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Strongly compatible generators of groups on Fréchet spaces

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We consider the linear Cauchy problem

$$\begin{cases} u_t = a(D)u, t \in \mathbb{R} \\ u(0) = u_0 \end{cases}, \tag{1}$$

where $a(D)\colon X \to X$ is a continuous linear operator on a Fréchet space X. By imposing a condition (which is neither stronger nor weaker than the equicontinuity of the powers of $a(D)$), we present the necessary and sufficient conditions for the generation of a uniformly continuous group on X, which provides the unique solution of (1). In addition, for every pseudodifferential operator $a(D)$ with constant coefficients defined on $\mathscr{F}L^2_{loc}$, which is a Fréchet space of distributions, we also provide the necessary and sufficient conditions such that the restriction $\{e^{t\,a(D)}\}_{t \geqslant 0}$ is a well defined semigroup on L^2 and $\mathscr{E}'$. We conclude that the heat equation solution on $\mathscr{F}L^2_{loc}$ for all $t \in \mathbb{R}$ extends the standard solution on Hilbert spaces for $t \geqslant 0$.

1. Introduction

If A is a pseudodifferential operator, e.g., $A = \frac{d}{dx}$, we may consider the associated Cauchy problem, i.e.,

$$\begin{cases} u_t = Au, t \in I \\ u(0) = u_0 \end{cases}, \tag{2}$$

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and try to solve it for a certain class of functions u_0 and a fixed interval of time I. When modeling biological, physical, and economic phenomena, evolution problems such as (2) arise naturally from partial differential equations (PDEs) by interpreting $(t,x) \mapsto u(t,x)$ as a vector-valued mapping $t \mapsto u(t,\cdot)$, e.g., $u(t,\cdot) \in L^2(\mathbb{R}^N)$.

A function $\mathbb{R} \times \mathbb{R}^N \ni (t,x) \mapsto u(t,x) \equiv \big(u(t)\big)(x) \in X$ is said to be a solution of (2) if it is differentiable on the temporal variable t, if it satisfies $\frac{d}{dt}u(t,x) = Au(t,x)$ for every (t,x), and if it satisfies the so-called initial condition, i.e., $u(0,x) = u_0(x)$ for every $x \in \mathbb{R}^N$, for a given function $u_0 \colon \mathbb{R}^N \to X$.

The main approach involves dealing with a closed linear operator $A \colon D(A) \subset X \to X$, which is densely defined on a Banach space X. This setting has yielded a rich theory over the last 50 years, which allows (2) to be solved by a strongly continuous semigroup $\big(T(t) \colon X \to X\big)_{t \geqslant 0}$ whenever some spectral conditions on A are fulfilled. However, many well-known topological vector spaces that arise during the analysis of PDEs are not normable, such as $C^1\big((-\infty, 0]\big)$ in equations with infinite delay. In this setting, a natural trade-off arises where the good topological properties on the space must be lost to obtain better properties on the operators. For instance, every linear differential operator with constant coefficients is bounded on the Schwartz space.

Let X be a Hausdorff locally convex space (HLCS). The map

$$t \mapsto \exp(tA)u_0 := \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n\right) u_0, \text{ for } u_0 \in X, \tag{3}$$

provides the unique solution of (2) on X for $t \geqslant 0$ whenever X is sequentially complete and $(A^n)_{n\in\mathbb{N}}$ is an equicontinuous family of bounded operators defined on X (see [15]).

The generation of a C_0-semigroup on HLCSs such as (3) has been addressed in previous studies by adding a hypothesis on the generator or on the phase space X. The C_0-semigroup was assumed to be quasi-equicontinuous by [2,12], and it was assumed to be locally equicontinuous and X was equipped with an auxiliary norm by [11]. Other studies considered the question in some particular Fréchet spaces, such as those by Dembart [5] (who considered the phase space as the space of the continuous functions defined on $[a,b]$ in a fixed topological vector space E) and by Frerick et al. [7] (by setting $X = \mathbb{K}^{\mathbb{N}}$, i.e., the collection of scalar sequences).

Hence, we can present some results regarding the generation of uniformly continuous groups (where the definition invokes stronger convergence rather than pointwise convergence) on Fréchet spaces. For example, we extend the main generation result obtained recently by [8] who did not establish the necessity implication for the exponential map convergence in the topology of bounded convergence. In addition, we provide further applications to linear Cauchy problems where $A = a(D)$ is a pseudodifferential operator on $\mathscr{F}L^2_{loc}$. Remarkably, we extend the analytic semigroup generated by the heat operator $-(1-\Delta)$ on L^2 to the group $\big(e^{-t\,(1-\Delta)}\big)_{t\in\mathbb{R}}$ on $\mathscr{F}L^2_{loc}$, thereby obtaining a distributional solution of the heat equation backwards in time.

The remainder of this paper is organized as follows. In Section 2, we establish the generation theorem (Theorem 2) on abstract Fréchet spaces for bounded linear operators, which have a simple compatibility property with respect to the Fréchet topology. In Section 3, we apply the results in Section 2 to evolution problems in $\mathscr{F}L^2_{loc}$, including the definition of $\mathscr{F}L^2_{loc}$, criteria for the regularization process, and the positive invariance on L^2.

2. Strongly compatible operators and generation theorem

Let $X = \big(X, (p_j)_{j\in\mathbb{N}}\big)$ be a Fréchet space and $\mathscr{L}(X)$ is the space of all continuous linear operators on X (see [6,15] and the references therein).

We require the appropriate compatibility between the operator A and the Fréchet topology on X such that its exponential operator is well defined and provides the solution of the associated Cauchy problem.

Definition 1. A linear operator $A\colon X \to X$ is said to be strongly compatible with $(p_j)_{j\in\mathbb{N}}$ and we write $A \in \mathscr{L}_{sc}(X)$ if for every $j \in \mathbb{N}$,

$$p_j^X(A) := \sup_{p_j(x)=1} p_j(Ax) < \infty. \tag{4}$$

If X is a Banach space, then $T \in \mathscr{L}_{sc}(X)$ if and only if $T \in \mathscr{L}(X)$. We note that the identity operator on X is always strongly compatible regardless of the choice of seminorms. Actually, the definition of $\mathscr{L}_{sc}(X)$ does not depend on a fixed system of seminorms and because we may not know all of the continuous seminorms on X explicitly, it is convenient and sufficient to only compute (4) for some well known fundamental family of seminorms on X. Hence, it is much simpler than the condition required by [1].

Proposition 1. *The countable family of seminorms* $\left(p_j^X\right)_{j\in\mathbb{N}}$ *defines a Fréchet topology on* $\mathscr{L}_{sc}(X)$.

Proof. First, we observe that if $A \in \mathscr{L}_{sc}(X)$, then

$$p_j(Ax) \leqslant p_j^X(A)p_j(x) \tag{5}$$

for every $x \in X$ and every j. Consequently, $p_j^X(A^n) \leqslant p_j^X(A)^n$ for every $n \in \mathbb{N}$.

Indeed, for a fixed $j \in \mathbb{N}$, if $x \in X$ satisfies $p_j(Ax) \neq 0$, then $x_0 = \dfrac{1}{p_j(x)}x$ satisfies $p_j(Ax_0) \leqslant p_j^X(A)$, and thus we deduce that $p_j(Ax) \leqslant p_j^X(A)\,p_j(x)$ for every $x \in X$.

In addition, (5) implies that $\left(p_j^X\right)_{j\in\mathbb{N}}$ is a separating family of seminorms, i.e., each $A \neq 0$ corresponds at least one $j \in \mathbb{N}$ with $p_j^X(A) \neq 0$.

Now, suppose that $(A_k)_{k\in\mathbb{N}}$ is a Cauchy sequence with respect to $\left(p_j^X\right)_{j\in\mathbb{N}}$. By (5), we find that $(A_kx)_k$ is a Cauchy sequence in X for every $x \in X$ such that the map $X \ni x \mapsto Ax := \lim_k A_kx$ is well defined and linear. We only need to observe that $p_j(Ax) = \lim_k p_j(A_kx) \leqslant \overline{\lim}_k\, p_j^X(A_k)p_j(x)$ in order to conclude that $A \in \mathscr{L}_{sc}(X)$. Thus, $\left(\mathscr{L}_{sc}(X), \left(p_j^X\right)_{j\in\mathbb{N}}\right)$ is a Fréchet space. $\square$

The property (4) implies that $p_j(Ax) = 0$ whenever $p_j(x) = 0$ for every j. According to [4], an operator that satisfies this property is said to be compatible with $(p_j)_j$, which was shown to be a natural condition for obtaining hyperbolicity.

Definition 2. A family $\{T(t) : t \in \mathbb{R}\} \subset \mathscr{L}(X)$ is said to be a C_0-group on X if $T(0) = \mathrm{id}_X$, $T(t+s) = T(t)T(s)$ and $T(\tau)x \xrightarrow[\tau\to 0]{X} x$, for every $s, t \in \mathbb{R}$ and $x \in X$. We write $T(\cdot)$ instead of $\{T(t) : t \in \mathbb{R}\}$.

The infinitesimal generator of this family $A\colon D(A) \subset X \to X$ is defined by

$$Ax := \lim_{t\to 0} \frac{T(t)x - x}{t},$$

where $x \in D(A)$ if and only if the limit given above exists.

In addition, if $T(t) \to I$ in $\mathscr{L}_{sc}(X)$ as $t \to 0$, then $T(\cdot) \subset \mathscr{L}_{sc}(X)$ is said to be a uniformly continuous group on X.

Consider the normed spaces $X_j := \left(X/p_j^{-1}(\{0\}), \|\cdot\|_j\right)$, where

$$\|[x]_j\|_j := \inf_{p_j(z)=0} p_j(x-z), \text{ for } [x]_j \text{ in } X/p_j^{-1}\left(\{0\}\right).$$

We say that X is a **quojection** if every X_j is complete. We may assume that $p_j \leqslant p_{j+1}$ for every j, so except for an identification argument, we have

$$X_1 \subset X_2 \subset \cdots \subset X.$$

In the following, we assume that X is a quojection unless stated otherwise.

Theorem 2. *Let $A\colon D(A) \subset X \to X$ be a linear operator. The following are equivalent:*

- **a.** *A is everywhere defined and is strongly compatible with $\big(X, (p_j)_{j\in\mathbb{N}}\big)$;*
- **b.** *A is the infinitesimal generator of a uniformly continuous group $T(\cdot)$ on $\big(X, (p_j)_{j\in\mathbb{N}}\big)$; in which case it is given by*

$$T(t) = e^{tA} := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \in \mathscr{L}_{sc}(X), \text{ for every } t \in \mathbb{R},$$

where the convergence is defined with respect to the $\mathscr{L}_{sc}(X)$-topology.

Proof. Let $A \in \mathscr{L}_{sc}(X)$ and $S_N := \sum_{n=0}^{N} \dfrac{(tA)^n}{n!} \in \mathscr{L}_{sc}(X)$. Given $\varepsilon > 0$,

$$p_j^X(S_N - S_M) \leqslant \sum_{n=M+1}^{N} \frac{1}{n!}\left(t\, p_j^X(A)\right)^n < \varepsilon,$$

for sufficiently large N, M. Clearly, e^{0A} is the identity of X. In addition, since $\sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$ is absolutely convergent, then we conclude that $e^{(s+t)A} = e^{sA}e^{tA}$ for all $t, s \in \mathbb{R}$ by the classical formula for the product of series. Moreover,

$$p_j^X(e^{tA} - \mathrm{id}_X) = p_j^X\left(\sum_{n=1}^{\infty} \frac{(tA)^n}{n!}\right) \leqslant \sum_{n=1}^{\infty} \frac{\left(t\, p_j^X(A)\right)^n}{n!} = e^{tp_j^X(A)} - 1,$$

and thus $\{e^{tA} : t \in \mathbb{R}\}$ is a uniformly continuous group on X.

In addition, by the definition of the generator, if $x \in X$ and $t \neq 0$, then we have

$$p_j\left(\frac{e^{tA}x - x}{t} - Ax\right) \leqslant \frac{1}{t}\sum_{n=2}^{\infty} \frac{\left(t\, p_j^X(A)\right)^n}{n!} p_j(x) = \left(\frac{e^{tp_j^X(A)} - 1}{t} - p_j^X(A)\right) p_j(x),$$

and thus A is the infinitesimal generator of $\{e^{tA} : t \in \mathbb{R}\}$.

For its reciprocal, we consider the Banach spaces $X_j := \big(X/p_j^{-1}(\{0\}), \|\cdot\|_j\big)$ as defined previously, and let $T_j(t)\colon X_j \to X_j$ be defined as

$$T_j(t)[x]_j := [T(t)x]_j \text{ for } [x]_j \in X_j.$$

Claim 1: $\{T_j(t) : t \in \mathbb{R}\}$ is a uniformly continuous group on X_j for every j.

We may write $\|T_j(t)[x]_j\|_j = \inf_{p_j(z)=0} p_j\Big(T(t)x - T(t)z - \big(z - T(t)z\big)\Big)$, which is dominated by

$$\inf_{p_j(z)=0} \left\{p_j^X\big(T(t)\big)p_j(x-z) + p_j(z) + p_j\big(T(t)z\big)\right\} = p_j^X\big(T(t)\big)\|[x]_j\|_j,$$

since $T(t)$ is strongly compatible with $(p_k)_{k\in\mathbb{N}}$ such that $T_j(t) \in \mathscr{L}(X_j)$.

Clearly, $T_j(0)$ is the identity operator on X_j. In addition, for $t, s \in \mathbb{R}$, we find that $T_j(t)\big(T_j(s)[x]_j\big) = [T(t) \circ T(s)x]_j = T_j(t+s)[x]_j$ and

$$\|T_j(t) - \mathrm{id}_{X_j}\|_{\mathscr{L}(X_j)} = \sup_{\|[x]_j\|_j = 1} \inf_{p_j(z)=0} p_j\big(T(t)x - x - z\big)$$

is dominated by $\sup\limits_{\|[x]_j\|_j=1} \inf\limits_{p_j(z)=0} p_j^X\big(T(t) - \mathrm{id}_X\big) p_j(x-z) = p_j^X\big(T(t) - \mathrm{id}_X\big) \xrightarrow[t \to 0]{\mathbb{R}} 0.$

The maps $x \mapsto \sigma_j(x) := [x]_j$ and $[x]_{j+1} \mapsto \pi_j\big([x]_{j+1}\big) := [x]_j$ are continuous and, by construction, we obtain $\big(T_j(t) \circ \pi_j\big)([x]_{j+1}) = \big(\pi_j \circ T_{j+1}(t)\big)([x]_{j+1})$.

It is natural to seek the infinitesimal generator of $T(\cdot)$ by using the infinitesimal generators A_j of $T_j(\cdot)$ in order to determine whether a linear operator $A\colon X \to X$ exists such that every $A_j\colon X_j \to X_j$ is simply the projection of A on X_j induced by σ_j; i.e., $[Ax]_j = A_j[x]_j$ holds for every j and $x \in X$. Indeed, this is the case.

__**Claim 2:**__ A unique linear operator $A\colon X \to X$ exists that changes

$$\begin{tikzcd} X \arrow[r, "A"] \arrow[d, "\sigma_j"'] & X \arrow[d, "\sigma_j"] \\ X_j \arrow[r, "A_j"'] & X_j \end{tikzcd} \tag{6}$$

into a commutative diagram for every $j \in \mathbb{N}$. In addition, $A \in \mathscr{L}_{sc}(X)$ and it is the infinitesimal generator of $T(\cdot)$.

Fix $x \in X$. Every σ_j is surjective, so we obtain a sequence $(z_j)_j$ in X that depends on x such that $\sigma_j(z_j) = A_j \circ \sigma_j(x)$ for every $j \in \mathbb{N}$, and thus

$$\sigma_j(z_j) = A_j \circ \sigma_j(x) = \pi_j\big(A_{j+1} \circ \sigma_{j+1}(x)\big) = \pi_j\big(\sigma_{j+1}(z_{j+1})\big) = \sigma_j(z_{j+1})$$

such that $p_l(z_j - z_k) = 0$ whenever $j,\ k \geqslant l$. Hence, we define a linear operator $A\colon X \to X$ by setting $Ax := \lim_{j \to \infty} z_j$, which satisfies

$$\sigma_j(Ax) = \sigma_j\left(\lim_{k \to \infty} z_k\right) = \lim_{\substack{k \to \infty \\ k \geqslant j}} \sigma_j(z_k - z_j) + \sigma_j(z_j) = \sigma_j(z_j) = (A_j \circ \sigma_j)(x).$$

$(p_j)_j$ is a separating family of seminorms, so $x \mapsto Ax$ is well defined and it is the unique linear operator that changes (6) into a commutative diagram. Moreover,

$$\begin{aligned} \sup_{p_j(x) \leqslant 1} p_j(Ax) &= \sup_{p_j(x) \leqslant 1} \left\{ \inf_{p_j(z)=0} p_j(Ax) - p_j(z) \right\} \\ &\leqslant \sup_{p_j(x) \leqslant 1} \left\{ \inf_{p_j(z)=0} p_j(Ax - z) \right\} = \sup_{p_j(x) \leqslant 1} \|[Ax]_j\|_j \\ &\leqslant \sup_{\|[x]_j\|_j \leqslant 1} \|A_j[x]_j\|_j < \infty \end{aligned}$$

and the last inequality holds because $\|[x]_j\|_j \leqslant p_j(x)$. Hence, $A \in \mathscr{L}_{sc}(X)$.

It is not difficult to see that these projections σ_j have a useful property, where if $[x_\lambda]_j \xrightarrow[X_j]{\lambda \in \Lambda} [0]_j$ for every j, then $(x_\lambda)_{\lambda \in \Lambda}$ is convergent in X and $x_\lambda \xrightarrow{\lambda \in \Lambda} 0$.

Finally, given $x \in X$, for every $j \in \mathbb{N}$, we have

$$\left[Ax - \frac{T(t)x - x}{t}\right]_j = [Ax]_j - \frac{[T(t)x]_j - [x]_j}{t} = A_j[x]_j - \frac{T_j(t)[x]_j - [x]_j}{t},$$

such that the net $\frac{T(t)x - x}{t}$ converges to Ax, for every $x \in X$. By the uniqueness of the infinitesimal generator, we conclude that

$$T(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = e^{tA}, \text{ for every } t \in \mathbb{R}. \quad \square$$

As found in the Banach spaces, groups with the same infinitesimal generator coincide.

Proposition 3. *If $T(\cdot)$ and $S(\cdot)$ are uniformly continuous groups on X with*

$$\lim_{t \to 0} \frac{T(t) - id_X}{t} = A = \lim_{t \to 0} \frac{S(t) - id_X}{t} \text{ in } \mathscr{L}_{sc}(X),$$

then $T(t) = S(t)$ for every $t \in \mathbb{R}$.

Corollary 4. *If $T(\cdot)$ is a uniformly continuous group on X, then:*

- **a.** *A unique operator A in $\mathscr{L}_{sc}(X)$ exists such that $T(t) = e^{tA}$;*
- **b.** *The operator A in part (a) is the infinitesimal generator of $T(\cdot)$;*
- **c.** *Nonnegative numbers ω_j exist such that $p_j^X\big(T(t)\big) \leqslant \exp(\omega_j t)$ for every $t \in \mathbb{R}$;*
- **d.** *The map $\mathbb{R} \ni t \mapsto T(t) \in \mathscr{L}_{sc}(X)$ is differentiable and*

$$\frac{dT(t)}{dt} = A \circ T(t) = T(t) \circ A, \text{ for every } t \in \mathbb{R}.$$

Consequently, the Cauchy problem

$$\begin{cases} T'(t) = AT(t), \ t \in \mathbb{R} \\ T(0) = id_X \end{cases}$$

possesses a unique solution whenever $A \in \mathscr{L}_{sc}(X)$ by Gronwall's inequality.

Remark 1. Let X be a sequentially complete HLCS. Several remarks can be made, as follows.

1) According to [15], the generation result was given as follows. Let $B \in \mathscr{L}(X)$. If for every continuous seminorm p on X, a continuous seminorm $q = q(p)$ on X exists such that

$$\sup_{k \in \mathbb{N}} p\big(B^k x\big) \leqslant q(x), \text{ for every } x \in X, \tag{7}$$

then the map

$$X \ni x \mapsto \sum_{k=0}^{\infty} \frac{tB^k}{k!} x, \text{ for every } t \geqslant 0,$$

is well defined and continuous.

If (7) holds, then $\{B^k\}_{k\in\mathbb{N}}$ is said to be an equicontinuous family with respect to k. This is a result of the generation of a C_0-semigroup with pointwise convergence. In addition, by Proposition 1, if $A \in \mathscr{L}_{sc}(X)$ then

$$p_j(A^k x) \leqslant [p_j^X(A)]^k\, p_j(x), \text{ for every } x \in X \text{ and for every } j, k \in \mathbb{N},$$

which may not have the uniformity of (7) on k. Moreover, nothing is assumed regarding the other seminorms except for $(p_j)_j$, so the conditions are not comparable. In fact, let $X = C(\mathbb{R})$ with the seminorms $p_j(\phi) := \sup_{|\xi|\leq j} |\phi(\xi)|$, $j \in \mathbb{N}$. If $A\colon X \to X$ is given by $A\phi = a\phi$, where $a(\xi) = |\xi|^2$, $\xi \in \mathbb{R}$, then $A^k\phi = |\xi|^{2k}\phi$, for any $k \in \mathbb{N}$, so we may choose ϕ_0 such that $p_j(A^k\phi_0) = j^{2k}$ for any $k \in \mathbb{N}$. Thus, given j, we cannot find a continuous seminorm q such that $p_j(A^k\phi_0) \leq q(\phi_0)$ for all $k \in \mathbb{N}$.
Moreover, Theorem 2 provides a complete characterization of a uniformly continuous group **with convergence in the space of operators**. It should also be noted that the reciprocal of Yosida's result requires that $B\colon D(B) \subset X \to X$ is known to be densely defined and that the resolvent $(n\, id_X - B)^{-1} \in \mathscr{L}(X)$ exists for every $n \in \mathbb{N}$.

2) According to [1], a C_0-semigroup $\{S(s) : s \geqslant 0\}$ in X is called a $(C_0, 1)$-semigroup if for every continuous seminorm p on X, a positive number σ_p and a continuous seminorm $q = q(p)$ on X exist such that

$$p\big(S(s)x\big) \leqslant e^{\sigma_p s} q(x), \text{ for every } x \in X \text{ and every } s \geqslant 0. \tag{8}$$

Results were then established regarding the generation of $(C_0, 1)$-semigroups (instead of uniformly bounded groups) and the perturbation of infinitesimal generators. By Corollary 4c, the uniformly continuous groups that we present in this study satisfy (8). However, we can also deal with a (possibly) smaller class of bounded linear operators on the phase space because we can extract a theory from this restriction that provides useful properties and applications. We aim to obtain the following properties based on the compatibility between the operator and the Fréchet topology of the phase space:
(i) To characterize all groups with their generators;
(ii) To provide a *simple* condition regarding the generators without invoking spectral theory, which is easier to verify based on applications than Babalola's results; and
(iii) To obtain a tool that works suitably well for constant coefficient pseudodifferential operators and such that each generates a group on some Fréchet space (i.e., $\mathscr{F}L^2_{loc}$), but that allows us to compare each group with standard approaches based on the usual Banach spaces, especially for solving evolution problems.

According to Babalola's results, given a linear operator $A\colon X \to X$, we need to compute the quotients $X_j = X/p_j^{-1}(0)$, evaluate the operator A on every X_j, and apply the spectral theory of A to every X_j. Finally, we can then determine whether A generates a semigroup. We consider that this procedure may not be simple to apply. In addition, the study conducted by [1] considered the Hille–Yosida theorems (the semigroup was generated under a pointwise convergence by [1]) and perturbation results, where a single application was presented by letting the phase space be the space of all smooth functions on the 1-torus.

3) Let Γ be a fundamental system of seminorms on X and let $\mathscr{L}_b(X)$ be the space $\mathscr{L}(X)$ equipped with the topology of uniform convergence on the bounded subsets of X, which is weaker than the $\mathscr{L}_{sc}(X)$-topology (in the sense that $\mathscr{L}_{sc}(X)$-convergence for nets implies $\mathscr{L}_b(X)$-convergence for nets). The main result given by [8] reads as follows: suppose that $\mu > 0$ exists with the property that for every $p \in \Gamma$, $q = q(\mu, p) \in \Gamma$ and $M = M(\mu, p) \geqslant 0$ exist such that

$$p(A^k x) \leqslant M\mu^k q(x), \text{ for every } x \in X \text{ and } k \in \mathbb{N}. \tag{9}$$

Then, A generates a uniformly continuous semigroup given by the exponential series expansion, with convergence in $\mathscr{L}_b(X)$.

If (4) holds, then (9) holds by setting $\mu = p_j^X(A)$, $q = p$ and $M = 1$. (4) is a stronger condition, but it is certainly easier to compute with and it is used to topologize $\mathscr{L}_{sc}(X)$ in an appropriate manner such that $\exp(A)$ is well defined as the exponential $\mathscr{L}_{sc}(X)$-series expansion. In addition, we obtained a complete characterization in Theorem 2, which was not achieved by [8].

4) The generation of C_0-semigroups on quojections was studied by [7], where the main application considered the following question posed by [3]: if $T(\cdot)$ is a C_0-semigroup on $\mathbb{K}^{\mathbb{N}}$, does a bounded operator A exist such that $T(\cdot)$ is represented pointwisely by the exponential series expansion of A? Answering this question is not our aim.

5) Other studies also dealt with C_0-semigroup generation under weak assumptions, such as the semigroup $T(\cdot)$ being quasi-equicontinuous (in the sense that a constant $\beta \geqslant 0$ exists such that $\left(e^{-\beta t}T(t)\right)_{t\geqslant 0}$ is an equicontinuous family with respect to t) as considered by [2,12], or the semigroup $T(\cdot)$ being locally equicontinuous (in the sense that for every $t \geqslant 0$ and every continuous seminorm p, a continuous seminorm $q = q(t,p)$ exists such that $p\big(T(s)x\big) \leqslant q(x)$ for every $0 \leqslant s \leqslant t$ and $x \in X$) as investigated by [11].

3. Some applications to PDEs

Clearly, geometric intuition has provided useful guidance when solving differential problems by seeking solutions on Hilbert spaces. However, we are convinced that it has hindered our understanding of many phenomena. We prefer to deal with weaker topologies to obtain continuous solutions of the linear Cauchy problems associated with certain pseudodifferential operators, thereby allowing us to study the meaning of the heat equation solution backward in time.

Let ϕ be a Schwartz function (and we write $\phi \in \mathscr{S}(\mathbb{R}^N,\mathbb{C})$) and its Fourier transform is given by

$$\big(\mathscr{F}\phi\big)(\xi) = \hat{\phi}(\xi) = \int\limits_{\mathbb{R}^N} e^{-2\pi i x\cdot\xi}\phi(x)\,dx,$$

and $\phi \mapsto \check{\phi}$ denotes the inverse Fourier transform. Let $\mathscr{S}'(\mathbb{R}^N,\mathbb{C})$ be the space of all continuous linear functionals defined on $\mathscr{S}(\mathbb{R}^N,\mathbb{C})$ and it is equipped with the $\bigstar$-weak topology (see [6,10]).

Definition 3. A pseudodifferential operator $a(D)\colon \mathscr{S}\big(\mathbb{R}^N\big) \to \mathscr{S}\big(\mathbb{R}^N\big)$ of order m on $\mathbb{R}^N$ with constant coefficients (or constant coefficients, m-ΨDO) is a linear map given by

$$\big(a(D)\phi\big)(x) := \big(a\,\hat{\phi}\big)^{\check{}}(x), x \in \mathbb{R}^N,$$

for every $\phi \in \mathscr{S}\big(\mathbb{R}^N\big)$, where $a \in C^\infty(\mathbb{R}^N)$ satisfies the property that for all multi-index α, a constant $c_\alpha > 0$ exists such that $\big|\partial^\alpha a(\xi)\big| \leqslant c_\alpha(1+|\xi|)^{m-|\alpha|}$ for every $\xi \in \mathbb{R}^N$.

We now present a Fréchet space, i.e., $\mathscr{F}L^2_{loc}$, which was introduced by Treves [14], where the elements are distributions of $\mathscr{D}'$ comprising the space of all continuous linear functionals defined on $C_c^\infty(\mathbb{R}^N)$. This is quite remarkable because when considering distributions, we usually deal with the $\bigstar$-weak topology, which is generally not metrizable. In addition, every distribution u with compact support (where we write $u \in \mathscr{E}'$) and every L^2 function belong to $\mathscr{F}L^2_{loc}$, so some good properties of the Fourier transform on L^2 are preserved and the Paley–Wiener–Schwartz theorem can be extensively invoked.

Let $\mathscr{F}L^2_{loc}$ be the completion of the metric space (E,d) constructed as follows. Let $E := \mathscr{F}^{-1}\big(\mathscr{S}'\cap L^2_{loc}\big)$ be endowed with the topology generated by the seminorms $\mathfrak{p}_j(u) := \|\widehat{u}\|_{L^2((B(0,j))}$ for $u \in E$ and $j \in \mathbb{N}$, which is actually a separating family, and thus the function

$$E \times E \ni (u,v) \mapsto d(u,v) := \sum_{j=1}^{\infty} 2^{-j} \frac{\mathfrak{p}_j(u-v)}{1+\mathfrak{p}_j(u-v)}$$

defines a metric on E. Clearly, its topology as a complete metric space is equivalent to that generated by the extended seminorms $\mathfrak{p}_j \colon \mathscr{F}L^2_{loc} \to [0,\infty)$, and thus $\mathscr{F}L^2_{loc} = \big(\mathscr{F}L^2_{loc}, (\mathfrak{p}_j)_{j\in\mathbb{N}}\big)$ is a Fréchet space.[3]

We provide further properties, as follows.

Proposition 5.

- **a.** *The Fourier transform $\mathscr{F} \colon \mathscr{F}L^2_{loc} \to L^2_{loc}$ is well defined and it is continuous;*
- **b.** *Every element of $\mathscr{F}L^2_{loc}$ is a distribution of $\mathscr{D}'\big(\mathbb{R}^N\big)$. Hence, $\mathscr{F}L^2_{loc}$ is a Fréchet space of distributions;*
- **c.** *L^2 and $\mathscr{E}'$ are topological subspaces of $\mathscr{F}L^2_{loc}$ and $\big(L^2, \|\cdot\|_{L^2}\big) \hookrightarrow \mathscr{F}L^2_{loc}$. In particular, every Sobolev space $\big(H^s, \|\cdot\|_s\big)$ is continuously embedded on $\mathscr{F}L^2_{loc}$, $s \geqslant 0$;*
- **d.** *Every constant coefficient m-ΨDO $a(D)$ induces a strongly compatible operator on $\big(\mathscr{F}L^2_{loc}, (\mathfrak{p}_j)_{j\in\mathbb{N}}\big)$ by setting*

$$a(D)[u] := \big[a(D)u\big], \text{ for } [u] \in \mathscr{F}L^2_{loc}.$$

Consequently, $\mathbb{R} \ni t \mapsto e^{a(D)t}u_0 \in \mathscr{F}L^2_{loc}$ provides the unique solution of

$$\begin{cases} u_t = a(D)u, t \in \mathbb{R} \\ u(0) = u_0 \in \mathscr{F}L^2_{loc} \end{cases};$$

- **e.** *$\big(\mathscr{F}L^2_{loc}, (\mathfrak{p}_j)_{j\in\mathbb{N}}\big)$ is a quojection.*

Proof. a. Let $[u] \in \mathscr{F}L^2_{loc}$. If $(u_l)_{l\in\mathbb{N}} \in [u]$, then $(\widehat{u_l})_{l\in\mathbb{N}}$ is a Cauchy sequence in L^2_{loc}, and thus a unique $w \in L^2_{loc}$ exists such that $\widehat{u_l} \overset{l\to\infty}{\longrightarrow} w$ in L^2_{loc} and we set $\widehat{[u]} := w$, which does not depend on the choice of $(u_l)_l$ in $[u]$. Thus, we define

$$C^\infty_c \ni \phi \mapsto \langle [u], \phi\rangle := \int_{\mathbb{R}^N} \widehat{[u]}(\xi)\phi(\xi)\, d\xi \ \in \mathbb{C}.$$

b. Let K be a compact subset of $\mathbb{R}^N$. If $\operatorname{supp}\phi \subset K \subset B(0,j)$, then

$$|\langle [u], \phi\rangle| \leqslant \|\widehat{[u]}\|_{L^2(B(0,j))}\, \|\phi\|_{L^2(K)} \leqslant |B(0,j)|^{1/2}\mathfrak{p}_j([u]) \sup_K |\phi|,$$

so $[u]$ is actually a distribution and we can write u instead of $[u]$.

c. We obtain the inclusion $\mathscr{E}' \subset E$ by the Paley–Wiener–Schwartz theorem and the embedding $\big(L^2, \|\cdot\|_{L^2}\big) \hookrightarrow E$ by the Plancherel theorem, and **c.** follows.

d. Let $a(D)$ be a constant coefficient m-ΨDO and $[u] \in \mathscr{F}L^2_{loc}$. If $|\xi| \leqslant j$, then

$$\mathscr{F}\big(a(D)u\big)(\xi) = \lim_{\substack{l\to\infty \\ L^2(B(0,j))}} \mathscr{F}\big(a(D)u_l\big)(\xi) = \lim_{\substack{l\to\infty \\ L^2(B(0,j))}} a(\xi)\,\widehat{u_l}(\xi) = a(\xi)\,\widehat{[u]}(\xi),$$

[3] According to [14], this is a reflexive Fréchet space, for which the dual space, $\mathscr{F}L^2_c$, is the inductive limit of a sequence of Hilbert spaces.

where $\widehat{[u]}$ is the limit in $L^2(B(0,j))$ of the Fourier transform of some sequence $(u_l)_{l\in\mathbb{N}} \in [u]$, by definition. The fact that $a(D)$ is strongly compatible with $\mathfrak{p}_j$ follows from

$$\mathfrak{p}_j\big(a(D)[u]\big) = \left(\int\limits_{|\xi|\leqslant j} |a(\xi)|^2 \left|\widehat{[u]}(\xi)\right|^2 d\xi \right)^{1/2} \leqslant \|a\|_{L^\infty(B(0,j))}\, \mathfrak{p}_j\big([u]\big).$$

For **e.**, let $X_j = \mathscr{F}L^2_{loc}/\mathfrak{p}_j^{-1}(\{0\})$. We claim that every $X_j \equiv L^2(B(0,j))$. By definition, if $[u] \in \mathscr{F}L^2_{loc}$, then

$$\big[[u]\big]_j := \{[u]+[v] : \mathfrak{p}_j([v]) = 0\} = \left\{ [f] \in \mathscr{F}L^2_{loc} : \widehat{[f]}(\xi) = \widehat{[u]}(\xi) \text{ a.e. } |\xi| \le j \right\}$$

and $\left\| \big[[u]\big]_j \right\|_{X_j} = \left\| \widehat{[u]} \right\|_{L^2(B(0,j))}$; therefore, we can identify $\big[[u]\big]_j$ with $\widehat{[u]}\Big|_{B(0,j)}$. □

$\mathscr{E}'(\mathbb{R}^N)$ is a subspace of $\mathscr{F}L^2_{loc}$, so may ask whether a semigroup $\{e^{ta(D)} : t \geqslant 0\}$ in $\mathscr{F}L^2_{loc}$ lets $\mathscr{E}'(\mathbb{R}^N)$ be invariant. We provide a sufficient condition for every N and the equivalence only for $N = 1$. In addition, due to the fact that $(L^2, \|\cdot\|_{L^2}) \hookrightarrow \mathscr{F}L^2_{loc}$, we provide a complete characterization of the semigroups $\{e^{ta(D)} : t \geqslant 0\}$ in $\mathscr{F}L^2_{loc}$ that let $L^2(\mathbb{R}^N)$ be invariant for any N.

If $z \in \mathbb{C}^N$, then we write $z = \xi + i\eta$, with $\xi = \Re z$ and $\eta = \Im z$ in $\mathbb{R}^N$.

Theorem 6. *Let $a(D)$ be a constant coefficient m-ΨDO with $m > 0$, $\xi \mapsto a(\xi)$ is its symbol, and $\{e^{ta(D)} : t \in \mathbb{R}\}$ is the group generated by $a(D)$ on $\mathscr{F}L^2_{loc}(\mathbb{R}^N)$, according to Proposition 5 (d).*

a. *Suppose that $a(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha$. If $m = 1$ and $\Re\, a_\alpha = 0$ whenever $|\alpha| = 1$, then*

$$e^{ta(D)}\big(\mathscr{E}'(\mathbb{R}^N)\big) \subset \mathscr{E}'(\mathbb{R}^N), \text{ for every } t \in \mathbb{R}. \tag{10}$$

In addition, if $N = 1$, then the converse holds.

b. *If a $C > 0$ exists such that $\Re\, a(\xi) \le -C|\xi|^m$ whenever $|\xi|$ is sufficiently large, then we have the following regularization effect:*

$$e^{ta(D)}\big(\mathscr{E}'(\mathbb{R}^N)\big) \subset \mathscr{S}(\mathbb{R}^N), \text{ for all } t > 0. \tag{11}$$

c. *We have the following positive invariance:*

$$e^{ta(D)}\big(L^2(\mathbb{R}^N)\big) \subset L^2(\mathbb{R}^N), \text{ for all } t \geqslant 0 \tag{12}$$

if and only if

$$\sup_{\xi\in\mathbb{R}^N} e^{t\,\Re\, a(\xi)} < \infty, \text{ for all } t \geqslant 0. \tag{13}$$

d. *If $\Re\, a(\xi) \leqslant 0$ whenever $|\xi|$ is sufficiently large, then (12) holds.*

Proof. a. If $|\alpha| = 1$, then $a_\alpha = ib_j$ with $b_j \in \mathbb{R}$ for every $j = 1, 2, \dots n$ and

$$a(D) = \sum_{|\alpha|\le 1} a_\alpha D^\alpha = a_0 + \sum_{j=1}^{n} ib_j (2\pi i)^{-1} \frac{\partial}{\partial x_j},$$

such that $\xi \mapsto a(\xi) = a_0 + \sum_{j=1}^{n} ib_j\xi_j$. Let $u \in \mathscr{E}'(\mathbb{R}^N)$, $\xi \in \mathbb{R}^N$ and $t \in \mathbb{R}$, then

$$\mathscr{F}\big(e^{ta(D)}u\big)(\xi) = e^{ta(\xi)}\hat{u}(\xi) = e^{ta_0}\left[e^{2\pi i(\frac{tb}{2\pi})\cdot\xi}\hat{u}(\xi)\right] = e^{ta_0}\mathscr{F}\big(\tau_{(tb/2\pi)}u\big)(\xi),$$

where $b = (b_1, \cdots, b_n)$ and τ_h denotes the translation by $h \in \mathbb{R}^N$. Hence, (10) holds, $e^{ta(D)} : \mathscr{E}'(\mathbb{R}^N) \to \mathscr{E}'(\mathbb{R}^N)$ is well defined and it coincides with e^{ta_0} times the translation by $tb/2\pi$.

Now, by setting $N = 1$, we suppose that (10) holds and let $u \in \mathscr{E}'(\mathbb{R})$.

First, by the Paley–Wiener–Schwartz theorem, $e^{ta(D)}u \in \mathscr{E}'(\mathbb{R})$ if and only if $\mathscr{F}\big(e^{ta(D)}u\big) : \mathbb{R} \to \mathbb{C}$ has an analytic extension $V = V_{(t,u)} : \mathbb{C} \to \mathbb{C}$ and the constants $C = C_{(t,u)}$, $R = R_{(t,u)} > 0$ and $L = L_{(t,u)} \in \mathbb{N}$ exist such that

$$|V(z)| \leqslant C(1 + |z|)^L e^{R|\Im z|}, \text{ for every } z \in \mathbb{C}.$$

Second, $\mathbb{C} \ni z \mapsto a(z) \in \mathbb{C}$ is a polynomial, so we deduce that $\mathbb{R} \ni \xi \mapsto \mathscr{F}\big(e^{ta(D)}u\big)(\xi) = e^{ta(\xi)}\hat{u}(\xi)$ admits a unique analytic extension, which is given by

$$\mathbb{C} \ni z \mapsto e^{ta(z)}\hat{u}(z) = V(z) \in \mathbb{C}.$$

Moreover, $\left|\mathscr{F}\big(e^{ta(D)}u\big)\right| = e^{t\,\Re\, a(z)}|\hat{u}(z)|$ holds for every $t \in \mathbb{R}$ and $z \in \mathbb{C}$. By combining this with the estimate of V, setting $u = \delta$ and $t = 1$, we obtain

$$e^{\Re\, a(z)} \leqslant C(1 + |z|)^L e^{R|\Im z|}, \text{ for every } z \in \mathbb{C}, \tag{14}$$

which holds if and only if the order m of $a(z)$ is equal to 1; otherwise, $z \mapsto \Re\, a(z)$ is a polynomial with degree $\geqslant 2$ and onto $\mathbb{R}$, which contradicts (14). Hence, we write $a(\xi) = a_0 + a_1\xi$ for some $a_0, a_1 \in \mathbb{C}$, and we claim that $\Re\, a_1 = 0$, based on which the result follows. By (14),

$$e^{\Re\, a(\xi)} = e^{\Re a_0}e^{\Re a_1\,\xi} \leq C(1 + |\xi|), \text{ for every } \xi \in \mathbb{R},$$

which cannot be true for all $\xi \in \mathbb{R}$ if $\Re\, a_1 \neq 0$.

b. Let $u \in \mathscr{E}'(\mathbb{R}^N)$ and $t \geqslant 0$, then $\mathbb{R}^N \ni \xi \mapsto \mathscr{F}\big(e^{ta(D)}u\big)(\xi) = e^{ta(\xi)}\hat{u}(\xi)$ is a C^∞ function. By the Paley–Wiener–Schwartz theorem,

$$|\mathscr{F}\big(e^{ta(D)}u\big)(\xi)| = e^{t\Re\, a(\xi)}|\hat{u}(\xi)| \leq C_u(1 + |\xi|)^{L_u}e^{t\Re\, a(\xi)}$$

and by hypothesis, the right-hand side of this inequality vanishes at infinity faster than any power of $|\xi|$. Now, it is easy to deduce that $\xi \mapsto \mathscr{F}\big(e^{ta(D)}u\big)(\xi)$ is a Schwartz function, and thus (b) follows.

c. For $u \in L^2(\mathbb{R}^N)$, $e^{ta(D)}u \in L^2(\mathbb{R}^N)$ if and only if $\mathscr{F}\big(e^{ta(D)}u\big)(\xi) = e^{ta(\xi)}\hat{u}$ belongs to $L^2(\mathbb{R}^N)$ if and only if $|e^{ta(\xi)}\hat{u}(\xi)| = e^{t\,\Re\, a(\xi)}|\hat{u}(\xi)|$ belongs to $L^2(\mathbb{R}^N)$.

Suppose that (13) holds. Let $u \in L^2(\mathbb{R}^N)$ and $t \geqslant 0$, then

$$\int_{\mathbb{R}^N} e^{2t\,\Re\, a(\xi)}|\hat{u}(\xi)|^2\, d\xi \leqslant \left(\sup_{\xi\in\mathbb{R}^N} e^{2t\,\Re\, a(\xi)}\right)\|\hat{u}\|_{L^2},$$

such that (12) is true. Conversely, suppose that (13) does not hold. Take $t \geqslant 0$, a sequence $(\xi_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$, and a collection of disjoint balls $B_n := B(\xi_n; r_n)$ such that $|\xi_n| \to \infty$, $e^{2t\,\Re\, a(\xi_n)} \geqslant 2^n/n$ and $e^{2t\,\Re\, a(\xi)} \geqslant 2^n/2n$ for every $\xi \in B_n$, and for every $n \in \mathbb{N}$. Let f_n be defined by

$$\mathbb{R}^N \ni \xi \mapsto f_n(\xi) := \frac{2^{-n/2}}{|B_n|^{1/2}} \chi_{B_n}(\xi),$$

where χ_{B_n} denotes the characteristic function of the set B_n, then the function $f := \sum_n f_n$ belongs to $L^2(\mathbb{R}^N)$ since

$$\int_{\mathbb{R}^N} f^2(\xi)\, d\xi = \int_{\mathbb{R}^N} \left(\sum_{n=1}^{\infty} f_n^2(\xi) \right) d\xi = \sum_{n=1}^{\infty} \int_{\mathbb{R}^N} f_n^2(\xi)\, d\xi = \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty.$$

However, $e^{ta(D)}\check{f}$ does not belong to $L^2(\mathbb{R}^N)$ because

$$\int_{\mathbb{R}^N} e^{2t\,\Re\, a(\xi)} |f(\xi)|^2\, d\xi = \sum_{n=1}^{\infty} \int_{B_n} e^{2t\,\Re\, a(\xi)} f_n^2(\xi)\, d\xi \geqslant \sum_{n=1}^{\infty} \frac{1}{2n} = \infty.$$

Hence, (12) does not hold for $u := \check{f} \in L^2(\mathbb{R}^N)$, and thus (c) and (d) follow. □

Therefore, in order to obtain a C_0-semigroup on $L^2(\mathbb{R}^N)$ that is generated by a linear differential operator with constant coefficients, we may replace the spectral conditions in the Hille–Yosida theorem (on Banach spaces) by the condition (13).

Proposition 7. *Consider the heat equation in $\mathbb{R}^N$:*

$$\begin{cases} u_t + u = \Delta u, t > 0 \\ u(0) = u_0 \end{cases} . \tag{15}$$

If $u_0 \in \mathscr{F}L^2_{loc}$, then the evolution problem (15) can be solved for every $t \in \mathbb{R}$ in a distributional sense. Moreover, if $u_0 \in L^2$, then this solution extends the analytic semigroup generated by $-(1-\Delta)$ on L^2 forward in time to a uniformly continuous group on $\mathscr{F}L^2_{loc}$ for all of real time.

Proof. First, $A := 1 - \Delta\colon H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N, \mathbb{C}) \to L^2(\mathbb{R}^N, \mathbb{C})$ is a linear operator in L^2 and a sectorial operator with $\Re\,\sigma(A) > 0$, and thus $-(1-\Delta)$ generates an analytic semigroup on L^2 indicated by $\{e^{-At} : t \geq 0\}$. In addition, the fractional power spaces associated with A are the usual Sobolev spaces H^s characterized by Bessel potentials: $H^s = \{u \in \mathscr{S}' : (1 + 4\pi^2|\xi|^2)^{s/2}\widehat{u} \in L^2\}$.

Second, the map $\xi \mapsto a(\xi) := -(1 + 4\pi^2|\xi|^2)$ is the symbol for the 2-ΨDO operator $a(D) := -(1 - \Delta) : \mathscr{F}L^2_{loc} \to \mathscr{F}L^2_{loc}$, which generates a continuous group on $\mathscr{F}L^2_{loc}$ denoted by $\{e^{a(D)\,t} : t \in \mathbb{R}\}$. In addition, by Corollary 6, $\{e^{a(D)\,t}\colon L^2 \to L^2\}_{t\geqslant 0}$ is a continuous semigroup.

Thus, we have obtained two semigroups in L^2 generated by the heat operator using two different generation approaches. However, we claim that they are the same semigroup, so the group on $\mathscr{F}L^2_{loc}$ extends the analytic semigroup defined on L^2. Let $t > 0$ and $u_0 \in L^2$. First, $\mathscr{F}(e^{-tA}u_0) = e^{-(1+4\pi^2|\xi|^2)\,t}\widehat{u_0}$ (as described by [9], page 34). Moreover, by the definition of $e^{a(D)\,t}$, we may apply the Fourier transform to this group to obtain $\mathscr{F}(e^{a(D)\,t}u_0) = e^{ta(\xi)}\widehat{u_0}$ and because they are elements of L^2, we conclude that both semigroups coincide on L^2 by the Plancherel theorem. □

An interesting consequence of this proof is that the heat equation (15) can be solved **backward in time** for any initial data $u_0 \in L^2 \subset \mathscr{F}L^2_{loc}$ in a distributional sense. Basically, for $u_0 \in L^2$, the regularity of $e^{-tA}u$ has three stages indexed by the time parameter: for $t < 0$, $e^{-t(1-\Delta)}u_0 \in \mathscr{F}L^2_{loc}$, i.e., the solution backward in time belongs to a space of very low regularity; if $t = 0$, there is nothing to add, where u_0 belongs to L^2;

and for $t > 0$, $e^{-t(1-\Delta)}u \in \bigcap_{s\in\mathbb{R}} H^s \subset C^\infty$, which is the regularization effect forward in time promoted by this sectorial operator.

The exponential factor in $\mathscr{F}\big(e^{-t(1-\Delta)}u_0\big) = e^{-(1+4\pi^2|\xi|^2)\,t}\widehat{u_0}$ explains how the regularity of (15) responds to the time parameter since

$$\int\limits_{\mathbb{R}^N} e^{-2t(1+4\pi^2|\xi|^2)}(1+|\xi|)^{2M} d\xi < \infty, \text{ for } t > 0 \text{ and } M \in \mathbb{N},$$

and

$$\lim_{|\xi|\to\infty} e^{-2t(1+4\pi^2|\xi|^2)}(1+|\xi|)^{2M} = \infty, \text{ for } t < 0 \text{ and } M \in \mathbb{N}.$$

The key point to note is that the fractional power spaces associated with $1-\Delta$ are completely characterized by a property that essentially connects Fourier analysis with the usual Hilbert spaces.

Example 1 *(The i derivative operator on $\mathbb{R}$).* If $A = i\frac{d}{dx} : H^1 \subset L^2 \to L^2$, then we cannot solve (2) using the mainstream approach of Banach spaces because A does not fulfill the spectral conditions of the Hille–Yosida theorem. In addition, $a(\xi) = -2\pi\xi$ is its symbol, so it generates a semigroup $\{e^{itd/dx} : t \geqslant 0\}$ on L^2 by Theorem 6 (c), which provides its unique solution on L^2.

Example 2 *(The positive power of the Laplace operator on $\mathbb{R}^n$).* Let $\alpha > 0$. The symbol $\xi \mapsto a(\xi) = -(4\pi^2|\xi|^2)^\alpha$ is associated with the operator $-(-\Delta)^\alpha$ such that $e^{-t(-\Delta)^\alpha}u \in \mathscr{S}(\mathbb{R}^n)$ for every $t > 0$ whenever $u \in \mathscr{E}'(\mathbb{R}^n)$ by Theorem 6 (b).

In particular, the solution of the Cauchy problem

$$\begin{cases} u_t = -(-\Delta)^\alpha u, \ t \in \mathbb{R} \\ u(0) = \delta, \end{cases}$$

belongs to $\mathscr{S}(\mathbb{R}^N)$ for every $t > 0$, where δ denotes the Dirac δ-distribution.

Example 3 *(The derivative operator on $\mathbb{R}$).* Consider the Cauchy problem

$$\begin{cases} u_t = u_x, t \in \mathbb{R} \\ u(0) = u_0 \in C^\infty \end{cases}. \tag{16}$$

Using the mainstream approach, we may impose three restrictions in order to solve (16): i) $t \geqslant 0$; ii) u_0 has a derivative u_0' and both are uniformly bounded continuous functions (we write $u_0, u_0' \in C_b(\mathbb{R},\mathbb{C})$); and iii) restrict the domain of $A = \frac{d}{dx}$ so it is a closed densely defined operator on $C_b(\mathbb{R},\mathbb{C})$. Thus, the C_0-semigroup generated by $\frac{d}{dx}$ is the translation semigroup (see [13]).

Furthermore, we can solve (16) in the (Fréchet) phase space $C^\infty(\mathbb{R},\mathbb{C})$ without any further assumptions. In addition, the group generated by $\frac{d}{dx}$ extends the C_0-semigroup above. Let $C^\infty_{\exp}$ be the set of all functions $\phi \in C^\infty$ such that for every $m \in \mathbb{Z}_+$ and $j \in \mathbb{N}$, a constant $M = M(\phi, m, j) > 0$ exists such that

$$\sup_{n\in\mathbb{N}} \sup_{|x|\leqslant j} \left| M^{-n} \frac{d^{n+m}}{dx^{n+m}}\phi(x) \right| < \infty.$$

Proposition 8. *Every $\phi \in C^\infty_{\exp}$ is a real analytic function. Moreover, we have the following:*

a. *$C^\infty_{\exp}$ is a dense subspace of $C^\infty(\mathbb{R})$;*

b. *The partial sums* $S_N := \sum_{n=0}^{N} \frac{t^n}{n!} \frac{d^n}{dx^n} \phi$ *converges in* $C^\infty(\mathbb{R})$ *to a function in* $C^\infty_{\exp}$ *for every* $\phi \in C^\infty_{\exp}$ *and* $t \in \mathbb{R}$, *where its limits are denoted by* $e^{t\frac{d}{dx}}\phi$;

c. $e^{t\frac{d}{dx}} : C^\infty_{\exp} \to C^\infty_{\exp}$ *is well defined and it is a bounded linear operator, and thus by density,* $e^{t\frac{d}{dx}} \in \mathscr{L}\big(C^\infty(\mathbb{R})\big)$;

d. *The family of operators* $\{e^{t\frac{d}{dx}} : t \in \mathbb{R}\}$ *is a uniformly continuous group on* $C^\infty(\mathbb{R})$ *such that*

$$\left(e^{t\frac{d}{dx}}\phi\right)(s) = \phi(s+t), \text{ for every } s \in \mathbb{R}.$$

Proof. $x \mapsto e^{-x^2}$ belongs to $C^\infty_{\exp}$, so we may argue that it is a mollifying function to obtain the proof. □

4. Final comments

If X is a Fréchet space and $A\colon X \to X$ is strongly compatible with it, then the operator $\exp(t\,A)$ is also strongly compatible and this solves the Cauchy problem

$$\begin{cases} u_t = Au, t \in \mathbb{R} \\ u(0) = u_0 \in X \end{cases}.$$

We have established criteria for identifying whether the semigroup generated by a constant coefficient m-ΨDO defined on $\mathscr{F}L^2_{loc}(\mathbb{R}^N)$ acts on L^2 and $\mathscr{E}'$. We also analyzed the regularization of initial data backward and forward by the solution group for the heat equation on $\mathscr{F}L^2_{loc}$, which extends the standard solution on Hilbert spaces for positive times, and this partially explains the regularization process performed by the exponential of the Laplacian operator.

The strong connection between the mainstream approach and the results obtained indicate that we may also consider hyperbolicity (see [4]), non-autonomous linear operators $A = A(t)$, the generation of analytic semigroups, and semilinear problems. Moreover, it is not clear how the $\mathscr{E}'$ equipped with its original topology is related to its topology as a subspace of $\mathscr{F}L^2_{loc}$.

Our future research will address these problems.

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