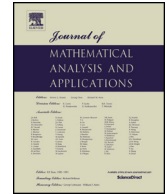




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Stability of linear impulsive neutral delay differential equations with constant coefficients

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ABSTRACT

In this article, we have developed a new method regarding asymptotic behavior and stability for the solutions of first order linear impulsive neutral delay differential equations with constant coefficients and constant delays. The asymptotic behavior of the solutions and the stability of the zero solution were investigated using a real root suitable for the characteristic equation. Furthermore, we obtained similar results for impulsive delay differential equations. Finally, three examples are given for the stability of the zero solution of the impulsive neutral delay differential equation.

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1. Introduction and preliminaries

In the last 30 years, studies on the impulsive delay differential equations (IDDEs) are available (see [1, 2, 8, 10, 12–14, 18, 22, 24, 31–35, 45, 48–50, 52, 55–58, 60] and references therein). There are enough mathematical models of many processes in various fields of science and technology. The theory of impulse neutral delay differential equations (INDDEs) is much more complex than the theory of IDDEs. Furthermore, due to theoretical and practical difficulties, the theory of INDDEs was developed very slowly compared to the theory of impulsive differential equations (for example, see [4, 11, 16, 21, 29, 36, 44, 46] and references therein) and the theory differential equations with neutral delay (for example, see [3, 15, 27, 28, 37–39, 47] and references therein). Much less we know about INDDEs and also is a very challenging problem. As it is known, it is very difficult, sometime maybe impossible, to get the explicit solutions for INDDEs. Therefore, research on the behavior of solutions of INDDEs is required. Note that such equations appear in networks with lossless transmission lines. Such networks occur, for example, on high-speed computers using lossless transmission lines to connect switching circuits together (see [15, 47]). For more information on neutral delay differential equations, see the books [9, 19, 20, 23, 26].

In [22], the authors obtained some results on the behavior of the solutions for IDDEs. Here, [22] is the first study to consider IDDEs. In later years, some authors have achieved very important results on the

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behavior of the solutions of IDDEs or INDDEs (see [1,2,5–7,12–14,17,18,25,30–35,45,49–60] and references therein).

In this paper, we have obtained the asymptotic results of the solutions of the linear INDDEs after defining a characteristic equation. We have also identified an exponential boundary of the solutions of the linear INDDEs and provided the conditions for the stability of the zero solutions. Our results are obtained by using a real root of the corresponding characteristic equation. We have also achieved similar results for the linear IDDEs. At last, three examples are given to demonstrate the efficiency of our results. The results were obtained using the methods in [27,37–39].

Consider the INDDE

$$[x(t) + cx(t - \sigma)]' = ax(t) + bx(t - \tau), \quad t \neq t_k, \quad t \geq 0, \quad (1.1)$$

$$\Delta x(t_k) = \ell_k, \quad k \in \mathbf{Z}^+ = \{1, 2, \dots\}, \quad (1.2)$$

where σ and τ are positive constants, a, b, c and ℓ_k are real constants, $x(t) \in \mathbf{R}$ and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$. The impulse times t_k satisfy $0 < t_1 < \dots < t_k < t_{k+1} < \dots$ and $\lim_{k \rightarrow \infty} t_k = \infty$.

Let's define the positive constant h , defined by

$$h = \max\{\sigma, \tau\}.$$

Together with (1.1), an initial condition is indicated, i.e.

$$x(t) = \phi(t), \quad -h \leq t \leq 0, \quad (1.3)$$

where the initial function ϕ is an any given continuous real-valued function on the interval $[-h, 0]$.

A real valued right continuous function $x(t)$ is said to be the *solution* of the initial value problem (1.1)-(1.3) if the following conditions are satisfied:

- $x(t)$ is continuous everywhere except the points t_k , $k \in \mathbf{Z}^+$,
- the function $x(t) + cx(t - \sigma)$ is continuously differentiable for $t \geq 0$ and $t \neq t_k$, $k \in \mathbf{Z}^+$,
- $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^+) = x(t_k)$, $k \in \mathbf{Z}^+$,
- $x(t)$ satisfies the differential equation (1.1) for $t \geq 0$, satisfies the impulsive conditions at $t = t_k$ and satisfies (1.3).

Consider the following linear IDDE, which is a special case

$$x'(t) = ax(t) + bx(t - \tau), \quad t \neq t_k, \quad t \geq 0, \quad (1.4)$$

$$\Delta x(t_k) = \ell_k, \quad k \in \mathbf{Z}^+ = \{1, 2, \dots\}. \quad (1.5)$$

This system can be obtained (as a special case) from the INDDE (1.1)-(1.2). In addition, the initial condition of the equation (1.4) is

$$x(t) = \phi(t), \quad -\tau \leq t \leq 0. \quad (1.6)$$

Along with (1.1), we associate its *characteristic equation*

$$\lambda(1 + ce^{-\lambda\sigma}) = a + be^{-\lambda\tau}. \quad (1.7)$$

Equation (1.7) is obtained from (1.1) by searching for solutions in the form $x(t) = e^{\lambda t}$ for $t \in \mathbf{R}$.

To obtain a mapping suitable for the asymptotic behavior of the solutions, we make the following observation. Let λ_0 be a real root of (1.7). Let's define

$$y(t) = e^{-\lambda_0 t} x(t) \quad \text{for all } t \in [-h, \infty). \quad (1.8)$$

Then, from (1.1) we obtain for all $t \geq 0$

$$[y(t) + ce^{-\lambda_0 \sigma} y(t - \sigma)]' = (a - \lambda_0) y(t) + be^{-\lambda_0 \tau} y(t - \tau) - c\lambda_0 e^{-\lambda_0 \sigma} y(t - \sigma). \quad (1.9)$$

In addition, the initial condition (1.3) can be made equivalent to

$$y(t) = e^{-\lambda_0 t} \phi(t) \quad \text{for } t \in [-h, 0]. \quad (1.10)$$

Furthermore, if $x(t)$ satisfies the impulsive conditions (1.2), then evidently

$$y(t_k) - y(t_k^-) = \ell_k e^{-\lambda_0 t_k}, \quad k \in \mathbf{Z}^+. \quad (1.11)$$

To prove our basic asymptotic criterion, we need the following helpful lemma.

Lemma 1.1. Assume that λ_0 is a real root of the characteristic equation (1.7) and let $t_k - \sigma$ be not impulsive points for all $k \in \mathbf{Z}^+$. Then y is the unique solution of the initial value problem (1.9)-(1.11) if and only if y is a solution of the following system

$$y(t) = e^{-\lambda_0 t} \phi(t), \quad \text{if } -h \leq t \leq 0$$

and

$$\begin{aligned} y(t) + ce^{-\lambda_0 \sigma} y(t - \sigma) &= \phi(0) + c\phi(-\sigma) + \sum_{i=1}^{n(t)} \ell_i e^{-\lambda_0 t_i} \\ &+ be^{-\lambda_0 \tau} \int_{-\tau}^0 e^{-\lambda_0 s} \phi(s) ds - c\lambda_0 e^{-\lambda_0 \sigma} \int_{-\sigma}^0 e^{-\lambda_0 s} \phi(s) ds \\ &+ (a - \lambda_0) \int_0^t y(s) ds + be^{-\lambda_0 \tau} \int_0^{t-\tau} y(s) ds \\ &- c\lambda_0 e^{-\lambda_0 \sigma} \int_0^{t-\sigma} y(s) ds, \quad \text{if } t \geq 0 \end{aligned} \quad (1.12)$$

where

$$n(t) = \max\{k \in \mathbf{Z}^+ : t_k \leq t\} \quad \text{and} \quad n(t) = 0 \quad \text{if } t < t_1.$$

Proof. Let y be a suitable solution of (1.9)-(1.11). Using the differential equation (1.9), for $t \in [t_k, t_{k+1})$, we get

$$\begin{aligned} y(t) + ce^{-\lambda_0 \sigma} y(t - \sigma) &= y(t_k) + ce^{-\lambda_0 \sigma} y(t_k - \sigma) + (a - \lambda_0) \int_{t_k}^t y(s) ds \\ &+ be^{-\lambda_0 \tau} \int_{t_k}^t y(s - \tau) ds - c\lambda_0 e^{-\lambda_0 \sigma} \int_{t_k}^t y(s - \sigma) ds. \end{aligned}$$

If we use (1.11), since $t_k - \sigma$ not impulsive points, from the last equation we obtain for $t \in [t_k, t_{k+1})$

$$\begin{aligned} y(t) + ce^{-\lambda_0 \sigma} y(t - \sigma) &= \ell_k e^{-\lambda_0 t_k} + y(t_k^-) + ce^{-\lambda_0 \sigma} y(t_k - \sigma) \\ &+ (a - \lambda_0) \int_{t_k}^t y(s) ds + be^{-\lambda_0 \tau} \int_{t_k}^t y(s - \tau) ds - c\lambda_0 e^{-\lambda_0 \sigma} \int_{t_k}^t y(s - \sigma) ds. \end{aligned} \quad (1.13)$$

We stress that the formula holds for $t \in [t_k, t_{k+1})$ only, but by backstepping we can express $y(t_k^-)$ using the analogous formula to (1.13) but for $t \in [t_{k-1}, t_k)$, as $t \rightarrow t_k^-$. Backstepping in this way we get:

$$\begin{aligned} y(t_k^-) + ce^{-\lambda_0 \sigma} y(t_k - \sigma) &= \ell_{k-1} e^{-\lambda_0 t_{k-1}} + y(t_{k-1}^-) + ce^{-\lambda_0 \sigma} y(t_{k-1} - \sigma) \\ &+ (a - \lambda_0) \int_{t_{k-1}}^{t_k} y(s) ds + be^{-\lambda_0 \tau} \int_{t_{k-1}}^{t_k} y(s - \tau) ds - c\lambda_0 e^{-\lambda_0 \sigma} \int_{t_{k-1}}^{t_k} y(s - \sigma) ds, \\ &\vdots \\ y(t_2^-) + ce^{-\lambda_0 \sigma} y(t_2 - \sigma) &= \ell_1 e^{-\lambda_0 t_1} + y(t_1^-) + ce^{-\lambda_0 \sigma} y(t_1 - \sigma) \\ &+ (a - \lambda_0) \int_{t_1}^{t_2} y(s) ds + be^{-\lambda_0 \tau} \int_{t_1}^{t_2} y(s - \tau) ds - c\lambda_0 e^{-\lambda_0 \sigma} \int_{t_1}^{t_2} y(s - \sigma) ds, \\ y(t_1^-) + ce^{-\lambda_0 \sigma} y(t_1 - \sigma) &= \phi(0) + c\phi(-\sigma) \\ &+ (a - \lambda_0) \int_0^{t_1} y(s) ds + be^{-\lambda_0 \tau} \int_0^{t_1} y(s - \tau) ds - c\lambda_0 e^{-\lambda_0 \sigma} \int_0^{t_1} y(s - \sigma) ds. \end{aligned}$$

By recursive substitution into (1.13) we get that in general, the solution $y(t)$ must satisfy:

$$\begin{aligned} y(t) + ce^{-\lambda_0 \sigma} y(t - \sigma) &= \phi(0) + c\phi(-\sigma) + \sum_{i=1}^{n(t)} \ell_i e^{-\lambda_0 t_i} \\ &+ (a - \lambda_0) \int_0^t y(s) ds + be^{-\lambda_0 \tau} \int_0^t y(s - \tau) ds - c\lambda_0 e^{-\lambda_0 \sigma} \int_0^t y(s - \sigma) ds, \end{aligned}$$

or, if the initial function is taken into account in the last two terms, from the last equation

$$\begin{aligned} y(t) + ce^{-\lambda_0 \sigma} y(t - \sigma) &= \phi(0) + c\phi(-\sigma) + \sum_{i=1}^{n(t)} \ell_i e^{-\lambda_0 t_i} \\ &+ be^{-\lambda_0 \tau} \int_{-\tau}^0 e^{-\lambda_0 s} \phi(s) ds - c\lambda_0 e^{-\lambda_0 \sigma} \int_{-\sigma}^0 e^{-\lambda_0 s} \phi(s) ds \\ &+ (a - \lambda_0) \int_0^t y(s) ds + be^{-\lambda_0 \tau} \int_0^{t-\tau} y(s) ds - c\lambda_0 e^{-\lambda_0 \sigma} \int_0^{t-\sigma} y(s) ds, \end{aligned}$$

for $t \geq 0$. Thus, y solution of the integral equation (1.12) is obtained. If y satisfies the integral equation (1.12), then y is the solution of (1.9)- (1.11). Indeed, from (1.12) we obtain

$$[y(t) + ce^{-\lambda_0\sigma}y(t-\sigma)]' = (a - \lambda_0)y(t) + be^{-\lambda_0\tau}y(t-\tau) - c\lambda_0e^{-\lambda_0\sigma}y(t-\sigma),$$

for $t \geq 0$ and $t \neq t_k$. We can show this easily

$$y(t_k^+) - y(t_k^-) = \ell_k e^{-\lambda_0 t_k}, \quad k \in \mathbf{Z}^+.$$

So, the proof of Lemma 1.1 is complete. $\square$

We can use Lemma 1.1 contained to derive the following corollary.

Corollary 1.2. Assume that λ_0 is a real root of the characteristic equation (1.7) and let $t_k - \sigma$ be not impulsive points for all $k \in \mathbf{Z}^+$. Then x is the unique solution of the initial value problem (1.1)-(1.3) if and only if the function y defined by

$$y(t) = e^{-\lambda_0 t} x(t) \quad \text{for } t \in [-h, \infty)$$

is the solution of the integral equation (1.12) which provides the initial condition

$$y(t) = e^{-\lambda_0 t} \phi(t) \quad \text{for } t \in [-h, 0].$$

At the end of this chapter, we will give three definitions (see, for example, [48]). The zero solution of (1.1)-(1.2) is said to be *stable* if for every $\epsilon > 0$, there exists a number $\delta = \delta(\epsilon) > 0$ such that, for any initial function ϕ with

$$\|\phi\| = \max_{-h \leq t \leq 0} |\phi(t)| < \delta$$

and $x(t) = x(t; 0, \phi)$ is any solution of (1.1)-(1.3), then $x(t; 0, \phi)$ satisfies

$$|x(t; 0, \phi)| < \epsilon \quad \text{for all } t \in [-h, \infty).$$

The zero solution of (1.1)-(1.2) is said to be *unstable* if it is not stable. Moreover, the zero solution of (1.1)-(1.2) is called *asymptotically stable* if it is stable and in addition there exists a number $\delta_0 > 0$ such that, for any initial function ϕ with $\|\phi\| < \delta_0$, then the solution $x(t; 0, \phi)$ of (1.1)-(1.3) satisfies

$$\lim_{t \rightarrow \infty} x(t; 0, \phi) = 0.$$

2. The asymptotic behavior of solutions

The following theorem is a fundamental asymptotic criterion for the solutions of the initial value problem (1.1)-(1.3).

Theorem 2.1. Suppose that Lemma 1.1 holds and assume that the root λ_0 has the property

$$\mu(\lambda_0) = |b|\tau e^{-\lambda_0\tau} + |c|e^{-\lambda_0\sigma}(1 + |\lambda_0|\sigma) + \sum_{i=1}^{\infty} |\ell_i|e^{-\lambda_0 t_i} < 1. \quad (2.1)$$

Then the solution x of (1.1)-(1.3) satisfies

$$\lim_{t \rightarrow \infty} [e^{-\lambda_0 t} x(t)] = \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)}, \quad (2.2)$$

where

$$L(\lambda_0; \phi) = \phi(0) + c\phi(-\sigma) + be^{-\lambda_0\tau} \int_{-\tau}^0 e^{-\lambda_0 s} \phi(s) ds - c\lambda_0 e^{-\lambda_0\sigma} \int_{-\sigma}^0 e^{-\lambda_0 s} \phi(s) ds \quad (2.3)$$

and

$$\beta(\lambda_0) = b\tau e^{-\lambda_0\tau} + ce^{-\lambda_0\sigma}(1 - \lambda_0\sigma). \quad (2.4)$$

Note: Property (2.1) guarantees that $0 < 1 + \beta(\lambda_0) < 2$.

We clearly see that $\lambda_0 = 0$ is a root of (1.7) with the property (2.1) if and only if the following conditions hold:

$$a + b = 0 \quad \text{and} \quad |b|\tau + |c| + \sum_{i=1}^{\infty} |\ell_i| < 1. \quad (2.5)$$

So, application of the Theorem 2.1 with $\lambda_0 = 0$ leads to the following corollary.

Corollary 2.2. Assume that the conditions in (2.5) are provided. Then the solution x of (1.1)-(1.3) satisfies

$$\lim_{t \rightarrow \infty} x(t) = \frac{\phi(0) + c\phi(-\sigma) + b \int_{-\tau}^0 \phi(s) ds}{1 + b\tau + c}.$$

Note: Because of (2.5) second condition ensures that $0 < 1 + b\tau + c < 2$.

Now, let's prove the following Theorem 2.1.

Proof. Property (2.1) implies $|\beta(\lambda_0)| + \sum_{i=1}^{\infty} |\ell_i| e^{-\lambda_0 t_i} \leq \mu(\lambda_0) < 1$. So, in particular, it holds $|\beta(\lambda_0)| < 1$. This implies that $0 < 1 + \beta(\lambda_0) < 2$.

Let x be the solution of (1.1)-(1.3). Let's define the function y by (1.8). An x solution of (1.1)-(1.3) will be equivalent to a y solution of the integral equation (1.12) which provides the initial condition (1.10). Then, since λ_0 is a real root of (1.7), we get for all $t \geq 0$,

$$\begin{aligned} y(t) + ce^{-\lambda_0\sigma} y(t - \sigma) &= L(\lambda_0; \phi) + \sum_{i=1}^{n(t)} \ell_i e^{-\lambda_0 t_i} \\ &+ (a - \lambda_0) \int_0^t y(s) ds + be^{-\lambda_0\tau} \int_0^{t-\tau} y(s) ds - c\lambda_0 e^{-\lambda_0\sigma} \int_0^{t-\sigma} y(s) ds, \\ y(t) + ce^{-\lambda_0\sigma} y(t - \sigma) &= L(\lambda_0; \phi) + \sum_{i=1}^{n(t)} \ell_i e^{-\lambda_0 t_i} \\ &+ (c\lambda_0 e^{-\lambda_0\sigma} - be^{-\lambda_0\tau}) \int_0^t y(s) ds + be^{-\lambda_0\tau} \int_0^{t-\tau} y(s) ds \\ &- c\lambda_0 e^{-\lambda_0\sigma} \int_0^{t-\sigma} y(s) ds, \end{aligned}$$

$$\begin{aligned}
y(t) + ce^{-\lambda_0\sigma}y(t-\sigma) &= L(\lambda_0; \phi) + \sum_{i=1}^{n(t)} \ell_i e^{-\lambda_0 t_i} \\
&\quad - be^{-\lambda_0\tau} \int_{t-\tau}^t y(s)ds + c\lambda_0 e^{-\lambda_0\sigma} \int_{t-\sigma}^t y(s)ds.
\end{aligned} \tag{2.6}$$

Next, we set for $t \geq -h$

$$z(t) = y(t) - \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)}.$$

In this case, it is reduced from the equation (2.6) to the following equation for $t \geq 0$

$$\begin{aligned}
z(t) + ce^{-\lambda_0\sigma}z(t-\sigma) &= \sum_{i=1}^{n(t)} \ell_i e^{-\lambda_0 t_i} \\
&\quad - be^{-\lambda_0\tau} \int_{t-\tau}^t z(s)ds + c\lambda_0 e^{-\lambda_0\sigma} \int_{t-\sigma}^t z(s)ds.
\end{aligned} \tag{2.7}$$

Moreover, (1.10) is written as for $t \in [-h, 0]$

$$z(t) = e^{-\lambda_0 t} \phi(t) - \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)}. \tag{2.8}$$

According to the definition of y and z , we need to show that equality (2.2), that is,

$$\lim_{t \rightarrow \infty} z(t) = 0. \tag{2.9}$$

Put

$$H(\lambda_0; \phi) = \max \left\{ 1, \max_{t \in [-h, 0]} \left| e^{-\lambda_0 t} \phi(t) - \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)} \right| \right\}.$$

Then, from (2.8) we get

$$|z(t)| \leq H(\lambda_0; \phi) \quad \text{for } -h \leq t \leq 0. \tag{2.10}$$

Now, we will prove the following inequality, i.e.

$$|z(t)| \leq H(\lambda_0; \phi) \quad \text{for } t \geq -h. \tag{2.11}$$

For the sake of contradiction suppose that there exists a point $\tilde{t} > 0$ such that $|z(\tilde{t})| > H(\lambda_0; \phi)$. Let

$$t^* = \inf \{ \tilde{t} : |z(\tilde{t})| > H(\lambda_0; \phi) \}.$$

By right continuity, either $|z(t^*)| = H(\lambda_0; \phi)$ if there is no impulsive point at t^* , or $|z(t^*)| \geq H(\lambda_0; \phi)$ as a consequence of a jump at t^* . Whatever the case, using right continuity, we have $|z(t)| \leq H(\lambda_0; \phi)$ for $-h \leq t < t^*$, where $|z(t^*)| = H(\lambda_0; \phi)$ if this occurs at a non-impulsive point. Then, using (2.1), by the integral representation of $z(t)$, which all solutions to (2.7), we have that

$$\begin{aligned}
|z(t^*)| &= \left| -ce^{-\lambda_0\sigma} z(t^* - \sigma) + \sum_{i=1}^{n(t^*)} \ell_i e^{-\lambda_0 t_i} \right. \\
&\quad \left. - be^{-\lambda_0\tau} \int_{t^*-\tau}^{t^*} z(s) ds + c\lambda_0 e^{-\lambda_0\sigma} \int_{t^*-\sigma}^{t^*} z(s) ds \right| \\
&\leq |c|e^{-\lambda_0\sigma} |z(t^* - \sigma)| + 1 \cdot \sum_{i=1}^{n(t^*)} |\ell_i| e^{-\lambda_0 t_i} \\
&\quad + |b|e^{-\lambda_0\tau} \int_{t^*-\tau}^{t^*} |z(s)| ds + |c|\lambda_0 e^{-\lambda_0\sigma} \int_{t^*-\sigma}^{t^*} |z(s)| ds \\
&\leq H(\lambda_0; \phi) \left(|c|e^{-\lambda_0\sigma} + \sum_{i=1}^{n(t^*)} |\ell_i| e^{-\lambda_0 t_i} + |b|\tau e^{-\lambda_0\tau} + |c|\lambda_0 \sigma e^{-\lambda_0\sigma} \right) \\
&\leq H(\lambda_0; \phi) \mu(\lambda_0) < H(\lambda_0; \phi)
\end{aligned}$$

and this gives us the desired contradiction, by the definition of t^* , since we proved $|z(t^*)| < H(\lambda_0; \phi)$, and we assumed $|z(t^*)| = H(\lambda_0; \phi)$ if t^* is a continuity point, or $|z(t^*)| \geq H(\lambda_0; \phi)$ if t^* is a discontinuity point. Thus, inequality (2.11) is provided.

Now, using (2.11), from (2.7) we derive for $t \geq 0$,

$$\begin{aligned}
|z(t)| &= \left| -ce^{-\lambda_0\sigma} z(t - \sigma) + \sum_{i=1}^{n(t)} \ell_i e^{-\lambda_0 t_i} \right. \\
&\quad \left. - be^{-\lambda_0\tau} \int_{t-\tau}^t z(s) ds + c\lambda_0 e^{-\lambda_0\sigma} \int_{t-\sigma}^t z(s) ds \right| \\
&\leq |c|e^{-\lambda_0\sigma} |z(t - \sigma)| + 1 \cdot \sum_{i=1}^{n(t)} |\ell_i| e^{-\lambda_0 t_i} \\
&\quad + |b|e^{-\lambda_0\tau} \int_{t-\tau}^t |z(s)| ds + |c|\lambda_0 e^{-\lambda_0\sigma} \int_{t-\sigma}^t |z(s)| ds \\
&\leq \left(\sum_{i=1}^{n(t)} |\ell_i| e^{-\lambda_0 t_i} + |b|\tau e^{-\lambda_0\tau} + |c|e^{-\lambda_0\sigma} (1 + |\lambda_0|)\sigma \right) H(\lambda_0; \phi) \\
&\leq \mu(\lambda_0) H(\lambda_0; \phi),
\end{aligned}$$

namely that, we have

$$|z(t)| \leq \mu(\lambda_0) H(\lambda_0; \phi) \quad \text{for } t \geq 0. \quad (2.12)$$

Using (2.1), (2.11) and (2.12), one can show, by an easy induction, from (2.7) we have

$$|z(t)| \leq [\mu(\lambda_0)]^n H(\lambda_0; \phi) \quad \text{for } t \geq nh - h \quad (n = 0, 1, \dots). \quad (2.13)$$

Because of (2.1), we obtain $\lim_{n \rightarrow \infty} [\mu(\lambda_0)]^n = 0$. Thus, from (2.13) we have that

$$\lim_{t \rightarrow \infty} z(t) = \lim_{t \rightarrow \infty} \left\{ e^{-\lambda_0 t} x(t) - \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)} \right\} = 0,$$

i.e. (2.9) holds. So, we get

$$\lim_{t \rightarrow \infty} [e^{-\lambda_0 t} x(t)] = \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)}.$$

The proof of Theorem 2.1 is complete. $\square$

3. An exponential boundary of the solutions and stability criterion

Theorem 3.1. Suppose that Lemma 1.1 holds and assume that $\mu(\lambda_0) < 1$, where $\mu(\lambda_0)$ is defined by (2.1). Let $\beta(\lambda_0)$ be defined by (2.4) and set

$$R(\lambda_0; \phi) = \max \left\{ 1, \max_{-h \leq t \leq 0} |\phi(t)|, \max_{-h \leq t \leq 0} [e^{-\lambda_0 t} |\phi(t)|] \right\}. \quad (3.1)$$

Then the solution x of (1.1)-(1.3) satisfies

$$|x(t)| \leq N(\lambda_0) R(\lambda_0; \phi) e^{\lambda_0 t}, \quad \text{for all } t \geq 0, \quad (3.2)$$

where

$$N(\lambda_0) = \mu(\lambda_0) + k(\lambda_0) \left(\frac{1 + \mu(\lambda_0)}{1 + \beta(\lambda_0)} \right) \quad (3.3)$$

and

$$k(\lambda_0) = 1 + |b| \tau e^{-\lambda_0 \tau} + |c| (1 + |\lambda_0| \sigma e^{-\lambda_0 \sigma}).$$

Note: Property (2.1) guarantees that $0 < 1 + \beta(\lambda_0) < 2$.

Moreover, the zero solution of (1.1)-(1.2) is:

(i) stable if $\lambda_0 = 0$ or, equivalently, if the (2.5) conditions are satisfied,

(ii) asymptotically stable if $\lambda_0 < 0$, and

(iii) unstable if $\lambda_0 > 0$.

Proof. Let x be the solution of (1.1)-(1.3) and let y and z be defined as given above, i.e. for $t \geq -h$

$$y(t) = e^{-\lambda_0 t} x(t) \quad \text{and} \quad z(t) = y(t) - \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)},$$

where $L(\lambda_0; \phi)$ is defined as in (2.3). Moreover, let $H(\lambda_0; \phi)$ be given as in the proof of Theorem 2.1., i.e.

$$H(\lambda_0; \phi) = \max \left\{ 1, \max_{t \in [-h, 0]} \left| e^{-\lambda_0 t} \phi(t) - \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)} \right| \right\}.$$

Then, as in the proof of Theorem 2.1, we can show that z satisfies inequality (2.12). By using of z , from (2.12) we get for $t \geq 0$

$$|y(t)| \leq \mu(\lambda_0) H(\lambda_0; \phi) + \frac{|L(\lambda_0; \phi)|}{1 + \beta(\lambda_0)}. \quad (3.4)$$

By using (2.1) and (3.1), from (2.3) we get

$$\begin{aligned} |L(\lambda_0; \phi)| &\leq |\phi(0)| + |c||\phi(-\sigma)| + |b|e^{-\lambda_0\tau} \int_{-\tau}^0 e^{-\lambda_0 s} |\phi(s)| ds \\ &\quad + |c||\lambda_0|e^{-\lambda_0\sigma} \int_{-\sigma}^0 e^{-\lambda_0 s} |\phi(s)| ds \\ &\leq (1 + |c| + |b|\tau e^{-\lambda_0\tau} + |c||\lambda_0|\sigma e^{-\lambda_0\sigma}) R(\lambda_0; \phi) \\ &= k(\lambda_0)R(\lambda_0; \phi). \end{aligned}$$

On the other hand, from the definition of $H(\lambda_0; \phi)$ we obtain

$$\begin{aligned} H(\lambda_0; \phi) &\leq \max \left\{ 1, R(\lambda_0; \phi) + \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)} \right\} = R(\lambda_0; \phi) + \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)} \\ &\leq R(\lambda_0; \phi) + \frac{k(\lambda_0)R(\lambda_0; \phi)}{1 + \beta(\lambda_0)} = \left(1 + \frac{k(\lambda_0)}{1 + \beta(\lambda_0)} \right) R(\lambda_0; \phi). \end{aligned}$$

So, by using (3.3), from (3.4) we have for $t \geq 0$

$$\begin{aligned} y(t) &\leq \mu(\lambda_0) \left(1 + \frac{k(\lambda_0)}{1 + \beta(\lambda_0)} \right) R(\lambda_0; \phi) + \frac{k(\lambda_0)R(\lambda_0; \phi)}{1 + \beta(\lambda_0)} \\ &= \left\{ \mu(\lambda_0) \left(1 + \frac{k(\lambda_0)}{1 + \beta(\lambda_0)} \right) + \frac{k(\lambda_0)}{1 + \beta(\lambda_0)} \right\} R(\lambda_0; \phi) \\ &= N(\lambda_0)R(\lambda_0; \phi). \end{aligned}$$

Finally, from the definition of y we get

$$|x(t)| \leq N(\lambda_0)R(\lambda_0; \phi)e^{\lambda_0 t}, \quad \text{for all } t \geq 0.$$

Thus, we have completed the proof of the first part of this theorem. Now let's prove the second part (stability criterion) of the theorem.

Assume that $\lambda_0 = 0$ and let $\phi \in C([-h, 0], \mathbf{R})$ be an any initial function and let x be the solution of (1.1)-(1.3). Then (3.2) holds and hence

$$|x(t)| \leq N(\lambda_0)R(\lambda_0; \phi) \quad \text{for all } t \geq 0.$$

Since $\|\phi\| = \max_{-h \leq t \leq 0} |\phi(t)| \leq R(\lambda_0; \phi)$ and $N(\lambda_0) > 1$, it follows that

$$|x(t)| \leq N(\lambda_0)R(\lambda_0; \phi) \quad \text{for all } t \geq -h.$$

For any $\epsilon > 0$, we choose $\delta = \frac{\epsilon}{N(\lambda_0)}$ such that $R(\lambda_0; \phi) < \delta$, we obtain that $\|\phi\| < \delta$. Then

$$|x(t)| \leq N(\lambda_0)R(\lambda_0; \phi) < N(\lambda_0)\delta = \epsilon.$$

From the last inequality, it is obtained that the zero solution of (1.1)-(1.2) is stable. Moreover, if $\lambda_0 < 0$, then again from the Theorem 3.1, we conclude that the above inequality for each $\phi \in C([-h, 0], \mathbf{R})$ is true, i.e.

$$|x(t)| \leq N(\lambda_0)R(\lambda_0; \phi) \quad \text{for all } t \geq 0.$$

Thus, as above, we concluded that the zero solution of (1.1)-(1.2) is stable. In addition, the inequality (3.2) guarantees that

$$\lim_{t \rightarrow \infty} x(t) = 0.$$

So, for $\lambda_0 < 0$ the zero solution of (1.1)-(1.2) is asymptotically stable. Finally, we assume that $\lambda_0 > 0$ and we will prove that the solution of the zero of (1.1)-(1.2) is unstable. For the contradiction, assume that the zero solution of (1.1)-(1.2) is stable. Then we can select the number $\delta > 0$ such that, for each $\phi \in C([-h, 0], \mathbf{R})$ with $\|\phi\| < \delta$, the solution x of (1.1)-(1.3) satisfies

$$|x(t)| < 1 \quad \text{for all } t \geq -h. \quad (3.5)$$

Set

$$\phi_0(t) = e^{\lambda_0 t} \quad \text{for } t \in [-h, 0].$$

We see that $\phi_0 \in C([-h, 0], \mathbf{R})$, from (2.3) we have

$$\begin{aligned} L(\lambda_0; \phi_0) &= \phi_0(0) + c\phi_0(-\sigma) + be^{-\lambda_0\tau} \int_{-\tau}^0 e^{-\lambda_0 s} \phi_0(s) ds \\ &\quad - c\lambda_0 e^{-\lambda_0\sigma} \int_{-\sigma}^0 e^{-\lambda_0 s} \phi_0(s) ds \\ &= 1 + b\tau e^{-\lambda_0\tau} + ce^{-\lambda_0\sigma}(1 - \lambda_0\sigma) = 1 + \beta(\lambda_0). \end{aligned}$$

Because of (2.1), we obtain

$$L(\lambda_0; \phi_0) = 1 + \beta(\lambda_0) > 0. \quad (3.6)$$

Next, we consider a number $\delta_0 > 0$ with $0 < \delta_0 < \delta$ and we put

$$\phi = \frac{\delta_0}{\|\phi_0\|} \phi_0.$$

Clearly, ϕ belongs to $C([-h, 0], \mathbf{R})$ and $\|\phi\| = \delta_0 < \delta$. Hence, for this initial function, the solution x of (1.1)-(1.3) satisfies (3.5). On the other hand, considering the linearity of the operator $L(\lambda_0; \cdot)$ and using (3.6), from (2.2) we obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} [e^{-\lambda_0 t} x(t)] &= \frac{L(\lambda_0; \phi)}{1 + \beta(\lambda_0)} \\ &= \frac{(\delta_0 / \|\phi_0\|) L(\lambda_0; \phi_0)}{1 + \beta(\lambda_0)} = \frac{\delta_0}{\|\phi_0\|} > 0. \end{aligned}$$

But, since $\lambda_0 > 0$, from (3.5) it follows that

$$\lim_{t \rightarrow \infty} [e^{-\lambda_0 t} x(t)] = 0.$$

We reached a contradiction here and the second part of the theorem was proved. Thus, the proof of Theorem 3.1 is now completed. $\square$

4. Application of the main results to the special case of the IDDEs

In this section, we will concentrate on the linear impulsive (non-neutral) delay differential system (1.4)-(1.6) and we shall apply our main results to this system.

The characteristic equation of (1.4) is

$$\lambda = a + be^{-\lambda\tau}. \quad (4.1)$$

Let λ_0 be a real root of (4.1). Let's define

$$y(t) = e^{-\lambda_0 t} x(t) \quad \text{for all } t \in [-\tau, \infty). \quad (4.2)$$

Then, from (1.4) we get for all $t \geq 0$

$$y'(t) = (a - \lambda_0)y(t) + be^{-\lambda_0\tau}y(t - \tau). \quad (4.3)$$

We can also write the initial condition (1.6) equivalent to

$$y(t) = e^{-\lambda_0 t} \phi(t) \quad \text{for } t \in [-\tau, 0]. \quad (4.4)$$

Furthermore, if $x(t)$ satisfies the impulsive conditions (1.5), then evidently

$$y(t_k) - y(t_k^-) = \ell_k e^{-\lambda_0 t_k}, \quad k \in \mathbf{Z}^+. \quad (4.5)$$

For the impulsive delay differential system (1.4)-(1.6), the following results hold.

Lemma 4.1. Assume that λ_0 be a real root of the characteristic equation (4.1). Then y is the unique solution of (4.3)-(4.5) if and only if y is a solution of the following system

$$y(t) = e^{-\lambda_0 t} \phi(t), \quad \text{if } -\tau \leq t \leq 0$$

and

$$\begin{aligned} y(t) = & \phi(0) + be^{-\lambda_0\tau} \int_{-\tau}^0 e^{-\lambda_0 s} \phi(s) ds + \sum_{i=1}^{n(t)} \ell_i e^{-\lambda_0 t_i} \\ & (a - \lambda_0) \int_0^t y(s) ds + be^{-\lambda_0\tau} \int_0^{t-\tau} y(s) ds, \quad \text{if } t \geq 0 \end{aligned} \quad (4.6)$$

where

$$n(t) = \max\{k \in \mathbf{Z}^+ : t_k \leq t\} \quad \text{and} \quad n(t) = 0 \quad \text{if } t < t_1.$$

Corollary 4.2. Assume that λ_0 be a real root of the characteristic equation (4.1). Then x is the unique solution of (1.4)-(1.6) if and only if the function y defined by

$$y(t) = e^{-\lambda_0 t} x(t) \quad \text{for all } t \in [-\tau, \infty)$$

is the solution of the integral equation (4.6) which provides the initial condition

$$y(t) = e^{-\lambda_0 t} \phi(t) \quad \text{for all } t \in [-\tau, 0].$$

Theorem 4.3. Assume that λ_0 be a real root of (4.1) with the property

$$\mu(\lambda_0) = |b|\tau e^{-\lambda_0 \tau} + \sum_{i=1}^{\infty} |\ell_i| e^{-\lambda_0 t_i} < 1. \quad (4.7)$$

Then the solution x of (1.4)-(1.6) satisfies

$$\lim_{t \rightarrow \infty} [e^{-\lambda_0 t} x(t)] = \frac{L(\lambda_0; \phi)}{1 + b\tau e^{-\lambda_0 \tau}},$$

where

$$L(\lambda_0; \phi) = \phi(0) + b e^{-\lambda_0 \tau} \int_{-\tau}^0 e^{-\lambda_0 s} \phi(s) ds.$$

Note: Property (4.7) guarantees that $0 < 1 + b\tau e^{-\lambda_0 \tau} < 2$.

Application of the Theorem 4.3 with $\lambda_0 = 0$ leads to the following corollary.

Corollary 4.4. Assume that

$$a + b = 0 \quad \text{and} \quad |b|\tau + \sum_{i=1}^{\infty} |\ell_i| < 1. \quad (4.8)$$

Then the solution x of (1.4)-(1.6) satisfies

$$\lim_{t \rightarrow \infty} x(t) = \frac{\phi(0) + b \int_{-\tau}^0 \phi(s) ds}{1 + b\tau}.$$

Note: Because of (4.8) second condition ensures that $0 < 1 + b\tau < 2$.

Theorem 4.5. Assume that λ_0 be a real root of (4.1) with the property (4.7) and set

$$R(\lambda_0; \phi) = \max \left\{ 1, \max_{-\tau \leq t \leq 0} |\phi(t)|, \max_{-\tau \leq t \leq 0} [e^{-\lambda_0 t} |\phi(t)|] \right\}.$$

Then the solution x of (1.4)-(1.6) satisfies

$$|x(t)| \leq N(\lambda_0) R(\lambda_0; \phi) e^{\lambda_0 t} \quad \text{for } t \geq 0,$$

where

$$N(\lambda_0) = \mu(\lambda_0) + (1 + \mu(\lambda_0)) \left(\frac{1 + |b|\tau e^{-\lambda_0 \tau}}{1 + b\tau e^{-\lambda_0 \tau}} \right).$$

Moreover, the zero solution of (1.4)-(1.6) is:

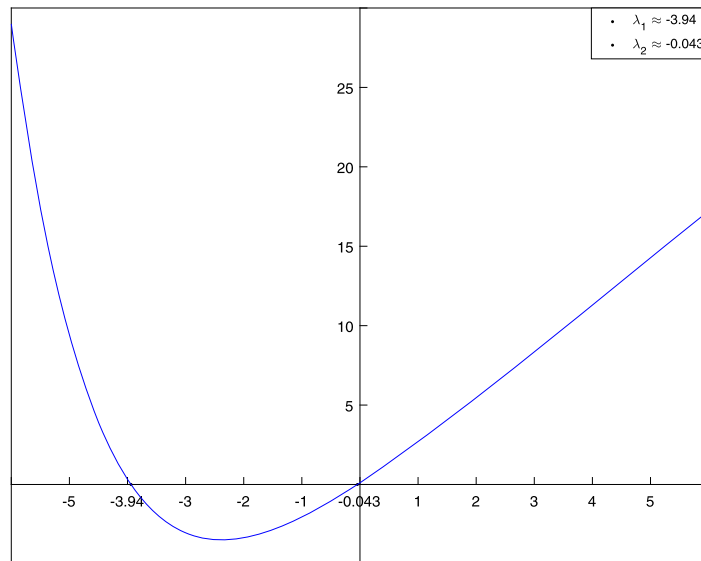


Fig. 5.1. Locations of the roots of (5.2).

- (i) stable if $\lambda_0 = 0$ or, equivalently, if the (4.8) conditions are satisfied,
- (ii) asymptotically stable if $\lambda_0 < 0$, and
- (iii) unstable if $\lambda_0 > 0$.

5. Examples

In the following examples, we will apply the stability criteria of the Theorem 3.1.

Example 5.1. Consider

$$\begin{aligned} [x(t) - \frac{1}{3e}x(t - \frac{1}{2})]' &= \frac{1}{3}x(t) - \frac{1}{e}x(t - \frac{1}{5}), \quad t \neq k, \quad t \geq 0, \\ \Delta x(t_k) &= \left(-\frac{1}{e}\right)^k, \quad k \in \mathbf{Z}^+, \\ x(t) &= \phi(t), \quad -\frac{1}{2} \leq t \leq 0, \end{aligned} \quad (5.1)$$

where $\phi \in C\left[-\frac{1}{2}, 0\right], \mathbf{R}$.

The characteristic equation of (5.1) is

$$\lambda \left(1 - \frac{1}{3e}e^{-\frac{\lambda}{2}}\right) = \frac{1}{3} - \frac{1}{e}e^{-\frac{\lambda}{5}}$$

or

$$\lambda \left(3 - e^{-\frac{\lambda}{2}-1}\right) = 1 - 3e^{-\frac{\lambda}{5}-1}. \quad (5.2)$$

We have $\lambda_1 \approx -3.94$ and $\lambda_2 \approx -0.043$ are real roots of (5.2). In Fig. 5.1, we give the locations of the roots of (5.2). Let $\lambda_0 = -3.94$. Then, the last term in $\mu(\lambda_0)$ is $\sum_{i=1}^{\infty} \left|(-\frac{1}{e})^i\right| e^{3.94i} = \sum_{i=1}^{\infty} e^{2.94i} \rightarrow \infty$. Therefore, Theorem 3.1 cannot be applied to equation (5.1). But, for $\lambda_0 = -0.043$, we get

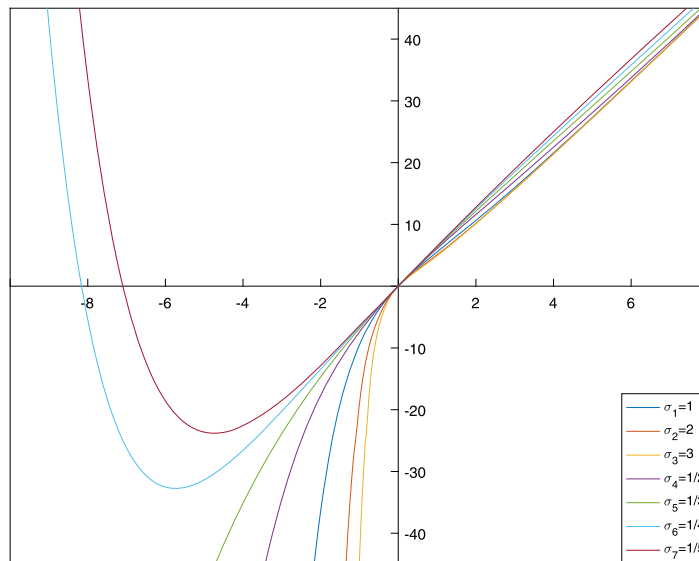


Fig. 5.2. Locations of the roots of (5.4).

$$\begin{aligned}\mu(\lambda_0) &= \mu(-0.043) = \frac{1}{5e}e^{0.043/5} + \frac{1}{3e}e^{0.043/2}(1 + 0.043/2) + \sum_{i=1}^{\infty} e^{-0.957i} \\ &\approx 0.83 < 1.\end{aligned}$$

Then the zero solution of (5.1) is asymptotically stable by Theorem 3.1.

Example 5.2. Consider

$$\begin{aligned}[x(t) - \frac{1}{3}x(t - \sigma)]' &= \frac{1}{2}x(t) - \frac{1}{2}x(t - \frac{1}{2}), \quad t \neq t_k, \quad t \geq 0, \\ \Delta x(t_k) &= \left(\frac{1}{4}\right)^k, \quad k \in \mathbf{Z}^+, \\ x(t) &= \phi(t), \quad -h \leq t \leq 0,\end{aligned}\tag{5.3}$$

where $\sigma > 0$ is any a constant, $h = \max\{\sigma, \frac{1}{2}\}$, $\phi \in C([-h, 0], \mathbf{R})$ and t_k are arbitrary impulsive points, such that $t_k - \sigma$ are not impulsive points for all $k \in \mathbf{Z}^+$.

The characteristic equation of (5.3) is

$$2\lambda(3 + e^{-\lambda\sigma}) = 3(1 - e^{-\frac{\lambda}{2}}).\tag{5.4}$$

We see easily that $\lambda = 0$ is real root of (5.4). In Fig. 5.2, we give for different σ values the locations of the roots of (5.4). Then for $\lambda_0 = 0$, we get

$$\mu(\lambda_0) = \mu(0) = \frac{1}{4} + \frac{1}{3} + \sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^i = \frac{11}{12} < 1.$$

Thus, the zero solution of (5.3) is stable by Theorem 3.1.

Example 5.3. Consider

$$[x(t) + \frac{1}{2}x(t - 1)]' = x(t) + \frac{1}{2}x(t - 1), \quad t \neq 2k, \quad t \geq 0,\tag{5.5}$$

$$\begin{aligned}\Delta x(t_k) &= k, \quad k \in \mathbf{Z}^+, \\ x(t) &= \phi(t), \quad -1 \leq t \leq 0,\end{aligned}$$

where $\phi \in C([-1, 0], \mathbf{R})$.

The characteristic equation of (5.5) is

$$\lambda(2 + e^{-\lambda}) = (2 + e^{-\lambda}). \quad (5.6)$$

We see easily that $\lambda = 1$ is unique real root of (5.6). Then for $\lambda_0 = 1$, we get

$$\mu(\lambda_0) = \mu(1) = \frac{1}{2e} + \frac{1}{e} + \sum_{i=1}^{\infty} ie^{-2i} \approx 0.73 < 1.$$

Thus, the zero solution of (5.5) is unstable by Theorem 3.1.

6. Conclusions

In this study, we have proved that there is a fundamental asymptotic criterion for the solutions of the initial value problem (1.1)-(1.3). We obtained a useful exponential boundary for the solutions of the initial value problem (1.1)-(1.3) and the stability of the zero solution of the equation (1.1) was shown. These results were obtained using an appropriate real root for the characteristic equation. We also gave three examples for stability.

As known, the solutions for delay or neutral differential equations are difficult to obtain. However, there have been studies on the solutions of delay or neutral differential equations (for example, see [40–42] and references therein). In [41], the authors obtained very interesting results for solutions to neutral differential equations with delay. That is, the authors showed a representation of the solutions of these equations using a piecewise defined matrix polynomial. Furthermore, by using [40–42], the stability, oscillatory, controllability and other behaviors of solutions of delay or neutral differential equations can be achieved. For example, by using [41], a necessary and sufficient condition was obtained for the relative controllability of the neutral differential equations with delay (see [43]). In recent years, the authors in [52–54] showed a representation of the solutions of IDDEs and obtained the stability of the trivial solution. With similar thinking, a representation of solutions of the INDDEs can be shown and the stability of the trivial solution can be ensured. But it seems to be more complex and more difficult than IDDEs. In this article, we have obtained a different method for stability without solving linear INDDEs. In other words, by converting the constructed equation into two integral equations, we determined the stability of the zero solution.

It would be interesting to use the same method for stability of second order linear impulsive delay differential equations. It will be the subject of a future study to extend the current results to second order linear impulsive delay differential equations with constant coefficients.

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