J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The Bochner-Schoenberg-Eberlein property for vector-valued Lipschitz algebras

Fatemeh Abtahi [a,*], Zeinab Kamali [b,c], Maryam Toutounchi [a]

[a] *Department of Mathematics, University of Isfahan, Isfahan, Iran*
[b] *Department of Mathematics, Isfahan (Khorasgan) Branch, Islamic Azad University, Isfahan, Iran*
[c] *School of Mathematics Institute for Research in Fundamental Sciences (IPM), Niavaran Building, Niavaran Square, Tehran, Iran*

ARTICLE INFO

Article history:
Received 30 November 2018
Available online xxxx
Submitted by L. Molnar

Keywords:
BSE-algebra
Lipschitz algebra
Metric space

ABSTRACT

Let (K, d) be a compact metric space, $0 < \alpha \leq 1$ and $\mathrm{Lip}_\alpha K$ the space of the Lipschitz functions on K. It is known that the Banach algebra $\mathrm{Lip}_\alpha K$ is a BSE-algebra. In this paper, for a commutative unital semisimple Banach algebra $\mathcal{A}$, we prove that the Banach algebra $\mathrm{Lip}_\alpha(K, \mathcal{A})$ of the $\mathcal{A}$-valued Lipschitz functions is a BSE-algebra if and only if $\mathcal{A}$ is so.

1. Introduction and preliminaries

The notion of BSE-algebras and BSE functions were first introduced and investigated by Takahasi and Hatori in 1990 [21] and subsequently by several authors for various kinds of Banach algebras, such as Fourier and Fourier-Stieltjes algebras [15], semigroup algebras [11], [12], [13], abstract Segal algebras [9] and Lau product algebras [16]. The interested reader is in addition referred to [3], [7], [9], [10] and [22]. The BSE-property also appeared in [14] and [23].

The acronym “BSE” stands for Bochner-Schoenberg-Eberlein and refers to a famous theorem, proved by Bochner and Schoenberg [2,19] for the additive group of real numbers. It was generalized by Eberlein [4] for a locally compact abelian group G, indicating the BSE-property of the group algebra $L^1(G)$; see [18] for a proof. In fact, this theorem characterizes the Fourier-Stieltjes transforms of the bounded Borel measures on locally compact abelian groups. This has led Takahasi and Hatori [21] to introduce the BSE-property for an arbitrary commutative and without order Banach algebra $\mathcal{A}$, as follows.

* Corresponding author.
E-mail addresses: f.abtahi@sci.ui.ac.ir, abtahif2002@yahoo.com (F. Abtahi), zekamath@yahoo.com (Z. Kamali), m.toutounchi@sci.ui.ac.ir (M. Toutounchi).

https://doi.org/10.1016/j.jmaa.2019.06.073
0022-247X/

Let $\Delta(\mathcal{A})$ be the character space of $\mathcal{A}$, the space consisting of all nonzero multiplicative linear functionals on $\mathcal{A}$. A bounded continuous function σ on $\Delta(\mathcal{A})$ is called a BSE-function if there exists a constant $C > 0$ such that for every finite number of complex numbers $c_1, ..., c_n$ and the same number of $\varphi_1, ..., \varphi_n$ in $\Delta(\mathcal{A})$, the inequality

$$\left|\sum_{j=1}^{n} c_j \sigma(\varphi_j)\right| \leq C \left\|\sum_{j=1}^{n} c_j \varphi_j\right\|_{\mathcal{A}^*} \tag{1.1}$$

holds. The BSE-norm of σ is denoted by $\|\sigma\|_{BSE}$ and is defined to be the infimum of all such C. The set of all BSE-functions is denoted by $C_{BSE}(\Delta(\mathcal{A}))$. It was shown in [21] that $C_{BSE}(\Delta(\mathcal{A}))$ is a commutative and semisimple Banach algebra, under the norm $\|\cdot\|_{BSE}$. Moreover $C_{BSE}(\Delta(\mathcal{A}))$ is embedded in $C_b(\Delta(\mathcal{A}))$, as a subalgebra. Note that for any $\sigma \in C_{BSE}(\Delta(\mathcal{A}))$, we have

$$\|\sigma\|_{BSE} = \sup\left\{\left|\sum_{i=1}^{n} c_i \sigma(\varphi_i)\right| : \left\|\sum_{i=1}^{n} c_i \varphi_i\right\|_{\mathcal{A}^*} \leq 1\right\}.$$

Here we provide some preliminaries, which we require throughout the paper. A commutative Banach algebra $\mathcal{A}$ is called without order if $a\mathcal{A} = \{0\}$ implies $a = 0$ $(a \in \mathcal{A})$. Following [17], a bounded linear operator T on a commutative and without order Banach algebra $\mathcal{A}$ is called a *multiplier* if it satisfies $T(ab) = aT(b)$, for all $a, b \in \mathcal{A}$. The set of all multipliers on $\mathcal{A}$ will be denoted by $M(\mathcal{A})$, which is a unital commutative Banach algebra, called the *multiplier algebra* of $\mathcal{A}$. By [17, Theorem 1.2.2], for any $T \in M(\mathcal{A})$ there exists a unique bounded continuous function $\widehat{T}$ on $\Delta(\mathcal{A})$ such that

$$\widehat{T(a)}(\varphi) = \widehat{T}(\varphi)\hat{a}(\varphi),$$

for all $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$. Let

$$\widehat{M(\mathcal{A})} = \{\widehat{T} : T \in M(\mathcal{A})\}.$$

A commutative and without order Banach algebra $\mathcal{A}$ is called a *BSE-algebra* (or has the *BSE-property*) if it satisfies the condition

$$C_{BSE}(\Delta(\mathcal{A})) = \widehat{M(\mathcal{A})}.$$

Let $\mathcal{A}$ be a commutative Banach algebra. Consider the Gelfand mapping

$$\mathcal{A} \to C_b(\Delta(\mathcal{A})) \qquad a \mapsto \widehat{a},$$

where $\widehat{a}$ is the Gelfand transform of a, defined as $\widehat{a}(\varphi) = \varphi(a)$ $(\varphi \in \Delta(\mathcal{A}))$. The commutative Banach algebra $\mathcal{A}$ is called semisimple if its Gelfand mapping is injective, or equivalently

$$\bigcap_{\varphi \in \Delta(\mathcal{A})} ker(\varphi) = \{0\}.$$

Note that every semisimple commutative Banach algebra is without order. Now let $\Phi : \Delta(\mathcal{A}) \to \mathbb{C}$ be a continuous function such that $\Phi \cdot \widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}$. We call Φ a multiplier of $\mathcal{A}$. In fact, it is another definition of a multiplier of a Banach algebra. Let

$$\mathcal{M}(\mathcal{A}) = \{\Phi : \Delta(\mathcal{A}) \to \mathbb{C} : \Phi \text{ is continuous and } \Phi \cdot \widehat{\mathcal{A}} \subseteq \widehat{\mathcal{A}}\}.$$

Please cite this article in press as: F. Abtahi et al., The Bochner-Schoenberg-Eberlein property for vector-valued Lipschitz algebras, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.06.073

In the presence of semisiplicity, this definition of multiplier is equivalent to the above definition, by considering $\Phi = \widehat{T}$. In fact, $\widehat{M(\mathcal{A})} = \mathcal{M}(\mathcal{A})$; see [17, p. 30] for more details. Thus a semisimple commutative Banach algebra $\mathcal{A}$ has the BSE-property if and only if

$$C_{BSE}(\Delta(\mathcal{A})) = \mathcal{M}(\mathcal{A}).$$

Let $(e_\alpha)_{\alpha\in I}$ be a bounded net in a commutative Banach algebra $\mathcal{A}$, satisfying the condition

$$\lim_\alpha \varphi(ae_\alpha) = \varphi(a),$$

for all $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$. Then $(e_\alpha)_{\alpha\in I}$ is called a bounded Δ-weak approximate identity for $\mathcal{A}$, in the sense of Jones-Lahr; see [8]. In [21, Corollary 5], the authors proved that $\mathcal{A}$ has a bounded Δ-weak approximate identity if and only if

$$\widehat{M(\mathcal{A})} \subseteq C_{BSE}(\Delta(\mathcal{A})).$$

It follows that all BSE-algebras possess a bounded Δ-weak approximate identity.

Let (X, d) be a metric space, $\mathcal{A}$ be a Banach algebra and $\alpha > 0$. Then a map $f : X \to \mathcal{A}$ is called bounded if

$$\|f\|_\infty = \sup_{x\in X} \|f(x)\| < \infty.$$

Recall from [5] and [6] that the Lipschitz constant of f is defined as

$$\rho_\alpha(f) = \sup_{x\neq y} \frac{\|f(x) - f(y)\|}{d(x, y)^\alpha}.$$

Furthermore, the vector-valued Lipschitz algebra $\mathrm{Lip}_\alpha(X, \mathcal{A})$ is the space consisting of all bounded maps $f : X \to \mathcal{A}$ such that $\rho_\alpha(f) < \infty$. Moreover, $\mathrm{Lip}_\alpha(X, \mathcal{A})$ is a Banach algebra, equipped with the norm

$$\|f\|_\alpha = \rho_\alpha(f) + \|f\|_\infty$$

and pointwise product. For convenience, throughout the paper we write $\mathrm{Lip}(X, \mathcal{A})$ instead of $\mathrm{Lip}_1(X, \mathcal{A})$.

In [15, Example 6.1], Kaniuth and Ülger proved that $\mathrm{Lip}_\alpha K$, $(0 < \alpha \leq 1)$, the Banach algebra of all complex-valued Lipschitz maps on a compact metric space (K, d) is a BSE algebra.

In this paper, we study the BSE-property for vector-valued Lipschitz algebra $\mathrm{Lip}_\alpha(K, \mathcal{A})$.

In section 2, we first study some basic properties of vector-valued Lipschitz algebra $\mathrm{Lip}_\alpha(X, \mathcal{A})$, inherited from $\mathcal{A}$, where (X, d) is a metric space and $\alpha > 0$. These results will be used in the rest of the paper.

In section 3, we prove that the BSE-property of $\mathcal{A}$ is guaranteed by the BSE-property of $\mathrm{Lip}_\alpha(K, \mathcal{A})$, where (K, d) is a compact metric space and $0 < \alpha \leq 1$. Then we investigate the reverse implication and show that whenever $\mathcal{A}$ is unital, $\mathrm{Lip}_\alpha(K, \mathcal{A})$ is a BSE algebra if and only if $\mathcal{A}$ is a BSE algebra. This provides examples of Banach algebras $\mathcal{A}$, for which $\mathrm{Lip}_\alpha(K, \mathcal{A})$ are not BSE-algebras. These examples justify the necessity of carrying out the present study.

2. Some basic properties of $\mathrm{Lip}_\alpha(X, \mathcal{A})$

In this section, we prove some primary, basic results and properties related to vector-valued Lipschitz algebras. Throughout the paper, $f_a : X \to \mathcal{A}$ $(a \in \mathcal{A})$ is the constant function on X, defined as $f_a(x) = a$ $(x \in X)$. All these functions belong to $\mathrm{Lip}_\alpha(X, \mathcal{A})$ and $\|f_a\|_\alpha = \|f_a\|_\infty = \|a\|$.

Proposition 2.1. *Let (X, d) be a metric space, $\mathcal{A}$ be a commutative Banach algebra and $\alpha > 0$. Then $\operatorname{Lip}_\alpha(X, \mathcal{A})$ is without order if and only if $\mathcal{A}$ is without order.*

Proof. Let $\mathcal{A}$ be without order and take a nonzero $f \in \operatorname{Lip}_\alpha(X, \mathcal{A})$. Thus there exists $x_0 \in X$ such that $f(x_0) \neq 0$. Since $\mathcal{A}$ is without order, there exists $b \in \mathcal{A}$ such that

$$f(x_0)b \neq 0.$$

Consider the constant function f_b on X. Then

$$(ff_b)(x_0) = f(x_0)f_b(x_0) = f(x_0)\, b \neq 0$$

and so $ff_b \neq 0$. It follows that $\operatorname{Lip}_\alpha(X, \mathcal{A})$ is without order.

Conversely, suppose that $\operatorname{Lip}_\alpha(X, \mathcal{A})$ is without order and take $a \in \mathcal{A}$ with $a \neq 0$. By the hypothesis, there exists $g \in \operatorname{Lip}_\alpha(X, \mathcal{A})$ such that $(f_a g)(x_0) \neq 0$, for some $x_0 \in X$. By taking $b = g(x_0)$, we obtain $ab \neq 0$. Therefore $\mathcal{A}$ is without order. □

Proposition 2.2. *Let (X, d) be a metric space, $\mathcal{A}$ be a commutative Banach algebra and $\alpha > 0$. Then $\operatorname{Lip}_\alpha(X, \mathcal{A})$ separates the points of X if and only if $\operatorname{Lip}_\alpha X$ separates the points of X.*

Proof. Suppose that $\operatorname{Lip}_\alpha X$ separates the points of X and $x, y \in X$ with $x \neq y$. Then there exists $f \in \operatorname{Lip}_\alpha X$ such that $f(x) \neq f(y)$. Choose a nonzero $a \in \mathcal{A}$ and define $g(x) = f(x)a$ $(x \in X)$. Then $g \in \operatorname{Lip}_\alpha(X, \mathcal{A})$ and $g(x) \neq g(y)$.

Conversely, suppose that $\operatorname{Lip}_\alpha(X, \mathcal{A})$ separates the points of X and take $x, y \in X$ with $x \neq y$. Thus there exists $f \in \operatorname{Lip}_\alpha(X, \mathcal{A})$ such that $f(x) \neq f(y)$. Define $g \in \operatorname{Lip}_\alpha X$ as $g(t) = \|f(t) - f(y)\|$ $(t \in X)$. Then

$$\begin{aligned}
\sup_{s,t\in X} \frac{|g(s) - g(t)|}{d(s,t)^\alpha} &= \sup_{s,t\in X} \frac{|\, \|f(s) - f(y)\| - \|f(t) - f(y)\|\, |}{d(s,t)^\alpha} \\
&\leq \sup_{s,t\in X} \frac{\|f(s) - f(t)\|}{d(s,t)^\alpha} \\
&= p_\alpha(f) < \infty,
\end{aligned}$$

which implies that $g \in \operatorname{Lip}_\alpha X$. Moreover, $g(y) = 0$ whereas

$$g(x) = \|f(x) - f(y)\| \neq 0.$$

This completes the proof. □

Remark 2.3. Note that if (X, d) is a metric space and $0 < \alpha \leq 1$, then $\operatorname{Lip}_\alpha X$ always separates the points of X; see [20, Lemma 3.1], for the proof of the case $\alpha = 1$. But this is not true for $\alpha > 1$. For example, consider $\mathbb{R}$ equipped with the usual Euclidean metric. Then $\operatorname{Lip}_\alpha \mathbb{R} = \operatorname{Cons}(\mathbb{R})$, the space consisting of all constant functions on $\mathbb{R}$, which does not separate the points of $\mathbb{R}$.

Proposition 2.2 together with Remark 2.3 yield the following result.

Corollary 2.4. *Let (X, d) be a metric space, $\mathcal{A}$ be a commutative Banach algebra and $0 < \alpha \leq 1$. Then $\operatorname{Lip}_\alpha(X, \mathcal{A})$ separates the points of X.*

Recall from [6, Theorem 2.9] and also [5, Corollary 2.3] that for a compact metric space (K, d), a commutative Banach algebra $\mathcal{A}$ and $0 < \alpha \leq 1$ we have

$$\Delta(\mathrm{Lip}_\alpha(K, \mathcal{A})) = K \times \Delta(\mathcal{A}).$$

Precisely, every character on $\mathrm{Lip}_\alpha(K, \mathcal{A})$ is of the form $(x, \varphi) := \varphi \circ \delta_x$, where φ and x run into $\Delta(\mathcal{A})$ and K, respectively and

$$(x, \varphi)(f) = \varphi \circ \delta_x(f) = \varphi(f(x)) \qquad (f \in \mathrm{Lip}_\alpha(K, \mathcal{A})).$$

Remark 2.5. It is worth to note that for any compact metric space (K, d) and $0 < \alpha \leq 1$, the function d^α defines a metric on K. Furthermore,

$$\mathrm{Lip}((K, d^\alpha), \mathcal{A}) = \mathrm{Lip}_\alpha((K, d), \mathcal{A}).$$

Thus from now to the end of the paper, we may assume without loss of generality that $\alpha = 1$.

Proposition 2.6. *Let (K, d) be a compact metric space and $\mathcal{A}$ be a commutative Banach algebra. Then $\mathrm{Lip}(K, \mathcal{A})$ has a bounded Δ-weak approximate identity if and only if $\mathcal{A}$ has one.*

Proof. Let $(f_\alpha)_{\alpha \in \mathcal{I}}$ be a bounded Δ-weak approximate identity for $\mathrm{Lip}(K, \mathcal{A})$. Then for each $x \in K$, the net $(f_\alpha(x))_{\alpha \in \mathcal{I}}$ is clearly a bounded Δ-weak approximate identity for $\mathcal{A}$.

Conversely, suppose that $\mathcal{A}$ has a bounded Δ-weak approximate identity, denoted by $(e_\alpha)_{\alpha \in \mathcal{I}}$. For any $\alpha \in \mathcal{I}$, consider the Lipschitz function $f_\alpha := f_{e_\alpha}$. Then it is easily verified that the net $(f_\alpha)_{\alpha \in \mathcal{I}}$ is a bounded Δ-weak approximate identity for $\mathrm{Lip}(K, \mathcal{A})$. $\square$

Proposition 2.7. *Let (K, d) be a compact metric space and $\mathcal{A}$ be a commutative Banach algebra. Then $\mathcal{A}$ is semisimple if and only if $\mathrm{Lip}(K, \mathcal{A})$ is semisimple.*

Proof. Suppose that $\mathcal{A}$ is semisimple and take $f, g \in \mathrm{Lip}(K, \mathcal{A})$ such that $f \neq g$. So there exists $x_0 \in K$ such that $f(x_0) \neq g(x_0)$. Since $\mathcal{A}$ is semisimple, there exists $\varphi \in \Delta(\mathcal{A})$ such that

$$\varphi(f(x_0)) \neq \varphi(g(x_0)).$$

It follows that

$$(x_0, \varphi)(f) \neq (x_0, \varphi)(g).$$

Thus $\Delta(\mathrm{Lip}(K, \mathcal{A}))$ separates the points of $\mathrm{Lip}(K, \mathcal{A})$ and so $\mathrm{Lip}(K, \mathcal{A})$ is semisimple. For the reverse implication, suppose that $\mathrm{Lip}(K, \mathcal{A})$ is semisimple and take $a, b \in \mathcal{A}$ such that $a \neq b$. Thus $f_a \neq f_b$ and by the hypothesis, there exist $x \in K$ and $\varphi \in \Delta(\mathcal{A})$ such that

$$(x, \varphi)(f_a) \neq (x, \varphi)(f_b).$$

It follows that

$$\varphi(a) = \varphi(f_a(x)) \neq \varphi(f_b(x)) = \varphi(b).$$

Consequently, $\Delta(\mathcal{A})$ separates the points of $\mathcal{A}$ and so $\mathcal{A}$ is semisimple. $\square$

Please cite this article in press as: F. Abtahi et al., The Bochner-Schoenberg-Eberlein property for vector-valued Lipschitz algebras, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.06.073

3. The BSE-property for $\mathrm{Lip}(K, \mathcal{A})$

In this section we investigate the problem how the BSE-properties of the algebras $\mathcal{A}$ and $\mathrm{Lip}(K, \mathcal{A})$ are related to each other.

Theorem 3.1. *Let (K, d) be a compact metric space and $\mathcal{A}$ be a commutative semisimple Banach algebra such that $\mathrm{Lip}(K, \mathcal{A})$ is a BSE-algebra. Then $\mathcal{A}$ is a BSE-algebra.*

Proof. Let $\mathrm{Lip}(K, \mathcal{A})$ be a BSE-algebra. By [21, Corollary 5] and Proposition 2.6 $\mathcal{A}$ has a bounded Δ-weak approximate identity and so

$$\widehat{M(\mathcal{A})} \subseteq C_{BSE}(\Delta(\mathcal{A})).$$

To prove the reverse of this inclusion, take $\sigma \in C_{BSE}(\Delta(\mathcal{A}))$ and $a_0 \in \mathcal{A}$. It is enough to detect an element $b_0 \in \mathcal{A}$ such that $\sigma\ \widehat{a_0} = \widehat{b_0}$. Define the function $\sigma_1 : K \times \Delta(\mathcal{A}) \to \mathbb{C}$, as

$$\sigma_1(x, \varphi) = \sigma(\varphi),$$

for all $x \in K$ and $\varphi \in \Delta(\mathcal{A})$. For every finite number of complex numbers $c_1, \cdots, c_n$ and the same number of $(x_1, \varphi_1), \cdots, (x_n, \varphi_n)$ in $K \times \Delta(\mathcal{A})$ we have

$$\begin{aligned}
\left|\sum_{j=1}^{n} c_j \sigma_1(x_j, \varphi_j)\right| &= \left|\sum_{j=1}^{n} c_j \sigma(\varphi_j)\right| \\
&\leq \|\sigma\|_{BSE} \left\|\sum_{j=1}^{n} c_j \varphi_j\right\|_{\mathcal{A}^*} \\
&= \|\sigma\|_{BSE} \sup_{\|a\| \leq 1} \left|\sum_{j=1}^{n} c_j \varphi_j(a)\right| \\
&= \|\sigma\|_{BSE} \sup_{\|f_a\|_1 \leq 1} \left|\sum_{j=1}^{n} c_j (x_j, \varphi_j)(f_a)\right| \\
&\leq \|\sigma\|_{BSE} \sup_{\|f\|_1 \leq 1} \left|\sum_{j=1}^{n} c_j (x_j, \varphi_j)(f)\right| \\
&= \|\sigma\|_{BSE} \left\|\sum_{j=1}^{n} c_j (x_j, \varphi_j)\right\|_{(\mathrm{Lip}(K, \mathcal{A}))^*}.
\end{aligned}$$

It follows that $\sigma_1 \in C_{BSE}(\Delta(\mathrm{Lip}(K, \mathcal{A})))$. Since $\mathrm{Lip}(K, \mathcal{A})$ is a BSE-algebra and $f_{a_0} \in \mathrm{Lip}(K, \mathcal{A})$, there exists $g \in \mathrm{Lip}(K, \mathcal{A})$ such that $\sigma_1\ \widehat{f_{a_0}} = \widehat{g}$. It follows that

$$\sigma_1(x, \varphi)\ \varphi(f_{a_0}(x)) = \varphi(g(x)),$$

for all $x \in X$ and $\varphi \in \Delta(\mathcal{A})$ and so

$$\sigma(\varphi)\varphi(a_0) = \varphi(g(x)). \tag{3.1}$$

Then the equality (3.1) implies that

$$\varphi(g(x)) = \varphi(g(y)) \quad (x, y \in K).$$

The simplicity of $\mathcal{A}$ implies that g is a constant function, as $g = f_{b_0}$, for some $b_0 \in \mathcal{A}$. Thus for all $x \in K$ and $\varphi \in \Delta(\mathcal{A})$ we have

$$\sigma(\varphi)\widehat{a_0}(\varphi) = \varphi(g(x)) = \varphi(f_{b_0}(x)) = \varphi(b_0) = \widehat{b_0}(\varphi).$$

Consequently $\sigma\widehat{a_0} = \widehat{b_0}$, which implies $\sigma \in \mathcal{M}(\mathcal{A})$, as claimed. $\square$

By Theorem 3.1, $\mathrm{Lip}(K, \mathcal{A})$ is not a BSE-algebra, whenever $\mathcal{A}$ is not BSE. One can actually construct examples of vector-valued Lipschitz algebras, which are not BSE-algebras.

Example 3.2. Let $\mathcal{A}$ be the Banach algebra $L^p(S, \mu)$ $(1 \leq p < \infty)$, whenever S is a totally ordered compact space with a regular bounded continuous measure μ on S, introduced in [1]. By [12, Theorem 3] and Theorem 3.1, $\mathrm{Lip}(K, \mathcal{A})$ is not a BSE-algebra.

Let $C^1[0, 1]$ be the space, consisting of all differentiable functions with continuous first derivative on $[0, 1]$. Then $C^1[0, 1]$ is a unital, semisimple and commutative Banach algebra equipped with the norm

$$\|f\|_{c^1} = \|f\|_\infty + \|f'\|_\infty$$

and pointwise product. Note that the character space of $C^1[0, 1]$ is homeomorphic with $[0, 1]$. In fact

$$\Delta(C^1[0, 1]) = \{\varphi_x : \ x \in [0, 1]\},$$

where $\varphi_x(f) = f(x)$ $(f \in C^1[0, 1])$.

Proposition 3.3. *The Banach algebra $C^1[0, 1]$ is not BSE.*

Proof. Define the sequence (f_n) of functions, belonging to $C^1[0, 1]$ as

$$f_n(x) = \left(x - \frac{1}{2}\right)^{1+\frac{1}{2n-1}} \qquad (x \in [0, 1]).$$

It is easily verified that for all $x \in [0, 1]$

$$\lim_{n\to\infty} \widehat{f_n}(\varphi_x) = \lim_{n\to\infty} f_n(x) = \left|x - \frac{1}{2}\right| = \widehat{f}(\varphi_x),$$

where $f(x) = |x - \frac{1}{2}|$ $(x \in [0, 1])$. Now [21, Theorem 4] implies that $f \in C_{BSE}(C^1[0, 1])$. However f is not differentiable at $x = \dfrac{1}{2}$. It follows that $f \notin C^1[0, 1]$ and so

$$C_{BSE}(C^1[0, 1]) \neq \widehat{C^1[0, 1]}.$$

Therefore $C^1[0, 1]$ is not a BSE-algebra. $\square$

The following result is a direct consequence of Theorem 3.1 and Proposition 3.3.

Corollary 3.4. *Let (K,d) be a compact metric space. Then $\mathrm{Lip}(K, C^1[0,1])$ is not a BSE algebra.*

In the sequel, we prove the converse of Theorem 3.1, for any unital Banach algebra $\mathcal{A}$. It is clear that $\mathcal{A}$ is unital if and only if $\mathrm{Lip}(K,\mathcal{A})$ is unital.

Theorem 3.5. *Let (K,d) be a compact metric space and $\mathcal{A}$ be a unital commutative semisimple Banach algebra. Then $\mathcal{A}$ is a BSE-algebra if and only if $\mathrm{Lip}(K,\mathcal{A})$ is a BSE-algebra.*

Proof. At first let $\mathrm{Lip}(K,\mathcal{A})$ be a BSE-algebra. Then by Theorem 3.1, $\mathcal{A}$ is a BSE algebra. Conversely, suppose that $\mathcal{A}$ is a BSE algebra. Since $\mathcal{A}$ is unital, [21, Corollary 5] implies that

$$\widehat{\mathrm{Lip}(K,\mathcal{A})} = \mathcal{M}(\mathrm{Lip}(K,\mathcal{A})) \subseteq C_{BSE}(K \times \Delta(\mathcal{A})).$$

For the reverse inclusion, take $\sigma \in C_{BSE}(K \times \Delta(\mathcal{A}))$. By [21, Theorem 4], there exists a net $\{f_\lambda\} \subseteq \mathrm{Lip}(K,\mathcal{A})$, bounded by $\beta > 0$, such that

$$\lim_\lambda \widehat{f_\lambda}(x,\varphi) = \sigma(x,\varphi) \qquad (x \in K, \varphi \in \Delta(\mathcal{A})). \tag{3.2}$$

We have to find a function $g \in \mathrm{Lip}(K,\mathcal{A})$ such that $\sigma = \widehat{g}$. For each $x \in K$, define $\sigma_x : \Delta(\mathcal{A}) \to \mathbb{C}$ as

$$\sigma_x(\varphi) = \sigma(x,\varphi) \qquad (\varphi \in \Delta(\mathcal{A})).$$

For any finitely many elements $\varphi_1, \cdots, \varphi_n$ of $\Delta(\mathcal{A})$ and complex numbers $c_1, \cdots, c_n$, we have

$$\begin{aligned}
\left|\sum_{j=1}^n c_j \sigma_x(\varphi_j)\right| &= \left|\sum_{j=1}^n c_j \sigma(x,\varphi_j)\right| \\
&\le \|\sigma\|_{BSE} \left\|\sum_{j=1}^n c_j (x,\varphi_j)\right\|_{(\mathrm{Lip}(K,\mathcal{A}))^*} \\
&= \|\sigma\|_{BSE} \sup_{\|f\|_1 \le 1} \left|\sum_{j=1}^n c_j (x,\varphi_j)(f)\right| \\
&= \|\sigma\|_{BSE} \sup_{\|f\|_1 \le 1} \left|\sum_{j=1}^n c_j \varphi_j(f(x))\right| \\
&\le \|\sigma\|_{BSE} \sup_{\|a\| \le 1} \left|\sum_{j=1}^n c_j \varphi_j(a)\right| \\
&= \|\sigma\|_{BSE} \left\|\sum_{j=1}^n c_j \varphi_j\right\|_{\mathcal{A}^*}.
\end{aligned}$$

Consequently $\sigma_x \in C_{BSE}(\Delta(\mathcal{A}))$ and also $\|\sigma_x\|_{BSE} \le \|\sigma\|_{BSE}$. Since $\mathcal{A}$ is a unital BSE-algebra, $\sigma_x \in \widehat{\mathcal{A}}$ and so there exists $a_x \in \mathcal{A}$ such that $\sigma_x = \widehat{a_x}$. It follows that

$$\sigma(x,\varphi) = \sigma_x(\varphi) = \varphi(a_x), \tag{3.3}$$

for all $\varphi \in \Delta(\mathcal{A})$. Now define the function $g : K \to \mathcal{A}$ as $g(x) := a_x$. Then g is well defined, by the semisimplicity of $\mathcal{A}$. Moreover, $g \in C(K \times \Delta(\mathcal{A}))$. Indeed, by Corollary 6 of [21], there exists positive number M such that

$$M\|a_x\| \le \|\widehat{a_x}\|_{BSE} \le \|a_x\|. \tag{3.4}$$

Thus we have

$$\begin{aligned}\|g\|_\infty = \sup_{x\in K}\|g(x)\| = \sup_{x\in K}\|a_x\| &\le \frac{1}{M}\sup_{x\in K}\|\widehat{a_x}\|_{BSE} \\ &= \frac{1}{M}\sup_{x\in K}\|\sigma_x\|_{BSE} \le \frac{1}{M}\|\sigma\|_{BSE} < \infty.\end{aligned}$$

Also we have

$$\sigma(x,\varphi) = \varphi(g(x)) = \widehat{g}(x,\varphi) \qquad ((x,\varphi) \in K \times \Delta(\mathcal{A})). \tag{3.5}$$

It is enough to show that $\rho_1(g) < \infty$. To that end, take $x, y \in K$ with $x \neq y$. For every finite number of complex numbers $c_1, \cdots, c_n$ and the same number of $\varphi_1, \cdots, \varphi_n \in \Delta(\mathcal{A})$ by (3.2) and (3.5) we have

$$\frac{\left|\left(\widehat{g(x) - g(y)}\right)\left(\sum_{i=1}^n c_i\varphi_i\right)\right|}{d(x,y)} = \lim_\lambda \frac{\left|\left(\widehat{f_\lambda(x) - f_\lambda(y)}\right)\left(\sum_{i=1}^n c_i\varphi_i\right)\right|}{d(x,y)}.$$

Moreover, by (3.4) for any λ we have

$$\begin{aligned}\frac{\left|\left(\widehat{f_\lambda(x) - f_\lambda(y)}\right)\left(\sum_{i=1}^n c_i\varphi_i\right)\right|}{d(x,y)} &\le \frac{\|\widehat{f_\lambda(x) - f_\lambda(y)}\|_{BSE}}{d(x,y)}\left\|\sum_{i=1}^n c_i\varphi_i\right\|_{\mathcal{A}^*} \\ &\le \frac{\|(f_\lambda(x) - f_\lambda(y))\|}{d(x,y)}\left\|\sum_{i=1}^n c_i\varphi_i\right\|_{\mathcal{A}^*} \\ &= \rho_1(f_\lambda)\left\|\sum_{i=1}^n c_i\varphi_i\right\|_{\mathcal{A}^*} \\ &\le \|f_\lambda\|_1\left\|\sum_{i=1}^n c_i\varphi_i\right\|_{\mathcal{A}^*} \\ &\le \beta\left\|\sum_{i=1}^n c_i\varphi_i\right\|_{\mathcal{A}^*}.\end{aligned}$$

Consequently

$$\frac{\|\widehat{g(x) - g(y)}\|_{BSE}}{d(x,y)} = \sup\left\{\frac{\left|\left(\widehat{g(x) - g(y)}\right)\left(\sum_{i=1}^n c_i\varphi_i\right)\right|}{d(x,y)} : \left\|\sum_{i=1}^n c_i\varphi_i)\right\|_{\mathcal{A}^*} \le 1\right\} \le \beta.$$

Again, by inequality (3.4), for any $x, y \in K$ with $x \neq y$, we get

$$\frac{\|(g(x)-g(y))\|}{d(x,y)} \leq \frac{\|\widehat{(g(x)-g(y))}\|_{BSE}}{Md(x,y)} \leq \frac{\beta}{M}.$$

This follows that

$$\rho_1(g) \leq \frac{\beta}{M} < \infty$$

and so $g \in \mathrm{Lip}(K, \mathcal{A})$. Therefore $\mathrm{Lip}(K, \mathcal{A})$ is a BSE algebra. □

Example 3.6. Let G be a non discrete locally compact abelian group. As it is shown in [21], the measure algebra $M(G)$ is not a BSE algebra. By Theorem 3.5, the Banach algebra $\mathrm{Lip}(K, M(G))$ is not BSE.

Acknowledgments

The authors express their sincere gratitude to the reviewers for their constructive comments and suggestions on the manuscript. Their invaluable comments and hints helped us to improve the main results and raise the quality of the paper. This research was partially supported by the Banach algebra Center of Excellence for Mathematics, University of Isfahan. The second author's research was supported in part by a grant from IAU and IPM (No. 95470066).

References

[1] J.W. Baker, J.S. Pym, H.L. Vasudea, Totally ordered measure spaces and their Lp algebras, Mathematika 29 (1982) 42–54.
[2] S. Bochner, A theorem on Fourier-Stieltjes integrals, Bull. Amer. Math. Soc. 40 (1934) 271–276.
[3] H.G. Dales, A. Ülger, Approximate identities in Banach function algebras, Studia Math. 226 (2015) 155–187.
[4] W.F. Eberlein, Characterizations of Fourier-Stieltjes transforms, Duke Math. J. 22 (1955) 465–468.
[5] K. Esmaeili, H. Mahyar, The character spaces and Šilov boundaries of vector-valued Lipschitz function algebras, Indian J. Pure Appl. Math. (2014) 977–988.
[6] T.G. Honary, A. Nikou, A.H. Sanatpour, On the character space of vector-valued Lipschitz algebras, Bull. Iranian Math. Soc. 40 (6) (2014) 1453–1468.
[7] J. Inoue, S.E. Takahasi, On characterizations of the image of Gelfand transform of commutative Banach algebras, Math. Nachr. 280 (2007) 105–126.
[8] C.A. Jones, C.D. Lahr, Weak and norm approximate identities are different, Pacific J. Math. 72 (1977) 99–104.
[9] Z. Kamali, M. Lashkarizadeh Bami, Bochner-Schoenberg-Eberlein property for abstract Segal algebras, Proc. Japan Acad. Ser. A Math. Sci. 89 (2013) 107–110.
[10] Z. Kamali, M. Lashkarizadeh Bami, The multiplier algebra and BSE property of the direct sum of Banach algebras, Bull. Aust. Math. Soc. (2013) 250–258.
[11] Z. Kamali, M. Lashkarizadeh, The Bochner-Schoenberg-Eberlein property for $L^1(\mathbb{R}^+)$, J. Fourier Anal. Appl. 20 (2) (2014) 225–233.
[12] Z. Kamali, M. Lashkarizadeh, The Bochner-Schoenberg-Eberlein property for totally ordered semigroup algebras, J. Fourier Anal. Appl. 22 (6) (2016) 1225–1234.
[13] Z. Kamali, A characterization of the L^∞-representation algebra $\mathfrak{R}(S)$ of a foundation semigroup and its application to BSE-algebras, Proc. Japan Acad. Ser. A Math. Sci. 92 (5) (2016) 59–63.
[14] E. Kaniuth, A.T. Lau, A. Ülger, Homomorphisms of commutative Banach algebras and extensions to multiplier algebras with applications to Fourier algebras, Studia Math. 183 (2007) 35–62.
[15] E. Kaniuth, A. Ülger, The Bochner-Schoenberg-Eberlein property for commutative Banach algebras, especially Fourier and Fourier-Stieltjes algebras, Trans. Amer. Math. Soc. 362 (2010) 4331–4356.
[16] E. Kaniuth, The Bochner-Schoenberg-Eberlein property and spectral synthesis for certain Banach algebra products, Canad. J. Math. 67 (2015) 827–847.
[17] R. Larsen, An Introduction to the Theory of Multipliers, Springer-Verlag, New York, 1971.
[18] W. Rudin, Fourier Analysis on Groups, Wiley Interscience, New York, 1984.
[19] I.J. Schoenberg, A remark on the preceding note by Bochner, Bull. Amer. Math. Soc. 40 (1934) 277–278.
[20] D.R. Sherbert, Banach algebras of Lipschitz functions, Pacific J. Math. 13 (4) (1963) 1387–1399.
[21] S.E. Takahasi, O. Hatori, Commutative Banach algebras which satisfy a Bochner-Schoenberg-Eberlein-type theorem, Proc. Amer. Math. Soc. 110 (1990) 149–158.
[22] S.E. Takahasi, O. Hatori, Commutative Banach algebras and BSE-inequalities, Math. Jpn. 37 (1992) 47–52.
[23] A. Ülger, Multipliers with closed range on commutative Banach algebras, Studia Math. 153 (2002) 59–80.