



Incompressible limit of the compressible nematic liquid crystal flows in a bounded domain with perfectly conducting boundary



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ABSTRACT

In this paper, we study the asymptotic behavior of the regular solution to a simplified Ericksen–Leslie model for the compressible nematic liquid crystal flow in a bounded smooth domain in \mathbb{R}^2 as the Mach number tends to zero. The evolution system consists of the compressible Navier–Stokes equations coupled with the transported heat flow for the averaged molecular orientation. We suppose that the Navier–Stokes equations are characterized by a Navier’s slip boundary condition, while the transported heat flow is subject to Neumann boundary condition. By deriving a differential inequality with certain decay property, the low Mach limit of the solutions is verified for all time, provided that the initial data are well-prepared.

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1. Introduction

Liquid crystals can form and remain in an intermediate phase of matter between liquids and solids. Liquid crystal materials generally have some common characteristics. Among these are rod-like molecular structure, rigidity of the long axis, and strong dipoles and easily polarizable substituents. Nematics, smectics, and columnar (or cholesteric) phases are the common forms of liquid crystals. Nematic liquid crystals exhibit long-range ordering in the sense that their rigid rod-like molecules arrange themselves with their long axes parallel to each other. Their molecules float around as in a liquid, but have the tendency to align along a preferred direction due to their orientation. The hydrodynamic theory of the nematic liquid crystals, due to Ericksen and Leslie, was developed during the period of 1958 through 1968 (see [8,23]). Since then, many remarkable developments have been made from both theoretical and applied aspects.

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In this paper, we mainly study the low Mach number limit for an initial boundary value problem of the following two-dimensional simplified Leslie-Erickson model, which describes the motion of a slightly compressible flow of nematic liquid crystals, in a bounded smooth domain $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} \partial_t \varrho^\varepsilon + \operatorname{div}(\varrho^\varepsilon \mathbf{u}^\varepsilon) = 0, \\ \partial_t(\varrho^\varepsilon \mathbf{u}^\varepsilon) + \operatorname{div}(\varrho^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) + \frac{1}{\varepsilon^2} \nabla P = \operatorname{div}(2\mu D(\mathbf{u}^\varepsilon)) + \lambda \nabla \operatorname{div} \mathbf{u}^\varepsilon - \nu \operatorname{div}(\nabla \mathbf{d}^\varepsilon \odot \nabla \mathbf{d}^\varepsilon - \frac{1}{2} |\nabla \mathbf{d}^\varepsilon|^2 \mathbb{I}), \\ \partial_t \mathbf{d}^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{d}^\varepsilon = \theta (\Delta \mathbf{d}^\varepsilon + |\nabla \mathbf{d}^\varepsilon|^2 \mathbf{d}^\varepsilon), \quad |\mathbf{d}^\varepsilon| = 1, \end{cases} \quad (1.1)$$

where $\varepsilon \in (0, 1]$ is the Mach number, $\varrho^\varepsilon : \Omega \times [0, +\infty) \rightarrow \mathbb{R}$ is the density function of the fluid, $\mathbf{u}^\varepsilon : \Omega \times [0, +\infty) \rightarrow \mathbb{R}^2$ represents velocity field of the fluid, $\mathbf{d}^\varepsilon : \Omega \times [0, +\infty) \rightarrow \mathbb{S}^2$ (\mathbb{S}^2 is the unit sphere in \mathbb{R}^3) represents the macroscopic/continuum molecule orientation of the nematic liquid crystal. $D(\mathbf{u}^\varepsilon) = (\nabla \mathbf{u}^\varepsilon + (\nabla \mathbf{u}^\varepsilon)^T)/2$, where $(\nabla \mathbf{u}^\varepsilon)^T$ denotes the transpose of the 2×2 matrix $\nabla \mathbf{u}^\varepsilon$. \mathbb{I} denotes the 2×2 identical matrix. The unusual term $\nabla \mathbf{d}^\varepsilon \odot \nabla \mathbf{d}^\varepsilon$ denotes the 2×2 matrix whose (i, j) -th entry is given by $\partial_i \mathbf{d}^\varepsilon \cdot \partial_j \mathbf{d}^\varepsilon$, for $1 \leq i, j \leq 2$, i.e., $\nabla \mathbf{d}^\varepsilon \odot \nabla \mathbf{d}^\varepsilon = (\nabla \mathbf{d}^\varepsilon)^T \nabla \mathbf{d}^\varepsilon$. The constants μ, λ, ν and θ are the shear viscosity, the bulk viscosity, the competition between kinetic energy and potential energy, and the microscopic elastic relation time for the molecular orientation field, respectively, which satisfied the following physical condition:

$$\mu > 0, \quad \mu + \lambda > 0, \quad \nu > 0, \quad \theta > 0.$$

$P = P(\varrho^\varepsilon)$ represents the pressure function and in this paper we consider the case of isentropic flows

$$P(\varrho^\varepsilon) = a(\varrho^\varepsilon)^\gamma \quad \text{with } a > 0, \gamma > 1.$$

We supplement the system with the following initial data

$$(\varrho^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon)(\mathbf{x}, 0) = (\varrho_0^\varepsilon(\mathbf{x}), \mathbf{u}_0^\varepsilon(\mathbf{x}), \mathbf{d}_0^\varepsilon(\mathbf{x})) \quad \text{in } \mathbf{x} \in \Omega. \quad (1.2)$$

We specify the velocity on the boundary with Navier's slip boundary condition, i.e.,

$$\mathbf{u}^\varepsilon \cdot \mathbf{n} = 0, \quad \tau \cdot \mathcal{T}(\mathbf{u}^\varepsilon, P(\varrho^\varepsilon)) \cdot \mathbf{n} + \alpha \mathbf{u}^\varepsilon \cdot \tau = 0 \quad \text{on } \mathbf{x} \in \partial\Omega, \quad (1.3)$$

with $\alpha \geq 0$ is a given function, where $\mathcal{T}(\mathbf{u}^\varepsilon, P(\varrho^\varepsilon)) := 2\mu D(\mathbf{u}^\varepsilon) + (\lambda \operatorname{div} \mathbf{u}^\varepsilon - \frac{1}{\varepsilon^2} P(\varrho^\varepsilon)) \mathbb{I}$ is the stress tensor, \mathbf{n} and τ are the normal and the tangent vectors on $\partial\Omega$, respectively. The boundary condition (1.3) can be used to describe the interaction between a fluid and a wall, and an example of such condition is the motion of oil in a pipe or container. For more description and the background on the this boundary condition, we refer the readers to [1, 42]. We also require the orientation on the boundary satisfying Neumann condition

$$\mathbf{n} \cdot \nabla \mathbf{d}^\varepsilon = \mathbf{0} \quad \text{for } \mathbf{x} \in \partial\Omega. \quad (1.4)$$

The above simplified Erickson–Leslie model (1.1), first introduced by Lin in [25], has recently attracted a lot of attention of applied mathematicians because of its physical application and mathematical challenges, see, for example, [12–16, 24, 30, 34, 38, 41] and their references therein. Mathematically, system (1.1)–(1.2) is a strongly coupled system between the compressible Navier–Stokes equations (the case $\mathbf{d} \equiv \mathbf{e}_3$, see e.g., [33, 35, 36, 42]) and the transported heat flows of harmonic map (the case $\varrho = \text{constant}$ and $\mathbf{u} \equiv \mathbf{0}$, see e.g., [39]), and thus, its mathematical analysis is full of challenges.

For system (1.1), the nonlinear term $|\nabla \mathbf{d}|^2 \mathbf{d}$ with the restriction $|\mathbf{d}| = 1$ causes significant mathematical difficulties. When the fluid is an incompressible viscous fluid (i.e., $\varrho \equiv 1$), there have been many attempts on rigorous mathematical analysis on it, see, for example, [10, 25, 27–29, 31, 34, 39, 41] and their references therein. If the supercritical nonlinear term $|\nabla \mathbf{d}|^2 \mathbf{d}$ in (1.1)₃ is replaced by the Ginzburg–Landau approximation $\frac{1}{\eta}(1 -$

$|\mathbf{d}|^2)\mathbf{d}$ ($\eta > 0$), thus the Dirichlet energy $\int_{\Omega} \frac{1}{2}|\nabla \mathbf{d}|^2 d\mathbf{x}$ for $d : \Omega \rightarrow \mathbb{S}^2$ is replaced by the Ginzburg-Landau energy $\int_{\Omega} (\frac{1}{2}|\nabla \mathbf{d}|^2 + \frac{1}{4\eta^2}(1 - |\mathbf{d}|^2)^2) d\mathbf{x}$ ($\varepsilon > 0$) for $\mathbf{d} : \Omega \rightarrow \mathbb{R}^3$, that is, equation (1.1)₃ is replaced by

$$\mathbf{d}_t + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} + \frac{1}{\eta^2}(1 - |\mathbf{d}|^2)\mathbf{d}.$$

In this situation, Lin and Liu [28,29] made some important analysis studies, such as the existence of weak/strong solutions and the partial regularity of the so-called suitable weak solutions for such Ginzburg-Landau approximation model. However, as the authors pointed out in [29], it is a challenging problem to study the limiting case as η tends to zero. During the past decade, progress has also been made on overcoming the difficulty caused by the nonlinearity $|\nabla \mathbf{d}|^2 \mathbf{d}$. In dimension two, the global existence and uniqueness of weak solutions are studied in [10,27]. When the space dimension is three, Lin and Wang [31] established the existence of global weak solutions to the Cauchy problem for the initial orientation $\mathbf{d}_0 \in \mathbb{S}_+^2$ (the up-hemisphere). For more studies on the incompressible liquid crystal flows, we refer the readers to see [34,39] and the references in [30].

When the fluid is allowed to be compressible, the nematic liquid crystal flows becomes more complicate. When the space dimension is two, Jiang et al. [16] established the global existence of weak solutions to system (1.1)–(1.2) under a restriction imposed on the initial energy including the case of small initial energy. For the three dimensional case, Jiang et al. [15] proved the global existence of weak solutions to the initial boundary problem of system (1.1) provided that the third component of initial orientation field satisfies smallness condition. Later, Lin et al. [26] further obtained the global existence of weak solutions to the initial boundary problem of system (1.1) for any large initial data with $\mathbf{d}_0 \in \mathbb{S}_+^2$, and they also showed the subsequential convergence of weak solutions. The local existence of strong solution to the initial value or initial boundary value problem of system (1.1) with nonnegative initial data were obtained in Huang, Wang and Wen [13]. Base on the results of the compressible Navier–Stokes equations, some blow-up criteria of local strong solution of system (1.1) were studied in [13,14]. The global existence and uniqueness of global strong solutions to the Cauchy problem of system (1.1) with initial data near the equilibrium state were established by Hu and Wu [12] in critical Besov spaces. The global existence and large time behavior of classical solution to the Cauchy problem of system (1.1) with large oscillations and vacuum initial data were showed by Li, Xu and Zhang [24]. Recently, based on the spectral analysis and Duhamel’s principle to control difficulties arising in the boundary of bounded domains, Wang and Yu [40] established the incompressible limit (i.e., the low Mach number limit) of the Ginzburg-Landau approximation of (1.1) in a sufficiently smooth bounded domain $\Omega \subseteq \mathbb{R}^3$. Later, by using an abstract result of Kato [20] to show that the energy of acoustic waves decays to zero, Kwon [22] obtained the incompressible limit of weak solutions of the compressible flows of nematic liquid crystals in the whole space \mathbb{R}^2 .

Physically, as the Mach number vanishes, the behaviors of compressible fluid flows would tend to the incompressible flows (cf. [32]). For the case of $\mathbf{d} \equiv \mathbf{e}_3$, incompressible limit problems have been investigated by many authors, starting with the work by Klainerman and Majda [21] for the Euler equations and Lions and Masmoudi [33] for the isentropic Navier–Stokes equations. Especially, Lions and Masmoudi [33] studied the incompressible limit for the weak solutions to the isentropic Navier–Stokes equations with following slip boundary condition

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \text{curl } \mathbf{u} = 0, \quad \text{on } \partial\Omega, \text{ where } \Omega \subset \mathbb{R}^2, \quad \text{or} \quad (1.5)$$

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \mathbf{n} \times \text{curl } \mathbf{u} = 0, \quad \text{on } \partial\Omega, \text{ where } \Omega \subset \mathbb{R}^3. \quad (1.6)$$

Similar results in the spirit of the analysis, presented by Lions and Masmoudi [33], are the recent progress by Feireisl and Novotný [9] for the full Navier–Stokes–Fourier system. It should be noticed that their estimates are not uniform in time. As for the all-time incompressible limit results for regular solutions, in

case of no-slip boundary conditions, i.e., $\mathbf{u}|_{\partial\Omega} = 0$ where $\Omega \subset \mathbf{R}^3$, and slightly compressible initial data, Bessaih [2] established the uniform estimates both in Mach number and time for regular solutions with almost incompressible initial data, and proved that the strong convergence to the solution of incompressible Navier–Stokes equations. In papers [36,37], the authors studied the incompressible limit of regular solutions to the compressible Navier–Stokes equations with slightly compressible initial data in a 2D bounded domain with the boundary condition (1.5), and with slightly compressible initial data in a 3D bounded domain with the boundary condition (1.3). Let us also mention that the corresponding incompressible limit for the full compressible Navier–Stokes equations and the compressible magnetohydrodynamic equations were investigated extensively in [4–7,11,17–19], and the references cited therein.

The aim of the present paper is to extend the result in [36] to the compressible Erickson–Leslie model (1.1), that is, to establish the global well-posedness of smooth solutions to the initial-boundary value problem (1.1)–(1.4) around a rest state, and furthermore, to verify rigorously the corresponding low Mach number limit as $\varepsilon \rightarrow 0$ (or say the incompressible limit) of solutions for all time. To present our main results clearly, we shall consider the flow with small density variation, that is, the density ϱ^ε varies slightly around the reference state $\bar{\varrho} = 1$, i.e.

$$\varrho^\varepsilon = 1 + \varepsilon\sigma^\varepsilon.$$

Then we can write the problem (1.1)–(1.4) in the form:

$$\begin{cases} \partial_t \sigma^\varepsilon + \operatorname{div}(\sigma^\varepsilon \mathbf{u}^\varepsilon) + \frac{1}{\varepsilon} \operatorname{div}(\mathbf{u}^\varepsilon) = 0, \\ \varrho^\varepsilon (\partial_t \mathbf{u}^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{u}^\varepsilon) + \frac{1}{\varepsilon} P'(1 + \varepsilon\sigma) \nabla \sigma^\varepsilon = \operatorname{div}(2\mu D(\mathbf{u}^\varepsilon)) + \lambda \nabla \operatorname{div} \mathbf{u}^\varepsilon - \nu (\nabla \mathbf{d}^\varepsilon)^T \Delta \mathbf{d}^\varepsilon, \\ \partial_t \mathbf{d}^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla) \mathbf{d}^\varepsilon = \theta (\Delta \mathbf{d}^\varepsilon + |\nabla \mathbf{d}^\varepsilon|^2 \mathbf{d}^\varepsilon), \quad |\mathbf{d}^\varepsilon| = 1, \\ (\varrho^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon)(\mathbf{x}, 0) = (\varrho_0^\varepsilon(\mathbf{x}), \mathbf{u}_0^\varepsilon(\mathbf{x}), \mathbf{d}_0^\varepsilon(\mathbf{x})) \quad \text{in } \mathbf{x} \in \Omega, \\ \mathbf{u}^\varepsilon \cdot \mathbf{n} = 0, \quad \tau \cdot \mathcal{T}(\mathbf{u}^\varepsilon, P(\varrho^\varepsilon)) \cdot \mathbf{n} + \alpha \mathbf{u}^\varepsilon \cdot \tau = 0 \quad \text{on } \mathbf{x} \in \partial\Omega, \\ \mathbf{n} \cdot \nabla \mathbf{d}^\varepsilon = \mathbf{0} \quad \text{on } \mathbf{x} \in \partial\Omega, \end{cases} \quad (1.7)$$

where we have used the equality $\operatorname{div}(\nabla \mathbf{d}^\varepsilon \odot \nabla \mathbf{d}^\varepsilon - \frac{1}{2} |\nabla \mathbf{d}^\varepsilon|^2 \mathbb{I}) = (\nabla \mathbf{d}^\varepsilon)^T \Delta \mathbf{d}^\varepsilon$.

The main results of the present paper read as follows.

Theorem 1.1. *There exists a positive constant β , such that if the initial data $\sigma_0^\varepsilon, \mathbf{u}_0^\varepsilon, \mathbf{d}_0^\varepsilon$ satisfy*

$$\|(\sigma_0^\varepsilon, \mathbf{u}_0^\varepsilon)\|_{H^2} + \|\mathbf{d}_0^\varepsilon - \mathbf{e}_3\|_{H^3} + \|(\sigma_t^\varepsilon, \mathbf{u}_t^\varepsilon, \nabla \mathbf{d}_t^\varepsilon)(0)\|_{H^1} + \|\varepsilon(\sigma_{tt}^\varepsilon, \mathbf{u}_{tt}^\varepsilon, \nabla \mathbf{d}_{tt}^\varepsilon)(0)\|_{L^2} \leq \beta,$$

with

$$\int_{\Omega} \sigma_0^\varepsilon dx = 0, \quad 1 + \varepsilon\sigma_0^\varepsilon \geq m \text{ for some positive constant } m \geq 0, \quad \mathbf{e}_3 = (0, 0, 1),$$

and the following compatibility conditions

$$\partial_t^i \mathbf{u}_0^\varepsilon \cdot \mathbf{n} = 0, \quad \tau \mathcal{T}(\partial_t^i \mathbf{u}_0^\varepsilon, P(\varrho_0^\varepsilon)) \cdot \mathbf{n} + \alpha \partial_t^i \mathbf{u}_0^\varepsilon \cdot \tau = 0, \quad \mathbf{n} \cdot \nabla \partial_t^i \mathbf{d}_0^\varepsilon = 0 \quad \text{for } \mathbf{x} \in \partial\Omega, \text{ and } i = 0, 1, 2,$$

hold, then for any $\varepsilon \in (0, \varepsilon_1)$ with $0 < \varepsilon_1 < 1$, the initial boundary value problem (1.7) admits a unique global-in-time solution $(\sigma^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon)$ in $\Omega \times \overline{\mathbb{R}}_+$, satisfying

$$\begin{aligned} \sigma^\varepsilon &\in C(\overline{\mathbb{R}}_+; H^2), & \mathbf{u}^\varepsilon &\in C(\overline{\mathbb{R}}_+; H^2) \cap L^2(\overline{\mathbb{R}}_+; H^3), & \mathbf{d}^\varepsilon - \mathbf{e}_3 &\in C(\overline{\mathbb{R}}_+; H^3) \cap L^2(\overline{\mathbb{R}}_+; H^4), \\ \sigma_t^\varepsilon &\in C(\overline{\mathbb{R}}_+; H^1), & \mathbf{u}_t^\varepsilon &\in C(\overline{\mathbb{R}}_+; H^1) \cap L^2(\overline{\mathbb{R}}_+; H^2), & \mathbf{d}_t^\varepsilon &\in C(\overline{\mathbb{R}}_+; H^2) \cap L^2(\overline{\mathbb{R}}_+; H^3), \\ \sigma_{tt}^\varepsilon &\in L^\infty(\overline{\mathbb{R}}_+; L^2), & \mathbf{u}_{tt}^\varepsilon &\in L^\infty(\overline{\mathbb{R}}_+; L^2) \cap L^2(\overline{\mathbb{R}}_+; H^1), & \mathbf{d}_{tt}^\varepsilon &\in L^\infty(\overline{\mathbb{R}}_+; H^1) \cap L^2(\overline{\mathbb{R}}_+; H^2), \end{aligned} \quad (1.8)$$

where $\overline{\mathbb{R}}_+ = [0, \infty)$. Furthermore, it holds that

$$\begin{aligned} & \sup_{0 \leq s \leq t} (\|(\sigma^\varepsilon, \mathbf{u}^\varepsilon)(s)\|_{H^2} + \|(\mathbf{d}^\varepsilon - \mathbf{e}_3)(s)\|_{H^3} + \|(\sigma_t^\varepsilon, \mathbf{u}_t^\varepsilon)(s)\|_{H^1} + \|\mathbf{d}_t^\varepsilon(s)\|_{H^2}) \\ & + \text{esssup}_{0 \leq s \leq t} (\|\varepsilon(\sigma_{tt}^\varepsilon, \mathbf{u}_{tt}^\varepsilon, \nabla \mathbf{d}_{tt}^\varepsilon)(s)\|_{L^2} + \|\mathbf{d}_{tt}^\varepsilon(s)\|_{L^2}) \leq C, \quad \forall t \in \mathbb{R}_+, \end{aligned}$$

where C is positive constant independent of ε .

Theorem 1.2. Let the assumptions in Theorem 1.1 be satisfied, and $(\sigma^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon)$ be the global-in-time solution established in Theorem 1.1. Assume that the initial data $(\mathbf{u}_0^\varepsilon, \mathbf{d}_0^\varepsilon) \rightarrow (\mathbf{v}_0, \mathbf{w}_0)$ as $\varepsilon \rightarrow 0$ in $H^s \times H^{s+1}$ for $0 \leq s < 2$. Then as $\varepsilon \rightarrow 0$, $(\mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon) \rightarrow (\mathbf{v}, \mathbf{w})$ in $C(\overline{\mathbb{R}}_{loc}^+; H^s \times H^{s+1})$ for any $0 \leq s < 2$. And there exists a function $\pi(x, t)$, such that $(\mathbf{v}, \mathbf{w}, \pi)$ is the unique smooth solution of the following initial boundary value problem for the incompressible nematic liquid crystal flows

$$\begin{cases} \operatorname{div} \mathbf{v} = 0, \\ \mathbf{v}_t - \mathbf{v} \cdot \nabla \mathbf{v} + \nabla \pi = \mu \Delta \mathbf{v} - \nu \nabla \cdot (\nabla \mathbf{w} \odot \nabla \mathbf{w} - \frac{1}{2} |\nabla \mathbf{w}|^2 \mathbb{I}), \\ \mathbf{w}_t - \mathbf{v} \cdot \nabla \mathbf{w} = \theta (\Delta \mathbf{w} + |\nabla \mathbf{w}|^2 \mathbf{w}), \\ (\mathbf{v}, \mathbf{w})(\mathbf{x}, 0) = (\mathbf{v}_0(\mathbf{x}), \mathbf{w}_0(\mathbf{x})) \quad \text{in } \mathbf{x} \in \Omega, \\ \mathbf{v} \cdot \mathbf{n} = 0, \quad \tau \cdot D(\mathbf{v}) \cdot \mathbf{n} + \alpha \mathbf{v} \cdot \tau = 0, \quad \mathbf{n} \cdot \nabla \mathbf{w} = 0 \quad \text{on } \mathbf{x} \in \partial\Omega. \end{cases}$$

Remark 1.3. In the assumption of Theorem 1.1, $\sigma_t^\varepsilon(0)$ is indeed defined by $-\operatorname{div}(\sigma_0^\varepsilon \mathbf{u}_0^\varepsilon) - \frac{1}{\varepsilon} \operatorname{div}(\mathbf{u}_0^\varepsilon)$ from the density equation (1.7)₁. Analogously, $\sigma_{tt}^\varepsilon(0)$, $\mathbf{u}_{tt}^\varepsilon(0)$, $\mathbf{u}_{tt}^\varepsilon(0)$, $\mathbf{d}_t^\varepsilon(0)$ and $\mathbf{d}_{tt}^\varepsilon(0)$ are defined recursively by (1.7)₁, (1.7)₂, (1.7)₃ and the initial data σ_0^ε , \mathbf{u}_0^ε and \mathbf{d}_0^ε . We also notice that the number ε_1 is depending only on $\Omega, \mu, \lambda, \nu, \theta$ and the initial data.

We now make some comments on the analysis of the paper. Roughly speaking, Theorems 1.1 and 1.2 are proved based on the uniform estimates of strong solutions in Sobolev norms which do not depend on the Mach number ε and time variable t . Due to the large parameter $\frac{1}{\varepsilon^2}$ in the momentum equations (1.7)₂, mathematically, it is difficult to obtain uniform estimates (of higher derivatives) in Mach number, which are necessary for the strong convergence to the background incompressible flows. Compared with the Cauchy or spatially periodic problem (cf. [22]), the presence of boundary here gives rise to some difficulties involved with controlling the boundary terms, in particular for the low Mach number limit, arise. To circumvent such difficulties, we estimate the vorticity and divergence of the velocity separately due to the Navier's slip boundary condition, and use the equality $\Delta \mathbf{u} = \nabla \operatorname{div} \mathbf{u} - \overline{\operatorname{curl}} \operatorname{curl} \mathbf{u}$ with $\overline{\operatorname{curl}} := (\partial_2, -\partial_1)^t$ and $\operatorname{curl} \mathbf{u} = \partial_1 u_2 - \partial_2 u_1$ to help us separate the divergence of the velocity and vorticity. Moreover, compared with the compressible Navier–Stokes equations, the strong coupling terms and strong nonlinear terms, such as $\operatorname{div}(\nabla \mathbf{d} \odot \nabla \mathbf{d})$, $\mathbf{u} \cdot \nabla \mathbf{d}$ and $|\nabla \mathbf{d}|^2 \mathbf{d}$, will bring out some new difficulties. Hence, we need to deal with the estimates involving the nonlinear terms carefully, especially on the higher order derivatives. By combining carefully all the temporal-spatial estimates, we obtain the uniform estimate with respect to the Mach number $\varepsilon \in (0, \bar{\varepsilon}]$ (where $0 < \bar{\varepsilon} \leq 1$) and time variable $t \in [0, \infty)$. We also notice that without the restriction of initial data, the corresponding incompressible limit is not necessarily valid (see [36]). In fact, as shown in [4], the diffusion effects of Navier–Stokes equations could prevent us from identifying the acoustic waves, from the compressible fluids in a bounded domain with Dirichlet boundary condition, are asymptotically damped due to the formation of a thin boundary layer. We remark that effect of a boundary layer on the propagation of the acoustic waves will not appear in the convergence of the solutions under initial assumptions on Theorem 1.1, the Navier's slip boundary condition (1.3) and the Neumann condition (1.4) in the present paper.

In the next section, we first show the uniform in ε and t estimates by deriving a differential inequality with certain decay property, and then we give the proof of Theorems 1.1 and 1.2. In what follows, we shall derive the uniform estimates in Mach number for all time. We will drop the superscript ε of σ^ε , \mathbf{u}^ε , \mathbf{d}^ε , etc. for the sake of simplicity. We denote by C (or C_i , $i = 1, 2, \dots$) the positive constants depending only on Ω , $\mu, \lambda, \nu, \theta$ and P , but not on ε . We denote by C_δ the constant depending only on δ .

2. Proof of Theorems 1.1 and 1.2

From the continuity equation (1.7)₁ and the boundary condition $\mathbf{u} \cdot \mathbf{n} = 0$, we deduce that

$$\frac{d}{dt} \int_{\Omega} \sigma d\mathbf{x} = - \int_{\Omega} \operatorname{div}(\sigma \mathbf{u} + \frac{1}{\varepsilon} \mathbf{u}) d\mathbf{x} = - \int_{\partial\Omega} (\sigma + \frac{1}{\varepsilon}) \mathbf{u} \cdot \mathbf{n} dS = 0,$$

which together with the condition $\int_{\Omega} \sigma_0 d\mathbf{x} = 0$ ensures that

$$\int_{\Omega} \sigma d\mathbf{x} = 0.$$

2.1. L^2 estimates of $\mathbf{d} - \mathbf{e}_3$, σ , \mathbf{u} and $\nabla \mathbf{d}$

By rewrite (1.7)₃ as

$$(\mathbf{d} - \mathbf{e}_3)_t + \mathbf{u} \cdot \nabla (\mathbf{d} - \mathbf{e}_3) = \theta(\Delta(\mathbf{d} - \mathbf{e}_3) + |\nabla(\mathbf{d} - \mathbf{e}_3)|^2 \mathbf{d}).$$

Multiplying the above equality with $\mathbf{d} - \mathbf{e}_3$ and integrating by parts, it follows that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{d} - \mathbf{e}_3\|_{L^2}^2 + \theta \|\nabla \mathbf{d}\|_{L^2}^2 &= - \int_{\Omega} \mathbf{u} \cdot \nabla (\mathbf{d} - \mathbf{e}_3) \cdot (\mathbf{d} - \mathbf{e}_3) d\mathbf{x} + \theta \int_{\Omega} |\nabla(\mathbf{d} - \mathbf{e}_3)|^2 \mathbf{d} \cdot (\mathbf{d} - \mathbf{e}_3) d\mathbf{x} \\ &\leq C \|\mathbf{u}\|_{H^2} \|\nabla \mathbf{d}\|_{L^2} \|\mathbf{d} - \mathbf{e}_3\|_{L^2} + \|\nabla \mathbf{d}\|_{L^2} \|\nabla \mathbf{d}\|_{H^2} \|\mathbf{d} - \mathbf{e}_3\|_{L^2} \\ &\leq \frac{\theta}{2} \|\nabla \mathbf{d}\|_{L^2}^2 + C \|\mathbf{d} - \mathbf{e}_3\|_{L^2}^2 (\|\mathbf{u}\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^2}^2), \end{aligned}$$

where we have used the fact that $|\mathbf{d}| = 1$ and the Young's inequality in the estimates above, which implies that

Lemma 2.1. *There exists a positive constant C_1 , such that*

$$\frac{d}{dt} \|\mathbf{d} - \mathbf{e}_3\|_{L^2}^2 + \theta \|\nabla \mathbf{d}\|_{L^2}^2 \leq C_1 \|\mathbf{d} - \mathbf{e}_3\|_{L^2}^2 (\|\mathbf{u}\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^2}^2).$$

By taking $\langle (1.7)_1, P'(1)\sigma \rangle$, it follows that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\sqrt{P'(1)}\sigma\|_{L^2}^2 - \frac{1}{\varepsilon} P'(1) \int_{\Omega} \mathbf{u} \cdot \nabla \sigma d\mathbf{x} &= - P'(1) \int_{\Omega} \sigma \operatorname{div}(\sigma \mathbf{u}) d\mathbf{x} \leq C \|\mathbf{u}\|_{H^1} \|\sigma\|_{H^1}^2 \\ &\leq \delta \|\mathbf{u}\|_{H^1}^2 + C \|\sigma\|_{H^1}^4. \end{aligned} \tag{2.1}$$

Multiplying (1.7)₂, (1.7)₃ by \mathbf{u} , $-\nu \Delta \mathbf{d}$ respectively, then summing up and integrating by parts, we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (\|\sqrt{\varrho} \mathbf{u}\|_{L^2}^2 + \nu \|\nabla \mathbf{d}\|_{L^2}^2) + \theta \nu \|\Delta \mathbf{d}\|_{L^2}^2 - \int_{\Omega} (\operatorname{div}(2\mu D(\mathbf{u})) + \lambda \nabla \operatorname{div} \mathbf{u}) \cdot \mathbf{u} \, dx + \frac{1}{\varepsilon} P'(1) \int_{\Omega} \nabla \sigma \cdot \mathbf{u} \, dx \\
&= \int_{\Omega} \frac{P'(1) - P'(1 + \varepsilon \sigma)}{\varepsilon} \nabla \sigma \cdot \mathbf{u} \, dx - \theta \nu \int_{\Omega} |\nabla \mathbf{d}|^2 \mathbf{d} \cdot \Delta \mathbf{d} \, dx \leq C(\|\sigma\|_{H^1}^2 \|\mathbf{u}\|_{H^1} + \|\nabla \mathbf{d}\|_{H^1}^4) \\
&\leq \delta \|\mathbf{u}\|_{H^1}^2 + C(\|\sigma\|_{H^1}^4 + \|\nabla \mathbf{d}\|_{H^1}^4),
\end{aligned} \tag{2.2}$$

where we have used the identity $\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d} - \frac{1}{2} |\nabla \mathbf{d}|^2 \mathbf{d}) = \nabla \mathbf{d} \cdot \Delta \mathbf{d}$, and the fact that $|\mathbf{d}| = 1$ yields the identity $\mathbf{d} \cdot \Delta \mathbf{d} = -|\nabla \mathbf{d}|^2$. Notice that by using the condition (1.7)₅ and the Korn's inequality (see Lemma 1.1 of [42]), one has

$$\begin{aligned}
& - \int_{\Omega} (\operatorname{div}(2\mu D(\mathbf{u})) + \lambda \nabla \operatorname{div} \mathbf{u}) \cdot \mathbf{u} \, dx \\
&= \int_{\Omega} (2\mu |D(\mathbf{u})|^2 + \lambda |\operatorname{div} \mathbf{u}|^2) \, dx - \int_{\partial \Omega} (\mathbf{u} \cdot \boldsymbol{\tau}) \boldsymbol{\tau} \cdot (2\mu D(\mathbf{u}) + \lambda \operatorname{div} \mathbf{u}) \cdot \mathbf{n} \, dS \\
&= \int_{\Omega} (2\mu |D(\mathbf{u})|^2 + \lambda |\operatorname{div} \mathbf{u}|^2) \, dx + \int_{\partial \Omega} \alpha (\mathbf{u} \cdot \boldsymbol{\tau})^2 \, dS \geq c_0 \|\mathbf{u}\|_{H^1}^2,
\end{aligned}$$

where c_0 is a positive constant. Putting the above estimates (2.1), (2.2) together and keeping the following Korn's inequality in mind (see [42])

$$\|\mathbf{F}\|_{H^1} \leq C \|\nabla \mathbf{F}\|_{L^2} \leq C(\|\operatorname{div} \mathbf{F}\|_{L^2} + \|\operatorname{curl} \mathbf{F}\|_{L^2}), \text{ for all } \mathbf{F} \in H^1(\Omega) \text{ with } \mathbf{F} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega, \tag{2.3}$$

where $\operatorname{curl} \mathbf{F} = \partial_1 F_2 - \partial_2 F_1$, then we obtain the following lemma by letting $\delta (\leq \frac{c_0}{4})$ small enough:

Lemma 2.2. *There exists a positive constant C_2 , such that*

$$\frac{d}{dt} \|(\sqrt{P'(1)} \sigma, \sqrt{\varrho} \mathbf{u}, \sqrt{\nu} \nabla \mathbf{d})\|_{L^2}^2 + c_0 \|\mathbf{u}\|_{H^1}^2 + \theta \nu \|\nabla \mathbf{d}\|_{H^1}^2 \leq C_2 (\|\sigma\|_{H^1}^4 + \|\nabla \mathbf{d}\|_{H^1}^4).$$

Defining two functions

$$\begin{aligned}
\Psi_0(t) &:= \|(\sigma, \sqrt{\varrho} \mathbf{u}, \mathbf{d} - \mathbf{e}_3, \sqrt{\nu} \nabla \mathbf{d})\|_{L^2}^2; \\
\Phi_0(t) &:= c_0 \|\mathbf{u}\|_{H^1}^2 + \theta \nu \|\nabla \mathbf{d}\|_{H^1}^2.
\end{aligned}$$

Then, we conclude from Lemmas 2.1 and 2.2 that there is a positive constant M_0 , such that

$$\frac{d}{dt} \Psi_0(t) + \Phi_0(t) \leq M_0 (\|\mathbf{d} - \mathbf{e}_3\|_{L^2}^2 (\|\mathbf{u}\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^2}^2) + \|\sigma\|_{H^1}^4 + \|\nabla \mathbf{d}\|_{H^1}^4). \tag{2.4}$$

2.2. Estimates of first order derivatives of σ , \mathbf{u} and $\nabla \mathbf{d}$

In this subsection, we derive the estimates of the first order temporal and spatial derivatives of $(\sigma, \mathbf{u}, \nabla \mathbf{d})$. We first differentiating (1.7)₂ with respect to t , it is easy to get that

$$\begin{aligned}
& (\varrho \mathbf{u}_t)_t - \operatorname{div}(2\mu D(\mathbf{u}_t)) - \lambda \nabla \operatorname{div} \mathbf{u}_t + \frac{1}{\varepsilon} P'(1) \nabla \sigma_t \\
&= \left(\frac{P'(1) - P'(1 + \varepsilon \sigma)}{\varepsilon} \nabla \sigma \right)_t - \varepsilon \sigma_t \mathbf{u} \cdot \nabla \mathbf{u} - \varrho (\mathbf{u}_t \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}_t) - \nu (\nabla \mathbf{d} \cdot \Delta \mathbf{d})_t.
\end{aligned} \tag{2.5}$$

Multiplying the above equation (2.5) by \mathbf{u} , integrating by parts and using the boundary conditions (1.7)₅, we deduce that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (2\mu \|D(\mathbf{u})\|_{L^2}^2 + \lambda \|\operatorname{div} \mathbf{u}\|_{L^2}^2) + \int_{\partial\Omega} \alpha (\mathbf{u} \cdot \boldsymbol{\tau})^2 dS + \frac{d}{dt} \int_{\Omega} \varrho \mathbf{u}_t \mathbf{u} d\mathbf{x} + \frac{1}{\varepsilon} P'(1) \int_{\Omega} \nabla \sigma_t \cdot \mathbf{u} d\mathbf{x} \\ &= \int_{\Omega} \left(\frac{P'(1) - P'(1 + \varepsilon \sigma)}{\varepsilon} \nabla \sigma \right)_t \cdot \mathbf{u} d\mathbf{x} + \int_{\Omega} (\varrho \mathbf{u}_t^2 - \varepsilon \sigma_t \mathbf{u} \cdot \nabla \mathbf{u} - \varrho (\mathbf{u}_t \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \nu (\nabla \mathbf{d} \cdot \Delta \mathbf{d})_t) \cdot \mathbf{u} d\mathbf{x} \\ &= - \int_{\Omega} P''(\varrho) \sigma_t \sigma \operatorname{div} \mathbf{u} d\mathbf{x} + \int_{\Omega} (\varrho \mathbf{u}_t^2 - \varepsilon \sigma_t \mathbf{u} \cdot \nabla \mathbf{u} - \varrho (\mathbf{u}_t \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \nu (\nabla \mathbf{d} \cdot \Delta \mathbf{d})_t) \cdot \mathbf{u} d\mathbf{x} \\ &\leq C (\|\mathbf{u}\|_{H^1}^2 + \|\mathbf{u}_t\|_{L^2}^2 + \|\sigma_t\|_{H^1}^2 (\|\sigma\|_{H^1}^2 + \|\mathbf{u}\|_{H^2}^4) + \|\mathbf{u}_t\|_{H^1}^2 \|\mathbf{u}\|_{H^1}^2 + \|\nabla \mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^2). \end{aligned} \quad (2.6)$$

On the other hand, we apply $\langle (1.7)_1, P'(1) \sigma_t \rangle$ and $\langle \nabla (1.7)_3, \nabla \mathbf{d}_t \rangle$ to infer that

$$\begin{aligned} & \|\sqrt{P'(1)} \sigma_t\|_{L^2}^2 + \frac{1}{\varepsilon} P'(1) \int_{\Omega} \sigma_t \operatorname{div} \mathbf{u} d\mathbf{x} = -P'(1) \int_{\Omega} \sigma_t \operatorname{div}(\sigma \mathbf{u}) d\mathbf{x} \\ & \leq C \|\sigma_t\|_{H^1} \|\sigma\|_{H^1} \|\mathbf{u}\|_{H^1}, \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} & \frac{\theta}{2} \frac{d}{dt} \|\nabla^2 \mathbf{d}\|_{L^2}^2 + \|\nabla \mathbf{d}_t\|_{L^2}^2 = \int_{\Omega} \nabla (-\mathbf{u} \cdot \nabla \mathbf{d} + \theta |\nabla \mathbf{d}|^2 \mathbf{d}) \cdot \nabla \mathbf{d}_t d\mathbf{x} \\ & \leq \frac{1}{2} \|\nabla \mathbf{d}_t\|_{L^2}^2 + C (\|\mathbf{u}\|_{H^2}^2 \|\nabla \mathbf{d}\|_{H^1}^2 + \|\nabla \mathbf{d}\|_{H^1}^6), \end{aligned} \quad (2.8)$$

respectively. Combining (2.6), (2.7) and (2.8) together, it follows that

Lemma 2.3. *There exists a positive constant C_3 , such that*

$$\begin{aligned} & \frac{d}{dt} (2\mu \|D(\mathbf{u})\|_{L^2}^2 + \lambda \|\operatorname{div} \mathbf{u}\|_{L^2}^2 + \theta \|\nabla^2 \mathbf{d}\|_{L^2}^2) + 2 \frac{d}{dt} \int_{\Omega} \varrho \mathbf{u}_t \mathbf{u} d\mathbf{x} + 2 \|\sqrt{P'(1)} \sigma_t\|_{L^2}^2 + \|\nabla \mathbf{d}_t\|_{L^2}^2 \\ & \leq C_3 \left(\|\mathbf{u}\|_{H^1}^2 + \|\mathbf{u}_t\|_{L^2}^2 + \|\sigma_t\|_{H^1}^2 (\|\sigma\|_{H^1}^2 + \|\mathbf{u}\|_{H^2}^4) + \|\mathbf{u}_t\|_{H^1}^2 \|\mathbf{u}\|_{H^1}^2 + \|\nabla \mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^2 \right. \\ & \quad \left. + \|\nabla \mathbf{d}\|_{H^1}^2 (\|\mathbf{u}\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^1}^4) \right). \end{aligned}$$

Now, we take $\langle \nabla (1.7)_1, \nabla \sigma \rangle$ to obtain that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla \sigma\|_{L^2}^2 + \frac{1}{\varepsilon} \int_{\Omega} \nabla \operatorname{div} \mathbf{u} \cdot \nabla \sigma d\mathbf{x} = - \int_{\Omega} ((\mathbf{u} \cdot \nabla) \nabla \sigma + \nabla \mathbf{u} \nabla \sigma + \nabla \sigma \operatorname{div} \mathbf{u} + \sigma \nabla \operatorname{div} \mathbf{u}) \nabla \sigma d\mathbf{x} \\ & \leq C (\|\mathbf{u}\|_{H^1} + \|\nabla \operatorname{div} \mathbf{u}\|_{L^2}) \|\sigma\|_{H^2}^2. \end{aligned} \quad (2.9)$$

To bound $\|\nabla \operatorname{div} \mathbf{u}\|_{L^2}$, let us rewrite the equation (1.7)₂ as

$$\varrho (\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) + \frac{1}{\varepsilon} P'(1 + \varepsilon \sigma) \nabla \sigma = (2\mu + \lambda) \nabla \operatorname{div} \mathbf{u} - \overline{\mu \operatorname{curl} \operatorname{curl} \mathbf{u}} - \nu \nabla \mathbf{d} \cdot \Delta \mathbf{d}, \quad (2.10)$$

where $\operatorname{curl} \mathbf{u} = \partial_1 u_2 - \partial_2 u_1$ and $\overline{\operatorname{curl}} = (\partial_2, -\partial_1)^t$. We take $\langle (2.10), P'(1)^{-1} \nabla \operatorname{div} \mathbf{u} \rangle$ to derive that

$$\begin{aligned}
& (2\mu + \lambda) \|\sqrt{P'(1)^{-1}} \nabla \operatorname{div} \mathbf{u}\|_{L^2}^2 - \frac{1}{\varepsilon} \int_{\Omega} \nabla \operatorname{div} \mathbf{u} \cdot \nabla \sigma \, d\mathbf{x} \\
&= \frac{1}{P'(1)} \left\{ \int_{\Omega} \left(\varrho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) - \nu \nabla \mathbf{d} \cdot \Delta \mathbf{d} + \frac{P'(1+\varepsilon\sigma) - P'(1)}{\varepsilon} \nabla \sigma \right) \cdot \nabla \operatorname{div} \mathbf{u} \, d\mathbf{x} + \mu \int_{\Omega} \overline{\operatorname{curl}} \operatorname{curl} \mathbf{u} \nabla \operatorname{div} \mathbf{u} \, d\mathbf{x} \right\} \\
&\leq C \|\nabla \operatorname{div} \mathbf{u}\|_{L^2} (\|\mathbf{u}_t\|_{L^2} + \|\mathbf{u}\|_{H^2} \|\mathbf{u}\|_{H^1} + \|\sigma\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^1} \|\nabla \mathbf{d}\|_{H^2}) + \frac{\mu}{P'(1)} \int_{\Omega} \overline{\operatorname{curl}} \operatorname{curl} \mathbf{u} \nabla \operatorname{div} \mathbf{u} \, d\mathbf{x}.
\end{aligned} \tag{2.11}$$

To bound the last term of the right-hand side of (2.11), notice that by using Lemma 2.1 of Clopeau, Mikelić and Robert [3], we can differentiate (1.7)₅ to get that

$$\operatorname{curl} \mathbf{u} = (2\chi - \frac{\alpha}{\mu}) \mathbf{u} \cdot \boldsymbol{\tau} \quad \text{on } \partial\Omega,$$

where χ is the curvature of $\partial\Omega$ given in a standard way by $\frac{d\boldsymbol{\tau}}{ds} = -\chi \mathbf{n}$. In view of the above boundary condition and the trace theorem, we find that

$$\begin{aligned}
\frac{\mu}{P'(1)} \int_{\Omega} \overline{\operatorname{curl}} \operatorname{curl} \mathbf{u} \nabla \operatorname{div} \mathbf{u} \, d\mathbf{x} &= \frac{\mu}{P'(1)} \int_{\partial\Omega} \operatorname{curl} \mathbf{u} \nabla \operatorname{div} \mathbf{u} \cdot \boldsymbol{\tau} \, dS \\
&= \frac{\mu}{P'(1)} \int_{\partial\Omega} (2\chi - \frac{\alpha}{\mu}) (\mathbf{u} \cdot \boldsymbol{\tau}) (\nabla \operatorname{div} \mathbf{u} \cdot \boldsymbol{\tau}) \, dS \\
&\leq \delta \|\nabla^2 \operatorname{div} \mathbf{u}\|_{L^2}^2 + C_{\delta} \|\mathbf{u}\|_{H^1}^2,
\end{aligned} \tag{2.12}$$

for some small enough positive number $\delta (< \frac{(2\mu+\lambda)P'(1)^{-1}}{4})$. Therefore, by using Cauchy inequality together with the estimates (2.9) and (2.10)–(2.12), we obtain the following lemma.

Lemma 2.4. *We have*

$$\begin{aligned}
& \frac{d}{dt} \|\nabla \sigma\|_{L^2}^2 + \gamma_1 \|\nabla \operatorname{div} \mathbf{u}\|_{L^2}^2 \\
&\leq \delta_1 (\|\nabla^2 \operatorname{div} \mathbf{u}\|_{L^2}^2 + \|\nabla \operatorname{div} \mathbf{u}\|_{L^2}^2) + C_{4,\delta_1} (\|\mathbf{u}\|_{H^1}^2 + \|\mathbf{u}_t\|_{L^2}^2 + \|\sigma\|_{H^2}^4 + \|\mathbf{u}\|_{H^2}^4 + \|\nabla \mathbf{d}\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^2}^2),
\end{aligned}$$

where γ_1 is a positive constant independent of ε , and δ_1 is small and to be chosen later.

Taking $\langle \partial_t(1.7)_1, P'(1)\sigma_t \rangle$, we obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\sqrt{P'(1)} \sigma_t\|_{L^2}^2 + \frac{1}{\varepsilon} P'(1) \int_{\Omega} \operatorname{div} \mathbf{u}_t \sigma_t \, d\mathbf{x} &= -P'(1) \int_{\Omega} ((\mathbf{u} \cdot \nabla \sigma)_t + (\sigma \operatorname{div} \mathbf{u})_t) \sigma_t \, d\mathbf{x} \\
&\leq \delta (\|\mathbf{u}\|_{H^1}^2 + \|\mathbf{u}_t\|_{H^1}^2) + C_{\delta} \|\sigma\|_{H^1}^2 \|\sigma_t\|_{H^1}^2,
\end{aligned} \tag{2.13}$$

while taking $\langle \partial_t(1.7)_2, \mathbf{u}_t \rangle$, we get

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\sqrt{\varrho} \mathbf{u}_t\|_{L^2}^2 + 2\mu \|D(\mathbf{u}_t)\|_{L^2}^2 + \lambda \|\operatorname{div} \mathbf{u}_t\|_{L^2}^2 + \int_{\partial\Omega} \alpha (\mathbf{u}_t \cdot \boldsymbol{\tau})^2 \, dS + \frac{1}{\varepsilon} P'(1) \int_{\Omega} \nabla \sigma_t \cdot \mathbf{u}_t \, d\mathbf{x} \\
&= \int_{\Omega} \left[\left(\frac{P'(1) - P'(1+\varepsilon\sigma)}{\varepsilon} \nabla \sigma \right)_t - (\varrho_t \mathbf{u}_t + \varepsilon \sigma_t \mathbf{u} \cdot \nabla \mathbf{u} + \varrho(\mathbf{u} \cdot \nabla \mathbf{u})_t + \nu (\nabla \mathbf{d} \Delta \mathbf{d})_t) \right] \cdot \mathbf{u}_t \, d\mathbf{x}
\end{aligned}$$

$$\leq \delta \|\mathbf{u}_t\|_{H^1}^2 + C_\delta (\|\sigma_t\|_{H^1}^2 (\|\sigma\|_{H^1}^2 + \|\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{u}\|_{H^1}^4) + \|\mathbf{u}_t\|_{H^1}^2 \|\mathbf{u}\|_{H^1}^2 + \|\nabla \mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^2). \quad (2.14)$$

Taking $\langle \partial_t(1.7)_3, \mathbf{d}_t \rangle$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{d}_t\|_{L^2}^2 + \theta \|\nabla \mathbf{d}_t\|_{L^2}^2 \\ & \leq \frac{1}{4} \|\mathbf{d}_t\|_{L^2}^2 + C (\|\mathbf{u}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^2 + \|\nabla \mathbf{d}_t\|_{H^1}^2 (\|\mathbf{u}\|_{H^1}^2 + \|\nabla \mathbf{d}\|_{H^1}^2) + \|\mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^4). \end{aligned}$$

Similarly, taking $\langle (1.7)_3, \mathbf{d}_t - \eta \Delta \mathbf{d} \rangle$, where η is a positive constant to be chosen later, we have

$$\begin{aligned} & \frac{\theta + \eta}{2} \frac{d}{dt} \|\nabla \mathbf{d}\|_{L^2}^2 + \|\mathbf{d}_t\|_{L^2}^2 + \theta \eta \|\Delta \mathbf{d}\|_{L^2}^2 \\ & \leq \frac{1}{4} \|\mathbf{d}_t\|_{L^2}^2 + \eta^2 \|\Delta \mathbf{d}\|_{L^2}^2 + C (\|\mathbf{u}\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^2 + \|\nabla \mathbf{d}\|_{H^1}^4). \end{aligned}$$

We choose $\eta = \frac{\theta}{2}$, then we get following estimate

$$\begin{aligned} & \frac{d}{dt} (\|\mathbf{d}_t\|_{L^2}^2 + \frac{3\theta}{2} \|\nabla \mathbf{d}\|_{L^2}^2) + \|\mathbf{d}_t\|_{L^2}^2 + \theta \|\nabla \mathbf{d}_t\|_{L^2}^2 + \frac{\theta^2}{2} \|\Delta \mathbf{d}\|_{L^2}^2 \\ & \leq C (\|\mathbf{u}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^2 + \|\nabla \mathbf{d}_t\|_{H^1}^2 (\|\mathbf{u}\|_{H^1}^2 + \|\nabla \mathbf{d}\|_{H^1}^2) + \|\mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^4 + \|\mathbf{u}\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^2 + \|\nabla \mathbf{d}\|_{H^1}^4). \end{aligned} \quad (2.15)$$

Taking $\langle \partial_t(1.7)_3, -\Delta \mathbf{d}_t \rangle$, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{d}_t\|_{L^2}^2 + \theta \|\Delta \mathbf{d}_t\|_{L^2}^2 = \int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{d} - |\nabla \mathbf{d}|^2 \mathbf{d})_t \cdot \Delta \mathbf{d}_t \, dx \\ & \leq \frac{\theta}{2} \|\Delta \mathbf{d}_t\|_{L^2}^2 + C (\|\mathbf{u}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^2 + \|\nabla \mathbf{d}_t\|_{H^1}^2 (\|\mathbf{u}\|_{H^1}^2 + \|\nabla \mathbf{d}\|_{H^1}^2) + \|\mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^4). \end{aligned} \quad (2.16)$$

Combining (2.13), (2.14), (2.15) and (2.16) together and taking δ appropriately small, and then applying (2.3) for \mathbf{u}_t , we obtain the following lemma.

Lemma 2.5. *We have*

$$\begin{aligned} & \frac{d}{dt} \left(\|\sqrt{P'(1)} \sigma_t\|_{L^2}^2 + \|\sqrt{\varrho} \mathbf{u}_t\|_{L^2}^2 + \|\mathbf{d}_t\|_{H^1}^2 + \frac{3\theta}{2} \|\nabla \mathbf{d}\|_{L^2}^2 \right) + \gamma_2 (\|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{d}_t\|_{H^1}^2 + \|\Delta \mathbf{d}_t\|_{L^2}^2) \\ & \leq \delta_2 \|\mathbf{u}\|_{H^1}^2 + C_{5, \delta_2} \left(\|\sigma_t\|_{H^1}^2 (\|\sigma\|_{H^1}^2 + \|\mathbf{u}_t\|_{L^2}^2 + \|\mathbf{u}\|_{H^1}^4) + \|\mathbf{u}_t\|_{H^1}^2 \|\mathbf{u}\|_{H^1}^2 + \|\nabla \mathbf{d}_t\|_{H^1}^2 (\|\mathbf{u}\|_{H^1}^2 + \|\nabla \mathbf{d}\|_{H^1}^2) \right. \\ & \quad \left. + \|\mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^4 + \|\nabla \mathbf{d}\|_{H^1}^6 \right), \end{aligned}$$

where γ_2 is a positive constant independent of ε , and δ_2 is small and to be chosen later.

Next, we estimate the vorticity of velocity, which is denoted by $\omega = \text{curl } \mathbf{u}$. By virtue of (1.7)₂, it is easy to see that ω satisfies the following system

$$\begin{cases} \varrho \omega_t + \varrho \mathbf{u} \cdot \nabla \omega - \mu \Delta \omega = g, \\ \omega = (2\chi - \frac{\alpha}{\mu}) \mathbf{u} \cdot \boldsymbol{\tau} \quad \text{on } \partial\Omega, \end{cases} \quad (2.17)$$

where

$$g = -\varrho\omega \operatorname{div} \mathbf{u} - \frac{\varepsilon}{\varrho} \nabla \sigma \times (\mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u} - \nu \nabla \mathbf{d} \cdot \Delta \mathbf{d}) + \operatorname{curl}(\nabla \mathbf{d} \cdot \Delta \mathbf{d}).$$

Multiplying (2.17)₁ by ω , and using the boundary condition (2.17)₂, we infer that

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{\varrho}\omega\|_{L^2}^2 + \mu \|\nabla \omega\|_{L^2}^2 = \int_{\Omega} g \omega d\mathbf{x} - \int_{\partial\Omega} \omega \nabla \omega \cdot \mathbf{n} dS. \quad (2.18)$$

Employing (2.17)₂ and the trace theorem, we arrive at

$$\int_{\partial\Omega} \omega \nabla \omega \cdot \mathbf{n} dS = \int_{\partial\Omega} \left((2\chi - \frac{\alpha}{\mu}) \mathbf{u} \cdot \boldsymbol{\tau} \right) \nabla \omega \cdot \mathbf{n} dS \leq \delta \|\omega\|_{H^2}^2 + C_{\delta} \|\mathbf{u}\|_{H^1}^2.$$

On the other hand, it is easy to see that

$$\int_{\Omega} g \omega d\mathbf{x} \leq \delta \|\omega\|_{H^1}^2 + C_{\delta} (\|\mathbf{u}\|_{H^2}^2 (\|\mathbf{u}\|_{H^1}^2 + \varepsilon^2 \|\sigma\|_{H^2}^2) + \varepsilon^2 \|\nabla \mathbf{d}\|_{H^2}^2 \|\nabla \mathbf{d}\|_{H^1}^2 \|\sigma\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^2}^2 \|\nabla \mathbf{d}\|_{H^1}^2).$$

Inserting the above two inequalities into (2.18), it follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\sqrt{\varrho}\omega\|_{L^2}^2 + \mu \|\nabla \omega\|_{L^2}^2 \\ & \leq \delta \|\omega\|_{H^2}^2 + C_{\delta} (\|\mathbf{u}\|_{H^1}^2 + \|\mathbf{u}\|_{H^2}^2 (\|\mathbf{u}\|_{H^1}^2 + \varepsilon^2 \|\sigma\|_{H^2}^2) + \varepsilon^2 \|\nabla \mathbf{d}\|_{H^2}^2 \|\nabla \mathbf{d}\|_{H^1}^2 \|\sigma\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^2}^2 \|\nabla \mathbf{d}\|_{H^1}^2). \end{aligned} \quad (2.19)$$

Taking $\langle \nabla(1.7)_3, \theta \Delta \nabla \mathbf{d} \rangle$, we obtain that

$$\begin{aligned} & \frac{\theta}{2} \frac{d}{dt} \|\nabla^2 \mathbf{d}\|_{L^2}^2 + \theta^2 \|\nabla \Delta \mathbf{d}\|_{L^2}^2 = \theta \int_{\Omega} (-\nabla(\mathbf{u} \cdot \nabla \mathbf{d}) + \nabla(|\nabla \mathbf{d}|^2 \mathbf{d})) \cdot \Delta \nabla \mathbf{d} d\mathbf{x} \\ & \leq \frac{\theta^2}{2} \|\Delta \nabla \mathbf{d}\|_{L^2}^2 + C(\|\nabla \mathbf{d}\|_{H^1}^2 (\|\mathbf{u}\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^2}^2) + \|\nabla \mathbf{d}\|_{H^1}^6), \end{aligned} \quad (2.20)$$

where we have use the boundary condition $\nabla \mathbf{d}_t \cdot \mathbf{n} = 0$ on $\partial\Omega$. Combining (2.19) and (2.20) together, and taking δ appropriately small, then we obtain the following lemma.

Lemma 2.6. *We have*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\sqrt{\varrho}\omega\|_{L^2}^2 + \theta \|\nabla^2 \mathbf{d}\|_{L^2}^2) + \mu \|\nabla \omega\|_{L^2}^2 + \theta^2 \|\Delta \nabla \mathbf{d}\|_{L^2}^2 \\ & \leq \delta_3 \|\omega\|_{H^2}^2 + C_{6,\delta_3} (\|\mathbf{u}\|_{H^1}^2 + \|\mathbf{u}\|_{H^2}^2 (\|\mathbf{u}\|_{H^1}^2 + \varepsilon^2 \|\sigma\|_{H^2}^2) + \varepsilon^2 \|\nabla \mathbf{d}\|_{H^2}^2 \|\nabla \mathbf{d}\|_{H^1}^2 \|\sigma\|_{H^2}^2 \\ & \quad + \|\nabla \mathbf{d}\|_{H^1}^2 (\|\mathbf{u}\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^1}^4)), \end{aligned}$$

where δ_3 is small and to be chosen later.

Now, assume δ_1 is a sufficiently small fixed constant, and to be chosen later, then C_{4,δ_1} is fixed. Defining two functions

$$\begin{aligned}
\Psi_1(t) &:= 2 \int_{\Omega} \varrho \mathbf{u} \cdot \mathbf{u}_t \, d\mathbf{x} + \|(\nabla \sigma, \sqrt{\varrho} \operatorname{curl} \mathbf{u})\|_{L^2}^2 + C_7 \|\mathbf{d}_t\|_{H^1}^2 + 2\mu \|D(\mathbf{u})\|_{L^2}^2 + \lambda \|\operatorname{div} \mathbf{u}\|_{L^2}^2 \\
&\quad + \frac{3\theta C_7}{2} \|\nabla \mathbf{d}\|_{L^2}^2 + 2\theta \|\nabla^2 \mathbf{d}\|_{L^2}^2 + C_7 \|(\sqrt{P'(1)} \sigma_t, \sqrt{\varrho} \mathbf{u}_t)\|_{L^2}^2; \\
\Phi_1(t) &:= \|\sqrt{P'(1)} \sigma_t\|_{L^2}^2 + \|\nabla \mathbf{d}_t\|_{L^2}^2 + \gamma_1 \|\nabla \operatorname{div} \mathbf{u}\|_{L^2}^2 + C_7 \gamma_2 (\|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{d}_t\|_{H^1}^2 + \|\Delta \mathbf{d}_t\|_{L^2}^2) \\
&\quad + \mu \|\nabla \operatorname{curl} \mathbf{u}\|_{L^2}^2 + \theta^2 \|\Delta \nabla \mathbf{d}\|_{L^2}^2,
\end{aligned}$$

where C_7 is a positive constant such that $C_7 \gamma_2 \geq 2(C_3 + C_{4,\delta_1}) + 1$ as least, so that the quality $\int_{\Omega} \varrho \mathbf{u} \cdot \mathbf{u}_t \, d\mathbf{x}$ can be controlled by $\|\sqrt{\varrho} \mathbf{u}\|_{L^2}$ and $C_7 \|\sqrt{\varrho} \mathbf{u}_t\|_{L^2}$, and the terms of $O(1) \|\mathbf{u}_t\|_{H^1}^2$ on the right sides in previous estimates can be eliminated by $\Phi_1(t)$. Notice that C_7 should be selected larger enough in order to complete the energy estimates, and once C_7 is fixed, we can choose $\delta_2 < \frac{\theta \nu}{100 C_7} < C_3$, and then C_5 is determined. Then we conclude from Lemmas 2.3–2.6 that for small ε , there are a positive constant M_1 and a sufficiently small constant δ_3 and a small fixed constant δ_1 , such that

$$\begin{aligned}
\frac{d}{dt} \Psi_1(t) + \Phi_1(t) &\leq \delta_1 (\|\nabla^2 \operatorname{div} \mathbf{u}\|_{L^2}^2 + \|\nabla \operatorname{div} \mathbf{u}\|_{L^2}^2) + \delta_3 \|\omega\|_{H^2}^2 + 2(C_3 + C_{4,\delta_1}) \|\mathbf{u}\|_{H^1}^2 + M_1 \left(\|\sigma_t\|_{H^1}^2 (\|\sigma\|_{H^1}^2 + \|\mathbf{u}\|_{H^2}^4) \right. \\
&\quad + \|\mathbf{u}_t\|_{H^1}^2 \|\mathbf{u}\|_{H^1}^2 + \|\nabla \mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^2 + \|\nabla \mathbf{d}\|_{H^1}^2 (\|\mathbf{u}\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^1}^4 + \|\nabla \mathbf{d}\|_{H^2}^2) + \|\sigma\|_{H^2}^4 \\
&\quad \left. + \|\mathbf{u}\|_{H^2}^4 + \|\nabla \mathbf{d}_t\|_{H^1}^2 (\|\mathbf{u}\|_{H^1}^2 + \|\nabla \mathbf{d}\|_{H^1}^2) + \|\mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^4 + \|\nabla \mathbf{d}\|_{H^2}^2 \|\nabla \mathbf{d}\|_{H^1}^2 \|\sigma\|_{H^2}^2 \right). \quad (2.21)
\end{aligned}$$

2.3. Estimates of second order derivatives of σ , \mathbf{u} and $\nabla \mathbf{d}$

We still need to estimate the spatial and temporal derivatives of second order to close the energy estimates. The strategy is similar as that of the first-order estimates.

Notice that $\operatorname{curl} \nabla = 0$, then we have

$$\begin{aligned}
\int_{\Omega} \overline{\operatorname{curl} \operatorname{curl} \mathbf{u}_t} \cdot \nabla \operatorname{div} \mathbf{u} \, d\mathbf{x} &= \int_{\partial \Omega} \omega_t \cdot (\mathbf{n} \times \nabla \operatorname{div} \mathbf{u}) \, dS \\
&= \int_{\partial \Omega} (2\chi - \frac{\alpha}{\mu}) (\mathbf{u}_t \cdot \boldsymbol{\tau}) (\mathbf{n} \times \nabla \operatorname{div} \mathbf{u}) \, dS \\
&\leq \delta \|\nabla^2 \operatorname{div} \mathbf{u}\|_{L^2}^2 + C_{\delta} \|\mathbf{u}_t\|_{H^1}^2.
\end{aligned}$$

Multiplying both sides of (2.5) by $\nabla \operatorname{div} \mathbf{u}$ and integrating, we obtain

$$\begin{aligned}
&\frac{2\nu + \lambda}{2} \frac{d}{dt} \|\nabla \operatorname{div} \mathbf{u}\|_{L^2}^2 - \frac{d}{dt} \int_{\Omega} \varrho \mathbf{u}_t \cdot \nabla \operatorname{div} \mathbf{u} \, d\mathbf{x} - \frac{P'(1)}{\varepsilon} \int_{\Omega} \nabla \sigma_t \cdot \nabla \operatorname{div} \mathbf{u} \, d\mathbf{x} \\
&= \int_{\Omega} \left[\left(\frac{P'(1 + \varepsilon \sigma) - P'(1)}{\varepsilon} \nabla \sigma \right)_t + \varepsilon \sigma_t \mathbf{u} \cdot \nabla \mathbf{u} + \varrho (\mathbf{u}_t \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}_t) + \nu (\nabla \mathbf{d} \cdot \Delta \mathbf{d})_t \right] \cdot \nabla \operatorname{div} \mathbf{u} \, d\mathbf{x} \\
&\quad - \int_{\Omega} \varrho \mathbf{u}_t \cdot \nabla \operatorname{div} \mathbf{u}_t \, d\mathbf{x} + \mu \int_{\Omega} \overline{\operatorname{curl} \operatorname{curl} \mathbf{u}_t} \cdot \nabla \operatorname{div} \mathbf{u} \, d\mathbf{x} \\
&\leq \delta \|(\nabla \operatorname{div} \mathbf{u}_t, \nabla \operatorname{div} \mathbf{u}, \nabla \sigma_t)\|_{L^2}^2 + C_{\delta} \left(\|\mathbf{u}_t\|_{H^1}^2 + \|\sigma_t\|_{H^1}^2 \|\sigma\|_{H^2}^2 + \|\mathbf{u}\|_{H^2}^2 (\varepsilon^2 \|\sigma_t\|_{H^1}^2 \|\mathbf{u}\|_{H^2}^2 + \|\mathbf{u}_t\|_{H^1}^2) \right. \\
&\quad \left. + \|\nabla \mathbf{d}\|_{H^2}^2 \|\nabla \mathbf{d}_t\|_{H^1}^2 \right).
\end{aligned}$$

Similarly, multiplying $\nabla(1.7)_1$ by $P'(1)\nabla\sigma_t$ and integrating, it follows that

$$\begin{aligned} & \|\sqrt{P'(1)}\nabla\sigma_t\|_{L^2}^2 + \frac{P'(1)}{\varepsilon} \int_{\Omega} \nabla\sigma_t \cdot \nabla \operatorname{div} \mathbf{u} d\mathbf{x} = -P'(1) \int_{\Omega} \nabla \operatorname{div}(\sigma\mathbf{u}) \cdot \nabla\sigma_t d\mathbf{x} \\ & \leq \delta \|\nabla\sigma_t\|_{L^2}^2 + C_{\delta} \|\mathbf{u}\|_{H^2}^2 \|\sigma\|_{H^2}^2. \end{aligned}$$

Putting the above two estimates together and choosing δ small enough, we get the following lemma.

Lemma 2.7. *We have*

$$\begin{aligned} & (2\mu + \lambda) \frac{d}{dt} \|\nabla \operatorname{div} \mathbf{u}\|_{L^2}^2 - 2 \frac{d}{dt} \int_{\Omega} \varrho \mathbf{u}_t \cdot \nabla \operatorname{div} \mathbf{u} d\mathbf{x} + \|\sqrt{P'(1)}\nabla\sigma_t\|_{L^2}^2 \\ & \leq \delta_4 \|(\nabla \operatorname{div} \mathbf{u}_t, \nabla \operatorname{div} \mathbf{u})\|_{L^2}^2 + C_{8,\delta_4} \left(\|\mathbf{u}_t\|_{H^1}^2 + \|\sigma_t\|_{H^1}^2 \|\sigma\|_{H^2}^2 + \|\mathbf{u}\|_{H^2}^2 (\varepsilon^2 \|\sigma_t\|_{H^1}^2 + \|\mathbf{u}\|_{H^2}^2 + \|\mathbf{u}_t\|_{H^1}^2) \right. \\ & \quad \left. + \|\nabla \mathbf{d}\|_{H^2}^2 \|\nabla \mathbf{d}_t\|_{H^1}^2 + \|\mathbf{u}\|_{H^2}^2 \|\sigma\|_{H^2}^2 \right), \end{aligned}$$

where δ_4 is small and to be chosen later.

Differentiating (1.7)₂ with respect to \mathbf{x} , we get

$$\begin{aligned} (2\mu + \lambda) \nabla^2 \operatorname{div} \mathbf{u} - \frac{1}{\varepsilon} P'(1) \nabla^2 \sigma &= \mu \nabla \overline{\operatorname{curl}} \operatorname{curl} \mathbf{u} + \nabla \left(\frac{P'(1 + \varepsilon\sigma) - P'(1)}{\varepsilon} \nabla \sigma \right) + \varrho (\nabla \mathbf{u}_t + \nabla (\mathbf{u} \cdot \nabla \mathbf{u})) \\ &+ \varepsilon \nabla \sigma (\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) + \nabla (\nabla \mathbf{d} \cdot \Delta \mathbf{d}). \end{aligned} \quad (2.22)$$

Multiply the above equality (2.22) by $P'(1)^{-1} \nabla^2 \operatorname{div} \mathbf{u}$ and integrate to get

$$\begin{aligned} & (2\mu + \lambda) \|\sqrt{P'(1)^{-1}} \nabla^2 \operatorname{div} \mathbf{u}\|_{L^2}^2 - \frac{1}{\varepsilon} \int_{\Omega} \nabla^2 \operatorname{div} \mathbf{u} \nabla^2 \sigma d\mathbf{x} \\ & \leq \delta \|\nabla^2 \operatorname{div} \mathbf{u}\|_{L^2}^2 + C_{\delta} \left(\|\nabla^2 \operatorname{curl} \mathbf{u}\|_{L^2}^2 + \|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{u}\|_{H^2}^4 + \|\sigma\|_{H^2}^4 + \|\sigma\|_{H^2}^2 (\|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{u}\|_{H^2}^4) + \|\nabla \mathbf{d}\|_{H^2}^4 \right), \end{aligned} \quad (2.23)$$

where we have used the fact that $\|\nabla \overline{\operatorname{curl}} \operatorname{curl} \mathbf{u}\|_{L^2} \leq \|\nabla^2 \operatorname{curl} \mathbf{u}\|_{L^2}$. Similarly, differentiate (1.7)₁ twice with respect to \mathbf{x} and then multiply the resulting equality with $\nabla^2 \sigma$, it follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla^2 \sigma\|_{L^2}^2 + \frac{1}{\varepsilon} \int_{\Omega} \nabla^2 \operatorname{div} \mathbf{u} \nabla^2 \sigma d\mathbf{x} \\ &= - \int_{\Omega} (\mathbf{u} \cdot \nabla (\nabla^2 \sigma) + 2 \nabla \mathbf{u} \cdot \nabla (\nabla \sigma) + \nabla^2 \mathbf{u} \cdot \nabla \sigma + \nabla^2 (\sigma \operatorname{div} \mathbf{u})) \cdot \nabla^2 \sigma d\mathbf{x} \\ & \leq \delta \|\mathbf{u}\|_{H^3}^2 + C_{\delta} \|\sigma\|_{H^2}^4. \end{aligned} \quad (2.24)$$

Thus, by putting the above estimates (2.23), (2.24) together and choosing δ suitably small, we obtain the following lemma.

Lemma 2.8. *We have*

$$(2\mu + \lambda) \|\sqrt{P'(1)^{-1}} \nabla^2 \operatorname{div} \mathbf{u}\|_{L^2}^2 + \frac{d}{dt} \|\nabla^2 \sigma\|_{L^2}^2 \\ \leq \delta_5 \|\mathbf{u}\|_{H^3}^2 + C_{9,\delta_5} \left(\|\nabla^2 \operatorname{curl} \mathbf{u}\|_{L^2}^2 + \|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{u}\|_{H^2}^4 + \|\sigma\|_{H^2}^4 + \|\sigma\|_{H^2}^2 (\|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{u}\|_{H^2}^4) + \|\nabla \mathbf{d}\|_{H^2}^4 \right),$$

where δ_5 is small and to be chosen later.

Now, differentiating (2.5) with respect to time t and dividing the resulting equality by ϱ , we obtain

$$\mathbf{u}_{tt} - \varrho^{-1} ((2\mu + \lambda) \nabla \operatorname{div} \mathbf{u}_t - \mu \overline{\operatorname{curl} \operatorname{curl}} \mathbf{u}_t) + \frac{1}{\varepsilon} \varrho^{-1} P'(1) \nabla \sigma_t \\ = \varrho^{-1} \left(\left(\frac{P'(1) - P'(1 + \varepsilon \sigma)}{\varepsilon} \nabla \sigma \right)_t - \varepsilon \sigma_t (\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) + (\nabla \mathbf{d} \cdot \Delta \mathbf{d})_t \right) - (\mathbf{u} \cdot \nabla \mathbf{u})_t. \quad (2.25)$$

Multiplying the above equality (2.25) by $\nabla \operatorname{div} \mathbf{u}_t$ and integrating, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\operatorname{div} \mathbf{u}_t\|_{L^2}^2 + (2\mu + \lambda) \|\sqrt{\varrho^{-1}} \nabla \operatorname{div} \mathbf{u}_t\|_{L^2}^2 - \frac{1}{\varepsilon} \int_{\Omega} \varrho^{-1} P'(1) \nabla \sigma_t \nabla \operatorname{div} \mathbf{u}_t \, d\mathbf{x} \\ \leq \delta \|\nabla \operatorname{div} \mathbf{u}_t\|_{L^2}^2 + C_{\delta} \left(\|\nabla \operatorname{curl} \mathbf{u}_t\|_{L^2}^2 + \|\sigma_t\|_{H^1}^2 (\|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{u}\|_{H^2}^4 + \|\sigma\|_{H^2}^2) \right. \\ \left. + \|\nabla \mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^2}^2 + \|\mathbf{u}_t\|_{H^1}^2 \|\mathbf{u}\|_{H^2}^2 \right). \quad (2.26)$$

Similarly, taking $\partial_t \nabla$ on (1.7)₁ to get

$$\nabla \sigma_{tt} + \mathbf{u} \cdot \nabla^2 \sigma_t + \mathbf{u}_t \cdot \nabla^2 \sigma + (\nabla \mathbf{u} \nabla \sigma)_t + (\nabla \sigma \operatorname{div} \mathbf{u})_t + (\sigma \nabla \operatorname{div} \mathbf{u})_t + \frac{1}{\varepsilon} \nabla \operatorname{div} \mathbf{u}_t = 0. \quad (2.27)$$

Multiplying the above equality (2.27) by $\varrho^{-1} P'(1) \nabla \sigma_t$ and integrating, it follows that

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{\varrho^{-1} P'(1)} \nabla \sigma_t\|_{L^2}^2 + \frac{1}{\varepsilon} \int_{\Omega} \varrho^{-1} P'(1) \nabla \sigma_t \nabla \operatorname{div} \mathbf{u}_t \, d\mathbf{x} \\ = \frac{P'(1)}{2} \int_{\Omega} ((\varrho^{-1})_t + \operatorname{div}(\varrho^{-1} \mathbf{u})) |\nabla \sigma_t|^2 \, d\mathbf{x} - P'(1) \int_{\Omega} \varrho^{-1} \nabla \sigma_t \cdot (\mathbf{u}_t \cdot \nabla^2 \sigma + (\nabla \mathbf{u} \nabla \sigma)_t + (\nabla \sigma \operatorname{div} \mathbf{u})_t + (\sigma \nabla \operatorname{div} \mathbf{u})_t) \, d\mathbf{x} \\ = P'(1) \int_{\Omega} \varrho^{-1} \operatorname{div}(\mathbf{u}) |\nabla \sigma_t|^2 \, d\mathbf{x} - P'(1) \int_{\Omega} \varrho^{-1} \nabla \sigma_t \cdot (\mathbf{u}_t \cdot \nabla^2 \sigma + (\nabla \mathbf{u} \nabla \sigma)_t + (\nabla \sigma \operatorname{div} \mathbf{u})_t + (\sigma \nabla \operatorname{div} \mathbf{u})_t) \, d\mathbf{x} \\ \leq \delta (\|\nabla \operatorname{div} \mathbf{u}_t\|_{L^2}^2 + \|\mathbf{u}\|_{H^3}^2 + \|\mathbf{u}_t\|_{H^1}^2) + C_{\delta} \left(\|\sigma_t\|_{H^1}^4 + \|\sigma\|_{H^2}^2 \|\sigma_t\|_{H^1}^2 \right). \quad (2.28)$$

Combining the above two estimates (2.26), (2.28) together, we get the following lemma.

Lemma 2.9. *We have*

$$\frac{d}{dt} \left(\|\operatorname{div} \mathbf{u}_t\|_{L^2}^2 + \|\sqrt{\varrho^{-1} P'(1)} \nabla \sigma_t\|_{L^2}^2 \right) + (2\mu + \lambda) \|\sqrt{\varrho^{-1}} \nabla \operatorname{div} \mathbf{u}_t\|_{L^2}^2 \\ \leq \delta_6 (\|\mathbf{u}_t\|_{H^2}^2 + \|\mathbf{u}\|_{H^3}^2) + C_{10,\delta_6} \left(\|\nabla \operatorname{curl} \mathbf{u}_t\|_{L^2}^2 + \|\sigma_t\|_{H^1}^2 (\|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{u}\|_{H^2}^4 + \|\sigma\|_{H^2}^2) \right. \\ \left. + \|\nabla \mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^2}^2 + \|\mathbf{u}_t\|_{H^1}^2 \|\mathbf{u}\|_{H^2}^2 \right),$$

where δ_6 is small and to be chosen later.

We now turn to estimate the derivative of $\operatorname{curl} \mathbf{u}$. To this end, we apply ∂_t on (2.17) to get that

$$\begin{cases} \varrho \omega_{tt} + \varrho \mathbf{u} \cdot \omega_t - \mu \Delta \omega_t = h, \\ \omega_t = (2\chi - \frac{\alpha}{\mu}) \mathbf{u}_t \cdot \boldsymbol{\tau} \end{cases} \quad \text{on } \partial\Omega, \quad (2.29)$$

where

$$h := -\varepsilon \sigma_t (\omega_t + \mathbf{u} \cdot \nabla \omega) - \varrho \mathbf{u}_t \nabla \omega + g_t.$$

Multiplying (2.29)₁ by ω_t and integrating, we have

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{\varrho} \omega_t\|_{L^2}^2 + \mu \|\nabla \omega_t\|_{L^2}^2 = \int_{\Omega} h \omega_t \, d\mathbf{x} + \mu \int_{\partial\Omega} \omega_t \nabla \omega_t \cdot \mathbf{n} \, dS.$$

Notice that the boundary condition of (2.29)₂ and the trace theorem give that

$$\begin{aligned} \int_{\partial\Omega} \omega_t \nabla \omega_t \cdot \mathbf{n} \, dS &= \int_{\Omega} (2\chi - \frac{\alpha}{\mu}) (\mathbf{u}_t \cdot \boldsymbol{\tau}) (\nabla \omega_t \cdot \mathbf{n}) \, dS \\ &\leq \delta \|\nabla \omega_t\|_{H^1}^2 + C_{\delta} \|\mathbf{u}_t\|_{H^1}^2. \end{aligned}$$

By using (2.32) below for $\nabla \omega_t$, we have

$$\|\nabla \omega_t\|_{H^1} \leq C(\|\Delta \omega_t\|_{L^2} + \|\nabla \omega_t \cdot \mathbf{n}\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|\nabla \omega_t\|_{L^2}).$$

Form (2.29)₁ again, we get

$$\mu \|\Delta \omega_t\|_{L^2}^2 \leq C(\|\mathbf{u}_{tt}\|_{H^1}^2 + \|h\|_{L^2}^2 + \|\mathbf{u}\|_{H^2}^2 \|\mathbf{u}_t\|_{H^1}^2).$$

Collecting all the above estimates, thus we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\sqrt{\varrho} \omega_t\|_{L^2}^2 + \mu \|\nabla \omega_t\|_{L^2}^2 \\ &\leq \delta (\|\mathbf{u}_{tt}\|_{H^1}^2 + \|\mathbf{u}_t\|_{H^2}^2 + \|\omega_t\|_{H^1}^2) + C_{\delta} \left(\|\mathbf{u}_t\|_{H^1}^2 + \varepsilon^2 \|\sigma_t\|_{H^1}^2 (\|\omega_t\|_{L^2}^2 + \|\mathbf{u}\|_{H^2}^2 \|\nabla \omega\|_{L^2}^2) + \|\mathbf{u}_t\|_{H^1}^2 \|\nabla \omega\|_{L^2}^2 \right. \\ &\quad \left. + \varepsilon^2 \|\sigma_t\|_{L^2}^2 \|\nabla \mathbf{u}\|_{H^1}^4 + \|\mathbf{u}_t\|_{H^1}^2 \|\mathbf{u}\|_{H^2}^2 + \varepsilon^4 \|\sigma_t\|_{H^1}^2 \|\sigma\|_{H^2}^2 \|\mathbf{u}\|_{H^2}^2 + \varepsilon^2 (\|\sigma_t\|_{H^1}^2 \|\nabla^2 \mathbf{u}\|_{H^1}^2 + \|\sigma\|_{H^2}^2 \|\mathbf{u}_t\|_{H^2}^2) \right) \\ &\quad + \int_{\Omega} \left(\frac{\varepsilon}{\varrho} \nabla \sigma \times (\nabla \mathbf{d} \cdot \Delta \mathbf{d}) + \operatorname{curl}(\nabla \mathbf{d} \cdot \Delta \mathbf{d}) \right)_t \omega_t \, d\mathbf{x}. \end{aligned} \quad (2.30)$$

Notice that the last term of the right-hand side can be estimate as follows

$$\begin{aligned} &\int_{\Omega} \left(\frac{\varepsilon}{\varrho} \nabla \sigma \times (\nabla \mathbf{d} \cdot \Delta \mathbf{d}) + \operatorname{curl}(\nabla \mathbf{d} \cdot \Delta \mathbf{d}) \right)_t \omega_t \, d\mathbf{x} \\ &= \int_{\Omega} \left(-\frac{\varepsilon^2}{\varrho} \sigma_t \nabla \sigma \times (\nabla \mathbf{d} \cdot \Delta \mathbf{d}) + \frac{\varepsilon}{\varrho} (\nabla \sigma \times (\nabla \mathbf{d} \cdot \Delta \mathbf{d}))_t + \operatorname{curl}(\nabla \mathbf{d} \cdot \Delta \mathbf{d})_t \right) \omega_t \, d\mathbf{x} \\ &\leq \delta \|\omega_t\|_{H^1}^2 + C_{\delta} \left(\varepsilon^4 \|\sigma_t\|_{H^1}^2 \|\sigma\|_{H^2}^2 \|\nabla \mathbf{d}\|_{H^2}^4 + \varepsilon^2 (\|\sigma_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^2}^4 + \|\sigma\|_{H^2}^2 \|\nabla \mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^2) \right. \\ &\quad \left. + \|\nabla \mathbf{d}_t\|_{H^2}^2 \|\nabla \mathbf{d}\|_{H^1}^2 + \|\nabla \mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^2}^2 \right). \end{aligned}$$

Hence, inserting the above estimate into (2.30), and then applying (2.3) for ω_t and choosing δ small enough, we obtain the following lemma.

Lemma 2.10. *We have*

$$\begin{aligned} & \frac{d}{dt} \|\sqrt{\varrho} \omega_t\|_{L^2}^2 + \mu \|\nabla \omega_t\|_{L^2}^2 \\ & \leq \delta_7 (\|\mathbf{u}_{tt}\|_{H^1}^2 + \|\mathbf{u}_t\|_{H^2}^2) + C_{11, \delta_7} \left(\|\mathbf{u}_t\|_{H^1}^2 + \|\sigma_t\|_{H^1}^2 (\|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{u}\|_{H^2}^4 + \|\mathbf{u}\|_{H^3}^2 + \|\sigma\|_{H^2}^2 (\|\mathbf{u}\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^2}^2)) \right. \\ & \quad \left. + \|\nabla \mathbf{d}\|_{H^2}^4) + \|\mathbf{u}_t\|_{H^2}^2 (\|\mathbf{u}\|_{H^2}^2 + \|\sigma\|_{H^2}^2) + \|\sigma\|_{H^2}^2 \|\nabla \mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^2 + \|\nabla \mathbf{d}_t\|_{H^2}^2 \|\nabla \mathbf{d}\|_{H^1}^2 + \|\nabla \mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^2}^2 \right), \end{aligned}$$

where δ_7 is small and to be chosen later.

On the other hand, multiplying (2.29) by $\omega_t - \eta \Delta \omega$, where $\eta > 0$ to be chosen later, and integrating, we obtain

$$\begin{aligned} & \frac{\mu}{2} \frac{d}{dt} \|\nabla \omega\|_{L^2}^2 + \|\sqrt{\varrho} \omega_t\|_{L^2}^2 + \eta \mu \|\Delta \omega\|_{L^2}^2 \\ & \leq \delta \|\omega\|_{H^2}^2 + C_\delta \|\mathbf{u}_t\|_{H^1}^2 + \frac{1}{4} \|\sqrt{\varrho} \omega_t\|_{L^2}^2 + C (\|\mathbf{u}\|_{H^2}^4 + \|\sigma\|_{H^2}^2 \|\mathbf{u}\|_{H^3}^2 + \|\sigma\|_{H^2}^2 \|\nabla \mathbf{d}\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^2}^4) \\ & \quad + \frac{1}{4} \|\sqrt{\varrho} \omega_t\|_{L^2}^2 + \eta^2 \|\varrho\|_{L^\infty} \|\Delta \omega\|_{L^2}^2 + \eta^2 \|\Delta \omega\|_{L^2}^2 + \frac{1}{4} (\|\sigma\|_{H^2}^2 \|\nabla \mathbf{d}\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^2}^4), \end{aligned}$$

where we have used the fact that the boundary condition yields

$$\begin{aligned} - \int_{\Omega} \Delta \omega \omega_t d\mathbf{x} &= \frac{1}{2} \frac{d}{dt} \|\nabla \omega\|_{L^2}^2 - \int_{\partial\Omega} (2\chi - \frac{\alpha}{\mu}) (\mathbf{u}_t \cdot \boldsymbol{\tau}) (\nabla \omega \cdot \mathbf{n}) dS \\ &\leq \frac{1}{2} \frac{d}{dt} \|\nabla \omega\|_{L^2}^2 + \delta \|\omega\|_{H^2}^2 + C_\delta \|\mathbf{u}_t\|_{H^1}^2. \end{aligned}$$

By choosing $\eta = \frac{\mu}{10} \leq \mu(2(1 + \|\varrho\|_{L^\infty}))^{-1}$, and applying the elliptic theory to the equation (2.17) to obtain that

$$\|\omega\|_{H^2}^2 \leq C (\|g\|_{L^2}^2 + \|\varrho\|_{H^2}^2 (\|\omega_t\|_{L^2}^2 + \|\mathbf{u}\|_{H^2}^2 \|\nabla \omega\|_{L^2}^2) + \|\mathbf{u} \cdot \boldsymbol{\tau}\|_{H^{\frac{3}{2}}(\partial\Omega)}^2),$$

and the trace theorem gives that

$$\|\mathbf{u} \cdot \boldsymbol{\tau}\|_{H^{\frac{3}{2}}(\partial\Omega)}^2 \leq C \|\mathbf{u}\|_{H^2}^2.$$

Then we obtain the following lemma for sufficiently small δ .

Lemma 2.11. *We have*

$$\begin{aligned} & \mu \frac{d}{dt} \|\nabla \omega\|_{L^2}^2 + \|\sqrt{\varrho} \omega_t\|_{L^2}^2 + \frac{\mu^2}{10} \|\Delta \omega\|_{L^2}^2 \\ & \leq \delta_8 \|\mathbf{u}\|_{H^2}^2 + C_{12, \delta_8} (\|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{u}\|_{H^2}^4 + \|\sigma\|_{H^2}^2 \|\mathbf{u}\|_{H^3}^2 + \|\sigma\|_{H^2}^2 \|\nabla \mathbf{d}\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^2}^4), \end{aligned}$$

where δ_8 is small and to be chosen later.

Taking $\langle \partial_t(1.7)_3, \mathbf{d}_{tt} - \eta \Delta \mathbf{d}_t \rangle$, it follows that

$$\begin{aligned} & \frac{\theta + \eta}{2} \frac{d}{dt} \|\nabla \mathbf{d}_t\|_{L^2}^2 + \|\mathbf{d}_{tt}\|_{L^2}^2 + \eta \theta \|\Delta \mathbf{d}_t\|_{L^2}^2 \\ & \leq \frac{1}{4} \|\mathbf{d}_{tt}\|_{L^2}^2 + \eta^2 \|\varepsilon \Delta \mathbf{d}_t\|_{L^2}^2 + C(\|\mathbf{u}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^2 + \|\mathbf{u}\|_{H^1}^2 \|\nabla \mathbf{d}_t\|_{H^1}^2 + \|\mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^4 + \|\nabla \mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^2). \end{aligned}$$

Similarly, by taking $\langle \nabla(1.7)_3, \Delta \nabla \mathbf{d}_t \rangle$, we get

$$\theta \frac{d}{dt} \|\Delta \nabla \mathbf{d}\|_{L^2}^2 + \frac{1}{2} \|\nabla^2 \mathbf{d}_t\|_{L^2}^2 \leq \frac{1}{8\theta^2} \|\Delta \nabla \mathbf{d}_t\|_{L^2}^2 + C \|\nabla \mathbf{d}\|_{H^1}^2 (\|\mathbf{u}\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^1}^4 + \|\nabla \mathbf{d}\|_{H^2}^2).$$

Notice that by differentiating (1.7)₃ with respect to \mathbf{x} and t , it follows that

$$\nabla \mathbf{d}_{tt} - \theta \Delta \nabla \mathbf{d}_t = \nabla(-\mathbf{u} \cdot \nabla \mathbf{d} + \theta |\nabla \mathbf{d}|^2 \mathbf{d})_t.$$

Applying the elliptic theory to the above equality to obtain that

$$\theta^2 \|\Delta \nabla \mathbf{d}_t\|_{L^2}^2 \leq \|\nabla \mathbf{d}_{tt}\|_{L^2}^2 + C \left(\|\nabla \mathbf{d}_t\|_{H^1}^2 (\|\mathbf{u}\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^1}^4 + \|\nabla \mathbf{d}\|_{H^2}^2) + \|\mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^2}^4 + \|\mathbf{u}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^2}^2 \right).$$

Hence, combining the above inequality together, and then choosing $\eta = \frac{\theta}{2}$, we get

Lemma 2.12. *There exists a positive constant C_{13} , such that*

$$\begin{aligned} & \frac{d}{dt} \left(\frac{3\theta}{2} \|\nabla \mathbf{d}_t\|_{L^2}^2 + \theta \|\Delta \nabla \mathbf{d}\|_{L^2}^2 \right) + \|\mathbf{d}_{tt}\|_{L^2}^2 + \frac{\theta^2}{2} \|\Delta \mathbf{d}_t\|_{L^2}^2 + \frac{1}{2} \|\nabla^2 \mathbf{d}_t\|_{L^2}^2 \\ & \leq \frac{1}{4} \|\nabla \mathbf{d}_{tt}\|_{L^2}^2 + C_{13} \left(\|\mathbf{u}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^2}^2 + \|\nabla \mathbf{d}_t\|_{H^1}^2 (\|\mathbf{u}\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^1}^4 + \|\nabla \mathbf{d}\|_{H^2}^2) + \|\mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^4 \right. \\ & \quad \left. + \|\nabla \mathbf{d}\|_{H^1}^2 (\|\mathbf{u}\|_{H^2}^2 + \|\nabla \mathbf{d}\|_{H^1}^4 + \|\nabla \mathbf{d}\|_{H^2}^2) \right). \end{aligned}$$

We now proceed to derive more estimates, to this end, we introduce the following notations.

Definition 2.13.

$$\begin{aligned} \Psi(t) &:= C_{14} \Psi_0(t) + 6C_{15} K \Psi_1(t) + \Psi_2(t) + \|(\varepsilon \sqrt{P'(1)} \sigma_{tt}, \varepsilon \sqrt{\varrho} \mathbf{u}_{tt}, \varepsilon \nabla \mathbf{d}_{tt})\|_{L^2}^2 + \|\mathbf{d}_{tt}\|_{L^2}^2; \\ \Phi(t) &:= C_{14} \Phi_0(t) + 6C_{15} K \Phi_1(t) + \Phi_2(t) + \|\varepsilon \sigma_{tt}\|_{L^2}^2 + \gamma_3 \|\varepsilon \mathbf{u}_{tt}\|_{H^1}^2 + \|\sigma\|_{H^2}^2 + 2\theta \|\nabla \mathbf{d}_{tt}\|_{L^2}^2 + \theta \|\varepsilon \Delta \mathbf{d}_{tt}\|_{L^2}^2, \end{aligned}$$

where

$$\begin{aligned} \Psi_2(t) &:= (2\mu + \lambda) \|\nabla \operatorname{div} \mathbf{u}\|_{L^2}^2 - 2 \int_{\Omega} \varrho \mathbf{u}_t \cdot \nabla \operatorname{div} \mathbf{u} \, dx + \|\nabla^2 \sigma\|_{L^2}^2 + \|\operatorname{div} \mathbf{u}_t\|_{L^2}^2 \\ & \quad + \|\sqrt{\varrho^{-1} P'(1)} \nabla \sigma_t\|_{L^2}^2 + C_{16} \|\sqrt{\varrho} \operatorname{curl} \mathbf{u}_t\|_{L^2}^2 + \frac{60KC_{15}}{\mu} \|\nabla \operatorname{curl} \mathbf{u}\|_{L^2}^2 + \frac{3\theta}{2} \|\nabla \mathbf{d}_t\|_{L^2}^2 + \theta \|\Delta \nabla \mathbf{d}\|_{L^2}^2; \\ \Phi_2(t) &:= \|\sqrt{P'(1)} \nabla \sigma_t\|_{L^2}^2 + (2\mu + \lambda) \|\sqrt{P'(1)^{-1}} \nabla^2 \operatorname{div} \mathbf{u}\|_{L^2}^2 + (2\mu + \lambda) \|\sqrt{\varrho^{-1}} \nabla \operatorname{div} \mathbf{u}_t\|_{L^2}^2 + \mu C_{16} \|\nabla \operatorname{curl} \mathbf{u}_t\|_{L^2}^2 \\ & \quad + \frac{60KC_{15}}{\mu^2} \|\sqrt{\varrho} \operatorname{curl} \mathbf{u}_t\|_{L^2}^2 + 6C_{15} \|\nabla^2 \operatorname{curl} \mathbf{u}\|_{L^2}^2 + \|\mathbf{d}_{tt}\|_{L^2}^2 + \frac{\theta^2}{2} \|\Delta \mathbf{d}_t\|_{L^2}^2 + \frac{1}{2} \|\nabla^2 \mathbf{d}_t\|_{L^2}^2. \end{aligned}$$

Here $\gamma_3, C_{14}, C_{15}, C_{16}$ are three positive constants to be determined later, and K is a positive constant such that for any $\omega = \operatorname{curl} \mathbf{u}$ satisfying the boundary condition (2.17) (see [37]),

$$\|\nabla^2 \omega\|_{L^2}^2 \leq K(\|\Delta \omega\|_{L^2}^2 + \|\mathbf{u}\|_{H^2}^2).$$

Then, we can conclude from Lemmas 2.7–2.12 that

$$\begin{aligned} \frac{d}{dt} \Psi_2(t) + \Phi_2(t) &\leq \delta_4(\|\nabla \operatorname{div} \mathbf{u}_t\|_{L^2}^2 + \|\nabla \operatorname{div} \mathbf{u}\|_{L^2}^2) + (\delta_5 + \delta_6)\|\mathbf{u}\|_{H^3}^2 + (\delta_6 + \delta_7)\|\mathbf{u}_t\|_{H^2}^2 + \delta_7\|\mathbf{u}_{tt}\|_{H^1}^2 \\ &\quad + \frac{60KC_{15}}{\mu} \delta_8 \|\mathbf{u}\|_{H^2}^2 + \left(\frac{60KC_{15}}{\mu} C_{8,\delta_4} + C_{9,\delta_5} + C_{11,\delta_7} + C_{12,\delta_8} \right) \|\mathbf{u}_t\|_{H^1}^2 \\ &\quad + C_{9,\delta_5} \|\nabla^2 \operatorname{curl} \mathbf{u}\|_{L^2}^2 + C_{10,\delta_6} \|\nabla \operatorname{curl} \mathbf{u}_t\|_{L^2}^2 + \frac{1}{4} \|\nabla \mathbf{d}_{tt}\|_{L^2}^2 + C\Phi(t)(\Psi(t) + \Psi^2(t)). \end{aligned} \quad (2.31)$$

By using the following fact to control the terms $\|\mathbf{u}\|_{H^2}$, $\|\mathbf{u}\|_{H^3}$ and $\|\mathbf{u}_t\|_{H^2}$

$$\|\mathbf{F}\|_{H^s} \leq \bar{C}(\|\operatorname{div} \mathbf{F}\|_{H^{s-1}} + \|\operatorname{curl} \mathbf{F}\|_{H^{s-1}} + \|\mathbf{F} \cdot \mathbf{n}\|_{H^{s-\frac{1}{2}}(\partial\Omega)} + \|\mathbf{F}\|_{H^{s-1}}) \quad (2.32)$$

for any $\mathbf{F} \in H^s(\Omega)$ with Ω be a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$ and outward normal \mathbf{n} , where \bar{C} is a positive constant independent of \mathbf{F} . Hence, we can derive from (2.21) and (2.31) that there exists a positive constant \bar{C} , such that

$$\begin{aligned} &\frac{d}{dt} (60C_{15}K\Psi_1(t) + \Psi_2(t)) + 60C_{15}K\Phi_1(t) + \Phi_2(t) \\ &\leq (\delta_4 + \bar{C}(\delta_6 + \delta_7))\|\nabla \operatorname{div} \mathbf{u}_t\|_{L^2}^2 + (\bar{C}(\delta_6 + \delta_7) + C_{10,\delta_6})\|\nabla \operatorname{curl} \mathbf{u}_t\|_{L^2}^2 + (60C_{15}K\delta_1 + \bar{C}(\delta_5 + \delta_6))\|\nabla^2 \operatorname{div} \mathbf{u}\|_{L^2}^2 \\ &\quad + (\bar{C}(\delta_5 + \delta_6) + C_{9,\delta_5})\|\nabla^2 \operatorname{curl} \mathbf{u}\|_{L^2}^2 + (\delta_4 + 60C_{15}K\delta_3 + \frac{60KC_{15}\bar{C}}{\mu} \delta_8 + 60KC_{15}\delta_1)\|\nabla \operatorname{div} \mathbf{u}\|_{L^2}^2 \\ &\quad + (60C_{15}K\delta_3 + \frac{60KC_{15}\bar{C}}{\mu} \delta_8)\|\nabla \operatorname{curl} \mathbf{u}\|_{L^2}^2 + \delta_7\|\mathbf{u}_{tt}\|_{H^1}^2 + (\frac{60KC_{15}\bar{C}}{\mu} C_{8,\delta_4} + C_{9,\delta_5} + C_{11,\delta_7} + C_{12,\delta_8})\|\mathbf{u}_t\|_{H^1}^2 \\ &\quad + (2(C_3 + C_{4,\delta_1}) + \bar{C}(\delta_5 + \delta_6) + 60C_{15}K\delta_3 + \frac{60KC_{15}\bar{C}}{\mu} \delta_8)\|\mathbf{u}\|_{H^1}^2 + \frac{1}{4} \|\nabla \mathbf{d}_{tt}\|_{L^2}^2 + C\Phi(t)(\Psi(t) + \Psi^2(t)). \end{aligned} \quad (2.33)$$

Let $\delta_i < \frac{1}{2}$ ($i = 1, 3, 4, 5, 6, 8$) sufficiently small such that $\delta_4 < \frac{c_0}{4}$ and $\max\{60KC_{15}\delta_1, 60KC_{15}\delta_3, \bar{C}(\delta_5 + \delta_6), \bar{C}\delta_6, \frac{60KC_{15}\bar{C}}{\mu} \delta_8\} < \frac{\delta_4}{4}$, then $C_{4,\delta_1} = C_4, C_{8,\delta_4} = C_8, C_{9,\delta_5} = C_9, C_{10,\delta_6} = C_{10}, C_{12,\delta_8} = C_{12}$ are determined. Let $\delta_7 \in (0, \frac{1}{2})$ such that $\bar{C}\delta_7 < \frac{\delta_4}{4}$ be select later, C_{14}, C_{15}, C_{16} are three big constants such that $C_{14}c_0 > 4(2(C_3 + C_4) + \bar{C} + 60C_{15}K + \frac{60KC_{15}\bar{C}}{\mu})$, $C_{15} > 2(\bar{C} + C_9)$ and $C_{16} > 2(\bar{C} + C_{10})$, then the higher spatial derivative of \mathbf{u} and \mathbf{u}_t on the right terms of (2.33) would be absorbed by the left-hand side of (2.33), it follows that

$$\begin{aligned} &\frac{d}{dt} (60C_{15}K\Psi_1(t) + \Psi_2(t)) + \frac{4}{5} (60C_{15}K\Phi_1(t) + \Phi_2(t)) \\ &\leq \delta_7\|\mathbf{u}_{tt}\|_{H^1}^2 + \frac{c_0C_{14}}{4} \|\mathbf{u}\|_{H^1}^2 + \frac{1}{4} \|\nabla \mathbf{d}_{tt}\|_{L^2}^2 + C\Phi(t)(\Psi(t) + \Psi^2(t)). \end{aligned} \quad (2.34)$$

In what follows, we shall give estimates of $\|\varepsilon\sigma_{tt}\|_{L^2}$, $\|\varepsilon\mathbf{u}_{tt}\|_{L^2}$ and $\|\varepsilon\nabla \mathbf{d}_{tt}\|_{L^2}$. Differentiating (1.7)₁ twice with respect to t , it follows that

$$\sigma_{ttt} + \mathbf{u} \cdot \sigma_t + 2\mathbf{u}_t \cdot \nabla \sigma_t + \mathbf{u}_{tt} \cdot \nabla \sigma + \sigma_{tt} \operatorname{div} \mathbf{u} + 2\sigma_t \operatorname{div} \mathbf{u}_t + \sigma \operatorname{div} \mathbf{u}_{tt} + \frac{1}{\varepsilon} \operatorname{div} \mathbf{u}_{tt} = 0.$$

Multiplying the above equality by $\varepsilon^2 P'(1)\sigma_{tt}$ and integrating, it follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\varepsilon \sqrt{P'(1)} \sigma_{tt}\|_{L^2}^2 - \varepsilon \int_{\Omega} P'(1) \nabla \sigma_{tt} \cdot \mathbf{u}_{tt} d\mathbf{x} \\ & \leq \delta (\|\operatorname{div} \mathbf{u}\|_{H^2}^2 + \|\mathbf{u}_t\|_{H^2}^2 + \|\varepsilon \mathbf{u}_{tt}\|_{H^1}^2) + C_{\delta} (\|\varepsilon \sigma_{tt}\|_{L^2}^2 + \|\sigma_t\|_{H^1}^2 + \|\sigma\|_{H^2}^2) \|\varepsilon \sigma_{tt}\|_{L^2}^2. \end{aligned} \quad (2.35)$$

Differentiating (1.7)₂ twice with respect to t , it is easy to get that

$$\varrho(\mathbf{u}_{ttt} - \mathbf{u} \cdot \nabla \mathbf{u}_{tt}) - \operatorname{div}(2\mu D(\mathbf{u}_{tt})) - \lambda \nabla \operatorname{div} \mathbf{u}_{tt} + \frac{1}{\varepsilon} P'(1) \nabla \sigma_{tt} = f, \quad (2.36)$$

with f is defined as

$$\begin{aligned} f = & \frac{P'(1) - P'(1 + \varepsilon \sigma)}{\varepsilon} \nabla \sigma_{tt} - 2P''(\varrho) \sigma_t \nabla \sigma_t - (\varepsilon P'''(\varrho) \sigma_t^2 + P''(\varrho) \sigma_{tt}) \nabla \sigma \\ & - \varrho(\mathbf{u}_{tt} \cdot \nabla \mathbf{u} + 2\mathbf{u}_t \cdot \nabla \mathbf{u}_t) - 2\varepsilon \sigma_t (\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u})_t - \varepsilon \sigma_{tt} (\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) - \nu (\nabla \mathbf{d} \cdot \Delta \mathbf{d})_{tt}. \end{aligned}$$

Multiplying (2.36) by $\varepsilon^2 \mathbf{u}_{tt}$ and integrating, it follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\varepsilon \sqrt{\varrho} \mathbf{u}_{tt}\|_{L^2}^2 + (2\mu + \lambda) \|\varepsilon \operatorname{div} \mathbf{u}_{tt}\|_{L^2}^2 + \lambda \|\varepsilon \operatorname{curl} \mathbf{u}_{tt}\|_{L^2}^2 + \varepsilon \int_{\Omega} P'(1) \nabla \sigma_{tt} \cdot \mathbf{u}_{tt} d\mathbf{x} \\ & \leq \delta \|\varepsilon \mathbf{u}_{tt}\|_{H^1}^2 + C_{\delta} \left(\|\sigma_t\|_{H^1}^2 \|\nabla \sigma_t\|_{L^2}^2 + \|\sigma_t\|_{H^1}^4 \|\nabla \sigma\|_{H^1}^2 + \|\varepsilon \sigma_{tt}\|_{L^2}^2 (\|\nabla \sigma\|_{H^1}^2 + \|\mathbf{u}_t\|_{H^1}^2) + \|\sigma_t\|_{H^1}^2 \|\varepsilon \mathbf{u}_{tt}\|_{L^2}^2 \right. \\ & \quad + \|\nabla \mathbf{u}\|_{H^1}^2 (\|\varepsilon \mathbf{u}_{tt}\|_{L^2}^2 + \|\varepsilon \sigma_{tt}\|_{L^2}^2 \|\mathbf{u}\|_{H^2}^2 + \|\sigma_t\|_{H^1}^2 \|\mathbf{u}_t\|_{H^1}^2) + \|\nabla \mathbf{u}_t\|_{L^2}^2 (\|\sigma_t\|_{H^1}^2 \|\mathbf{u}\|_{H^2}^2 + \|\mathbf{u}_t\|_{H^1}^2) \\ & \quad \left. + \|\varepsilon \nabla \mathbf{d}_{tt}\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^2}^2 + \|\nabla \mathbf{d}_t\|_{H^1}^2 \right), \end{aligned} \quad (2.37)$$

where we have used the fact that the boundary condition $\mathbf{u}_{tt} \cdot \mathbf{n} = 0$ on $\partial\Omega$ implies the following estimate

$$\begin{aligned} \int_{\Omega} \frac{P'(1) - P'(1 + \varepsilon \sigma)}{\varepsilon} \nabla \sigma_{tt} \cdot \varepsilon^2 \mathbf{u}_{tt} d\mathbf{x} &= - \int_{\Omega} \frac{P'(1) - P'(1 + \varepsilon \sigma)}{\varepsilon} \sigma_{tt} \cdot \varepsilon^2 \operatorname{div} \mathbf{u}_{tt} d\mathbf{x} - \int_{\Omega} P''(\varrho) \sigma_{tt} \nabla \sigma \cdot \varepsilon^2 \mathbf{u}_{tt} d\mathbf{x} \\ &\leq \delta \|\varepsilon \mathbf{u}_{tt}\|_{H^1}^2 + C_{\delta} \|\varepsilon \sigma_{tt}\|_{L^2}^2 \|\sigma\|_{H^2}^2. \end{aligned}$$

Taking $\langle (1.7)_3, \mathbf{d}_{tt} \rangle$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\mathbf{d}_{tt}\|_{L^2}^2 + \theta \|\nabla \mathbf{d}_{tt}\|_{L^2}^2 = \int_{\Omega} (-\mathbf{u} \cdot \nabla \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d})_{tt} \cdot \mathbf{d}_{tt} d\mathbf{x} \\ & \leq \frac{1}{16} \|\mathbf{d}_{tt}\|_{L^2}^2 + C \left(\|\varepsilon \mathbf{u}_{tt}\|_{L^2}^2 \|\nabla \mathbf{d}\|_{H^1}^2 + \|\mathbf{u}\|_{H^2}^2 \|\varepsilon \nabla \mathbf{d}_{tt}\|_{L^2}^2 + \|\mathbf{u}_t\|_{H^1}^2 \|\nabla \mathbf{d}_t\|_{H^1}^2 + \|\varepsilon \mathbf{d}_{tt}\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^4 \right. \\ & \quad \left. + \|\varepsilon \nabla \mathbf{d}_{tt}\|_{L^2}^2 \|\nabla \mathbf{d}\|_{H^2}^2 + \|\nabla \mathbf{d}_t\|_{H^1}^4 + \|\mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^2 \right). \end{aligned} \quad (2.38)$$

Taking $\langle \partial_t^2 (1.7)_3, \varepsilon^2 \Delta \mathbf{d}_{tt} \rangle$, it follows that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\varepsilon \nabla \mathbf{d}_{tt}\|_{L^2}^2 + \theta \|\varepsilon \Delta \mathbf{d}_{tt}\|_{L^2}^2 \\ & \leq \delta \|\varepsilon \Delta \mathbf{d}_{tt}\|_{L^2}^2 + C_{\delta} \left(\|\varepsilon \mathbf{u}_{tt}\|_{L^2}^2 \|\nabla \mathbf{d}\|_{H^1}^2 + \|\mathbf{u}\|_{H^2}^2 \|\varepsilon \nabla \mathbf{d}_{tt}\|_{L^2}^2 + \|\mathbf{u}_t\|_{H^1}^2 \|\nabla \mathbf{d}_t\|_{H^1}^2 + \|\varepsilon \mathbf{d}_{tt}\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^4 \right. \\ & \quad \left. + \|\varepsilon \nabla \mathbf{d}_{tt}\|_{L^2}^2 \|\nabla \mathbf{d}\|_{H^2}^2 + \|\nabla \mathbf{d}_t\|_{H^1}^4 + \|\mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}_t\|_{H^1}^2 \|\nabla \mathbf{d}\|_{H^1}^2 \right). \end{aligned} \quad (2.39)$$

Combining (2.35), (2.37), (2.38) and (2.39) together, letting δ small enough, applying (2.3) for \mathbf{u}_{tt} , and then using the Definition 2.13, we obtain the following lemma.

Lemma 2.14. *We have*

$$\begin{aligned} & \frac{d}{dt} (\|\varepsilon \sqrt{P'(1)} \sigma_{tt}\|_{L^2}^2 + \|\varepsilon \sqrt{\varrho} \mathbf{u}_{tt}\|_{L^2}^2 + \|\mathbf{d}_{tt}\|_{L^2}^2 + \|\varepsilon \nabla \mathbf{d}_{tt}\|_{L^2}^2) + \gamma_3 \|\varepsilon \mathbf{u}_{tt}\|_{H^1}^2 + 2\theta \|\nabla \mathbf{d}_{tt}\|_{L^2}^2 + \theta \|\varepsilon \Delta \mathbf{d}_{tt}\|_{L^2}^2 \\ & \leq \delta_8 (\|\mathbf{u}_t\|_{H^2}^2 + \|\mathbf{u}\|_{H^3}^2) + \frac{1}{8} \|\mathbf{d}_{tt}\|_{L^2}^2 + C\Phi(t)(\Psi(t) + \Psi^2(t)). \end{aligned}$$

In order to close the established estimates, we have to control the term $\|\varepsilon \sigma_{tt}\|_{L^2}$ and $\|\sigma\|_{H^2}$. To this end, we can calculate from $\nabla(1.7)_1$ that

$$\begin{aligned} \|\varepsilon \sigma_{tt}\|_{L^2}^2 & \leq C\varepsilon^2 (\|\mathbf{u}_t\|_{H^1}^2 \|\sigma\|_{H^2}^2 + \|\sigma_t\|_{H^1}^2 \|\mathbf{u}\|_{H^2}^2) + \|\operatorname{div} \mathbf{u}_t\|_{L^2}^2 \\ & \leq C(\varepsilon^2 \Phi(t) \Psi(t) + \|\operatorname{div} \mathbf{u}_t\|_{L^2}^2). \end{aligned} \quad (2.40)$$

Moreover, it follows from (1.7)₂, the condition $\int_{\Omega} \sigma \, dx = 0$, and the Poincaré's inequality that

$$\begin{aligned} \|\sigma\|_{H^2}^2 & \leq C \|\nabla \sigma\|_{H^1}^2 \leq C\varepsilon (\|\varrho\|_{H^1}^2 \|\mathbf{u}_t\|_{H^1}^2 + \|\varrho\|_{H^1}^2 \|\mathbf{u}\|_{H^2}^2 + \|\mathbf{u}\|_{H^3}^2) \\ & \leq C\varepsilon (\|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{u}\|_{H^3}^2) + C\varepsilon (\|\sigma\|_{H^1}^2 \|\mathbf{u}_t\|_{H^1}^2 + \|\sigma\|_{H^1}^2 \|\mathbf{u}\|_{H^2}^2) \\ & \leq C\varepsilon (\|\mathbf{u}_t\|_{H^1}^2 + \|\mathbf{u}\|_{H^3}^2) + C\varepsilon \Phi(t)(\Psi(t) + \Psi^2(t)). \end{aligned} \quad (2.41)$$

By applying (2.32) to estimate $\|\mathbf{u}_t\|_{H^2}$ and $\|\mathbf{u}\|_{H^3}$ again, and then letting δ_7, δ_8 sufficiently small, we can derive from (2.40), (2.41), (2.4) and (2.34), and Lemma 2.14 that there exists a positive constant $\varepsilon \in (0, \varepsilon_2]$ such that

$$\frac{d}{dt} \Psi(t) + \frac{1}{2} \Phi(t) \leq C_{17} \Phi(t)(\Psi(t) + \Psi^2(t)),$$

where $C_{17} > 1$ is a constant independent of ε . Then we obtain the following uniform estimate by using similar method in [42] (see also [36]).

Proposition 2.15. *Let $\Omega \subseteq \mathbb{R}^2$ be a simply connected, bounded domain with smooth boundary $\partial\Omega$. Let (σ, \mathbf{u}) be a solution to the initial boundary value problem (1.7) in $\Omega \times (0, T)$ with $\frac{1}{4} \leq 1 + \varepsilon \sigma \leq 4$, $\forall (x, t) \in \Omega \times (0, T)$, $\varepsilon \in (0, \varepsilon_2]$. Suppose that*

$$\Psi(0) \leq \frac{\zeta}{C_{17}}, \quad \zeta \in (0, \frac{1}{2}].$$

Then we have

$$\Psi(t) \leq \frac{\zeta}{C_{17}}, \quad t \in [0, T].$$

Remark 2.16. We notice that C_{17} in Proposition 2.15 is a positive constant independent of ε .

We now turn to give the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We first notice that by using standard method (see e.g. [15, 42]), one has the local existence theorem of system (1.7) for all $\varepsilon \in (0, 1]$. To do it, we shall choose $\varepsilon = 1$ without the loss of generality. First, we can linearize system (1.7) and show the existence of the approximate system,

$$\begin{cases} \partial_t \sigma + \operatorname{div}(\sigma \tilde{\mathbf{u}}) + \operatorname{div}(\tilde{\mathbf{u}}) = 0, \\ \tilde{\varrho} \partial_t \mathbf{u} - \operatorname{div}(2\mu D(\mathbf{u})) - \lambda \nabla \operatorname{div} \mathbf{u} = -P'(\tilde{\varrho}) \nabla \tilde{\sigma} - \tilde{\varrho} \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}} - \nu (\nabla \tilde{\mathbf{d}})^T \Delta \tilde{\mathbf{d}}, \\ \partial_t \tilde{\mathbf{d}} - \theta \Delta \tilde{\mathbf{d}} = -\tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{d}} + \theta |\nabla \tilde{\mathbf{d}}|^2 \tilde{\mathbf{d}}, \quad |\tilde{\mathbf{d}}| = 1, \\ (\sigma, \mathbf{u}, \mathbf{d})(\mathbf{x}, 0) = (\sigma_0(\mathbf{x}), \mathbf{u}_0(\mathbf{x}), \mathbf{d}_0(\mathbf{x})) \quad \text{in } \mathbf{x} \in \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0, \quad \tau \cdot \mathcal{T}(\mathbf{u}, P(\tilde{\varrho})) \cdot \mathbf{n} + \alpha \mathbf{u} \cdot \tau = 0 \quad \text{on } \mathbf{x} \in \partial\Omega, \\ \mathbf{n} \cdot \nabla \tilde{\mathbf{d}} = \mathbf{0} \quad \text{on } \mathbf{x} \in \partial\Omega, \end{cases} \quad (2.42)$$

where $\tilde{\sigma}$, $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{d}}$ are known functions with $\tilde{\mathbf{u}} \cdot \mathbf{n}|_{\partial\Omega} = 0$, $(\tau \cdot \mathcal{T}(\tilde{\mathbf{u}}, P(\tilde{\varrho})) \cdot \mathbf{n} + \alpha \tilde{\mathbf{u}} \cdot \tau)|_{\partial\Omega} = 0$ and $\mathbf{n} \cdot \nabla \tilde{\mathbf{d}}|_{\partial\Omega} = \mathbf{0}$. $\tilde{\varrho} = 1 + \tilde{\sigma}$. Assume that

$$\begin{aligned} \tilde{\sigma} &\in C(0, T; H^2), & \tilde{\mathbf{u}} &\in C(0, T; H^2) \cap L^2(0, T; H^3), & \tilde{\mathbf{d}} - \mathbf{e}_3 &\in C(0, T; H^3) \cap L^2(0, T; H^4), \\ \tilde{\sigma}_t &\in C(0, T; H^1), & \tilde{\mathbf{u}}_t &\in C(0, T; H^1) \cap L^2(0, T; H^2), & \tilde{\mathbf{d}}_t &\in C(0, T; H^2) \cap L^2(0, T; H^3), \\ \tilde{\sigma}_{tt} &\in L^\infty(0, T; L^2), & \tilde{\mathbf{u}}_{tt} &\in L^\infty(0, T; L^2) \cap L^2(0, T; H^1), & \tilde{\mathbf{d}}_{tt} &\in L^\infty(0, T; H^1) \cap L^2(0, T; H^2), \end{aligned}$$

with

$$\begin{aligned} &\sup_{0 \leq t \leq T} \left(\|(\tilde{\sigma}, \tilde{\mathbf{u}})(t)\|_{H^2} + \|(\tilde{\mathbf{d}} - \mathbf{e}_3)(t)\|_{H^3} + \|(\tilde{\sigma}_t, \tilde{\mathbf{u}}_t)(t)\|_{H^1} + \|\tilde{\mathbf{d}}_t(t)\|_{H^2} \right) \\ &+ \sup_{0 \leq t \leq T} \left(\|(\tilde{\sigma}_{tt}, \tilde{\mathbf{u}}_{tt}, \nabla \tilde{\mathbf{d}}_{tt})(t)\|_{L^2} + \|\tilde{\mathbf{d}}_{tt}(t)\|_{L^2} + \|\tilde{\mathbf{u}}_t\|_{L^2(0, T; H^2)} \right) \\ &+ \|\tilde{\mathbf{u}}_{tt}\|_{L^2(0, T; H^1)} + \|\tilde{\mathbf{d}}_t\|_{L^2(0, T; H^3)} + \|\tilde{\mathbf{u}}_{tt}\|_{L^2(0, T; H^2)} \leq M(T) \quad \text{for some } T > 0. \end{aligned}$$

Since (2.42)₁–(2.42)₃ are uncoupled, we are able to solve them by different methods. We can solve (2.42)₁ by the method of characteristics (see, e.g., [32]), and (2.42)₂ and (2.42)₃ by the standard Galerkin's method (see, e.g., [36]), and the regularities by energy estimates. Then, we can use the Schauder's fixed point theorem to show that there exists $0 < T_1 := T_1(\sigma_0, \mathbf{u}_0, \mathbf{d}_0, m) \leq T$ such that the initial-boundary problem (1.7) admits a unique solution $(\sigma, \mathbf{u}, \mathbf{d})$ in $\Omega \times [0, T_1]$, satisfying

$$\begin{aligned} \sigma &\in C(0, T_1; H^2), & \mathbf{u} &\in C(0, T_1; H^2) \cap L^2(0, T_1; H^3), & \mathbf{d} - \mathbf{e}_3 &\in C(0, T_1; H^3) \cap L^2(0, T_1; H^4), \\ \sigma_t &\in C(0, T_1; H^1), & \mathbf{u}_t &\in C(0, T_1; H^1) \cap L^2(0, T_1; H^2), & \mathbf{d}_t &\in C(0, T_1; H^2) \cap L^2(0, T_1; H^3), \\ \sigma_{tt} &\in L^\infty(0, T_1; L^2), & \mathbf{u}_{tt} &\in L^\infty(0, T_1; L^2) \cap L^2(0, T_1; H^1), & \mathbf{d}_{tt} &\in L^\infty(0, T_1; H^1) \cap L^2(0, T_1; H^2). \end{aligned} \quad (2.43)$$

Moreover, σ satisfies that

$$\frac{1}{4} \leq 1 + \varepsilon \sigma(\mathbf{x}, t) \leq 4 \quad \forall (\mathbf{x}, t) \in \bar{\Omega} \times [0, T_1].$$

In what follows, we shall use the uniform a priori estimate established in Proposition 2.15 to continue the local solution $(\sigma^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon)$ globally in time by applying the standard extension techniques (see, for example, [36, 42] for the compressible Navier–Stokes equations), and obtain therefore a global solution. In fact, from the a priori estimates obtained by previous Lemmas and Propositions, the embedding $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$ and the fact that $\|\sigma^\varepsilon(\cdot, t)\|_{H^2} \leq C\Psi(t)$, one can get that there exists a positive constant C_{18} , such that if

$$\Psi(t) \leq C_{18},$$

then

$$\frac{1}{2} \leq 1 + \varepsilon \sigma^\varepsilon(\mathbf{x}, t) \leq 2 \quad \forall \mathbf{x} \in \bar{\Omega}.$$

Hence, suppose that $\Psi(0) \leq C_{19} := \min\{C_{18}, \frac{\zeta}{2C_{17}}\}$, then $\sigma_0^\varepsilon(\mathbf{x})$ satisfies that

$$\frac{1}{2} \leq 1 + \varepsilon \sigma_0^\varepsilon(\mathbf{x}) \leq 2.$$

By using the local existence theorem, there exists $T_1 > 0$ such that the initial-boundary value problem (1.7) admits a unique solution $(\sigma^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon)$ in $\Omega \times [0, T_1]$, and satisfying (2.43). Therefore, by using Proposition 2.15, we deduce that

$$\Psi(t) \leq C_{20} \quad \forall t \in [0, T_1],$$

where C_{20} is a positive constant independent of ε , from which

$$\frac{1}{2} \leq 1 + \varepsilon \sigma^\varepsilon(\mathbf{x}, T_1) \leq 2 \quad \forall \mathbf{x} \in \overline{\Omega}.$$

Then, we can use $(\sigma^\varepsilon(\mathbf{x}, T_1), \mathbf{u}^\varepsilon(\mathbf{x}, T_2), \mathbf{b}^\varepsilon(\mathbf{x}, T_3))$ and initial data, repeat the steps above, one can get that the problem (1.7) admits a unique solution $(\sigma^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon)$ in $\Omega \times [T_1, 2T_1]$. Applying this extension technique and the uniform estimate about temporal variable in Proposition 2.15, we complete the proof of Theorem 1.1. \square

Proof of Theorem 1.2. By employing the uniform estimate given in Proposition 2.15, we get that solution $(\sigma^\varepsilon, \mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon)$ to (1.7) belongs to (1.8). Due to the Sobolev embedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$, one gets that σ^ε is bounded in $\Omega \times \overline{\mathbb{R}}_+$. This implies that

$$\rho^\varepsilon = 1 + \varepsilon \sigma^\varepsilon \rightarrow 1 \text{ as } \varepsilon \rightarrow 0^+,$$

for all $(\mathbf{x}, t) \in \Omega \times \overline{\mathbb{R}}_+$. Moreover, the uniform bounds of $(\mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon)$ of (1.8) and the Aubin-Lions Lemma yield that, there exists a subsequence of $(\mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon)$, still denoted by $(\mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon)$, such that

$$\begin{aligned} (\mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon) &\rightarrow (\mathbf{u}, \mathbf{d}) && \text{in } L^\infty(\overline{\mathbb{R}}_+; H^s \times H^{s+1}), \forall 0 < s < 2, \\ (\mathbf{u}^\varepsilon, \mathbf{d}^\varepsilon) &\rightarrow (\mathbf{u}, \mathbf{d}) && \text{in } L^2(\overline{\mathbb{R}}_+; H^2 \times H^3), \end{aligned}$$

and

$$(\mathbf{u}_t^\varepsilon, \mathbf{d}_t^\varepsilon) \rightarrow (\mathbf{u}_t, \mathbf{d}_t) \quad \text{in } L^2(\overline{\mathbb{R}}_+; H^1 \times H^2)$$

as $\varepsilon \rightarrow 0^+$. Therefore, pass to the limit on (1.7) as $\varepsilon \rightarrow 0^+$, and (\mathbf{u}, \mathbf{d}) is the solution of the corresponding incompressible nematic liquid crystal flows. This completes the proof of Theorem 1.2. \square

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References

- [1] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, II, *Comm. Pure Appl. Math.* 17 (1964) 35–92.
- [2] H. Bessaih, Limite de modeles de fluides compressible, *Prot. Math.* 52 (1995) 441–463.

- [3] T. Clopeau, A. Mikelić, R. Robert, On the vanishing viscosity limit for the 2D incompressible Navier–Stokes equations with the friction type boundary conditions, *Nonlinearity* 11 (1998) 1625–1636.
- [4] B. Desjardins, E. Grenier, P.L. Lions, N. Masmoudi, Incompressible limit for solutions of the isentropic Navier–Stokes equations with Dirichlet boundary conditions, *J. Math. Pures Appl.* 78 (1999) 461–471.
- [5] C. Dou, S. Jiang, Q. Ju, Global existence and the low Mach number limit for the compressible magnetohydrodynamic equations in a bounded domain with perfectly conducting boundary, *Z. Angew. Math. Phys.* 64 (2013) 1661–1678.
- [6] C. Dou, S. Jiang, Y. Ou, Low Mach number limit of full Navier–Stokes equations in a 3D bounded domain, *J. Differential Equations* 258 (2015) 379–398.
- [7] C. Dou, Q. Ju, Low mach number limit for the compressible magnetohydrodynamic equations in a bounded domain for all time, *Commun. Math. Sci.* 12 (2014) 661–679.
- [8] J. Ericksen, Continuum theory of nematic liquid crystals, *Res. Mechanica* 21 (1987) 381–392.
- [9] E. Feireisl, A. Novotný, Singular Limits in Thermodynamics of Viscous Fluids, *Advances in Mathematical Fluid Mechanics*, Birkhäuser, Basel, Switzerland, 2009.
- [10] M. Hong, Global existence of solutions of the simplified Ericksen–Leslie system in dimension two, *Calc. Var. Partial Differential Equations* 40 (2011) 15–36.
- [11] X. Hu, D. Wang, Low Mach number limit of viscous compressible magnetohydrodynamic flows, *SIAM J. Math. Anal.* 41 (2009) 1272–1294.
- [12] X. Hu, H. Wu, Global solution to the three-dimensional compressible flow of liquid crystal, *SIAM J. Math. Anal.* 252 (2013) 2678–2699.
- [13] T. Huang, C. Wang, H. Wen, Strong solutions of the compressible nematic liquid crystal flow, *J. Differential Equations* 252 (2012) 2222–2265.
- [14] T. Huang, C. Wang, H. Wen, Blow up criterion for compressible nematic liquid crystal flows in dimension three, *Arch. Ration. Mech. Anal.* 204 (2012) 285–311.
- [15] F. Jiang, S. Jiang, D. Wang, On multi-dimensional compressible flows of nematic liquid crystals with large initial energy in a bounded domain, *J. Funct. Anal.* 265 (2013) 3369–3397.
- [16] F. Jiang, S. Jiang, D. Wang, Global weak solutions to the equations of compressible flow of nematic liquid crystals in two dimensions, *Arch. Ration. Mech. Anal.* 214 (2014) 1100–1150.
- [17] S. Jiang, Q. Ju, F. Li, Incompressible limit of the compressible magnetohydrodynamic equations with periodic boundary conditions, *Comm. Math. Phys.* 297 (2010) 371–400.
- [18] S. Jiang, Q. Ju, F. Li, Low Mach number limit for the multi-dimensional full magnetohydrodynamic equations, *Nonlinearity* 25 (2012) 1351–1365.
- [19] S. Jiang, Y. Ou, Incompressible limit of the non-isentropic Navier–Stokes equations with well-prepared initial data in three-dimensional bounded domains, *J. Math. Pures Appl.* 96 (2011) 1–28.
- [20] T. Kato, Wave operators and similarity for some non-selfadjoint operators, *Math. Ann.* 162 (1966) 258–279.
- [21] S. Klainerman, A. Majda, Singular perturbations of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, *Comm. Pure Appl. Math.* 34 (1981) 481–524.
- [22] Y. Kwon, Incompressible limit for the compressible flows of nematic liquid crystals in the whole space, *Adv. Math. Phys.* 2015 (2015) 427865, <https://doi.org/10.1155/2015/427865>.
- [23] F. Leslie, Theory of Flow Phenomenon in Liquid Crystals, *The Theory of Liquid Crystals*, vol. 4, Academic Press, London-New York, 1979, pp. 1–81.
- [24] J. Li, Z. Xu, J. Zhang, Global well-posedness with large oscillations and vacuum to the three-dimensional equations of compressible nematic liquid crystal flows, *arXiv:1204.4966v1 [math.AP]*, 2012.
- [25] F. Lin, Nonlinear theory of defects in nematic liquid crystals; phase transition and flow phenomena, *Comm. Pure Appl. Math.* 42 (1989) 789–814.
- [26] J. Lin, S. Lai, C. Wang, Global finite energy weak solutions to the compressible nematic liquid crystal flow in dimension three, *SIAM J. Math. Anal.* 47 (2015) 2952–2983.
- [27] F. Lin, J. Lin, C. Wang, Liquid crystal flow in two dimensions, *Arch. Ration. Mech. Anal.* 197 (2010) 297–336.
- [28] F. Lin, C. Liu, Nonparabolic dissipative systems modeling the flow of liquid crystals, *Comm. Pure Appl. Math.* 48 (1995) 501–537.
- [29] F. Lin, C. Liu, Partial regularities of the nonlinear dissipative systems modeling the flow of liquid crystals, *Discrete Contin. Dyn. Syst. Ser. A* 2 (1996) 1–23.
- [30] F. Lin, C. Wang, Recent developments of analysis for hydrodynamic flow of nematic liquid crystals, *Philos. Trans. R. Soc. Ser. A* 372 (2029) (2014), <https://doi.org/10.1098/rsta.2013.0361>.
- [31] F. Lin, C. Wang, Global existence of weak solutions of the nematic liquid crystal flow in dimension three, *Comm. Pure Appl. Math.* 69 (2016) 1532–1571.
- [32] P.L. Lions, *Mathematical Topics in Fluid Dynamics*, vol. 1. Incompressible Models, Oxford Univ. Press, London, 1996.
- [33] P.L. Lions, N. Masmoudi, Incompressible limit for a viscous compressible fluid, *J. Math. Pures Appl.* 77 (1998) 585–627.
- [34] Q. Liu, Space-time derivative estimates of the Koch–Tataru solutions to the nematic liquid crystal system in Besov spaces, *J. Differential Equations* 258 (2015) 4368–4397.
- [35] A. Matsumura, T. Nishida, The initial value problems for the equations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.* 20 (1980) 68–104.
- [36] Y. Ou, Incompressible limites of the Navier–Stokes equations for all time, *J. Differential Equations* 247 (2009) 3295–3314.
- [37] Y. Ou, D. Ren, Incompressible limit of global strong solutions to 3-D barotropic Navier–Stokes equations with well-prepared initial data and Navier’s slip boundary conditions, *J. Math. Anal. Appl.* 420 (2014) 1316–1336.
- [38] I. Stewart, *The Static and Dynamic Continuum Theory of Liquid Crystals*, Taylor & Francis, London, New York, 2004.
- [39] C. Wang, Well-posedness for the heat flow of harmonic maps and the liquid crystal flow with rough initial data, *Arch. Ration. Mech. Anal.* 200 (2011) 1–19.

- [40] D. Wang, C. Yu, Incompressible limit for the compressible flow of liquid crystals, *J. Math. Fluid Mech.* 16 (2014) 771–786.
- [41] H. Wen, S. Ding, Solutions of incompressible hydrodynamic flow of liquid crystals, *Nonlinear Anal. Real World Appl.* 12 (2011) 1510–1531.
- [42] W. Zajackowski, On the nonstationary motion of a compressible barotropic viscous fluid with boundary slip condition, *J. Appl. Anal.* 4 (1998) 167–204.