in the free version of our model (with $\phi = 0$), the particle density increases linearly in time, cf. (1.1). Then Theorem 2.8 shows that the repulsion satisfying item (c) of Assumption 2.1 yields a significant attenuation of this increase. This qualitatively corresponds to the case of $\alpha \geq 1$ in (1.4) – (1.6) where the repulsion from the same patch is more essential. Hence, one might speculate that under the mentioned condition in item (c) the almost homogeneity of the environment (encrypted in b) yields an almost homogeneous distribution of entities. Or – in other words – the homogeneous distribution is stable to small perturbations of the homogeneity of the environment. However, this might be only a guess as we failed to get any mathematical result in this direction. Moreover, the result of Theorem 2.8 cannot be directly transferred to the kinetic equation (2.38), (2.39). The reason for this is that in the proof we crucially use the fact that the states μ_t satisfy (in the weak sense, see (2.3)) the Kolmogorov equation (2.2) with L given in (2.8). At the same time, there is no analogy of such L related to the kinetic equation, see also comments to Theorem 2.10.

2.7.4. Theorem 2.9

The existence and uniqueness stated in this theorem are quite expectable, and its most interesting result is the bounds (2.42). Assume that the initial distribution of the entities is tuned in such a way that $b^+ = b^- = b$, see (2.41). Then the solution is obtained explicitly in the form

$$\varrho_t(x) = \varrho_0(x) + \frac{1}{\langle \phi \rangle} \log(1 + bt),$$

which means that $\varrho_t(x) - \varrho_0(x)$ is independent of x . By (2.40) the mentioned tuning consists in satisfying

$$\forall x \in \mathbb{R}^d \quad b(x) \exp(-(\phi * \varrho_0)(x)) = \text{const},$$

and thus could be considered as the result of the interplay between b and ϕ mentioned above. If $b^+ > b^-$, then the function $\omega_-(t)$ is bounded in time whereas $\omega_+(t)$ increases linearly. This resembles the situation in (1.4) with $\alpha < 1$. Note that ϕ in Theorem 2.9 is not supposed to satisfy item (c) of Assumption 2.1. Unfortunately, we failed to get more precise bounds at the expense of further restriction imposed on ϕ .

2.7.5. Theorem 2.10

This statement establishes a relationship between the mesoscopic evolution $\varrho_0 \rightarrow \varrho_t$ obtained from the kinetic equation and the Markov evolution of micro-states $\mu_0 \rightarrow \mu_t$. Its main message is that the former evolution approximates the latter one in the sense of Definition 2.5. And its main drawback is that this

relationship is restricted in time to $[0, T]$ with T found explicitly, see the proof of the theorem below. Here one has to keep in mind that the evolution of the Poisson states $\pi_{\vartheta_0} \rightarrow \pi_{\vartheta_t}$ corresponding to the mentioned evolution of the densities need not be Markov, that is, it cannot be obtained from equations like that in (2.2). The second thing to be aware of is that $q_{t,\varepsilon}$ in (2.43) with $\varepsilon \in (0, 1)$ need not be correlation functions. Because of this, their continuation in time were impossible that yields the drawback mentioned above.

3. The evolution of states

In this section, we prove Theorem 2.6 and Corollary 2.7. We begin by proving that (2.35) has a unique solution on a bounded time interval.

3.1. Finite time horizon

For $\vartheta \in \mathbb{R}$ and $\vartheta' > \vartheta$, we set

$$T(\vartheta', \vartheta) = \frac{\vartheta' - \vartheta}{b} \exp\left(\vartheta - \langle \phi \rangle e^{\vartheta'}\right) = \frac{\vartheta' - \vartheta}{\beta(\vartheta')} e^{\vartheta}, \quad (3.1)$$

see (2.33).

Lemma 3.1. *For each $\vartheta_0 \in \mathbb{R}$ and $\vartheta > \vartheta_0$, the problem in (2.35) with $k_0 \in \mathcal{K}_{\vartheta_0}$ has a unique solution on the time interval $[0, T(\vartheta, \vartheta_0))$.*

Proof. For an arbitrary $\vartheta_1 \in \mathbb{R}$ and $\vartheta_2 > \vartheta_1$, by means of (2.28) one can define a bounded operator $(L^\Delta)_{\vartheta_2\vartheta_1}^2 : \mathcal{K}_{\vartheta_1} \rightarrow \mathcal{K}_{\vartheta_2}$ the operator norm of which can be estimated similarly as in (2.29) and (2.30). Clearly, for each $\vartheta' \in (\vartheta_1, \vartheta_2)$, it satisfies

$$\|(L^\Delta)_{\vartheta_2\vartheta_1}^2\| \leq \frac{\beta(\vartheta')\beta(\vartheta_1)e^{-\vartheta'-\vartheta_1}}{e^2(\vartheta_2-\vartheta')(\vartheta'-\vartheta_1)} \leq \frac{[\beta(\vartheta_2)]^2 e^{-2\vartheta_1}}{e^2(\vartheta_2-\vartheta')(\vartheta'-\vartheta_1)}. \quad (3.2)$$

Note that the definition of $(L^\Delta)_{\vartheta_2\vartheta_1}^2$ is independent of the choice of ϑ' , which we use to estimate the norm of this operator. In a similar way, one defines $(L^\Delta)_{\vartheta_2\vartheta_1}^n$ for all $n \in \mathbb{N}$. To estimate its norm, we introduce

$$\vartheta^l = \vartheta_1 + l\epsilon, \quad \epsilon = (\vartheta_2 - \vartheta_1)/n.$$

Then like in (3.2) we obtain

$$\|(L^\Delta)_{\vartheta_2\vartheta_1}^n\| \leq \left(\frac{n}{e}\right)^n \frac{1}{[T(\vartheta_2, \vartheta_1)]^n}. \quad (3.3)$$

Let us consider $(L^\Delta)_{\vartheta_2\vartheta_1}^0$ as the corresponding (continuous) embedding operator, cf. (2.26). Then, for each $\vartheta' \in (\vartheta_1, \vartheta_2)$, $n \in \mathbb{N}_0$ and $m \leq n$, we have that

$$(L^\Delta)_{\vartheta_2\vartheta_1}^n = (L^\Delta)_{\vartheta_2\vartheta'}^m (L^\Delta)_{\vartheta'\vartheta_1}^{n-m} = (L^\Delta)_{\vartheta_2\vartheta'}^0 (L^\Delta)_{\vartheta'\vartheta_1}^n. \quad (3.4)$$

In view of (2.32) and (2.34), these equalities imply

$$\forall n \in \mathbb{N}_0 \quad (L^\Delta)_{\vartheta_2\vartheta_1}^n : \mathcal{K}_{\vartheta_1} \rightarrow \mathcal{D}_{\vartheta_2}, \quad (3.5)$$

$$\forall n \in \mathbb{N} \quad (L^\Delta)_{\vartheta_2\vartheta_1}^n = L_{\vartheta_2}^\Delta (L^\Delta)_{\vartheta_2\vartheta_1}^{n-1}.$$

Consider

$$Q_{\vartheta_2\vartheta_1}(t) = \sum_{n \geq 0} \frac{t^n}{n!} (L^\Delta)_{\vartheta_2\vartheta_1}^n. \quad (3.6)$$

By (3.3) we conclude that the series in (3.6) converges in the operator norm topology – uniformly on compact subsets of $[0, T(\vartheta_2, \vartheta_1))$ – and thus defines a continuous function

$$[0, T(\vartheta_2, \vartheta_1)) \ni t \mapsto Q_{\vartheta_2\vartheta_1}(t) \in \mathcal{C}_{\vartheta_2\vartheta_1}, \quad (3.7)$$

where $\mathcal{C}_{\vartheta_2\vartheta_1}$ denotes the Banach space of all bounded linear operators acting from $\mathcal{K}_{\vartheta_1}$ to $\mathcal{K}_{\vartheta_2}$. Its operator norm can be estimated with the help of (3.3), that yields

$$\|Q_{\vartheta_2\vartheta_1}(t)\| \leq \frac{T(\vartheta_2, \vartheta_1)}{T(\vartheta_2, \vartheta_1) - t}. \quad (3.8)$$

In a similar way, by (3.4) and the second equality in (3.5) one obtains that

$$\forall t \in (0, T(\vartheta_2, \vartheta_1)) \quad \frac{d}{dt} Q_{\vartheta_2\vartheta_1}(t) = L_{\vartheta_2\vartheta'}^\Delta Q_{\vartheta'\vartheta_1}(t) = L_{\vartheta_2}^\Delta Q_{\vartheta_2\vartheta_1}(t), \quad (3.9)$$

where the choice of $\vartheta' \in (\vartheta_1, \vartheta_2)$ depends on the value of t and should be made in such a way that $t < T(\vartheta', \vartheta_1)$, which is possible since $T(\vartheta', \vartheta_1)$ continuously depends on ϑ' . By (3.9) we conclude that the map in (3.7) is continuously differentiable on $(0, T(\vartheta_2, \vartheta_1))$. We combine all these facts and obtain that

$$k_t = Q_{\vartheta\vartheta_0}(t)k_0 \quad (3.10)$$

is the solution in question, see Definition 2.4. Assume now that \hat{k}_t is another solution of (2.35) with the same initial condition. Then $u_t = \hat{k}_t - k_t \in \mathcal{K}_\vartheta$ also solves this problem but with the zero initial condition. Let $\tau \geq 0$ be such that $u_t = 0$ for $t \leq \tau$. Take $t \in (\tau, T(\vartheta, \vartheta_0))$ and write

$$I_{\vartheta'\vartheta} u_t = \int_{\tau}^t L_{\vartheta'\vartheta}^\Delta u_s ds = L_{\vartheta'\vartheta}^\Delta \int_{\tau}^t u_s ds, \quad (3.11)$$

which obviously holds with any $\vartheta' > \vartheta$. Here $I_{\vartheta'\vartheta} : \mathcal{K}_\vartheta \rightarrow \mathcal{K}_{\vartheta'}$ is the embedding operator, cf. (2.26). Now we repeat in (3.11) the same equality for u_s , and then repeat this again due times to get

$$\begin{aligned} I_{\vartheta'\vartheta} u_t &= (L^\Delta)_{\vartheta'\vartheta}^2 \int_{\tau}^t \left(\int_{\tau}^{t_1} u_s ds \right) dt_1 \\ &= (L^\Delta)_{\vartheta'\vartheta}^n \int_{\tau}^t \int_{\tau}^{t_1} \cdots \int_{\tau}^{t_{n-2}} \left(\int_{\tau}^{t_{n-1}} u_s ds \right) dt_1 \cdots dt_{n-1}, \quad n \in \mathbb{N}. \end{aligned} \quad (3.12)$$

Then by means of (3.3) we obtain from (3.12) that

$$\|I_{\vartheta'\vartheta} u_t\|_{\vartheta'} \leq \frac{1}{n!} \left(\frac{n}{e} \right)^n \left(\frac{t - \tau}{T(\vartheta', \vartheta)} \right)^n \sup_{s \in [\tau, t]} \|u_s\|_{\vartheta}.$$

Since n here is an arbitrary integer, we have that $I_{\vartheta'\vartheta}u_t = 0$ (hence, $u_t = 0$) for all those values of $t \in (\tau, T(\vartheta, \vartheta_0))$, for which $t - \tau < T(\vartheta', \vartheta)$ with a given $\vartheta' > \vartheta$. In such a way, one proves that $u_t = 0$ for all $t \in [0, T(\vartheta, \vartheta_0))$, which completes the whole proof. \square

Remark 3.2. Since $Q_{\vartheta_2\vartheta_1}(t)$ is defined by the series in (3.6), by (2.34) one can prove that, for each $\vartheta'_1 \leq \vartheta_1$ and $\vartheta'_2 \in (\vartheta'_1, \vartheta_2)$, the follows holds

$$\forall k \in \mathcal{K}_{\vartheta'_1} \quad Q_{\vartheta_2\vartheta_1}(t)k = Q_{\vartheta'_2\vartheta'_1}(t)k,$$

whenever $t < \min\{T(\vartheta_2, \vartheta_1); T(\vartheta'_2, \vartheta'_1)\}$.

3.2. The identification

The main result of this section is given in the following statement. Recall that $\mathcal{K}_{\vartheta}^*$ is defined in (2.36).

Lemma 3.3. For each $t < T(\vartheta_2, \vartheta_1)/2$, the operator defined in (3.6) has the property $Q_{\vartheta_2\vartheta_1}(t) : \mathcal{K}_{\vartheta_1}^* \rightarrow \mathcal{K}_{\vartheta_2}^*$.

By this lemma it will follow that the solution k_t obtained in Lemma 3.1 for $t < T(\vartheta_2, \vartheta_1)/2$ satisfies all the conditions of Proposition 2.3 – and hence is the correlation function of a unique $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$ – whenever k_0 has the same property. Its proof is performed in a number of steps. First we introduce an auxiliary model by replacing $b(x)$ by an integrable $b^\sigma(x)$, $\sigma > 0$. For this new model, by repeating the steps made in the proof of Lemma 3.1 we construct $Q_{\vartheta_2\vartheta_1}^\sigma$, for which we prove that $Q_{\vartheta_2\vartheta_1}^\sigma(t) : \mathcal{K}_{\vartheta_1}^* \rightarrow \mathcal{K}_{\vartheta_2}^*$. Then we show that

$$\langle\langle G, Q_{\vartheta_2\vartheta_1}^\sigma(t)k \rangle\rangle \rightarrow \langle\langle G, Q_{\vartheta_2\vartheta_1}(t)k \rangle\rangle, \quad \text{as } \sigma \rightarrow 0^+. \quad (3.13)$$

3.2.1. The auxiliary model

For $\sigma > 0$, we set

$$\psi_\sigma(x) = \exp(-\sigma|x|^2), \quad b^\sigma(x) = b(x)\psi_\sigma(x). \quad (3.14)$$

Let $L^{\Delta, \sigma}$ stand for the corresponding operator defined in (2.28) with b replaced by b^σ . Clearly, this $L^{\Delta, \sigma}$ satisfies the estimates in (2.33) and (3.3) with the same right-hand sides, that allows one to define $Q_{\vartheta_2\vartheta_1}^\sigma(t)$ by the series as in (3.6) with L^Δ replaced by $L^{\Delta, \sigma}$. Likewise, one defines also $L_{\vartheta'}^{\Delta, \sigma}$ and $L_{\vartheta'\vartheta}^{\Delta, \sigma}$. Note that the operator norm of $Q_{\vartheta_2\vartheta_1}^\sigma(t)$ satisfies (3.8) with the same right-hand side.

The Banach space predual to \mathcal{K}_ϑ is

$$\mathcal{G}_\vartheta = \{G : \Gamma_0 \rightarrow \mathbb{R} : |G|_\vartheta < \infty\}, \quad |G|_\vartheta := \int_{\Gamma_0} |G(\eta)| e^{\vartheta|\eta|} \lambda(d\eta). \quad (3.15)$$

Like in (2.26) we then have

$$\mathcal{G}_{\vartheta'} \hookrightarrow \mathcal{G}_\vartheta, \quad \text{for } \vartheta' > \vartheta. \quad (3.16)$$

Let \widehat{L}^σ be defined by the relation

$$\langle\langle \widehat{L}^\sigma G, k \rangle\rangle = \langle\langle G, L^{\Delta, \sigma} k \rangle\rangle. \quad (3.17)$$

Then we set, cf. (2.31),

$$\widehat{D}_\vartheta = \{G \in \mathcal{G}_\vartheta : |\cdot|G \in \mathcal{G}_\vartheta\}.$$

This allows one to define $\widehat{L}_\vartheta^\sigma = (\widehat{L}^\sigma, \widehat{D}_\vartheta)$, and also bounded operators $(\widehat{L}^\sigma)_{\vartheta_1\vartheta_2}^n : \mathcal{G}_{\vartheta'} \rightarrow \mathcal{G}_\vartheta$, $n \in \mathbb{N}$, the operator norms of which satisfy the estimates in (3.3) with the same right-hand side. Thereafter, we define, cf. (3.6),

$$H_{\vartheta_1\vartheta_2}^\sigma(t) = \sum_{n \geq 0} \frac{t^n}{n!} (\widehat{L}^\sigma)_{\vartheta_1\vartheta_2}^n, \quad \vartheta_2 > \vartheta_1. \quad (3.18)$$

Like in the proof of Lemma 3.1 one shows that the series in (3.18) defines a continuous map

$$[0, T(\vartheta_2, \vartheta_1)) \ni t \mapsto H_{\vartheta_1\vartheta_2}^\sigma(t) \in \widehat{\mathcal{C}}_{\vartheta_1\vartheta_2}, \quad (3.19)$$

where $\widehat{\mathcal{C}}_{\vartheta_1\vartheta_2}$ is the Banach space of all bounded linear operators acting from $\mathcal{G}_{\vartheta_2}$ to $\mathcal{G}_{\vartheta_1}$. The operator norm of $H_{\vartheta_1\vartheta_2}^\sigma(t)$ satisfies (3.8) with the same right-hand side. By the very construction, cf. (3.17), we have that

$$\forall t \in [0, T(\vartheta_2, \vartheta_1)) \quad \langle\langle H_{\vartheta_1\vartheta_2}^\sigma(t)G, k \rangle\rangle = \langle\langle G, Q_{\vartheta_2\vartheta_1}^\sigma(t)k \rangle\rangle, \quad (3.20)$$

holding for all $k \in \mathcal{K}_{\vartheta_1}$ and $G \in \mathcal{G}_{\vartheta_2}$.

In the sequel, we will use the following property of \widehat{L}^σ . To describe it, we derive from (3.17) and (2.28), (2.29) the explicit form of this operator. To this end, we employ (2.27) and obtain

$$(\widehat{L}^\sigma G)(\eta) = \int_{\mathbb{R}^d} b^\sigma(x) \sum_{\xi \subset \eta} e(t_x; \xi) e(\tau_x; \eta \setminus \xi) G(\eta \setminus \xi \cup x) dx. \quad (3.21)$$

Note that, for a broad class of functions $G : \Gamma_0 \rightarrow \mathbb{R}$, e.g., for $G \in B_{\text{bs}}(\Gamma_0)$, the computations in (3.21) can be performed explicitly as the integral is convergent and the sum is finite. In Appendix, we prove that

$$L^\sigma K G = K \widehat{L}^\sigma G, \quad (3.22)$$

where K is defined in (2.17). Then, for each $G \in B_{\text{bs}}(\Gamma_0)$ and $n \in \mathbb{N}$, we obtain from (3.22) the following

$$(L^\sigma)^n K G = K (\widehat{L}^\sigma)^n G.$$

3.2.2. Approximations

Now we aim at proving that

$$\forall \sigma > 0 \quad \forall t \in [0, T(\vartheta_2, \vartheta_1)) \quad \langle\langle G, Q_{\vartheta_2\vartheta_1}^\sigma(t)k \rangle\rangle \geq 0, \quad (3.23)$$

whenever $k \in \mathcal{K}_{\vartheta_1}^*$ and $G \in B_{\text{bs}}^*(\Gamma_0)$. Note that $B_{\text{bs}}(\Gamma_0) \subset \mathcal{G}_\vartheta$ for any real ϑ , see Definition 2.2. To prove (3.23) we approximate k by $k^{\Lambda, N}$, Λ and N being a compact subset of \mathbb{R}^d and integer, respectively. The meaning of this is that $Q_{\vartheta_2\vartheta_1}^\sigma(t)k^{\Lambda, N}$ has the desired positivity by construction. Then the proof of (3.23) will be done by showing the corresponding convergence as $\Lambda \rightarrow \mathbb{R}^d$ and $N \rightarrow +\infty$.

For a compact Λ , let Γ_Λ denote the set of all $\gamma \in \Gamma$ such that $\gamma \subset \Lambda$. Clearly, $\Gamma_\Lambda \subset \Gamma_0$ and $\Gamma_\Lambda \in \mathcal{B}(\Gamma)$. Set $\mathcal{B}(\Gamma_\Lambda) = \{A \in \mathcal{B}(\Gamma) : A \subset \Gamma_\Lambda\}$ and $p_\Lambda(\gamma) = \gamma \cap \Lambda$, $\gamma \in \Gamma$. Then the formula

$$\mu^\Lambda(A) = \mu(p_\Lambda^{-1}(A)), \quad A \in \mathcal{B}(\Gamma_\Lambda) \quad (3.24)$$

defines a probability measure on the measurable space $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ – the projection of μ on Γ_Λ . Here $p_\Lambda^{-1}(A) = \{\gamma \in \Gamma : p_\Lambda(\gamma) \in A\}$. It is possible to show, see [14], that for each $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$ and any compact

Λ , μ^Λ is absolutely continuous with respect to λ . For $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$, let $R_\mu^\Lambda : \Gamma_\Lambda \rightarrow \mathbb{R}$ be the corresponding Radon-Nikodym derivative. The correlation function k_μ and R_μ^Λ are related to each other by

$$k_\mu(\eta) = \int_{\Gamma_\Lambda} R_\mu^\Lambda(\eta \cup \xi) \lambda(d\xi), \quad \eta \in \Gamma_\Lambda. \quad (3.25)$$

For a given $N \in \mathbb{N}$ and $\eta \in \Gamma_0$, we then set

$$R_0^{\Lambda, N}(\eta) = \begin{cases} R_\mu^\Lambda(\eta) & \text{if } \eta \in \Gamma_\Lambda \text{ and } |\eta| \leq N; \\ 0 & \text{otherwise.} \end{cases} \quad (3.26)$$

Clearly, we have that $R_0^{\Lambda, N}(\eta) \geq 0$ and $R_0^{\Lambda, N} \in \mathcal{G}_\vartheta$ for any $\vartheta \in \mathbb{R}$. Set, cf. (3.25),

$$q_0^{\Lambda, N}(\eta) = \int_{\Gamma_0} R_0^{\Lambda, N}(\eta \cup \xi) \lambda(d\xi), \quad \eta \in \Gamma_0. \quad (3.27)$$

Then, for $G \in B_{\text{bs}}^*(\Gamma_0)$, by (2.17) and (2.27) we have that

$$\langle\langle G, q_0^{\Lambda, N} \rangle\rangle = \int_{\Gamma_0} (KG)(\eta) R_0^{\Lambda, N}(\eta) \lambda(d\eta) \geq 0. \quad (3.28)$$

At the same time, by (3.25), (3.26) and (3.27) we have

$$0 \leq q_0^{\Lambda, N}(\eta) \leq k_\mu(\eta), \quad (3.29)$$

holding for λ -almost all $\eta \in \Gamma_0$.

By (3.26) $R_0^{\Lambda, N}$ (up to normalization) is the density of a state in which the number of entities does not exceed N , and hence is finite. We allow it to evolve $R_0^{\Lambda, N} \rightarrow R_t^{\Lambda, N}$, where $R_t^{\Lambda, N}$ is obtained from the Fokker-Planck equation

$$\frac{d}{dt} R_t^{\Lambda, N} = L^\dagger R_t^{\Lambda, N}, \quad R_t^{\Lambda, N}|_{t=0} = R_0^{\Lambda, N},$$

where L^\dagger is defined by

$$\int_{\Gamma_0} (L^\sigma F)(\eta) R(\eta) \lambda(d\eta) = \int_{\Gamma_0} F(\eta) (L^\dagger R)(\eta) \lambda(d\eta). \quad (3.30)$$

In Appendix, we show that

$$L^\dagger = A + B, \quad (3.31)$$

$$(AR)(\eta) = -\Psi_\sigma(\eta) R(\eta), \quad \Psi_\sigma(\eta) := \int_{\mathbb{R}^d} b^\sigma(x) e(\tau_x; \eta) dx,$$

$$(BR)(\eta) = \sum_{x \in \eta} b^\sigma(x) e(\tau_x; \eta \setminus x) R(\eta \setminus x).$$

Note that the reason to substitute b by b^σ is to make convergent the integral in the second line of (3.31). Note also that according to our assumptions and (3.14) we have that

$$\Psi_\sigma(\eta) \leq \bar{b} \left(\frac{\pi}{\sigma} \right)^{d/2}, \quad (3.32)$$

which means that A is a bounded multiplication operator. Now we define L^\dagger in \mathcal{G}_0 , introduced in (3.15) with $\vartheta = 0$. For $R \in \mathcal{G}_0$, by (3.31) we get

$$\begin{aligned} |BR|_0 &\leq \int_{\Gamma_0} \left(\sum_{x \in \eta} b^\sigma(x) e(\tau_x; \eta \setminus x) |R(\eta \setminus x)| \right) \lambda(d\eta) \\ &= \int_{\Gamma_0} \Psi_\sigma(\eta) |R(\eta)| \lambda(d\eta) \leq \bar{b} \left(\frac{\pi}{\sigma} \right)^{d/2} |R|_0. \end{aligned} \quad (3.33)$$

This means that L^\dagger is bounded and $\|L^\dagger\| \leq 2\bar{b}(\pi/\sigma)^{d/2}$. Let \mathcal{G}_0^+ be the cone of positive elements of \mathcal{G}_0 . For $R \in \mathcal{G}_0^+$, by making the same calculations as in (3.33) we obtain

$$\int_{\Gamma_0} (L^\dagger R)(\eta) \lambda(d\eta) = 0$$

Therefore, L^\dagger generates a stochastic semigroup $S^\dagger = \{S^\dagger(t)\}_{t \geq 0}$ on \mathcal{G}_0 . This semigroup has a useful property which we describe now. Recall that \mathcal{G}_ϑ is defined in (3.15) and that $\mathcal{G}_\vartheta \hookrightarrow \mathcal{G}_0$ whenever $\vartheta > 0$, see (3.16).

Lemma 3.4. *For each $\vartheta > 0$ and $t > 0$, we have that $S^\dagger(t) : \mathcal{G}_\vartheta \rightarrow \mathcal{G}_\vartheta$.*

Proof. We employ the Thieme-Voigt perturbation theory in the form adapted to the present context formulated in [4, Proposition 3.2, page 421]. According to item (iv) of this statement, we have to show that there exist positive c and ε such that, for all positive $R \in \mathcal{G}_\vartheta$, the following holds

$$\int_{\Gamma_0} e^{\vartheta|\eta|} (L^\dagger R)(\eta) \lambda(d\eta) \leq c \int_{\Gamma_0} e^{\vartheta|\eta|} R(\eta) \lambda(d\eta) - \varepsilon \int_{\Gamma_0} \Psi_\sigma(\eta) R(\eta) \lambda(d\eta). \quad (3.34)$$

Set $F_\vartheta(\eta) = e^{\vartheta|\eta|}$. Then by (3.30) we have

$$\begin{aligned} \text{LHS}(3.34) &= \int_{\Gamma_0} (L^\sigma F_\vartheta)(\eta) R(\eta) \lambda(d\eta) \\ &= (e^\vartheta - 1) \int_{\Gamma_0} \left(\int_{\mathbb{R}^d} b^\sigma(x) e(\tau_x; \eta) dx \right) e^{\vartheta|\eta|} R(\eta) \lambda(d\eta) \\ &\leq \bar{c}(\sigma, \vartheta) \int_{\Gamma_0} e^{\vartheta|\eta|} R(\eta) \lambda(d\eta). \end{aligned}$$

Here $\bar{c}(\sigma, \vartheta) := (e^\vartheta - 1)\bar{b}(\pi/\sigma)^d$, cf. (3.32). Then (3.34) holds with $\varepsilon \leq (\sigma/\pi)^d \bar{b}^{-1}$ and $c \geq 1 + \bar{c}(\sigma, \vartheta)$. \square

By (3.34) and the Grönwall inequality we obtain that

$$\int_{\Gamma_0} e^{\vartheta|\eta|} \left(S^\dagger(t) R_0^{\Lambda, N} \right) (\eta) \lambda(d\eta) \leq \exp \left(\vartheta N + \bar{c}(\sigma, \vartheta) t \right), \quad (3.35)$$

holding for all $t > 0$, see (3.26).

Lemma 3.5. *Let $\mu_0 \in \mathcal{P}_{\text{exp}}(\Gamma)$ and $\vartheta_0 \in \mathbb{R}$ be such that $k_{\mu_0} \in \mathcal{K}_{\vartheta_0}$. Let also $\vartheta > \vartheta_0$ be fixed. Finally, let $R_0^{\Lambda, N}$ be calculated according to (3.27), (3.26) with this μ_0 . Then, for each $G \in B_{\text{bs}}(\Gamma_0)$ and all $t < T(\vartheta, \vartheta_0)$, $\sigma > 0$, compact Λ and $N \in \mathbb{N}$, the following holds*

$$\langle\langle G, Q_{\vartheta\vartheta_0}^\sigma(t) q_0^{\Lambda, N} \rangle\rangle = \langle\langle KG, S^\dagger(t) R_0^{\Lambda, N} \rangle\rangle, \quad (3.36)$$

where KG is as in (2.17).

Proof. For a given $G \in B_{\text{bs}}(\Gamma_0)$ and $t < T(\vartheta, \vartheta_0)$, denote

$$f_G(t) = \langle\langle G, Q_{\vartheta\vartheta_0}^\sigma(t) q_0^{\Lambda, N} \rangle\rangle, \quad g_G(t) = \langle\langle KG, S^\dagger(t) R_0^{\Lambda, N} \rangle\rangle. \quad (3.37)$$

For this t , one finds $\vartheta_1 \in (\vartheta_0, \vartheta)$ such that $t < T(\vartheta_1, \vartheta_0)$. Then, for each $n \in \mathbb{N}$, by (3.9) we obtain

$$f_G^{(n)}(t) := \frac{d^n f_G(t)}{dt^n} = \langle\langle G, (L^{\Delta, \sigma})_{\vartheta, \vartheta_1}^n Q_{\vartheta_1\vartheta_0}^\sigma(t) q_0^{\Lambda, N} \rangle\rangle. \quad (3.38)$$

We take now any $\tau < T(\vartheta, \vartheta_0)$ and then pick $\vartheta_1 \in (\vartheta_0, \vartheta)$ such that $\tau < T(\vartheta_1, \vartheta_0)$. Then we apply in (3.38) the estimates in (3.3), (3.8) and (3.29) to obtain

$$\sup_{t \in [0, \tau]} \left| \frac{d^n f_G(t)}{dt^n} \right| \leq C_f n^n M_f^n, \quad (3.39)$$

$$C_f := |G|_\vartheta \sup_{t \in [0, \tau]} \|Q_{\vartheta_1\vartheta_0}^\sigma(t) q_0^{\Lambda, N}\|_{\vartheta_1} \leq \frac{T(\vartheta_1, \vartheta_0)}{T(\vartheta_1, \vartheta_0) - \tau} |G|_\vartheta \|k_{\mu_0}\|_{\vartheta_0},$$

$$M_f = 1/eT(\vartheta_1, \vartheta_0).$$

Likewise, for the derivatives of g_G , we get

$$g_G^{(n)}(t) := \frac{d^n g_G(t)}{dt^n} = \langle\langle KG, (L^\dagger)^n S^\dagger(t) R_0^{\Lambda, N} \rangle\rangle. \quad (3.40)$$

Recall that G here belongs to $B_{\text{bs}}(\Gamma_0)$, see Definition 2.2. Let $C_G > 0$ be such that $|G(\eta)| \leq C_G$ for all η . Then

$$|(KG)(\eta)| \leq C_G \exp(\omega|\eta|), \quad \omega := \log 2.$$

We use this estimate in (3.40) together with (3.35) and the fact that L^\dagger – the generator of S^\dagger – is bounded, to obtain that, for the same τ as in (3.39), the following holds

$$\sup_{t \in [0, \tau]} \left| \frac{d^n g_G(t)}{dt^n} \right| \leq C_g M_g^n, \quad (3.41)$$

$$C_g := C_G \exp \left(\omega N + \bar{c}(\sigma, \omega) \tau \right),$$

$$M_f = 2\bar{b} \left(\frac{\pi}{\sigma} \right)^{d/2}.$$

By (3.41) we conclude that g_G is a real analytic function on each $[0, \tau]$. At the same time, by the Denjoy-Carleman theorem [6] one gets that f_G is quasi-analytic on each $[0, \tau]$, $\tau < T(\vartheta, \vartheta_0)$. These two facts imply that $h_G = f_G - g_G$ is also quasi-analytic on each $[0, \tau]$ with τ chosen as mentioned above. Now by (3.17) and (3.38) we get

$$f_G^{(n)}(0) = \langle\langle (\widehat{L}^\sigma)^n G, q_0^{\Lambda, N} \rangle\rangle. \quad (3.42)$$

Likewise, by (3.30) and (3.40) we have

$$g_G^{(n)}(0) = \langle\langle (L^\sigma)^n KG, R_0^{\Lambda, N} \rangle\rangle.$$

Now we apply here (3.22), then (3.28) and arrive at

$$g_G^{(n)}(0) = \langle\langle K(\widehat{L}^\sigma)^n G, R_0^{\Lambda, N} \rangle\rangle = \langle\langle (\widehat{L}^\sigma)^n G, q_0^{\Lambda, N} \rangle\rangle,$$

which by (3.42) yields that $h_G^{(n)}(0) = 0$ holding for all $n \in \mathbb{N}$. Since h_G is quasi-analytic, this implies that $h_G(t) = 0$ for all $t < T(\vartheta, \vartheta_0)$, that by (3.37) completes the proof. \square

Corollary 3.6. *Let $q_0^{\Lambda, N}$ and $\vartheta, \vartheta_0, t$ be as in Lemma 3.5. Then, for each $t < T(\vartheta, \vartheta_0)$, $k_t^{\Lambda, N} := Q_{\vartheta\vartheta_0}^\sigma(t)q_0^{\Lambda, N}$ has the positivity property as in (2.23), and hence $k_t^{\Lambda, N}(\eta) \geq 0$ for λ -almost all η .*

Proof. By (3.26) it follows that $S^\dagger R_0^{\Lambda, N} \subset \mathcal{G}_0^+$ as S^\dagger is stochastic. Then the proof follows by (2.17), cf. (3.28). \square

In the sequel, we use one more property of $Q_{\vartheta\vartheta_0}^\sigma(t)$. Recall that W_x is defined in (2.28).

Lemma 3.7. *Let $q_0^{\Lambda, N}$ and $\vartheta, \vartheta_0, t$ be as in Lemma 3.5 and Corollary 3.6. Then, for each $x \in \mathbb{R}^d$ and $G \in B_{\text{bs}}^*(\Gamma_0)$, it follows that*

$$\langle\langle G, k_t^{\Lambda, N} \rangle\rangle \geq \langle\langle G, e(\tau_x; \cdot) W_x k_t^{\Lambda, N} \rangle\rangle. \quad (3.43)$$

Hence

$$k_t^{\Lambda, N}(\eta) \geq e(\tau_x; \eta) (W_x k_t^{\Lambda, N})(\eta), \quad (3.44)$$

holding for all $x \in \mathbb{R}^d$ and λ -almost all $\eta \in \Gamma_0$.

Proof. Until the end of this proof t, x and all other variables are fixed. Then by (2.28) and (2.27) we have

$$\begin{aligned} \text{RHS}(3.43) &= \int_{\Gamma_0} G(\eta) e(\tau_x; \eta) \left(\int_{\Gamma_0} e(t_x; \xi) k_t^{\Lambda, N}(\eta \cup \xi) \lambda(d\xi) \right) \lambda(d\eta) \\ &= \int_{\Gamma_0} k_t^{\Lambda, N}(\eta) \left(\sum_{\xi \subset \eta} e(t_x; \xi) e(\tau_x; \eta \setminus \xi) G(\eta \setminus \xi) \right) \lambda(d\eta) \end{aligned} \quad (3.45)$$

$$= \int_{\Gamma_0} V_x(\eta)(t) R_t^{\Lambda, N}(\eta) \lambda(d\eta),$$

where the last equality follows by (3.36), $R_t^{\Lambda, N} = S^\dagger(t) R_0^{\Lambda, N}$ and

$$V_x(\eta) = \sum_{\zeta \subset \eta} \sum_{\xi \subset \zeta} e(t_x; \xi) e(\tau_x; \zeta \setminus \xi) G(\zeta \setminus \xi).$$

Proceeding as in the proof of (4.32) (see Appendix below) we show that

$$V_x(\eta) = e(\tau_x; \eta)(KG)(\eta).$$

We use this in (3.45) to get that

$$\text{LHS(3.43)} - \text{RHS(3.43)} = \langle\langle KG, [1 - e(\tau_x; \cdot)] R_t^{\Lambda, N} \rangle\rangle \geq 0,$$

which yields (3.43). The latter inequality follows by: (a) the positivity of KG , see (2.17); (b) the positivity of $1 - e(\tau_x; \eta)$, see (2.28); (c) the fact that $R_t^{\Lambda, N} = S^\dagger(t) R_0^{\Lambda, N}$, S^\dagger being a stochastic semigroup and (3.26). The proof of (3.44) is obvious. \square

3.2.3. Taking the limits

By Corollary 3.6 we have that $k_t^{\Lambda, N}$ has the positivity in question. Thus, we first fix $\sigma > 0$ and pass to the limit $\Lambda \rightarrow \mathbb{R}^d$ and $N \rightarrow +\infty$. A sequence of compact subsets $\{\Lambda_n\}_{n \in \mathbb{N}}$ is said to be cofinal if: (i) $\Lambda_n \subset \Lambda_{n+1}$ for each n ; (ii) each $x \in \mathbb{R}^d$ is eventually contained in its members. The proof of the next statement in a sense repeats the proof of [17, Lemma 5.9]

Lemma 3.8. *Let $\mu_0, \vartheta, \vartheta_0$ and t be as in Lemma 3.5. Then, for any cofinal sequence $\{\Lambda_n\}_{n \in \mathbb{N}}$ and $G \in B_{\text{bs}}(\Gamma_0)$, we have that*

$$\lim_{n \rightarrow \infty} \left(\lim_{N \rightarrow \infty} \langle\langle G, k_t^{\Lambda_n, N} \rangle\rangle \right) = \langle\langle G, k_t^\sigma \rangle\rangle, \quad k_t^\sigma := Q_{\vartheta \vartheta_0}^\sigma(t) k_{\mu_0}.$$

Proof. For $k_t^{\Lambda_n, N} = Q_{\vartheta \vartheta_0}^\sigma(t) q_0^{\Lambda_n, N}$, by (3.20) we have

$$\langle\langle G, k_t^{\Lambda_n, N} \rangle\rangle = \langle\langle H_{\vartheta_0 \vartheta}^\sigma(t) G, q_0^{\Lambda_n, N} \rangle\rangle, \quad \langle\langle G, k_t^\sigma \rangle\rangle = \langle\langle H_{\vartheta_0 \vartheta}^\sigma(t) G, k_{\mu_0} \rangle\rangle.$$

Set $G_t = H_{\vartheta_0 \vartheta}^\sigma(t) G$ and then write

$$\delta(n, N) = \langle\langle G_t, k_{\mu_0} \rangle\rangle - \langle\langle G_t, q_0^{\Lambda_n, N} \rangle\rangle = \langle\langle G_t, k_{\mu_0} - q_0^{\Lambda_n, N} \rangle\rangle =: J_n^{(1)} + J_{n, N}^{(2)}.$$

Here

$$\begin{aligned} J_n^{(1)} &= \int_{\Gamma_0} G_t(\eta) k_{\mu_0}(\eta) (1 - \mathbb{1}_{\Gamma_{\Lambda_n}}(\eta)) \lambda(d\eta), \\ J_{n, N}^{(2)} &= \int_{\Gamma_0} G_t(\eta) \left(k_{\mu_0}(\eta) \mathbb{1}_{\Gamma_{\Lambda_n}}(\eta) - q_0^{\Lambda_n, N}(\eta) \right) \lambda(d\eta), \end{aligned} \tag{3.46}$$

and $\mathbb{1}_{\Gamma_{\Lambda_n}}$ is the indicator of Γ_{Λ_n} . To prove that, for each n , $J_{n, N}^{(2)} \rightarrow 0$ as $N \rightarrow +\infty$, we rewrite it as follows

$$\begin{aligned}
J_{n,N}^{(2)} &= \int_{\Gamma_0} \left[G_t(\eta) \int_{\Gamma_{\Lambda_n}} R_{\mu_0}^{\Lambda_n}(\eta \cup \xi) \mathbb{1}_{\Gamma_{\Lambda_n}}(\eta) (1 - I_N(\eta \cup \xi)) \lambda(d\xi) \right] \lambda(d\eta) \\
&= \int_{\Gamma_0} G_t(\eta) \int_{\Gamma_0} R_{\mu_0}^{\Lambda_n}(\eta \cup \xi) \mathbb{1}_{\Gamma_{\Lambda_n}}(\eta \cup \xi) (1 - I_N(\eta \cup \xi)) \lambda(d\xi) \lambda(d\eta) \\
&= \int_{\Gamma_{\Lambda_n}} \sum_{\xi \subset \eta} G_t(\xi) R_{\mu_0}^{\Lambda_n}(\eta) (1 - I_N(\eta)) \lambda(d\eta) \\
&= \sum_{m=N+1}^{\infty} \frac{1}{m!} \int_{(\Lambda_n)^m} R_{\mu_0}^{\Lambda_n}(\{x_1, \dots, x_m\}) \\
&\quad \times \sum_{k=0}^m \sum_{\{i_1, \dots, i_k\} \subset \{1, \dots, m\}} G_t^{(k)}(x_{i_1}, \dots, x_{i_k}) dx_1 \cdots dx_m,
\end{aligned} \tag{3.47}$$

where $I_N(\eta) = 1$ whenever $|\eta| \leq N$ and $I_N(\eta) = 0$ otherwise. Recall that we assume $k_{\mu_0} \in \mathcal{K}_{\vartheta_0}$. Then by (3.25) and (2.25) it follows that

$$R_{\mu_0}^{\Lambda_n}(\{x_1, \dots, x_m\}) \leq k_{\mu_0}(\{x_1, \dots, x_m\}) \leq \|k_{\mu_0}\|_{\vartheta_0} e^{\vartheta_0 m},$$

holding for all m . By means of this estimate we obtain from (3.47) the following

$$\begin{aligned}
|J_{n,N}^{(2)}| &\leq \|k_0\|_{\vartheta_0} \\
&\quad \times \sum_{m=N+1}^{\infty} \frac{1}{m!} e^{\vartheta_0 m} \int_{(\Lambda_n)^m} \sum_{k=0}^m \sum_{\{i_1, \dots, i_k\} \subset \{1, \dots, m\}} |G_t^{(k)}(x_{i_1}, \dots, x_{i_k})| dx_1 \cdots dx_m \\
&\leq \|k_0\|_{\vartheta_0} \sum_{m=N+1}^{\infty} \frac{1}{m!} e^{\vartheta_0 m} \sum_{k=0}^m \frac{m!}{k!(m-k)!} \|G_t^{(k)}\|_{L^1((\mathbb{R}^d)^k)} |\Lambda_n|^{m-k},
\end{aligned} \tag{3.48}$$

where $|\Lambda|$ stands for the Lebesgue measure of Λ . The sum over m in the last line of (3.48) is the remainder of the series

$$\begin{aligned}
&\sum_{m=0}^{\infty} \sum_{k=0}^m \frac{e^{\vartheta_0 k}}{k!} \|G_t^{(k)}\|_{L^1((\mathbb{R}^d)^k)} \frac{e^{\vartheta_0(m-k)}}{(m-k)!} |\Lambda_n|^{m-k} \\
&= \sum_{k=0}^{\infty} \frac{e^{\vartheta_0 k}}{k!} \|G_t^{(k)}\|_{L^1((\mathbb{R}^d)^k)} \sum_{m=0}^{\infty} \frac{e^{\vartheta_0 m}}{m!} |\Lambda_n|^m = |G_t|_{\vartheta_0} \exp(e^{\vartheta_0} |\Lambda_n|).
\end{aligned}$$

Hence, by (3.48) we obtain that

$$\delta_n := \lim_{N \rightarrow +\infty} \delta(n, N) = J_n^{(1)}.$$

Then it remains to show that $\delta_n \rightarrow 0$. By (3.46) we have

$$|J_n^{(1)}| \leq \sum_{p=1}^{\infty} \frac{1}{p!} \int_{(\mathbb{R}^d)^p} |G_t^{(p)}(x_1, \dots, x_p)| k_{\mu_0}^{(p)}(x_1, \dots, x_p) \sum_{l=1}^p \mathbb{1}_{\Lambda_N^c}(x_l) dx_1 \cdots dx_p.$$

Since $k_{\mu_0} \in \mathcal{K}_{\vartheta_0}$ and $G_t^{(p)}$ and $k_{\mu_0}^{(p)}$ are symmetric, we rewrite the above estimate as

$$|J_n^{(1)}| \leq \|k_{\mu_0}\|_{\vartheta_0} \sum_{p=1}^{\infty} \frac{p}{p!} e^{\vartheta_0 p} \int_{\Lambda_n^c} \int_{(\mathbb{R}^d)^{p-1}} |G_t^{(p)}(x_1, \dots, x_p)| dx_1 \cdots dx_p.$$

For each $t < T(\vartheta, \vartheta_0)$, one finds $\epsilon > 0$ such that $t < T(\vartheta + \epsilon, \vartheta_0 + \epsilon)$, see (3.1). We fix these t and ϵ . Since $G_0 \in B_{bs}(\Gamma_0)$, and hence $G_0 \in \mathcal{G}_{\vartheta+\epsilon}$, by (3.19) and (3.20) we have that $G_t \in \mathcal{G}_{\vartheta_0+\epsilon}$. We apply this in the estimate above and obtain

$$|J_n^{(1)}| \leq \frac{\|k_0\|_{\vartheta_0}}{\epsilon} \Delta_n, \quad (3.49)$$

$$\Delta_n := \sum_{p=1}^{\infty} \frac{1}{p!} e^{(\vartheta_0+\epsilon)p} \int_{\Lambda_n^c} \int_{(\mathbb{R}^d)^{p-1}} |G_t^{(p)}(x_1, \dots, x_p)| dx_1 \cdots dx_p.$$

Furthermore, for each $M \in \mathbb{N}$, we have

$$\Delta_n \leq \Delta_{n,M}^{(1)} + \Delta_M^{(2)}, \quad (3.50)$$

$$\Delta_{n,M}^{(1)} := \sum_{p=1}^M \frac{1}{p!} e^{(\vartheta_0+\epsilon)p} \int_{\Lambda_n^c} \int_{(\mathbb{R}^d)^{p-1}} |G_t^{(p)}(x_1, \dots, x_p)| dx_1 \cdots dx_p,$$

$$\Delta_M^{(2)} := \sum_{p=M+1}^{\infty} \frac{1}{p!} e^{(\vartheta_0+\epsilon)p} \int_{(\mathbb{R}^d)^p} |G_t^{(p)}(x_1, \dots, x_p)| dx_1 \cdots dx_p.$$

Fix some $\varepsilon > 0$, and then pick M such that $\Delta_M^{(2)} < \varepsilon/2$, which is possible since

$$\sum_{p=1}^{\infty} \frac{1}{p!} e^{(\vartheta_0+\epsilon)p} \int_{(\mathbb{R}^d)^p} |G_t^{(p)}(x_1, \dots, x_p)| dx_1 \cdots dx_p = |G_t|_{\vartheta_0+\epsilon},$$

as $G_t \in \mathcal{G}_{\vartheta_0+\epsilon}$. At the same time,

$$\forall p \in \mathbb{N} \quad g_p(x) := \int_{(\mathbb{R}^d)^{p-1}} |G_t^{(p)}(x, x_2, \dots, x_p)| dx_2 \cdots dx_p \in L^1(\mathbb{R}^d).$$

Since the sequence $\{\Lambda_n\}$ is exhausting, for M satisfying $\Delta_M^{(2)} < \varepsilon/2$, there exists n_1 such that, for $n > n_1$, the following holds

$$\Delta_{n,M}^{(1)} = \sum_{p=1}^M \frac{1}{p!} e^{(\vartheta_0+\epsilon)p} \int_{\Lambda_n^c} g_p(x) dx < \frac{\varepsilon}{2}.$$

By (3.50) this yields $\Delta_n < \varepsilon$ for all $n > n_1$, which by (3.49) completes the proof. \square

Our next aim is to prove (3.13).

Lemma 3.9. Let $\mu_0, \vartheta, \vartheta_0$ be as in Lemma 3.8. Then for each $G \in B_{\text{bs}}(\Gamma_0)$ and $t < T(\vartheta, \vartheta_0)/2$, it follows that

$$\langle\langle G, Q_{\vartheta\vartheta_0}^\sigma(t)k_{\mu_0} \rangle\rangle \rightarrow \langle\langle G, Q_{\vartheta\vartheta_0}(t)k_{\mu_0} \rangle\rangle, \quad \text{as } \sigma \rightarrow 0^+. \quad (3.51)$$

Proof. For each $\vartheta_1, \vartheta_2 \in (\vartheta_0, \vartheta)$, $\vartheta_2 > \vartheta_1$, we may write

$$\begin{aligned} [Q_{\vartheta\vartheta_0}(t) - Q_{\vartheta\vartheta_0}^\sigma(t)]k_{\mu_0} &= \int_0^t \frac{d}{ds} [Q_{\vartheta\vartheta_1}^\sigma(t-s)Q_{\vartheta_1\vartheta_0}(s)]k_{\mu_0} ds \\ &= \int_0^t Q_{\vartheta\vartheta_2}^\sigma(t-s)D_{\vartheta_2\vartheta_1}^\sigma k_s ds, \end{aligned} \quad (3.52)$$

where $k_s = Q_{\vartheta_1\vartheta_0}(s)k_{\mu_0}$ (see (3.10)), D^σ is equal to L^Δ (see (2.28)) with $b(x)$ replaced by $(1 - \psi_\sigma(x))b(x)$ and t satisfies

$$t < \min\{T(\vartheta, \vartheta_2); T(\vartheta_1, \vartheta_0)\}. \quad (3.53)$$

Now we take $G \in B_{\text{bs}}(\Gamma_0) \subset \mathcal{G}_\vartheta$ and obtain by (3.52) the following

$$\Upsilon_\sigma(t) := \langle\langle G, Q_{\vartheta\vartheta_0}(t)k_{\mu_0} \rangle\rangle - \langle\langle G, Q_{\vartheta\vartheta_0}^\sigma(t)k_{\mu_0} \rangle\rangle = \int_0^t \langle\langle G_{t-s}, D_{\vartheta_2\vartheta_1}^\sigma k_s \rangle\rangle ds, \quad (3.54)$$

where $G_{t-s} = H_{\vartheta_2\vartheta}^\sigma(t-s)G$, see (3.20). Then by (2.28), (2.29) and (2.27) we rewrite (3.54) as follows

$$\begin{aligned} \Upsilon_\sigma(t) &= \int_{\mathbb{R}^d} (1 - \psi_\sigma(x))b(x)g_t(x)dx, \\ g_t(x) &:= \int_0^t \int_{\Gamma_0} G_{t-s}(\eta \cup x)e(\tau_x; \eta)(W_x k_s)(\eta)ds\lambda(d\eta). \end{aligned} \quad (3.55)$$

Then by Lebesgue's dominated convergence theorem it follows that the proof of (3.51) will be done if we show that: (a) for each $t < T(\vartheta, \vartheta_0)/2$, it is possible to choose ϑ_2 and ϑ_1 in such a way that t also satisfies (3.53); (b) for these t, ϑ_2 and ϑ_1 , g_t defined in the second line of (3.55) is absolutely integrable in x . To get (a) we proceed as in [17], see the very end of the proof of Lemma 5.2 in that paper. Set $\vartheta_1 = (\vartheta + \vartheta_0)/2$ and $\vartheta_2 = \vartheta_1 + \varepsilon\beta(\vartheta)e^{-\vartheta}$ with $\varepsilon > 0$ satisfying $\vartheta_2 < \vartheta$. For this choice, by (3.1) we have

$$T(\vartheta_1, \vartheta_0) = \frac{\vartheta - \vartheta_0}{2\beta(\vartheta_1)}e^{\vartheta_0} \geq \frac{1}{2}T(\vartheta, \vartheta_0) > t,$$

since β is an increasing function, see (2.33). At the same time,

$$T(\vartheta, \vartheta_2) = \frac{\vartheta - \vartheta_1}{\beta(\vartheta)}e^{\vartheta_2} - \varepsilon e^{-(\vartheta - \vartheta_2)} > \frac{1}{2}T(\vartheta, \vartheta_0) - \varepsilon.$$

As t is fixed, one can take positive $\varepsilon < \frac{1}{2}T(\vartheta, \vartheta_0) - t$, that yields (a). To get (b) we first use (2.29) and then (3.8), which for $s \leq t$ and t satisfying (3.53) yields

$$|W_x(k_s)(\eta)| \leq \|k_s\|_{\vartheta_1} e^{\vartheta_1|\eta|} \exp(\langle \phi \rangle e^{\vartheta_1}) \leq c(t, k_{\mu_0}) e^{\vartheta_1|\eta|},$$

$$c(t, k_{\mu_0}) := \frac{T(\vartheta_1, \vartheta_0) \|k_{\mu_0}\|_{\vartheta_0}}{T(\vartheta_1, \vartheta_0) - t} \exp(\langle \phi \rangle e^{\vartheta}).$$

We use this estimate to get the following

$$\int_{\mathbb{R}^d} |g_t(x)| dx \leq c(t, k_{\mu_0}) \int_0^t h(s) ds, \quad (3.56)$$

where

$$\begin{aligned} h(s) &= \int_{\mathbb{R}^d} \int_{\Gamma_0} |G_s(\eta \cup x) e^{\vartheta_1|\eta|} dx \lambda(d\eta) \\ &= e^{-\vartheta_1} \int_{\Gamma_0} |\eta| e^{-(\vartheta_2 - \vartheta_1)|\eta|} |G_s(\eta)| e^{\vartheta_2|\eta|} \lambda(d\eta) \\ &\leq \frac{|G_s|_{\vartheta_2}}{e^{1+\vartheta_1}(\vartheta_2 - \vartheta_1)} \leq \frac{T(\vartheta, \vartheta_2) |G|_{\vartheta}}{(T(\vartheta, \vartheta_2) - t) e^{1+\vartheta_1}(\vartheta_2 - \vartheta_1)}, \end{aligned}$$

where we used the fact that the operator norm of $H_{\vartheta_2\vartheta}^\sigma(s)$ can be estimated according to (3.8). Now we apply the latter estimate in (3.56) to get the integrability in question, that completes the proof. \square

3.2.4. Proof of Lemma 3.3

To prove the lemma we have to show that $Q_{\vartheta_2\vartheta_1}k \in \mathcal{K}_{\vartheta_2}^*$ whenever $k \in \mathcal{K}_{\vartheta_1}^*$. By (2.28) and (2.21) we conclude that

$$\frac{d}{dt} k_t(\varnothing) = (L^\Delta k_t)(\varnothing) = 0,$$

which means that $(Q_{\vartheta_2\vartheta_1}(t)k)(\varnothing) = k(\varnothing) = 1$. Then it remains to prove that $Q_{\vartheta_2\vartheta_1}(t)k$ satisfies (2.23) with an arbitrary $G \in B_{\text{bs}}^*(\Gamma_0)$, which holds by Corollary 3.6 and then by Lemmas 3.8 and 3.9.

3.3. The proof of Theorem 2.6

By Lemma 3.3 $k_t = Q_{\vartheta\vartheta_0}(t)k_{\mu_0}$, $t < T(\vartheta, \vartheta_0)/2$ satisfies (2.21) and is the correlation function of a unique state $\mu_t \in \mathcal{P}_{\text{exp}}(\Gamma)$. Thus, to prove Theorem 2.6 we have to construct the continuation of this k_t to all $t > 0$ that satisfies the estimate stated in (i) and the equation (2.21) in the form stated in (ii). First we show that k_t with $t < T(\vartheta, \vartheta_0)/2$ satisfies the estimate in question. To this end we use the correlation function k_t^0 of the state μ_t^0 corresponding to the free immigration case described by (2.8) with $\phi = 0$. This correlation function can be found explicitly, see [13, eq. (2.21)]. In our case, it is

$$k_t^0(\eta) = \sum_{\xi \subset \eta} e(\varphi_t; \xi) k_{\mu_0}(\eta \setminus \xi), \quad \varphi_t(x) = b(x)t. \quad (3.57)$$

By direct inspection one gets that k_t^0 satisfies the equation

$$\frac{d}{dt} k_t^0(\eta) = (L^{0,\Delta} k_t^0)(\eta) = \sum_{x \in \eta} b(x) k_t^0(\eta \setminus x), \quad (3.58)$$

as well as the estimate stated in (i).

Lemma 3.10. *For all $t < T(\vartheta, \vartheta_0)/2$ and λ -almost all $\eta \in \Gamma_0$, $k_t = Q_{\vartheta\vartheta_0}(t)k_{\mu_0}$ satisfies*

$$k_t(\eta) \leq k_t^0(\eta). \quad (3.59)$$

Proof. By (3.58) one concludes that $L^{0,\Delta}$ satisfies the estimate as in (2.33) with $\beta(\vartheta)$ replaced by $\beta^0(\vartheta) = \bar{b} \leq \beta(\vartheta)$. This can be used to define bounded operators $(L^{0,\Delta})_{\vartheta_2\vartheta_1}^n$ satisfying (3.3), (3.4) and (3.5), and then $Q_{\vartheta_2\vartheta_1}^0(t)$, $t < T^0(\vartheta_2, \vartheta_1)$, defined by (3.6) with the use of these $(L^{0,\Delta})_{\vartheta_2\vartheta_1}^n$. Here

$$T^0(\vartheta_2, \vartheta_1) := \frac{\vartheta_2 - \vartheta_1}{\bar{b}} e^{-\vartheta_1}. \quad (3.60)$$

Note that $T^0(\vartheta_2, \vartheta_1) > T(\vartheta_2, \vartheta_1)$, see (3.1). Then

$$k_t^0 = Q_{\vartheta\vartheta_0}^0(t)k_{\mu_0} \in \mathcal{K}_\vartheta, \quad t < T^0(\vartheta, \vartheta_0).$$

On the other hand, by (3.57) we have that $k_t^0 \in \mathcal{K}_{\vartheta_t}$ with $\vartheta_t = \log(e^{\vartheta_0} + \bar{b}t)$. By (3.60) one can show that $t < T^0(\vartheta, \vartheta_0)$ implies $\vartheta_t < \vartheta$, which means that k_t lies in a smaller member of the scale $\{\mathcal{K}_\vartheta\}$ than \mathcal{K}_ϑ .

To prove that (3.59) holds we fix $t < T(\vartheta, \vartheta_0)/2$ and then similarly as in (3.52) write

$$k_t^0 - k_t = Q_{\vartheta\vartheta_0}^0(t)k_{\mu_0} - Q_{\vartheta\vartheta_0}(t)k_{\mu_0} = \int_0^t Q_{\vartheta\vartheta_2}^0(t-s)D_{\vartheta_2\vartheta_1}Q_{\vartheta_1\vartheta_0}(s)k_{\mu_0}ds, \quad (3.61)$$

where ϑ_2 and $\vartheta_1 < \vartheta_2$ are chosen in (ϑ_0, ϑ) in such a way that t satisfies (3.53), see the proof of Lemma 3.9. Moreover, $D = L^{0,\Delta} - L^\Delta$ and hence D is a positive operator as follows by Lemma 3.7, see also (3.58) and (2.28). By (3.58) $L^{0,\Delta}$ is positive; hence, so is also $Q_{\vartheta\vartheta_2}^0(t-s)$. At the same time, by Lemma 3.3 $Q_{\vartheta_1\vartheta_0}(s)k_{\mu_0} \in \mathcal{K}_{\vartheta_1}^*$ and thus is also positive, cf. Corollary 3.6. Then the right-hand side of (3.61) is positive, that yields (3.59). \square

Corollary 3.11. *For each $t < T(\vartheta, \vartheta_0)/2$, we have that $k_t = Q_{\vartheta\vartheta_0}(t)k_{\mu_0} \in \mathcal{K}_{\vartheta_t}$.*

Proof. The proof follows by (3.59) and the fact that $k_t^0 \in \mathcal{K}_\vartheta$ for these values of t . \square

The value of ϑ_0 in the latter statements is predetermined by the choice of the initial state μ_0 . At the same time, the choice of ϑ can be optimized to make the time interval in these statements as long as possible. By simple calculations we then get that

$$\sup_{\vartheta > \vartheta_0} T(\vartheta, \vartheta_0) = \tau(\vartheta_0) := \frac{\delta}{\bar{b}} \exp\left(\vartheta_0 - \frac{1}{\delta}\right), \quad (3.62)$$

where positive $\delta = \delta(\vartheta_0)$ is the unique solution of the equation

$$\delta e^\delta = \exp(-\vartheta_0 - \log\langle\phi\rangle). \quad (3.63)$$

Note that the latter implies

$$\tau(\vartheta) = \frac{1}{\bar{b}\langle\phi\rangle} \exp\left(-\delta(\vartheta) - \frac{1}{\delta(\vartheta)}\right). \quad (3.64)$$

Proof of Theorem 2.6. By (3.9) one readily obtains that the family $\{Q_{\vartheta_2\vartheta_1}(t) : \vartheta_1 \in \mathbb{R}, \vartheta_2 > \vartheta_1, t < T(\vartheta_2, \vartheta_1)\}$ has the property

$$Q_{\vartheta_2\vartheta_1}(t+s) = Q_{\vartheta_2\vartheta'}(t)Q_{\vartheta'\vartheta_1}(s), \quad \vartheta' \in (\vartheta_1, \vartheta_2), \quad (3.65)$$

where positive s and t ought to satisfy: $s < T(\vartheta', \vartheta_1)$, $t < T(\vartheta_2, \vartheta')$, $t+s < T(\vartheta_2, \vartheta_1)$.

Take some $\epsilon \in (0, 1/2)$ and set $s_1 = \epsilon\tau(\vartheta_0)$, see (3.62). For $t \leq s_1$, by Lemma 3.3 and Corollary 3.11 it follows that $k_t = Q_{\vartheta^0\vartheta_0}(t)k_{\mu_0}$ with $\vartheta^0 := \vartheta_0 + \delta(\vartheta_0)$ satisfies the estimate stated in claim (i). Recall that $\vartheta_t = \log(e^{\vartheta_0} + \bar{b}t)$ and thus $\vartheta^0 > \vartheta_t$ for all $t \leq s_1$. Now we fix some $T \leq s_1$ and then take a positive $\varepsilon < T$. For each $0 < t \leq T - \varepsilon$, by the mentioned estimate and (2.32) it follows that $k_t \in \mathcal{K}_{\vartheta_{T-\varepsilon}}$. By (3.9) and then by (2.34) we have that

$$\begin{aligned} \frac{d}{dt}k_t &= L_{\vartheta^0\vartheta_{T-\varepsilon}}^\Delta k_t = L_{\vartheta_{T-\varepsilon/2}\vartheta_{T-\varepsilon}}^\Delta k_t \\ &= L_{\vartheta_T\vartheta_{T-\varepsilon}}^\Delta k_t = L_{\vartheta_T}^\Delta k_t. \end{aligned} \quad (3.66)$$

To prove the continuity and continuous differentiability stated in (ii) we use again (3.65). Let positive s be such that $t+s \leq T - \varepsilon$ with ε as in (3.66). For $\vartheta > \vartheta_{s_1}$ and t as in (3.66), we have that $t < T(\vartheta, \vartheta_0)$. Let also $s < T(\vartheta^0, \vartheta)$ with this ϑ . Then

$$k_{t+s} = Q_{\vartheta^0\vartheta_0}(t+s)k_{\mu_0} = Q_{\vartheta^0\vartheta}(s)Q_{\vartheta\vartheta_0}(t)k_{\mu_0} = Q_{\vartheta^0\vartheta}(s)k_t = Q_{\vartheta_T\vartheta_{T-\varepsilon}}(s)k_t. \quad (3.67)$$

Here we have taken into account that $k_t \in \mathcal{K}_{\vartheta_{T-\varepsilon}}$ and the property mentioned in Remark 3.2. Of course, the latter equality in (3.67) makes sense for sufficiently small s . Then the continuity and continuous differentiability follow by (3.67) and (3.9). Thus, both claims (i) and (ii) hold true for $t \leq s_1$.

Set $\vartheta_1^* = \vartheta_{s_1}$ and $\vartheta^1 = \vartheta_1^* + \delta(\vartheta_1^*)$. By Corollary 3.11 it follows that $k_{s_1} \in \mathcal{K}_{\vartheta_1^*}$. For each $k \in \mathcal{K}_{\vartheta_1^*}$, we have that $Q_{\vartheta^1\vartheta_1^*}(t)k \in \mathcal{K}_{\vartheta^1}$ for $t < \tau(\vartheta_1^*)$. Keeping this in mind we then set

$$k_{s_1+t} = Q_{\vartheta^1\vartheta_1^*}(t)k_{s_1}, \quad t \in [0, \tau(\vartheta_1^*)]. \quad (3.68)$$

For t such that $s_1 + t < \tau(\vartheta_0)$, by (3.65) and (3.68) it follows that $k_{s_1+t} = Q_{\vartheta^1\vartheta_0}(s_1+t)k_{\mu_0}$. In view of the latter, we set

$$k_t = \begin{cases} Q_{\vartheta_1^*\vartheta_0}(t)k_{\mu_0}, & \text{for } t \leq s_1; \\ Q_{\vartheta^1\vartheta_1^*}(t-s_1)k_{s_1}, & \text{for } t \in [s_1, s_1 + \tau(\vartheta_1^*)]. \end{cases}$$

As above, we show that this k_t has all the properties stated in both claims (i) and (ii) for $t \leq s_1 + s_2$, with $s_2 := \epsilon\tau(\vartheta_1^*)$. Then we repeat the same procedure again and again. That is, we set $s_n = \epsilon\tau(\vartheta_{n-1}^*)$,

$$\vartheta_n^* = \vartheta_{s_1+\dots+s_n} = \log(e^{\vartheta_0} + \bar{b}(s_1 + \dots + s_n)), \quad (3.69)$$

and $\vartheta^n = \vartheta_n^* + \delta(\vartheta_n^*)$. Thereafter, for a given m and $t \in [s_1 + \dots + s_m, s_1 + \dots + s_{m+1}]$, we set

$$k_t = Q_{\vartheta^m, \vartheta_m^*}(t - (s_1 + \dots + s_m))k_{s_1+\dots+s_m},$$

and prove that this k_t has all the properties in question. To complete the proof we have to show that the intervals just mentioned cover $[0, +\infty)$, which means that

$$\sum_{n=1}^{\infty} s_n = \epsilon \sum_{n=1}^{\infty} \tau(\vartheta_n^*) = +\infty.$$

Assume that $\sum_{n=1}^{\infty} s_n = \bar{s} < \infty$. Then, for each n , ϑ_n^* satisfies $\vartheta_1^* \leq \vartheta_n^* \leq \bar{\vartheta} := \log(e^{\vartheta_0} + \bar{s})$, see (3.69). At the same time, the assumed convergence yields $\tau(\vartheta_n^*) \rightarrow 0$ as $n \rightarrow +\infty$, which by (3.64) and (3.63) implies $|\vartheta_n^*| \rightarrow +\infty$ as $n \rightarrow +\infty$, that contradicts the boundedness just mentioned. \square

3.4. The proof of Theorem 2.8

For a given compact Λ , let m_Λ be the minimal number of the balls Δ_x that cover Λ . Then there exist compact Λ_l , $l = 1, \dots, m_\Lambda$ such that:

$$(a) \sup_{x, y \in \Lambda_l} |x - y| \leq r; \quad (b) \quad \Lambda \subset \bigcup_{l=1}^{m_\Lambda} \Lambda_l, \quad (3.70)$$

and hence

$$\mu_t(N_\Lambda) \leq \sum_{l=1}^{m_\Lambda} \mu_t(N_{\Lambda_l}). \quad (3.71)$$

Then it is enough to prove (2.37) for some (hence, for each) Λ_l , for which we set $F(\gamma) = e^{\phi_* N_{\Lambda_l}(\gamma)}$, cf. (2.15). By Theorem 2.6 and Corollary 2.7 we know that the correlation function k_{μ_t} lies in $\mathcal{K}_{\vartheta_t}$. Hence, one can calculate $f(t) := \mu_t(F)$, see (2.15). Note that $f(t) = \mu_t^{\Lambda_l}(F)$, cf. (3.24). Then, cf. (3.30), by (2.8) we have

$$\begin{aligned} \frac{d}{dt} f(t) &= \int_{\Gamma_{\Lambda_l}} F(\gamma) (L^\dagger R_t^{\Lambda_l})(\gamma) \lambda(d\gamma) = \int_{\Gamma_{\Lambda_l}} (LF)(\gamma) R_t^{\Lambda_l}(\gamma) \lambda(d\gamma) \\ &= \int_{\Gamma_{\Lambda_l}} \left(\int_{\mathbb{R}^d} b(x) e(\tau_x; \gamma) [F(\gamma \cup x) - F(\gamma)] dx \right) R_t^{\Lambda_l}(\gamma) \lambda(d\gamma) \\ &= \int_{\Gamma_{\Lambda_l}} \left(\int_{\Lambda_l} b(x) (e^{\phi_*} - 1) e(\tau_x; \gamma) F(\gamma) dx \right) R_t^{\Lambda_l}(\gamma) \lambda(d\gamma) \\ &\leq (e^{\phi_*} - 1) \bar{b} \int_{\Gamma_{\Lambda_l}} \Upsilon_{\Lambda_l}(\gamma) R_t^{\Lambda_l}(\gamma) \lambda(d\gamma) \\ &\leq (e^{\phi_*} - 1) \bar{b} |\Lambda_l| \leq (e^{\phi_*} - 1) \bar{b} v. \end{aligned} \quad (3.72)$$

Here

$$\Upsilon_{\Lambda_l}(\gamma) := \int_{\Lambda_l} \exp \left(- \sum_{y \in \gamma} [\phi(x - y) - \phi_*] \right) dx \leq |\Lambda_l|,$$

since, for $\gamma \in \Lambda_l$, by item (a) in (3.70) and item (c) of Assumption 2.1 we have that $\phi(x - y) \geq \phi_*$. Also by (a) in (3.70) it follows that Λ_l is contained in a ball Δ_z with some z , that yields $|\Lambda_l| \leq v$. The latter was used in the last line of (3.72). Now by (3.72) and (2.15) we have

$$f(t) \leq f(0) + (e^{\phi_*} - 1)\bar{b}vt \leq C_{\Delta_0}^{\phi_*}(\mu_0) + (e^{\phi_*} - 1)\bar{b}vt, \quad t \geq 0,$$

where we used the fact that all Δ_z have the same volume. Now by Jensen's inequality we obtain

$$\phi_*\mu_t(N_{\Lambda_t}) \leq \log f(t),$$

that by (3.71) yields (2.37).

4. The mesoscopic evolution

In this section, we prove Theorems 2.9 and 2.10.

4.1. The proof of Theorem 2.9

It is convenient to pass in (2.38) to a new unknown $u_t \in L^\infty(\mathbb{R}^d)$ defined by

$$u_t(x) = \langle \phi \rangle [\varrho_t(x) - \varrho_0(x)]. \quad (4.1)$$

Then ϱ_t solves (2.38) if and only if u_t solves

$$\frac{d}{dt}u_t(x) = \hat{b}(x) \exp\left(-(\hat{\phi} * u_t)(x)\right), \quad u_t|_{t=0} = 0, \quad (4.2)$$

that can also be rewritten in the form

$$u_t(x) = \hat{b}(x) \int_0^t \exp\left(-(\hat{\phi} * u_s)(x)\right) ds. \quad (4.3)$$

Here

$$\hat{b}(x) = \langle \phi \rangle b(x) e^{-(\phi * \varrho_0)(x)}, \quad \hat{\phi}(x) = \phi(x) / \langle \phi \rangle. \quad (4.4)$$

For a given $T > 0$, set

$$\mathcal{U}_T = C([0, T] \rightarrow L^\infty(\mathbb{R}^d))$$

$$\mathcal{U}_T^+ = \{u \in \mathcal{U}_T : u_t(x) \geq 0, \text{ for all } t \in [0, T] \text{ and a.a. } x \in \mathbb{R}^d\},$$

and equip \mathcal{U}_T with the norm

$$\|u\|_T = \sup_{t \in [0, T]} \|u_t\|_{L^\infty(\mathbb{R}^d)}.$$

Then we define $V : \mathcal{U}_T \rightarrow \mathcal{U}_T$ be setting

$$(V(u))_t(x) = \hat{b}(x) \int_0^t \exp\left(-(\hat{\phi} * u_s)(x)\right) ds, \quad (4.5)$$

and rewrite the problem in (4.3) on the time interval $[0, T]$ in the form

$$u = V(u). \quad (4.6)$$

Clearly,

$$V : \mathcal{U}_T^+ \rightarrow \mathcal{V}_T^+ = \{u \in \mathcal{U}_T^+ : u_0 = 0, \|u\|_T \leq b^+ T\},$$

where b^+ is the same as in (2.40). By the inequality $|e^{-\alpha} - e^{-\alpha'}| \leq |\alpha - \alpha'|$ that holds for all $\alpha, \alpha' \geq 0$, one shows that V is a contraction whenever

$$b^+ T < 1, \quad (4.7)$$

considered as a condition on T . In this case, the problem in (4.2) has a unique solution on $[0, T]$ which is the fixed point $u \in \mathcal{V}_T^+$ of V . For each $t \in [0, T]$, as a positive element of $L^\infty(\mathbb{R}^d)$ u_t satisfies $\omega_-(t) \leq u_t(x) \leq \omega_+(t)$ holding almost everywhere on \mathbb{R}^d . Here ω_\pm are to be continuously differentiable functions such that $\omega_\pm(0) = 0$. To find them we write

$$\frac{d}{dt}\omega_-(t) \leq \frac{d}{dt}u_t(x) \leq \frac{d}{dt}\omega_+(t), \quad (4.8)$$

which together with the zero initial condition will yield the bounds in question. On the other hand, for these bounds by (4.2) we have, see (4.4) and (2.40),

$$b^- e^{-\omega_+(t)} \leq \frac{d}{dt}u_t(x) \leq b^+ e^{-\omega_-(t)}. \quad (4.9)$$

Now we combine (4.8) with (4.9) and obtain that ω_\pm ought to satisfy

$$\frac{d}{dt}\omega_-(t) = b^- e^{-\omega_+(t)}, \quad \frac{d}{dt}\omega_+(t) = b^+ e^{-\omega_-(t)}. \quad (4.10)$$

For $b_+ = b_- =: b$, the solution of this system, and thereby of (4.2), is

$$\omega_+(t) = \omega_-(t) = u_t(x) = \log(1 + bt).$$

For $b_+ > b_-$, by (4.10) we get

$$\frac{d^2}{dt^2}\omega_-(t) = \frac{d^2}{dt^2}\omega_+(t),$$

which yields

$$\left(\frac{d}{dt}\omega_+(t) - \frac{d}{dt}\omega_-(t)\right) = \left(\frac{d}{dt}\omega_+(t) - \frac{d}{dt}\omega_-(t)\right)|_{t=0} = b^+ - b^-.$$

In view of the zero initial condition, the latter yields in turn

$$\omega_+(t) = \omega_-(t) + (b^+ - b^-)t.$$

We plug this in the first equation in (4.10) that turns it into an equation for ω_- , the solution of which is clearly given in the second line of (2.41). Thereafter, ω_+ is obtained from the formula above. Thus, with the help of (4.1) we have proved the existence of the unique solution of the kinetic equation in (2.38) – satisfying the bounds stated in Theorem 2.9 – on the time interval $[0, T]$. Our aim now is to continue it to all $t > 0$. To this end, for $t > 0$ we rewrite (4.3)

$$\begin{aligned}
u_{T+t}(x) &= \hat{b}(x) \int_0^T \exp\left(-(\hat{\phi} * u_s)(x)\right) ds + \hat{b}(x) \int_0^t \exp\left(-(\hat{\phi} * u_{T+s})(x)\right) ds \\
&= u_T(x) + \hat{b}(x) \int_0^t \exp\left(-(\hat{\phi} * u_{T+s})(x)\right) ds,
\end{aligned} \tag{4.11}$$

where we have taken into account that $u_T(x) = (V(u))_T(x)$, as follows from the consideration above. Now we introduce

$$u_t^{(1)}(x) = u_{T+t}(x) - u_T(x),$$

and rewrite (4.11) in the form, cf. (4.5), (4.6),

$$\begin{aligned}
u^{(1)} &= V^{(1)}(u^{(1)}), \\
V^{(1)}(u^{(1)})_t(x) &:= \hat{b}^{(1)}(x) \int_0^t \exp\left(-(\hat{\phi} * u_s^{(1)})(x)\right) ds, \\
\hat{b}^{(1)}(x) &:= \hat{b}(x) \exp\left(-(\hat{\phi} * u_T)(x)\right).
\end{aligned}$$

Similarly as above, we establish that $V^{(1)} : \mathcal{U}_T^+ \rightarrow \mathcal{V}_T^+$ and is a contraction on \mathcal{V}_T^+ with the same T as in (4.7). This yields the existence of its unique fixed point $u^{(1)} \in \mathcal{V}_T^+$. Then the continuation on the time interval $[T, 2T]$ is

$$u_{T+t}(x) = u_T(x) + u_t^{(1)}(x).$$

Since u_t on both intervals $[0, T]$ and $[T, 2T]$ solves the same differential equation, it satisfies the same bounds found from this equation, i.e.,

$$\omega_-(t) \leq u_t(x) \leq \omega_+(t), \quad t \in [0, 2T], \tag{4.12}$$

where ω_{\pm} are as in (2.41). The continuations of u_t beyond $[0, 2T]$ are constructed by repeating the same procedure. The bounds in (2.42) are readily obtained from (4.12) by (4.1). This completes the proof of the theorem.

4.2. The proof of Theorem 2.10

In the proof of this theorem we mostly follow the line of arguments used in the proof of Theorem 3.9 in [3].

4.2.1. The rescaled evolution

Here we construct the evolution $q_{0,\varepsilon} \rightarrow q_{t,\varepsilon}$ mentioned in the theorem. For a conceptual background of this approach, see the corresponding parts of [3,7] and the references therein.

Let L_{ε}^{Δ} be the operator defined in (2.28) in which ϕ is multiplied by $\varepsilon \in (0, 1]$. Next, define the rescaling operator R_{ε} that acts as $(R_{\varepsilon}k)(\eta) = \varepsilon^{-\eta}k(\eta)$. In general, for a correlation function, k_{μ} , $R_{\varepsilon}k_{\mu}$ need not be the correlation function of any state. At the same time, $R_{\varepsilon}^{-1}k_{\mu}$ is the correlation function of the ‘thinning’ μ_{ε} of μ defined by its Bogoliubov functional (2.10), (2.11)

$$B_{\mu_\varepsilon}(\theta) = B_\mu(\varepsilon\theta) = \int_{\Gamma} \prod_{x \in \gamma} (1 + \varepsilon\theta(x)) \mu(d\gamma),$$

as the map $\theta \mapsto \varepsilon\theta$ preserves Θ . Then we set

$$L^{\varepsilon, \Delta} = R_\varepsilon^{-1} L_\varepsilon^\Delta R_\varepsilon. \quad (4.13)$$

Next, denote, cf. (2.28),

$$\tau_x^\varepsilon(y) = \exp(-\varepsilon\phi(x-y)), \quad t_x^\varepsilon(y) = \frac{1}{\varepsilon}(\tau_x^\varepsilon(y) - 1). \quad (4.14)$$

Note that

$$\tau_x^\varepsilon(y) \rightarrow 1, \quad t_x^\varepsilon(y) \rightarrow t_x^0(y) := -\phi(x-y), \quad \text{as } \varepsilon \rightarrow 0^+, \quad (4.15)$$

and also, cf. (1.3)

$$\frac{1}{\varepsilon} \int_{\mathbb{R}^d} (1 - e^{-\varepsilon\phi(x)}) dx \leq \langle \phi \rangle. \quad (4.16)$$

Now let W_x^ε be defined as in (2.28) with t_x replaced by t_x^ε . In view of (4.16), $(W_x^\varepsilon k)(\eta)$ satisfies (2.29) with the same (hence ε -independent) right-hand side. This means that $L^{\varepsilon, \Delta} k$ satisfies (2.30), which allows one to define the unbounded operators $L_{\vartheta}^{\varepsilon, \Delta} = (L^{\varepsilon, \Delta}, \mathcal{D}_{\vartheta})$, $\vartheta \in \mathbb{R}$ and the bounded operators $L_{\vartheta\vartheta_0}^{\varepsilon, \Delta}$, $\vartheta > \vartheta_0$ exactly as in the case of $\varepsilon = 1$, see subsection 2.4. Therefore, one can construct the family $\{Q_{\vartheta\vartheta_0}^\varepsilon(t) : \vartheta_0 \in \mathbb{R}, \vartheta > \vartheta_0, t \in [0, T(\vartheta, \vartheta_0))\}$ with $T(\vartheta, \vartheta_0)$ as in (3.1), see also (3.6). Then, for $q_{0,\varepsilon} \in \mathcal{K}_{\vartheta_0}$, we set

$$q_{t,\varepsilon} = Q_{\vartheta\vartheta_0}^\varepsilon(t)q_{0,\varepsilon}, \quad t < T(\vartheta, \vartheta_0). \quad (4.17)$$

This $q_{t,\varepsilon}$ satisfies, cf. (2.35),

$$\frac{d}{dt} q_{t,\varepsilon} = L_{\vartheta}^{\varepsilon, \Delta} q_{t,\varepsilon}. \quad (4.18)$$

By (4.13), (4.14) and (4.17) we have that

$$q_{t,1} = k_t = Q_{\vartheta\vartheta_0}(t)k_{\mu_0}, \quad (4.19)$$

where k_t is as in Theorem 2.6.

4.2.2. The Vlasov evolution

Let W_x^0 be defined as in (2.28) with t_x replaced by t_x^0 , see (4.15). Then $W_x^0 k$ also satisfies (2.29). Then the Vlasov operator is defined as, cf. (4.15),

$$(L^{0, \Delta} k)(\eta) = \sum_{x \in \eta} b(x)(W_x^0 k)(\eta \setminus x). \quad (4.20)$$

Analogously as above, we define the family of operators $\{Q_{\vartheta\vartheta_0}^0(t) : \vartheta_0 \in \mathbb{R}, \vartheta > \vartheta_0, t \in [0, T(\vartheta, \vartheta_0))\}$ with $T(\vartheta, \vartheta_0)$ as in (3.1). Then, for $q_{0,0} \in \mathcal{K}_{\vartheta_0}$, $q_{0,t} = Q_{\varepsilon\vartheta_0}^0(t)q_{0,0} \in \mathcal{K}_{\vartheta}$ is the unique solution of the problem

$$\frac{d}{dt} q_{t,0} = L_{\vartheta}^{0, \Delta} q_{t,0}, \quad q_{t,0}|_{t=0} = q_{0,0}, \quad (4.21)$$

on the time interval $[0, T(\vartheta, \vartheta_0))$.

Lemma 4.1. Let $q_{0,0}$ in (4.21) be the correlation function of the Poisson state π_{ϱ_0} with $k_{\pi_{\varrho_0}} \in \mathcal{K}_{\vartheta_0}$. Then $q_{t,0} = Q_{\vartheta\vartheta_0}^0(t)q_{0,0}$ can be continued to all $t > 0$, and this continuation is the correlation function of the Poisson state π_{ϱ_t} with ϱ_t that solves the kinetic equation (2.38), the existence and properties of which were established in Theorem 2.9.

Proof. In fact, to prove this statement we have to check whether $k_{\pi_{\varrho_t}} = e(\varrho_t; \cdot)$ satisfies the first equality in (4.21). By (4.15) and (2.18) we have that

$$W_x e(\varrho_t; \eta) = \int_{\Gamma_0} e(t_x^0; \xi) e(\varrho_t; \eta \cup \xi) \lambda(d\xi) = e(\varrho_t; \eta) \exp(-(\phi * \varrho_t)(x)).$$

We plug this in (4.20) and then get

$$\begin{aligned} L^{0,\Delta} e(\varrho_t; \eta) &= \sum_{x \in \eta} b(x) \exp(-(\phi * \varrho_t)(x)) e(\varrho_t; \eta \setminus x) \\ &= \sum_{x \in \eta} \frac{d}{dt} \varrho_t(x) e(\varrho_t; \eta \setminus x), \end{aligned}$$

see (2.38), which completes the proof. \square

4.2.3. The proof of Theorem 2.10

As we assume that μ_0 is Poisson approximable, see Definition 2.5, there exist $q_{0,\varepsilon} \in \mathcal{K}_{\vartheta_0}$, $\varepsilon \in [0, 1]$ such that: (a) $q_{0,1} = k_{\mu_0}$; (b) $q_{0,0} = k_{\pi_{\varrho_0}} = e(\varrho_0; \cdot)$; (c) $\|q_{0,\varepsilon} - e(\varrho_0; \cdot)\|_{\vartheta_0} \rightarrow 0$ as $\varepsilon \rightarrow 0^+$. Let $q_{t,\varepsilon}$ be as in (4.17), (4.18) with this $q_{0,\varepsilon}$. By (4.19) we have that $q_{t,1} = k_t$; hence, it remains to prove that there exist $T > 0$ and $\vartheta > \vartheta_0$ such that the convergence stated in (2.43) does hold. For ϑ_0 as just discussed, let $\delta(\vartheta_0)$ and $\tau(\vartheta_0)$ be as in (3.63) and (3.64), respectively. Then we set $\vartheta = \vartheta_0 + \delta(\vartheta_0)$, $T = \tau(\vartheta_0)/2$ and then write

$$Q_{\vartheta\vartheta_0}^\varepsilon(t)q_{0,\varepsilon} - Q_{\vartheta\vartheta_0}^0(t)e(\varrho_0; \cdot) = J_\varepsilon^1(t) + J_\varepsilon^2(t), \quad t \leq T, \quad (4.22)$$

$$J_\varepsilon^1(t) = [Q_{\vartheta\vartheta_0}^\varepsilon(t) - Q_{\vartheta\vartheta_0}^0(t)] e(\varrho_0; \cdot),$$

$$J_\varepsilon^2(t) = Q_{\vartheta\vartheta_0}^\varepsilon(t) (q_{0,\varepsilon} - e(\varrho_0; \cdot)).$$

Similarly as in (3.52) we write

$$J_\varepsilon^1(t) = \int_0^t Q_{\vartheta\vartheta_2}^\varepsilon(t-s) D_{\vartheta_2\vartheta_1}^\varepsilon e(\varrho_s; \cdot) ds \quad (4.23)$$

with

$$D^\varepsilon = L^{\varepsilon,\Delta} - L^{0,\Delta}. \quad (4.24)$$

In the right-hand side of (4.23), we have taken into account that $Q_{\vartheta_1\vartheta_0}^0(s)e(\varrho_0; \cdot) = e(\varrho_s; \cdot)$ with $e(\varrho_s; \cdot) \in \mathcal{K}_{\vartheta_1}$. The numbers ϑ_1 and ϑ_2 will be chosen later. By (2.42) and (2.41), for $\varrho_0 \leq e^{\vartheta_0}$, one can show that $\varrho_t(x) \leq e^{\vartheta_0} + \bar{b}t$, holding for all $t > 0$ and both cases $b^+ > b^-$ and $b^+ = b^-$. Then in (4.23) we have that $e(\varrho_s; \cdot) \in \mathcal{K}_{\vartheta_T}$, that holds for all $s \leq t \leq T$. Thus, we set $\vartheta_1 = \vartheta_T$. In view of the continuity of $T(\vartheta', \vartheta)$ in both arguments, see (3.1), for each fixed $t \leq T$, one can pick $\vartheta_2 \in (\vartheta_T, \vartheta)$ in such a way that $t-s < T(\vartheta, \vartheta_2)$, holding for all $s \in [0, t]$, whenever the following is satisfied

$$T < T(\vartheta, \vartheta_T), \quad (4.25)$$

for the choice $T = \tau(\vartheta_0)/2$ made above. In Appendix below we prove that (4.25) does hold. Then we fix $t \leq T$, pick ϑ_2 as mentioned above, and then use the estimate

$$\|Q_{\vartheta\vartheta_2}^\varepsilon(t-s)\| \leq \frac{T(\vartheta, \vartheta_2)}{T(\vartheta, \vartheta_2) - t}$$

where we employed (3.8) as $L^{\varepsilon, \Delta}$ satisfies (3.3) with the same right-hand side. Now we take into account that $\|e(\varrho_s; \dots)\|_{\vartheta_T} \leq 1$ and apply this and the latter estimate in (4.23). This yields

$$\|J_\varepsilon^1(t)\|_\vartheta \leq \frac{tT(\vartheta, \vartheta_2)}{T(\vartheta, \vartheta_2) - t} \|D_{\vartheta_2\vartheta_T}^\varepsilon\|. \quad (4.26)$$

In view of (4.24), to estimate $\|D_{\vartheta_2\vartheta_T}^\varepsilon\|$ we have to consider $W_x^\varepsilon - W_0^\varepsilon$. By means of the standard inequality

$$\left| \prod_{x \in \xi} a_x - \prod_{x \in \xi} b_x \right| \leq \sum_{x \in \xi} |a_x - b_x| \prod_{y \in \xi \setminus x} \max\{|a_x|, |b_x x|\}, \quad a_x, b_x \in \mathbb{R},$$

and by (2.25) for $k \in \mathcal{K}_{\vartheta_T}$ we get, see (2.28) and (4.14) and (4.15),

$$\begin{aligned} |[(W_x^\varepsilon - W_x^0)k](\eta)| &\leq \|k\|_{\vartheta_T} e^{\vartheta_T |\eta|} \int_{\Gamma_0} \left| \prod_{y \in \xi} t_x^\varepsilon(y) - \prod_{y \in \xi} t_x^0(y) \right| \exp(\vartheta_T |\xi|) \lambda(d\xi) \\ &\leq \varepsilon \bar{\phi} \|k\|_{\vartheta_T} e^{\vartheta_T |\eta|} \int_{\Gamma_0} |\xi| e(\phi(x - \cdot); \xi) \exp(\vartheta_T |\xi|) \lambda(d\xi) \\ &= \varepsilon \bar{\phi} \langle \phi \rangle \|k\|_{\vartheta_T} e^{\vartheta_T |\eta| + \vartheta_T} \exp(e^{\vartheta_T} \langle \phi \rangle). \end{aligned}$$

By means of this estimate we finally get

$$\|D_{\vartheta_2\vartheta_T}^\varepsilon\| \leq \varepsilon \frac{\bar{\phi} \langle \phi \rangle \bar{b}}{e(\vartheta_2 - \vartheta_T)} \exp(e^{\vartheta_T} \langle \phi \rangle). \quad (4.27)$$

Now we recall that $\vartheta = \vartheta_0 + \delta(\vartheta_0)$ and then $T(\vartheta, \vartheta_0) = \tau(\vartheta_0)$. At the same time, $T = \tau(\vartheta_0)/2$, which by (3.8) yields

$$\|Q_{\vartheta\vartheta_0}^\varepsilon(t)\| \leq \|Q_{\vartheta\vartheta_0}^\varepsilon(T)\| \leq 2.$$

We apply this estimate in the last line of (4.22) and then obtain

$$\|J_\varepsilon^2(t)\| \leq 2 \|q_{0,\varepsilon} - e(\varrho_0; \cdot)\|. \quad (4.28)$$

Then the proof of the convergence in (2.43) follows by (4.22), (4.26), (4.27) and (4.28).

Acknowledgment

The author is grateful to M. Röckner for valuable comments. The research presented in this article was financially supported by National Science Centre, Poland, grant 2017/25/B/ST1/00051, that is cordially acknowledged by the author.

Appendix

The proof of (2.28)

For $\gamma \in \Gamma$, let $\eta \Subset \gamma$ mean that $\eta \subset \gamma$ and $\eta \in \Gamma_0$. Then, see [14, eq. (4.18)], a generalization of (2.19) that relates a state $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$ to its correlation function k_μ reads

$$\int_{\Gamma} \left(\sum_{\eta \Subset \gamma} G(\eta) \right) \mu(d\gamma) = \int_{\Gamma_0} G(\eta) k_\mu(\eta) \lambda(d\eta),$$

holding for all appropriate $G : \Gamma_0 \rightarrow \mathbb{R}$. Then by (2.4) and (2.8), for a given $\mu \in \mathcal{P}_{\text{exp}}(\Gamma)$ and $\theta \in \Theta$, we have

$$\begin{aligned} \text{LHS(2.20)} &= \int_{\Gamma} \int_{\mathbb{R}^d} b(x) \theta(x) \prod_{y \in \gamma} [1 + t_x(y) + \tau_x(y) \theta(y)] dx \mu(d\gamma) \\ &= \int_{\mathbb{R}^d} b(x) \theta(x) \left(\int_{\Gamma} \left(\sum_{\eta \Subset \gamma} e(\hat{\theta}_x; \eta) \right) \mu(d\gamma) \right) dx \\ &= \int_{\mathbb{R}^d} b(x) \theta(x) \left(\int_{\Gamma_0} e(\hat{\theta}_x; \eta) k_\mu(\eta) \lambda(d\eta) \right) dx \\ &= \int_{\Gamma_0} \int_{\mathbb{R}^d} b(x) \theta(x) k_\mu(\eta) \prod_{y \in \eta} [t_x(y) + \tau_x(y) \theta(y)] dx \lambda(d\eta) \\ &= \int_{\mathbb{R}^d} b(x) \left(\int_{\Gamma_0} k_\mu(\eta) \left(\sum_{\xi \subset \eta} e(t_x; \xi) e(\tau_x; \eta \setminus \xi) e(\theta; \eta \cup x \setminus \xi) \right) \lambda(d\eta) \right) \\ &= \int_{\Gamma_0} \int_{\Gamma_0} \int_{\mathbb{R}^d} b(x) k_\mu(\eta \cup \xi) e(t_x; \xi) e(\tau_x; \eta) e(\theta; \eta \cup x) dx \lambda(d\eta) \lambda(d\xi) \\ &= \int_{\Gamma_0} \left(\sum_{x \in \eta} b(x) e(\tau_x; \eta \setminus x) \right. \\ &\quad \times \left. \left(\int_{\Gamma_0} e(t_x; \xi) k_\mu(\eta \setminus x \cup \xi) \lambda(d\xi) \right) \right) e(\theta; \eta) \lambda(d\eta) \\ &= \int_{\Gamma_0} (L^\Delta k_\mu)(\eta) e(\theta; \eta) \lambda(d\eta), \end{aligned} \tag{4.29}$$

where L^Δ in the last line of (4.29) is given in (2.28). Then the proof follows by (2.20).

The proof of (3.22)

For $F = KG$, we have that

$$F(\eta \cup x) = \sum_{\xi \subset \eta} G(\xi) + \sum_{\xi \subset \eta} G(\xi \cup x) = F(\eta) + \sum_{\xi \subset \eta} G(\xi \cup x).$$

Then

$$\text{LHS(3.22)}(\eta) = \int_{\mathbb{R}^d} b^\sigma(x) e(\tau_x; \eta) \left(\sum_{\xi \subset \eta} G(\xi \cup x) \right) dx. \quad (4.30)$$

By (3.21), we also have

$$\text{RHS(3.22)}(\eta) = \int_{\mathbb{R}^d} b^\sigma(x) U_x(\eta) dx, \quad (4.31)$$

where

$$\begin{aligned} U_x(\eta) &= \sum_{\zeta \subset \eta} \sum_{\xi \subset \zeta} e(t_x; \xi) e(\tau_x; \zeta \setminus \xi) G(\zeta \setminus \xi \cup x) \\ &= \sum_{\xi \subset \eta} e(t_x; \xi) \sum_{\zeta \subset \eta \setminus \xi} e(\tau_x; \zeta) G(\zeta \cup x) \\ &= \sum_{\zeta \subset \eta} e(\tau_x; \zeta) G(\zeta \cup x) \sum_{\xi \subset \eta \setminus \zeta} e(t_x; \xi) \\ &= \sum_{\zeta \subset \eta} e(\tau_x; \zeta) e(\tau_x; \eta \setminus \zeta) G(\zeta \cup x) = e(\tau_x; \eta) \sum_{\zeta \subset \eta} G(\zeta \cup x). \end{aligned} \quad (4.32)$$

In the latter line, we used the following, see the last line in (2.28),

$$\sum_{\xi \subset \eta} e(t_x; \xi) = e(1 + t_x; \eta) = e(\tau_x; \eta).$$

Now we use (4.32) in (4.31) and then by (4.30) conclude that (3.22) holds.

The proof of (3.31)

By (2.8) we have

$$\begin{aligned} \text{LHS(3.30)} &= - \int_{\Gamma_0} \Psi_\sigma(\eta) F(\eta) R(\eta) \lambda(d\eta) \\ &\quad + \int_{\Gamma_0} \left(\int_{\mathbb{R}^d} b^\sigma(x) e(\tau_x; \eta) F(\eta \cup x) dx \right) R(\eta) \lambda(d\eta). \end{aligned} \quad (4.33)$$

The second line of (4.33) by (2.27) can be transformed to

$$\int_{\Gamma_0} \left(\sum_{x \in \eta} b^\sigma(x) e(\tau_x; \eta) R(\eta \setminus x) \right) F(\eta) \lambda(d\eta) = \int_{\Gamma_0} F(\eta) (BR)(\eta) \lambda(d\eta).$$

Then by means of (3.30) we conclude that L^\dagger acts as described in (3.31).

The proof of (4.25)

By (3.62) we have that

$$T = \tau(\vartheta_0)/2 = \frac{\delta}{2b} \exp\left(\vartheta_0 - \frac{1}{\delta}\right), \quad (4.34)$$

with $\delta > 0$ satisfying (3.63). Then

$$e^{\vartheta_T} = e^{\vartheta_0} + \bar{b}T = e^{\vartheta_0} \left(1 + \frac{\delta}{2} \exp\left(-\frac{1}{\delta}\right)\right) =: e^{\vartheta_0} v(\delta). \quad (4.35)$$

On the other hand, by (3.1) and then by (4.35) and (4.34) we have

$$\begin{aligned} T(\vartheta, \vartheta_T) &= \frac{\vartheta_0 + \delta - \vartheta_T}{\bar{b}} \exp\left(\vartheta_T - \frac{1}{\delta}\right) \\ &= \frac{\delta - \log v(\delta)}{\bar{b}} \exp\left(\vartheta_0 - \frac{1}{\delta}\right) v(\delta) = 2T \left(1 - \frac{1}{\delta} \log v(\delta)\right) v(\delta). \end{aligned}$$

Then (4.25) turns into the following

$$2 \left(1 - \frac{1}{\delta} \log v(\delta)\right) v(\delta) > 1,$$

which is obviously the case as $1 < v(\delta) < \exp(\delta/2)$ for each $\delta > 0$.

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