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Existence of weak and regular solutions for Keller–Segel system with degradation coupled to fluid equations

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ABSTRACT

We establish the global well-posedness for the following chemotaxis-fluid system

$$\begin{cases} \partial_t n + u \cdot \nabla n = \Delta n - \nabla \cdot (n \nabla c) - \mu n^q, \\ \partial_t c + u \cdot \nabla c = \Delta c - c + n, \\ \partial_t u + \kappa (u \cdot \nabla) u + \nabla P = \Delta u - n \nabla \phi, \\ \nabla \cdot u = 0, \end{cases}$$

in $\mathbb{R}^d$, $d = 2, 3$, where $\mu > 0$, $q > 2 - \frac{1}{d}$ and $\kappa \in \{0, 1\}$. For either $q \geq 2$, $(\kappa, d) = (1, 2)$ or $q > 2$, $(\kappa, d) = (0, 3)$, we prove the global existence of regular solutions. In case that $q > 2 - \frac{1}{d}$ and $\kappa = 0$, very weak solutions are constructed as well.

1. Introduction

In the seminal paper of Keller and Segel [11], a mathematical model was proposed to describe a directed movement of biological individuals toward higher concentrations of a chemical signal. The original model is simplified to focus only on the effect of chemotaxis, but in fact, the movement of bacteria may be influenced by complex factors. For example, the bacteria living inside a viscous fluid, such as Bacillus subtilis or Escherichia coli, can not be free from the influence of surrounding fluid [17,24]. Moreover, it is known that coral fertilization results from the chemotactic behavior of sperm [6,12,18].

Motivated by some studies about chemotaxis models involving chemotaxis-fluid interaction for cases that the chemical is produced by the bacteria [4,5,7,16,20,26], we consider a chemotaxis system coupled with fluid equations on the whole space such that for $T < \infty$

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$$\begin{cases} \partial_t n + u\cdot\nabla n = \Delta n - \nabla\cdot(n\nabla c) - \mu n^q, & (x,t)\in\mathbb{R}^d\times(0,T),\\ \partial_t c + u\cdot\nabla c = \Delta c - c + n, & (x,t)\in\mathbb{R}^d\times(0,T),\\ \partial_t u + \kappa(u\cdot\nabla)u + \nabla P = \Delta u - n\nabla\phi, & (x,t)\in\mathbb{R}^d\times(0,T),\\ \nabla\cdot u = 0, & (x,t)\in\mathbb{R}^d\times(0,T), \end{cases} \tag{1.1}$$

where $\mu > 0$, $q > 2 - \frac{1}{d}$ are given and $\kappa \in \{0, 1\}$, $d = 2, 3$. Here, the unknowns n, c, u denote the population density of the bacteria, the concentration of chemical substance, and the fluid velocity field, respectively. The unknown P represents the associated pressure, through buoyant forces. The function $\phi = \phi(x)$ denotes the potential function, e.g., the gravitational force or centrifugal force.

The aim of this paper is to verify the global existence of solutions to (1.1) depending on the power of superlinear degradation term, $q > 2 - \frac{1}{d}$. Before we go to our main results, for a clearer understanding, it is worthy to mention some related works. The general chemotaxis-fluid system is written as

$$\begin{cases} \partial_t n + u\cdot\nabla n = \Delta n - \nabla\cdot(n\nabla c) + f(n), & (x,t)\in\Omega\times(0,T),\\ \partial_t c + u\cdot\nabla c = \Delta c - c + n, & (x,t)\in\Omega\times(0,T),\\ \partial_t u + \kappa(u\cdot\nabla)u + \nabla P = \Delta u - n\nabla\phi + g(x,t), & (x,t)\in\Omega\times(0,T),\\ \nabla\cdot u = 0, & (x,t)\in\Omega\times(0,T), \end{cases} \tag{1.2}$$

where $g(x, t)$ is an external force and Ω is a bounded smooth domain on $\mathbb{R}^d$ with no-flux boundary conditions. To the authors' knowledge, there have been a few results about the above problem (1.2). In the two dimensional case, Tao and Winkler [21] showed that the quadratic degradation prevents the solution from blow-up. More precisely, when $f(n) = rn - \mu n^2$ and $\kappa = 1$, they obtained a global classical solution and showed decay properties of solutions if $r = 0$. In the three dimensional case, they also proved that if $\mu \ge 23$, $\kappa = 0$, then (1.2) possesses a global classical solution for any $r \ge 0$ (see [20]).

Neglecting the fluid interaction, we can simplify (1.2) to the following classical Keller–Segel model

$$\begin{cases} \partial_t n = \Delta n - \nabla\cdot(n\nabla c) + f(n), & (x,t)\in\Omega\times(0,T),\\ \tau\partial_t c = \Delta c - c + n, & (x,t)\in\Omega\times(0,T). \end{cases} \tag{1.3}$$

For parabolic-elliptic case ($\tau = 0$), Winkler [27] proved the global existence of very weak solutions to (1.3) provided that f satisfies for $q > 2 - \frac{1}{d}$

$$\begin{cases} f(n) \le a - bn^q, \ \text{for all } n \ge 0, \ \text{with } a \ge 0, \ b > 0,\\ f(n) \ge -C_0(n + n^q), \ \text{for all } n \ge 0, \ \text{with } C_0 > 0. \end{cases} \tag{1.4}$$

Instead of the natural notion, he introduced a sub- and a super-solution of the equation of n because one integral identity is not obtained from limiting procedure of the regularized solutions. In particular, as to the super-solution, the notion of entropy solution commonly used in higher order thin film equations is adapted (see [19]). Readers are referred to [14,22,28,29] for related results to the global existence or blow-up of solutions of similar parabolic-elliptic systems. For $\tau = 1$, through technical computations linked to the parabolic-parabolic problem, Viglialoro [25] achieved very weak global solutions provided (1.4). Indeed, the global solvability problem has rarely been investigated for a chemotaxis-fluid model at which the degradation is weaker than quadratic. It seems, due to the presence of fluid velocity fields, that control of n and c is much harder since we need to obtain proper estimates of u as well. The main contribution of this paper is devoted to proving the existence of global-in-time solutions to (1.1) depending on $q > 2 - \frac{1}{d}$.

We first note that the initial data can be assumed as follows:

Assumption 1.1. The initial data (n_0, c_0, u_0) satisfy that for $d = 2, 3$

$$\begin{cases} n_0 \in (L^1 \cap H^2)(\mathbb{R}^d), \quad c_0 \in (L^q \cap H^3)(\mathbb{R}^d), \quad u_0 \in H^3(\mathbb{R}^d), & x \in \mathbb{R}^d, \\ n_0(x) \geq 0, \quad c_0(x) \geq 0, & x \in \mathbb{R}^d. \end{cases}$$

Assumption 1.2. For given $q > 2 - \frac{1}{d}$, $d = 2, 3$, a number r satisfies

$$\begin{cases} 2 < r < 2q & \text{if } d = 2, \\ \frac{5}{3} \leq r < \frac{3q}{2} & \text{if } d = 3 \end{cases} \tag{1.5}$$

and let $r' := \frac{dr}{d+r}$. The initial data (n_0, c_0, u_0) satisfy that

$$\begin{cases} n_0 \in (L^1 \cap H^2)(\mathbb{R}^2),\ c_0 \in (L^{\frac{r}{2}} \cap W^{1,r} \cap W^{1,r'} \cap H^3)(\mathbb{R}^2),\ u_0 \in (W^{2,r'} \cap H^3)(\mathbb{R}^2), & \text{if } d = 2, \\ n_0 \in (L^1 \cap H^2)(\mathbb{R}^3),\ c_0 \in (L^{\frac{3r}{5}} \cap W^{1,r} \cap H^3)(\mathbb{R}^3),\ u_0 \in (W^{2,r'} \cap H^3)(\mathbb{R}^3), & \text{if } d = 3, \\ n_0(x) \geq 0, \quad c_0(x) \geq 0, & x \in \mathbb{R}^d. \end{cases}$$

Our first main result is the existence of regular solutions for (1.1) which reads as follows:

Theorem 1.1 *(Global existence of regular solution). Let $\mu > 0$. Suppose that either $q \geq 2$, $(\kappa, d) = (1, 2)$ or $q > 2$, $(\kappa, d) = (0, 3)$. If the initial data (n_0, c_0, u_0) satisfy Assumption 1.1, then the problem (1.1) possesses the unique regular solution (n, c, u) of (1.1) satisfying for any $T < \infty$*

$$(n, c, u) \in L^\infty(0, T; H^2(\mathbb{R}^d) \times H^3(\mathbb{R}^d) \times H^3(\mathbb{R}^d)),$$
$$(\nabla n, \nabla c, \nabla u) \in L^2(0, T; H^2(\mathbb{R}^d) \times H^3(\mathbb{R}^d) \times H^3(\mathbb{R}^d)).$$

Remark 1.1. As mentioned earlier, it was shown in [20] that for a bounded domain in $\mathbb{R}^3$, if $\mu \geq 23$, then the solution becomes regular. According to our computations, it seems that for the case of the whole space, the solution becomes regular if $\mu \geq 23$, by following the method of proof in [20]. We are, however, not going to try to prove the case with such restriction on μ in this paper.

Secondly, we construct γ-very weak solutions for $q > 2 - \frac{1}{d}$ in case that the fluid equation is the Stokes system (see Definition 2.3 for the notion of γ-very weak solutions).

Theorem 1.2 *(Global existence of γ-very weak solutions). Let $T > 0$, $\mu > 0$, $\kappa = 0$, $q > 2 - \frac{1}{d}$ for $d = 2, 3$. If the initial data (n_0, c_0, u_0) satisfy Assumption 1.2 and $\gamma \in \left(0, q - \frac{2(d-1)}{d}\right)$, then the problem (1.1) possesses at least one γ-very weak solution (n, c, u) in $\mathbb{R}^d \times (0, T)$.*

Remark 1.2. Our method of proof in Theorem 1.2 is mainly based on the construction of solutions established in [25,27]. Compared to [25,27], main difficulty is caused by the presence of fluid. In particular, the estimate of ∇c is quite technical and we used suitable decompositions of solutions and inductive arguments for a decomposition for time variable (see the proof of Lemma 4.2 for details).

The rest of this paper is organized as follows. In Section 2, as a preparatory step, we introduce the notations, definitions and recall some useful lemmas. A local-well-posedness and blow-up criterion of system (1.1) will be also presented in Section 2. Using some useful a priori estimates, Theorem 1.1 will be proved in Section 3. In order to prove Theorem 1.2, we consider a regularization of the system (1.1) by using a parameter $\varepsilon \in (0, 1)$. Section 4 is devoted to deriving ε-independent estimates for the regularized system and verifying the existence of weak solutions. Owing to the compactness argument, we show that the solution of the regularized system converges to the solution of (1.1) in a weak sense. In Appendix, the proof of a blow-up criterion will be provided.

2. Preliminaries

Let us first briefly introduce some notations as follows:

- $W^{k,p}(\mathbb{R}^d) := \{f \in L^1_{\text{loc}}(\mathbb{R}^d) : D^\alpha f \in L^p(\mathbb{R}^d),\ 0 \le |\alpha| \le k\}$ for $1 \le p \le \infty$.
- $H^k(\mathbb{R}^d) := W^{k,2}(\mathbb{R}^d)$.
- $\|f\|_{L^q(0,T;W^{k,p}(\mathbb{R}^d))} := \big\| \|f(\cdot,t)\|_{W^{k,p}(\mathbb{R}^d)} \big\|_{L^q(0,T)}$ for $1 \le p,q \le \infty$.
- $\|f\|_{L^p(\mathbb{R}^d\times(0,T))} := \|f\|_{L^p(0,T;L^p(\mathbb{R}^d))}$.
- Γ is the heat kernel such that $\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}$ for $x \in \mathbb{R}^d, x \ne 0$ and $t > 0$.
- E is the fundamental solution of the Laplace equation in $\mathbb{R}^d$ such that

$$E(x) = \begin{cases} \frac{1}{2\pi}\log|x| & \text{for } d = 2, \\ 7 - \frac{1}{4\pi|x|} & \text{for } d = 3. \end{cases}$$

- Throughout this paper, C denotes a generic constant which may differ from line to line.

In the following two lemmas, we recall known maximal estimates of the heat equations (see [13,15]) and the Stokes system (see [9,10]).

Lemma 2.1. *Let $d \in \mathbb{N}$, $T > 0$, $z_0 \in W^{k,p}(\mathbb{R}^d)$ for $1 < p < \infty$ and $k = 1, 2$. If $f \in L^p(0,T;W^{k-1,p}(\mathbb{R}^d))$, then there exists a unique solution $z \in L^p(0,T;W^{k,p}(\mathbb{R}^d))$ for the following heat equation:*

$$\begin{cases} z_t - \Delta z = \nabla \cdot f, & (x,t) \in \mathbb{R}^d \times (0,T), \\ z(x,0) = z_0(x), & x \in \mathbb{R}^d. \end{cases}$$

Moreover, z satisfies

$$\|z\|_{L^p(0,T;W^{k,p}(\mathbb{R}^d))} \le C\left(\|f\|_{L^p(0,T;W^{k-1,p}(\mathbb{R}^d))} + \|z_0\|_{W^{k,p}(\mathbb{R}^d)}\right). \tag{2.6}$$

Lemma 2.2. *Let $d \ge 2$, $T > 0$, $z_0 \in W^{2,p}(\mathbb{R}^d)$ and $f \in L^q(0,T;L^p(\mathbb{R}^d))$ for $1 < p,q < \infty$. Suppose that (z,P) is the solution of the following Stokes system:*

$$\begin{cases} z_t - \Delta z + \nabla P = f, & (x,t) \in \mathbb{R}^d \times (0,T), \\ z(x,0) = z_0(x), & x \in \mathbb{R}^d. \end{cases}$$

Then, the following estimate is satisfied:

$$\begin{aligned} &\|z_t\|_{L^q(0,T;L^p(\mathbb{R}^d))} + \|\Delta z\|_{L^q(0,T;L^p(\mathbb{R}^d))} + \|\nabla P\|_{L^q(0,T;L^p(\mathbb{R}^d))} \\ &\le C\left(\|f\|_{L^q(0,T;L^p(\mathbb{R}^d))} + \|z_0\|_{W^{2,p}(\mathbb{R}^d)}\right). \end{aligned} \tag{2.7}$$

Furthermore, we provide other maximal estimates of the heat equation.

Lemma 2.3. *Let $d \in \mathbb{N}$, $T > 0$, $z_0 \in W^{2,p}(\mathbb{R}^d)$ and $f \in L^q(0,T;L^p(\mathbb{R}^d))$ for $1 < p,q < \infty$. Suppose that z is the solution of the following heat equation:*

$$\begin{cases} z_t - \Delta z = f, & (x,t) \in \mathbb{R}^d \times (0,T), \\ z(x,0) = z_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{2.8}$$

Then, the following estimates are satisfied:

$$\|z_t\|_{L^q(0,T;L^p(\mathbb{R}^d))} + \|z\|_{L^q(0,T;W^{2,p}(\mathbb{R}^d))} \le C\left(\|f\|_{L^q(0,T;L^p(\mathbb{R}^d))} + \|z_0\|_{W^{2,p}(\mathbb{R}^d)}\right), \tag{2.9}$$

$$\|\nabla z\|_{L^q(0,T;L^p(\mathbb{R}^d))} \le C\left(T^{\frac{1}{2}}\|f\|_{L^q(0,T;L^p(\mathbb{R}^d))} + T^{\frac{1}{q}}\|\nabla z_0\|_{L^p(\mathbb{R}^d)}\right). \tag{2.10}$$

Proof. The details of the proof of (2.9) can be found in [13,15]. We shall only prove (2.10). Let $z(x,t)$ be a solution of (2.8). Using the heat kernel Γ, $z(x,t)$ can be represented by

$$z(x,t) = \int_{\mathbb{R}^d} \Gamma(x-y,t)z_0(y)dy + \int_0^t \int_{\mathbb{R}^d} \Gamma(x-y,t-s)f(y,s)dyds,$$

$$\nabla z(x,t) = \int_{\mathbb{R}^d} \Gamma(x-y,t)\nabla z_0(y)dy + \int_0^t \int_{\mathbb{R}^d} \nabla\Gamma(x-y,t-s)f(y,s)dyds =: A_1(x,t) + A_2(x,t).$$

Now, we will use the well-known estimates for the heat kernel such that

$$\|\Gamma(\cdot,t)\|_{L^1(\mathbb{R}^d)} = 1 \quad \text{and} \quad \|\nabla\Gamma(\cdot,t)\|_{L^1(\mathbb{R}^d)} \le Ct^{-\frac{1}{2}} \quad \text{for } t > 0. \tag{2.11}$$

Using Young's convolution inequality, Hölder's inequality and (2.11), we have for $t \in (0,T)$

$$\begin{aligned} \|A_1\|_{L^q(0,t;L^p(\mathbb{R}^d))} &= \left\| \|(\Gamma * \nabla z_0)(\cdot,s)\|_{L^p(\mathbb{R}^d)} \right\|_{L^q(0,t)} \\ &\le \left\| \|\Gamma(\cdot,s)\|_{L^1(\mathbb{R}^d)} \|\nabla z_0\|_{L^p(\mathbb{R}^d)} \right\|_{L^q(0,t)} \\ &= \left(\int_0^t \|\Gamma(\cdot,s)\|^q_{L^1(\mathbb{R}^d)} \|\nabla z_0\|^q_{L^p(\mathbb{R}^d)} ds \right)^{\frac{1}{q}} \\ &\le \sup_{s\in(0,t]} \|\Gamma(\cdot,s)\|_{L^1(\mathbb{R}^d)} t^{\frac{1}{q}} \|\nabla z_0\|_{L^p(\mathbb{R}^d)} \\ &= t^{\frac{1}{q}} \|\nabla z_0\|_{L^p(\mathbb{R}^d)}. \end{aligned} \tag{2.12}$$

On the other hand, we have

$$\|\nabla\Gamma(\cdot,t-s) * f(\cdot,s)\|_{L^p(\mathbb{R}^d)} \le \|\nabla\Gamma(\cdot,t-s)\|_{L^1(\mathbb{R}^d)} \|f(\cdot,s)\|_{L^p(\mathbb{R}^d)} \le C(t-s)^{-\frac{1}{2}} \|f(\cdot,s)\|_{L^p(\mathbb{R}^d)},$$

which implies

$$\begin{aligned} \|A_2\|_{L^q(0,t;L^p(\mathbb{R}^d))} &\le C \left\| s^{-\frac{1}{2}} * \|f(\cdot,s)\|_{L^p(\mathbb{R}^d)} \right\|_{L^q(0,t)} \\ &\le C\|s^{-\frac{1}{2}}\|_{L^1(0,t)} \|f\|_{L^q(0,t;L^p(\mathbb{R}^d))} \\ &= Ct^{\frac{1}{2}} \|f\|_{L^q(0,t;L^p(\mathbb{R}^d))}. \end{aligned} \tag{2.13}$$

Combining (2.12) and (2.13), we complete the proof. □

Remark 2.1. Suppose that z is the solution of (2.8) with $z_0 = 0$. Due to the Sobolev embedding and (2.10), we obtain

$$\|\nabla z\|_{L^q(0,t;L^{p^*}(\mathbb{R}^d))} \le C\|f\|_{L^q(0,t;L^p(\mathbb{R}^d))}, \tag{2.14}$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$ with $1 \le p < d$.

Next, we state the Gagliardo-Nirenberg inequality on the whole space (see [2, Theorem 1.3.7]).

Lemma 2.4 *(The Gagliardo-Nirenberg inequality). Let $d \ge 1$, $1 \le q, r \le \infty$ and let j, m be two integers, $0 \le j < m$. If*

$$\frac{1}{p} - \frac{j}{d} = \left(\frac{1}{r} - \frac{m}{d}\right)\theta + \frac{1-\theta}{q}$$

for some $\theta \in [\frac{j}{m}, 1]$, then there exists $C(d, m, j, \theta, q, r) > 0$ such that

$$\sum_{|\zeta|=j} \|D^\zeta f\|_{L^p(\mathbb{R}^d)} \le C \left(\sum_{|\zeta|=m} \|D^\zeta f\|_{L^r(\mathbb{R}^d)}\right)^\theta \|f\|_{L^q(\mathbb{R}^d)}^{(1-\theta)} \quad \text{for every } f \in \mathcal{D}(\mathbb{R}^d). \tag{2.15}$$

The following lemma is the weak version of Young's convolution inequality (see [8, Proposition 8.9]).

Lemma 2.5. *Let $d \ge 1$, $1 < p, q, r < \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. If $f \in L^p(\mathbb{R}^d)$ and $g \in$ weak $L^q(\mathbb{R}^d)$, then $f * g \in L^r(\mathbb{R}^d)$ and there exists $C > 0$ independent of f and g such that*

$$\|f * g\|_{L^r(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}[g]_{L^q(\mathbb{R}^d)}, \tag{2.16}$$

where weak $L^q(\mathbb{R}^d)$ is the set of all g such that $[g]_{L^q(\mathbb{R}^d)} < \infty$ associated with

$$[g]_{L^q(\mathbb{R}^d)} = \left(\sup_{\alpha>0} \alpha^q \lambda_g(\alpha)\right)^{1/q}, \quad \lambda_g(\alpha) = m(\{x : |g(x)| > \alpha\}), \quad m : \text{Lebesgue measure.}$$

We first establish the local existence of regular solutions (n, c, u) to (1.1). Since its verification is rather standard, we skip its details (see e.g. [3]).

Theorem 2.1 *(Local existence). Let $m \ge 3$, $q \ge 1$, $d = 2, 3$ and $\kappa \in \mathbb{R}$. Assume that $\|\nabla^\ell \phi\|_{L^\infty(\mathbb{R}^d)} < \infty$ for $1 < |\ell| < m$. There exists $T_{\max} > 0$, the maximal time of existence, such that, if the initial data $(n_0, c_0, u_0) \in H^{m-1}(\mathbb{R}^d) \times H^m(\mathbb{R}^d) \times H^m(\mathbb{R}^d)$, then there exists a unique regular solution (n, c, u) of (1.1) satisfying for any $t < T_{\max}$*

$$(n, c, u) \in L^\infty(0, t; H^{m-1}(\mathbb{R}^d) \times H^m(\mathbb{R}^d) \times H^m(\mathbb{R}^d)),$$
$$(\nabla n, \nabla c, \nabla u) \in L^2(0, t; H^{m-1}(\mathbb{R}^d) \times H^m(\mathbb{R}^d) \times H^m(\mathbb{R}^d)).$$

Next, we establish a blow-up criterion. The proof will be given in the appendix.

Theorem 2.2 *(A Blow-up criterion). Let $q \ge 1$. Suppose that ϕ and the initial data (n_0, c_0, u_0) satisfy all the assumptions presented in Theorem 2.1. Then for either $(\kappa, d) = (1, 2)$ or $(\kappa, d) = (0, 3)$, if $T_{\max} > 0$ is the maximal time of existence, then*

$$\limsup_{t\nearrow T_{\max}}\int_0^t \|\nabla c\|_{L^{d+2}(\mathbb{R}^d)}^{d+2} = \infty. \tag{2.17}$$

In case of the Stokes system ($\kappa = 0$), we define a γ-very weak solution by introducing two notions of weak solutions. Firstly, we define the notion of a very weak sub-solution.

Definition 2.1. Let $T > 0$. A pair (n, c) of nonnegative functions and a vector field u, belonging to

$$n \in L^1(\mathbb{R}^d \times (0,T)),\ \ c \in L^1(0,T; W^{1,1}(\mathbb{R}^d)),\ \ u \in L^1(0,T; W^{1,1}(\mathbb{R}^d;\mathbb{R}^d)),$$

are called a very weak sub-solution of (1.1) in $\mathbb{R}^d \times (0,T)$ if the followings are satisfied:

(i) $n\nabla c$, n^q, nu and cu belong to $L^1(\mathbb{R}^d \times (0,T))$;

(ii) $$-\int_0^T\int_{\mathbb{R}^d} n\partial_t\varphi - \int_{\mathbb{R}^d} n_0\varphi(\cdot,0) \le \int_0^T\int_{\mathbb{R}^d} n\Delta\varphi + \int_0^T\int_{\mathbb{R}^d} n\nabla c\cdot\nabla\varphi - \int_0^T\int_{\mathbb{R}^d}\mu n^q\varphi + \int_0^T\int_{\mathbb{R}^d} nu\cdot\nabla\varphi,$$
for all nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d \times [0,T))$;

(iii) $$-\int_0^T\int_{\mathbb{R}^d} c\partial_t\varphi - \int_{\mathbb{R}^d} c_0\varphi(\cdot,0) = -\int_0^T\int_{\mathbb{R}^d}\nabla c\nabla\varphi - \int_0^T\int_{\mathbb{R}^d} c\varphi + \int_0^T\int_{\mathbb{R}^d} n\varphi + \int_0^T\int_{\mathbb{R}^d} cu\cdot\nabla\varphi,$$
for all nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d \times [0,T))$;

(iv) $$-\int_0^T\int_{\mathbb{R}^d} u\cdot\partial_t\varphi - \int_{\mathbb{R}^d} u_0\cdot\varphi(\cdot,0) = -\int_0^T\int_{\mathbb{R}^d}\nabla u\cdot\nabla\varphi - \int_0^T\int_{\mathbb{R}^d} n\nabla\phi\cdot\varphi,$$
for all nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d \times [0,T);\mathbb{R}^d)$ satisfying $\nabla\cdot\varphi \equiv 0$.

Next, we adopt the notion of a weak γ-entropy super-solution.

Definition 2.2. Let $T > 0$ and γ be a fixed number with $\gamma \in (0,1)$. A pair (n, c) of nonnegative functions and a vector field u, belonging to

$$n \in L^\gamma(\mathbb{R}^d \times (0,T)),\ \ c \in L^1(0,T; W^{1,2}(\mathbb{R}^d)),\ \ u \in L^1(0,T; L^\infty(\mathbb{R}^d;\mathbb{R}^d)),$$

are called a weak γ-entropy super-solution of (1.1) in $\mathbb{R}^d \times (0,T)$ if the followings are satisfied:

(i) $n^{\gamma-2}|\nabla n|^2$, $n^\gamma\nabla c$, $n^{q+\gamma-1}$, $n^\gamma u$ belong to $L^1(\mathbb{R}^d \times (0,T))$;

(ii) $$-\int_0^T\int_{\mathbb{R}^d} n^\gamma\partial_t\varphi - \int_{\mathbb{R}^d} n_0^\gamma\varphi(\cdot,0) \ge \gamma(1-\gamma)\int_0^T\int_{\mathbb{R}^d} n^{\gamma-2}|\nabla n|^2\varphi + \int_0^T\int_{\mathbb{R}^d} n^\gamma\Delta\varphi + \gamma\int_0^T\int_{\mathbb{R}^d} n^\gamma\nabla c\cdot\nabla\varphi$$

$$-\gamma\int_0^T\int_{\mathbb{R}^d}\mu n^{q+\gamma-1}\varphi + \gamma(\gamma-1)\int_0^T\int_{\mathbb{R}^d}\varphi n^{\gamma-1}\nabla n\cdot\nabla c + \int_0^T\int_{\mathbb{R}^d} n^\gamma u\cdot\nabla\varphi,$$

for all nonnegative $\varphi \in C_0^\infty(\mathbb{R}^d \times [0,T))$.

Finally, we define the notion of a γ-very weak solution of (1.1).

Definition 2.3. Let $T > 0$ and γ be a fixed number with $\gamma \in (0, 1)$. A pair (n, c) of nonnegative functions and a vector field u are called a γ-very weak solution of (1.1) in $\mathbb{R}^d \times (0, T)$ if it is both a very weak sub-solution and a weak γ-entropy super-solution of (1.1) in $\mathbb{R}^d \times (0, T)$.

3. Global regular solutions

We first note that the total mass of n is non-increasing.

Lemma 3.1. *Let $T > 0$, $q > 0$ and (n, c, u) be a regular solution of (1.1) in $\mathbb{R}^d \times (0, T)$, $d = 2, 3$. Then, there exists $C = C(\|n_0\|_{L^1(\mathbb{R}^d)}) > 0$ such that for all $t \in (0, T)$*

$$\sup_{t\in(0,T)} \int_{\mathbb{R}^d} n(\cdot, t) + \int_0^t \int_{\mathbb{R}^d} n^q \leq C. \tag{3.18}$$

Proof. From integration of the first equation of (1.1) over $\mathbb{R}^d$, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^d} n + \mu \int_{\mathbb{R}^d} n^q = 0. \tag{3.19}$$

Integrating (3.19) in time, we have

$$\int_{\mathbb{R}^d} n(\cdot, t) + \mu \int_0^t \int_{\mathbb{R}^d} n^q = \int_{\mathbb{R}^d} n_0.$$

This completes the proof. $\square$

Remark 3.1. We note, due to (3.18), that for $q > 2$

$$\int_0^t \int_{\mathbb{R}^d} n^2 \leq \int_0^t \|n\|^{2\theta_0}_{L^1(\mathbb{R}^d)} \|n\|^{2(1-\theta_0)}_{L^q(\mathbb{R}^d)} \leq C, \tag{3.20}$$

where $\theta_0 = \frac{q-2}{2(q-1)} \in (0, 1)$.

In the next two lemmas, using Lemma 3.1, we obtain some energy estimates of c and u.

Lemma 3.2. *Let $T > 0$, $q > 1$ and (n, c, u) be a regular solution of (1.1) in $\mathbb{R}^d \times (0, T)$, $d = 2, 3$. Then, there exists $C = C(\|n_0\|_{L^1(\mathbb{R}^d)}, \|c_0\|_{L^q(\mathbb{R}^d)}, q) > 0$ such that for all $t \in (0, T)$*

$$\sup_{t\in(0,T)} \int_{\mathbb{R}^d} c^q + \int_0^t \int_{\mathbb{R}^d} |\nabla c^{\frac{q}{2}}|^2 \leq C. \tag{3.21}$$

Proof. Multiplying the second equation of (1.1) by c^{q-1} and integrating over $\mathbb{R}^d$, we have

$$\frac{1}{q}\frac{d}{dt} \int_{\mathbb{R}^d} c^q + \frac{4(q-1)}{q^2} \int_{\mathbb{R}^d} |\nabla c^{\frac{q}{2}}|^2 + \int_{\mathbb{R}^d} c^q \leq \int_{\mathbb{R}^d} nc^{q-1} \leq \frac{1}{2} \int_{\mathbb{R}^d} c^q + C \int_{\mathbb{R}^d} n^q. \tag{3.22}$$

Integrating (3.22) in time, we have

$$\int_{\mathbb{R}^d} c^q(\cdot,t) + \frac{4(q-1)}{q}\int_0^t\int_{\mathbb{R}^d} |\nabla c^{\frac{q}{2}}|^2 + \frac{q}{2}\int_0^t\int_{\mathbb{R}^d} c^q \le C\int_0^t\int_{\mathbb{R}^d} n^q + \int_{\mathbb{R}^d} c_0^q.$$

Therefore, (3.18) ensures that this lemma holds. □

Lemma 3.3. *Let $T>0$, $q>2$ and (n,c,u) be a regular solution of (1.1) in $\mathbb{R}^d\times(0,T)$, $d=2,3$. Suppose that either $(\kappa,d)=(1,2)$ or $(\kappa,d)=(0,3)$. Then, there exists $C>0$ such that for all $t\in(0,T)$*

$$\sup_{(x,t)\in\mathbb{R}^2\times(0,T)} |u(x,t)| \le C \quad \textit{and} \quad \sup_{t\in(0,T)}\int_{\mathbb{R}^3} |u(\cdot,t)|^6 \le C, \tag{3.23}$$

where C depends on T and the initial values of n,u satisfying Assumption 1.1.

Proof. Multiplying u to the third equation of (1.1) and integrating over $\mathbb{R}^d$, we compute

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |u|^2 + \int_{\mathbb{R}^d} |\nabla u|^2 \le C\int_{\mathbb{R}^d} n|u| \le C\|n\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)}. \tag{3.24}$$

Letting $y(t) := \|u(\cdot,t)\|_{L^2(\mathbb{R}^d)}$, (3.24) turns into

$$y'(t) \le C\|n\|_{L^2(\mathbb{R}^d)}.$$

Thus, owing to (3.18), integration in time yields

$$\|u\|_{L^\infty(0,t;L^2(\mathbb{R}^d))} + \|\nabla u\|_{L^2(\mathbb{R}^d\times(0,t))} \le C \quad \text{for } t\in(0,T), \tag{3.25}$$

where C depends on T. Next, we shall use the vorticity equation. For notational convention, we denote by $w := \nabla\times u$ the vorticity of the velocity field. Taking curl to the third equation of (1.1), we find the vorticity equation such that

$$\partial_t w + \kappa u\cdot\nabla w = \Delta w - \nabla\times(n\nabla\phi). \tag{3.26}$$

In case of $(\kappa,d)=(1,2)$, we multiply (3.26) by $|w|^{q-2}w$ for $q>2$ and integrate over $\mathbb{R}^2$ to obtain

$$\begin{aligned}
\frac{1}{q}\frac{d}{dt}\int_{\mathbb{R}^2} |w|^q + \frac{4(q-1)}{q^2}\int_{\mathbb{R}^2} \left|\nabla|w|^{\frac{q}{2}}\right|^2 &\le C\int_{\mathbb{R}^2} n\left|\nabla|w|^{q-1}\right| \\
&\le C\int_{\mathbb{R}^2} n\left|\nabla|w|^{\frac{q}{2}}\right||w|^{\frac{q-2}{2}} \\
&\le C\|n\|_{L^q(\mathbb{R}^2)}\left\|\nabla|w|^{\frac{q}{2}}\right\|_{L^2(\mathbb{R}^2)}\left\||w|^{\frac{q-2}{2}}\right\|_{L^{\frac{2q}{q-2}}(\mathbb{R}^2)} \\
&= C\|n\|_{L^q(\mathbb{R}^2)}\left\|\nabla|w|^{\frac{q}{2}}\right\|_{L^2(\mathbb{R}^2)}\|w\|_{L^q(\mathbb{R}^2)}^{\frac{q-2}{2}} \\
&\le C\int_{\mathbb{R}^2} n^q + \frac{2(q-1)}{q^2}\int_{\mathbb{R}^2}\left|\nabla|w|^{\frac{q}{2}}\right|^2 + C\int_{\mathbb{R}^2}|w|^q.
\end{aligned}$$

Therefore, we have

$$\sup_{t\in(0,T)}\int_{\mathbb{R}^2}|w(\cdot,t)|^q \le C \quad \text{for any } \; q>2. \tag{3.27}$$

Then, we have for $t\in(0,T)$ and for any $q>2$

$$\|u\|_{L^\infty(\mathbb{R}^2\times(0,t))} \le C\|u\|^{\frac{q-2}{2(q-1)}}_{L^\infty(0,t;L^2(\mathbb{R}^2))}\|\nabla u\|^{\frac{q}{2(q-1)}}_{L^\infty(0,t;L^q(\mathbb{R}^2))} \le C\|w\|^{\frac{q}{2(q-1)}}_{L^\infty(0,t;L^q(\mathbb{R}^2))} \le C,$$

where we used the Gagliardo-Nirenberg inequality (2.15), (3.25), (3.27) and

$$\|\nabla u\|_{L^\infty(0,t;L^q(\mathbb{R}^d))} \le C\|w\|_{L^\infty(0,t;L^q(\mathbb{R}^d))} \quad \text{for } d=2,3. \tag{3.28}$$

In case of $(\kappa,d)=(0,3)$, multiplying (3.26) by w and integrating over $\mathbb{R}^3$, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|w|^2+\int_{\mathbb{R}^3}|\nabla w|^2 &= -\int_{\mathbb{R}^3} w\nabla\times(n\nabla\phi)\\
&\le \int_{\mathbb{R}^3}|\nabla w\times(n\nabla\phi)|\\
&\le C\|n\|_{L^2(\mathbb{R}^3)}\|\nabla w\|_{L^2(\mathbb{R}^3)}\\
&\le \frac{1}{2}\|\nabla w\|^2_{L^2(\mathbb{R}^3)}+C\|n\|^2_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Using (3.20), we have

$$\sup_{t\in(0,T)}\int_{\mathbb{R}^3}|w(\cdot,t)|^2 \le C, \tag{3.29}$$

which, together with (3.25), (3.28) and (3.29), helps to infer that

$$\|u\|_{L^\infty(0,t;L^6(\mathbb{R}^3))} \le C\|\nabla u\|_{L^\infty(0,t;L^2(\mathbb{R}^3))} \le C\|w\|_{L^\infty(0,t;L^2(\mathbb{R}^3))} \le C.$$

This completes the proof. □

With the aid of two above lemmas, we control Δc, which plays a significant role for the case that $q>2$.

Lemma 3.4. *Let $T>0$, $q>2$ and (n,c,u) be a regular solution of (1.1) in $\mathbb{R}^d\times(0,T)$, $d=2,3$. Suppose that $p>q$ if $d=2$ or $q<p<6$ if $d=3$. Then, there exists $C>0$ such that for all $t\in(0,T)$*

$$\int_{\mathbb{R}^d} n(\cdot,t)^{p-1}+\int_{\mathbb{R}^d} c(\cdot,t)^p \le C, \tag{3.30}$$

$$\int_0^t\int_{\mathbb{R}^d}|\Delta c|^p \le C, \tag{3.31}$$

where C depends on T and the initial values of n,c,u satisfying Assumption 1.1.

Proof. From now on, we denote $Q_t := \mathbb{R}^d \times (0,t)$, $d = 2, 3$. First, testing the first equation of (1.1) by n^{p-2} for $p > 2$, due to Young's inequality, the integration over $\mathbb{R}^d \times (0,t)$ implies

$$\begin{aligned}
&\int_{\mathbb{R}^d} n(\cdot,t)^{p-1} + \frac{4(p-2)}{p-1}\int_0^t\int_{\mathbb{R}^d} |\nabla n^{\frac{p-1}{2}}|^2 + \mu(p-1)\int_0^t\int_{\mathbb{R}^d} n^{p+q-2} \\
&= -(p-2)\int_0^t\int_{\mathbb{R}^d} n^{p-1}\Delta c + \int_{\mathbb{R}^d} n_0^{p-1} \\
&\le (p-1)\int_0^t\int_{\mathbb{R}^d} n^p + \int_0^t\int_{\mathbb{R}^d} |\Delta c|^p + \int_{\mathbb{R}^d} n_0^{p-1}.
\end{aligned}$$

Invoking the maximal estimate (2.9), there exists $C_p > 0$ such that

$$\|\Delta c\|^p_{L^p(Q_t)} \le C_p\left(\|n\|^p_{L^p(Q_t)} + \|c\|^p_{L^p(Q_t)} + \|u\cdot\nabla c\|^p_{L^p(Q_t)} + \|c_0\|^p_{W^{2,p}(\mathbb{R}^d)}\right). \tag{3.32}$$

Now, we use Hölder's inequality and the Gagliardo-Nirenberg inequality (2.15) to obtain for $q > 2$

$$\|u\cdot\nabla c\|_{L^p(\mathbb{R}^2)} \le \|u\|_{L^\infty(\mathbb{R}^2)}\|\nabla c\|_{L^p(\mathbb{R}^2)} \le C\|u\|_{L^\infty(\mathbb{R}^2)}\|c\|^{(1-\theta_1)}_{L^q(\mathbb{R}^2)}\|\Delta c\|^{\theta_1}_{L^p(\mathbb{R}^2)}, \tag{3.33}$$

$$\|u\cdot\nabla c\|_{L^p(\mathbb{R}^3)} \le \|u\|_{L^6(\mathbb{R}^3)}\|\nabla c\|_{L^{\frac{6p}{6-p}}(\mathbb{R}^3)} \le C\|u\|_{L^6(\mathbb{R}^3)}\|c\|^{(1-\theta_2)}_{L^q(\mathbb{R}^3)}\|\Delta c\|^{\theta_2}_{L^p(\mathbb{R}^3)}, \tag{3.34}$$

where $\theta_1 = \frac{1/2+1/q-1/p}{1+1/q-1/p}$, $\theta_2 = \frac{(6-p)/6p-1/q-1/3}{1/p-1/q-2/3}$ and $\theta_1, \theta_2 \in (0,1)$. In (3.33) and (3.34), we used the well-known estimate

$$\|D^2 f\|_{L^q(\mathbb{R}^d)} \le C\|\Delta f\|_{L^q(\mathbb{R}^d)} \quad \text{for } 1 < q < \infty. \tag{3.35}$$

Here, we note that for $d = 3$, p satisfies $2 < p < 6$. Using (3.21) and (3.23), if $d = 2$, (3.33) implies

$$\begin{aligned}
\|u\cdot\nabla c\|^p_{L^p(Q_t)} &\le C\int_0^t \|u\|^p_{L^\infty(\mathbb{R}^2)}\|c\|^{(1-\theta_1)p}_{L^q(\mathbb{R}^2)}\|\Delta c\|^{\theta_1 p}_{L^p(\mathbb{R}^2)} \\
&\le C\int_0^t \|\Delta c\|^{\theta_1 p}_{L^p(\mathbb{R}^2)} \\
&\le \frac{1}{2C_p}\|\Delta c\|^p_{L^p(Q_t)} + CT.
\end{aligned} \tag{3.36}$$

Similarly, if $d = 3$, from (3.34) we have

$$\begin{aligned}
\|u\cdot\nabla c\|^p_{L^p(Q_t)} &\le C\int_0^t \|u\|^p_{L^6(\mathbb{R}^3)}\|c\|^{(1-\theta_2)p}_{L^q(\mathbb{R}^3)}\|\Delta c\|^{\theta_2 p}_{L^p(\mathbb{R}^3)} \\
&\le C\int_0^t \|\Delta c\|^{\theta_2 p}_{L^p(\mathbb{R}^3)} \\
&\le \frac{1}{2C_p}\|\Delta c\|^p_{L^p(Q_t)} + CT.
\end{aligned} \tag{3.37}$$

Then from (3.32) with (3.36) and (3.37), we have

$$\|\Delta c\|_{L^p(Q_t)}^p \le 2C_p \left(\|n\|_{L^p(Q_t)}^p + \|c\|_{L^p(Q_t)}^p + CT + \|c_0\|_{W^{2,p}(\mathbb{R}^d)}^p \right), \tag{3.38}$$

which helps to derive that

$$\begin{aligned}
&\int_{\mathbb{R}^d} n(\cdot,t)^{p-1} + \frac{4(p-2)}{p-1} \int_0^t \int_{\mathbb{R}^d} |\nabla n^{\frac{p-1}{2}}|^2 + \mu(p-1) \int_0^t \int_{\mathbb{R}^d} n^{p+q-2} \\
&\le (p-1) \int_0^t \int_{\mathbb{R}^d} n^p + \int_0^t \int_{\mathbb{R}^d} |\Delta c|^p + \int_{\mathbb{R}^d} n_0^{p-1} \\
&\le (p-1+2C_p) \int_0^t \int_{\mathbb{R}^d} n^p + 2C_p \int_0^t \int_{\mathbb{R}^d} c^p + C\left(T + \|c_0\|_{W^{2,p}(\mathbb{R}^d)}^p + \|n_0\|_{L^{p-1}(\mathbb{R}^d)}^{p-1} \right).
\end{aligned}$$

We employ Young's inequality and the interpolation inequality such that for $p > q$

$$(p-1+2C_p) \int_0^t \int_{\mathbb{R}^d} n^p \le C \int_0^t \int_{\mathbb{R}^d} n^q + \frac{\mu(p-1)}{4} \int_0^t \int_{\mathbb{R}^d} n^{p+q-2}. \tag{3.39}$$

Then, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^d} n(\cdot,t)^{p-1} + \frac{4(p-2)}{p-1} \int_0^t \int_{\mathbb{R}^d} |\nabla n^{\frac{p-1}{2}}|^2 + \frac{3\mu(p-1)}{4} \int_0^t \int_{\mathbb{R}^d} n^{p+q-2} \\
&\le C \int_0^t \int_{\mathbb{R}^d} n^q + 2C_p \int_0^t \int_{\mathbb{R}^d} c^p + C\left(T + \|c_0\|_{W^{2,p}(\mathbb{R}^d)}^p + \|n_0\|_{L^{p-1}(\mathbb{R}^d)}^{p-1} \right).
\end{aligned} \tag{3.40}$$

On the other hand, testing the second equation of (1.1) by c^{p-1} and applying Young's inequality, the integration over $\mathbb{R}^d \times (0,t)$ implies that

$$\begin{aligned}
&\frac{4C_p}{p} \int_{\mathbb{R}^d} c(\cdot,t)^p + \frac{16(p-1)C_p}{p^2} \int_0^t \int_{\mathbb{R}^d} |\nabla c^{\frac{p}{2}}|^2 + 4C_p \int_0^t \int_{\mathbb{R}^d} c^p \\
&= 4C_p \int_0^t \int_{\mathbb{R}^d} nc^{p-1} + \frac{4C_p}{p} \int_{\mathbb{R}^d} c_0^p \\
&\le C \int_0^t \int_{\mathbb{R}^d} n^p + C_p \int_0^t \int_{\mathbb{R}^d} c^p + C \int_{\mathbb{R}^d} c_0^p \\
&\le \frac{\mu(p-1)}{4} \int_0^t \int_{\mathbb{R}^d} n^{p+q-2} + C \int_0^t \int_{\mathbb{R}^d} n^q + C_p \int_0^t \int_{\mathbb{R}^d} c^p + C \int_{\mathbb{R}^d} c_0^p.
\end{aligned} \tag{3.41}$$

Combining (3.40) and (3.41), it is direct that

$$
\begin{aligned}
&\int_{\mathbb{R}^d} n(\cdot,t)^{p-1} + \frac{4C_p}{p}\int_{\mathbb{R}^d} c(\cdot,t)^p + \frac{4(p-2)}{p-1}\int_0^t\int_{\mathbb{R}^d} |\nabla n^{\frac{p-1}{2}}|^2 \\
&+ \frac{16(p-1)C_p}{p^2}\int_0^t\int_{\mathbb{R}^d} |\nabla c^{\frac{p}{2}}|^2 + \frac{\mu(p-1)}{2}\int_0^t\int_{\mathbb{R}^d} n^{p+q-2} + C_p\int_0^t\int_{\mathbb{R}^d} c^p \\
&\le C\int_0^t\int_{\mathbb{R}^d} n^q + C\left(T + \|c_0\|^p_{W^{2,p}(\mathbb{R}^d)} + \|n_0\|^{p-1}_{L^{p-1}(\mathbb{R}^d)}\right),
\end{aligned} \tag{3.42}
$$

which guarantees (3.30). The boundedness of $\|\Delta c\|_{L^p(Q_t)}$ is a direct consequence of combining (3.18), (3.38), (3.39) and (3.42). This completes the proof. $\square$

In case that $q = 2$ in two spatial dimensions, we can obtain L^2-estimate of Δc.

Lemma 3.5. *Let $T > 0$, $q = 2$, $\kappa = 1$ and (n, c, u) be a regular solution of (1.1) in $\mathbb{R}^2 \times (0, T)$. Then, there exists $C > 0$ such that for all $t \in (0, T)$*

$$
\sup_{t\in(0,T)}\int_{\mathbb{R}^2} |\nabla c(\cdot,t)|^2 + \int_0^t\int_{\mathbb{R}^2} |\Delta c|^2 \le C, \tag{3.43}
$$

where C depends on T and the initial values of n, c, u satisfying Assumption 1.1.

Proof. Multiplying $-\Delta c$ to the second equation of (1.1) and integrating over $\mathbb{R}^2$, the Gagliardo-Nirenberg inequality (2.15) and Hölder's inequality imply

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\nabla c|^2 + \int_{\mathbb{R}^2} |\Delta c|^2 + \int_{\mathbb{R}^2} |\nabla c|^2 &\le \int_{\mathbb{R}^2} |\nabla u||\nabla c|^2 + \int_{\mathbb{R}^2} n|\Delta c| \\
&\le \|\nabla u\|_{L^2(\mathbb{R}^2)}\|\nabla c\|^2_{L^4(\mathbb{R}^2)} + \frac{1}{4}\int_{\mathbb{R}^2} |\Delta c|^2 + C\int_{\mathbb{R}^2} n^2 \\
&\le C\|\nabla u\|_{L^2(\mathbb{R}^2)}\|\nabla c\|_{L^2(\mathbb{R}^2)}\|\Delta c\|_{L^2(\mathbb{R}^2)} + \frac{1}{4}\int_{\mathbb{R}^2} |\Delta c|^2 + C\int_{\mathbb{R}^2} n^2 \\
&\le C\|\nabla u\|^2_{L^2(\mathbb{R}^2)}\|\nabla c\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2}\int_{\mathbb{R}^2} |\Delta c|^2 + C\int_{\mathbb{R}^2} n^2.
\end{aligned} \tag{3.44}
$$

By (3.18) and (3.25), it follows from (3.44) that

$$
\sup_{t\in(0,T)}\int_{\mathbb{R}^2} |\nabla c(\cdot,t)|^2 + \int_0^t\int_{\mathbb{R}^2} |\Delta c|^2 \le C
$$

as desired. $\square$

We are now ready to provide the proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose that $T_{\max} < \infty$, where $T_{\max}$ is the maximal time of existence given in Theorem 2.1. For $q > 2$ and $d = 2, 3$, we claim that

$$\limsup_{t \nearrow T_{\max}} \int_0^t \|\nabla c\|^{d+2}_{L^{d+2}(\mathbb{R}^d)} \leq C. \tag{3.45}$$

This is contrary to (2.17), which automatically implies $T_{\max} = \infty$. Therefore, it suffices to prove (3.45). In order to achieve it, we let $T < T_{\max}$ and use (3.30), (3.31). Then, the Gagliardo-Nirenberg inequality (2.15), Hölder's inequality and (3.35) imply that for all $t \in (0, T)$

$$\int_0^t \|\nabla c\|^{d+2}_{L^{d+2}(\mathbb{R}^d)} \leq C \int_0^t \|\Delta c\|^{\frac{d+2}{2}}_{L^{d+2}(\mathbb{R}^d)} \|c\|^{\frac{d+2}{2}}_{L^{d+2}(\mathbb{R}^d)} \leq C t^{\frac{1}{2}} \left(\int_0^t \|\Delta c\|^{d+2}_{L^{d+2}(\mathbb{R}^d)} \right)^{\frac{1}{2}} \leq C.$$

On the other hand, for $q = 2$ and $d = 2$, we show that

$$\limsup_{t \nearrow T_{\max}} \int_0^t \|\nabla c\|^4_{L^4(\mathbb{R}^2)} \leq C.$$

Applying the Gagliardo-Nirenberg inequality (2.15) to (3.43), we obtain

$$\int_0^t \|\nabla c\|^4_{L^4(\mathbb{R}^2)} \leq C \int_0^t \|\Delta c\|^2_{L^2(\mathbb{R}^2)} \|\nabla c\|^2_{L^2(\mathbb{R}^2)} \leq C \sup_{t \in (0,T)} \int_{\mathbb{R}^2} |\nabla c(\cdot, t)|^2 \int_0^t \int_{\mathbb{R}^2} |\Delta c|^2 \leq C.$$

Since $T < T_{\max}$ is arbitrary, we complete the proof. $\square$

4. Global γ-very weak solutions for $q > 2 - \frac{1}{d}$

In this section, we consider a modified system of (1.1) in terms of perturbation parameter $\varepsilon \in (0, 1)$ at which there exists a regular solution. By an adaptation of maximal estimates concerning parabolic equations, we will derive some ε-independent estimates for this regular solution.

Let $T > 0$. For $\varepsilon \in (0, 1)$, let us consider an approximated system of (1.1)

$$\begin{cases} \partial_t n_\varepsilon + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta n_\varepsilon - \nabla \cdot (n_\varepsilon \nabla c_\varepsilon) - \mu {n_\varepsilon}^q - \varepsilon n_\varepsilon^\beta, & (x,t) \in \mathbb{R}^d \times (0,T), \\ \partial_t c_\varepsilon + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - c_\varepsilon + n_\varepsilon, & (x,t) \in \mathbb{R}^d \times (0,T), \\ \partial_t u_\varepsilon + \nabla P_\varepsilon = \Delta u_\varepsilon - n_\varepsilon \nabla \phi, & (x,t) \in \mathbb{R}^d \times (0,T), \\ \nabla \cdot u_\varepsilon = 0, & (x,t) \in \mathbb{R}^d \times (0,T), \\ n_\varepsilon(x,0) = n_0(x),\ c_\varepsilon(x,0) = c_0(x),\ u_\varepsilon(x,0) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{4.46}$$

where $q > 2 - \frac{1}{d}$, $\beta > 2$, $d = 2, 3$. Here, we suppose that initial data (n_0, c_0, u_0) satisfy Assumption 1.2. By Theorem 1.1, (4.46) has a unique global regular solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ in $\mathbb{R}^d \times (0, T)$. If $q \geq 2$ for $d = 2$ or $q > 2$ for $d = 3$, then the system does not need to be regularized since it possesses a regular solution. Therefore, throughout this section, we assume that q satisfies

$$\begin{cases} \frac{3}{2} < q < 2 & \text{if } d = 2, \\ \frac{5}{3} < q \leq 2 & \text{if } d = 3. \end{cases} \tag{4.47}$$

In order to proceed to derive ε-independent estimates, we first note that for all $t \in (0, T)$, n_ε satisfies

$$\int_{\mathbb{R}^d} n_\varepsilon(\cdot,t)+\mu\int_0^t\int_{\mathbb{R}^d} n_\varepsilon^q+\varepsilon\int_0^t\int_{\mathbb{R}^d} n_\varepsilon^\beta=\int_{\mathbb{R}^d} n_0(x), \tag{4.48}$$

which implies that

$$\|n_\varepsilon\|_{L^\infty(0,t;L^1(\mathbb{R}^d))}\le C \quad \text{and} \quad \|n_\varepsilon\|_{L^q(\mathbb{R}^d\times(0,t))}\le C, \tag{4.49}$$

where C depend on $\|n_0\|_{L^1(\mathbb{R}^d)}$.

Lemma 4.1. *Let $d=2,3$, $T>0$ and q satisfy (4.47). Suppose that $(n_\varepsilon,c_\varepsilon,u_\varepsilon)$ be a regular solution of (4.46) in $\mathbb{R}^d\times(0,T)$. Then, there exists $C=C(\|n_0\|_{L^1(\mathbb{R}^d)},\|u_0\|_{W^{2,r'}(\mathbb{R}^d)},T)>0$ independent of ε such that for all $t\in(0,T)$*

$$\int_0^t\int_{\mathbb{R}^d}|\nabla u_\varepsilon|^r\le C, \tag{4.50}$$

where $\frac{d}{d-1}<r<\frac{dq}{d-1}$ and $r'=\frac{dr}{d+r}$.

Proof. Let $1<r'<\frac{dq}{d+q-1}$ for $d=2,3$ and $r>0$ be the Sobolev exponent of r', namely $r=\frac{dr'}{d-r'}\in(\frac{d}{d-1},\frac{dq}{d-1})$. We invoke (4.49) and Hölder's inequality to obtain for $t\in(0,T)$

$$\int_0^t\|n_\varepsilon\|_{L^{r'}(\mathbb{R}^d)}^r\le\int_0^t\|n_\varepsilon\|_{L^1(\mathbb{R}^d)}^{r(1-\theta)}\|n_\varepsilon\|_{L^q(\mathbb{R}^d)}^{r\theta}\le C\left(\int_0^t\|n_\varepsilon\|_{L^q(\mathbb{R}^d)}^q\right)^{\frac{r\theta}{q}}t^{1-\frac{r\theta}{q}}\le Ct^{1-\frac{r\theta}{q}}, \tag{4.51}$$

where $\theta=\frac{qr'-q}{qr'-r'}$ and we used that $r\theta<q$. From the equation of u_ε in (4.46), we observe via the maximal estimate (2.7) of the Stokes system that $\Delta u_\varepsilon\in L^r(0,t;L^{r'}(\mathbb{R}^d))$ and

$$\|\Delta u_\varepsilon\|_{L^r(0,T;L^{r'}(\mathbb{R}^d))}\le C\|n_\varepsilon\|_{L^r(0,T;L^{r'}(\mathbb{R}^d))}+\|u_0\|_{W^{2,r'}(\mathbb{R}^d)}.$$

By the Sobolev embedding and (4.51), we end up with $\nabla u_\varepsilon\in L^r(\mathbb{R}^d\times(0,t))$ and

$$\int_0^t\int_{\mathbb{R}^d}|\nabla u_\varepsilon|^r\le C\int_0^t(\int_{\mathbb{R}^d}|\Delta u_\varepsilon|^{r'})^{\frac{r}{r'}}\le C.$$

This completes the proof. □

Lemma 4.2. *Let $d=2,3$, $T>0$, and q satisfy (4.47). Suppose that $(n_\varepsilon,c_\varepsilon,u_\varepsilon)$ be a regular solution of (4.46) in $\mathbb{R}^d\times(0,T)$. Then, there exists $C>0$ independent of ε such that for all $t\in(0,T)$*

$$\int_0^t\int_{\mathbb{R}^d}|\nabla c_\varepsilon|^r\le C, \tag{4.52}$$

where

$$\begin{cases} C=C(\|n_0\|_{L^1(\mathbb{R}^2)},\|c_0\|_{(L^{\frac{r}{2}}\cap W^{1,r}\cap W^{1,r'})(\mathbb{R}^2)},\|u_0\|_{L^\infty(\mathbb{R}^2)},T)>0 & \text{for } 2<r<2q \quad \text{if } d=2,\\ C=C(\|n_0\|_{L^1(\mathbb{R}^3)},\|c_0\|_{(L^{\frac{3}{5}r}\cap W^{1,r})(\mathbb{R}^3)},\|u_0\|_{L^\infty(\mathbb{R}^3)},T)>0 & \text{for } \frac{5}{3}\le r<\frac{3q}{2} \quad \text{if } d=3.\end{cases}$$

Proof. We recall the Green tensor, $S = (S_{ij})_{i,j=1}^d$, of the Stokes system in $\mathbb{R}^d$, which is given by

$$S_{ij}(x,t) = \Gamma(x,t)\delta_{ij} + \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\int_{\mathbb{R}^d} \Gamma(y,t)E(x-y)dy \ \text{ for } \ 1 \le i,j \le d, \ \ t \in (0,T),$$

where Γ is the heat kernel and E is the fundamental solution of the Laplace equation in $\mathbb{R}^d$. It is known that for any integers l, $k \ge 0$

$$|D_x^l \partial_t^k S(x,t)| \le \frac{C}{(|x| + \sqrt{t})^{d+l+2k}}. \tag{4.53}$$

Therefore, $u_\varepsilon(x,t)$ can be written as

$$\begin{aligned} u_\varepsilon(x,t) &= \int_{\mathbb{R}^d} S(x-y,t)u_0(y)dy - \int_0^t \int_{\mathbb{R}^d} S(x-y,t-s)(n_\varepsilon \nabla\phi)(y,s)dyds \\ &=: v_0(x,t) + v_\varepsilon(x,t). \end{aligned} \tag{4.54}$$

Recalling the semigroup property of Stokes system (see e.g. [1]), the uniform boundedness for $v_0(x,t)$ is obtained as

$$\|v_0\|_{L^\infty(\mathbb{R}^d\times(0,t))} = \|S * u_0\|_{L^\infty(\mathbb{R}^d\times(0,t))} \le C\|u_0\|_{L^\infty(\mathbb{R}^d)}. \tag{4.55}$$

On the other hand, testing the second equation of (4.46) by $c_\varepsilon^{\ell-1}$ for $1 < \ell \le q$, due to (4.49), we see that

$$\int_{\mathbb{R}^d} c_\varepsilon(\cdot,t)^\ell + \frac{4(\ell-1)}{\ell}\int_0^t\int_{\mathbb{R}^d} |\nabla {c_\varepsilon}^{\frac{\ell}{2}}|^2 + \frac{\ell}{2}\int_0^t\int_{\mathbb{R}^d} {c_\varepsilon}^\ell \le C\int_0^t\int_{\mathbb{R}^d} {n_\varepsilon}^q + \int_{\mathbb{R}^d} c_0^\ell \le C.$$

Then, we obtain

$$c_\varepsilon \in L^\infty(0,t;L^\ell(\mathbb{R}^d)) \ \text{ and } \ \nabla c_\varepsilon^{\frac{\ell}{2}} \in L^2(\mathbb{R}^d \times (0,t)) \quad \text{for any } \ 1 < \ell \le q. \tag{4.56}$$

For further estimates on c_ε, we rewrite the second equation of (4.46), due to divergence free condition of u_ε, as follows:

$$\partial_t c_\varepsilon - \Delta c_\varepsilon = -\nabla\cdot(u_\varepsilon c_\varepsilon) - c_\varepsilon + n_\varepsilon.$$

For given c_0 and v_0, we solve the following equation

$$\begin{cases} \partial_t w_\varepsilon - \Delta w_\varepsilon = -\nabla\cdot(v_0 c_\varepsilon) + n_\varepsilon - c_\varepsilon, & (x,t) \in \mathbb{R}^d \times [0,t), \\ w_\varepsilon(x,0) = c_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{4.57}$$

We define $\tilde{w_\varepsilon} := c_\varepsilon - w_\varepsilon$, and we then see that $\tilde{w_\varepsilon}$ solves

$$\begin{cases} \partial_t \tilde{w_\varepsilon} - \Delta \tilde{w_\varepsilon} = -\nabla\cdot(v_\varepsilon c_\varepsilon), & (x,t) \in \mathbb{R}^d \times [0,t), \\ \tilde{w_\varepsilon}(x,0) = 0, & x \in \mathbb{R}^d, \end{cases} \tag{4.58}$$

where we used that $u_\varepsilon = v_0 + v_\varepsilon$.

- (The case $d = 2$) From Hölder's inequality and (4.53), we have for $t \in (0,T)$

$$
\begin{aligned}
\|\nabla v_\varepsilon(\cdot,t)\|_{L^{\frac{4}{3}}(\mathbb{R}^2)} &\le \int_0^t \|\nabla S(\cdot,t-s)*(n_\varepsilon\nabla\phi)(\cdot,s)\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}ds \\
&\le \int_0^t \frac{C}{(t-s)^{\frac{1}{2}}}\|n_\varepsilon(\cdot,s)\|_{L^{\frac{4}{3}}(\mathbb{R}^2)}ds \\
&\le \int_0^t \frac{C}{(t-s)^{\frac{1}{2}}}\|n_\varepsilon(\cdot,s)\|_{L^1(\mathbb{R}^2)}^{\frac{3q-4}{4(q-1)}}\|n_\varepsilon(\cdot,s)\|_{L^q(\mathbb{R}^2)}^{\frac{q}{4(q-1)}}ds \\
&\le C\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}}\|n_\varepsilon(\cdot,s)\|_{L^q(\mathbb{R}^2)}^{\frac{q}{4(q-1)}}ds \\
&\le Ct^\sigma\|n_\varepsilon\|_{L^q(\mathbb{R}^2\times(0,t))}^{\frac{q}{4(q-1)}} \le Ct^\sigma,
\end{aligned}
\tag{4.59}
$$

where $\sigma = \frac{2q-3}{4(q-1)}$ and $q > \frac{3}{2}$. Therefore, by the Sobolev embedding and (4.59), we have

$$
\|v_\varepsilon\|_{L^\infty(0,t;L^4(\mathbb{R}^2))} \le C\|\nabla v_\varepsilon\|_{L^\infty(0,t;L^{\frac{4}{3}}(\mathbb{R}^2))} \le Ct^\sigma, \tag{4.60}
$$

where C is independent of t. Now, we shall show that

$$
\|\nabla w_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} \le C\left(1+t^{\frac{1}{r}}+t^{\frac{1}{r}-\frac{\theta}{q}}+t^{\frac{r}{2(r-1)}}\right), \tag{4.61}
$$

where $C = C(\|n_0\|_{L^1(\mathbb{R}^2)}, \|c_0\|_{W^{1,r}(\mathbb{R}^2)}, \|u_0\|_{L^\infty(\mathbb{R}^2)}) > 0$, $r \in (2, 2q)$, $r' = \frac{2r}{2+r}$ and $\theta = \frac{qr'-q}{qr'-r'}$. Since (4.57) is linear, we can obtain the estimate of w_ε, by using n_ε, c_ε and $\nabla\cdot(v_0c_\varepsilon)$, one by one. We recall (4.51) such that

$$
\|n_\varepsilon\|_{L^r(0,t;L^{r'}(\mathbb{R}^2))} \le Ct^{\frac{1}{r}-\frac{\theta}{q}}, \tag{4.62}
$$

where $r \in (2, 2q)$, $r' = \frac{2r}{2+r}$ and $\theta = \frac{qr'-q}{qr'-r'}$. Using the Gagliardo-Nirenberg inequality (2.15) and (4.56), we compute

$$
\int_0^t \|c_\varepsilon^{\frac{\ell}{2}}\|_{L^4(\mathbb{R}^2)}^4 \le C\int_0^t \|c_\varepsilon^{\frac{\ell}{2}}\|_{L^2(\mathbb{R}^2)}^2\|\nabla c_\varepsilon^{\frac{\ell}{2}}\|_{L^2(\mathbb{R}^2)}^2 \le C.
$$

Therefore, with the aid of (4.55), we see that

$$
\|c_\varepsilon\|_{L^{2\ell}(\mathbb{R}^2\times(0,t))} + \|v_0c_\varepsilon\|_{L^{2\ell}(\mathbb{R}^2\times(0,t))} \le C \quad \text{for any } 1<\ell\le q. \tag{4.63}
$$

Using (4.62) and (4.63), the maximal estimates for w_ε imply for $r \in (2, 2q)$

$$
\begin{aligned}
&\|\nabla w_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} \\
&\le C\left(\|v_0c_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} + \|n_\varepsilon\|_{L^r(0,t;L^{r'}(\mathbb{R}^2))} + \|c_\varepsilon\|_{L^r(0,t;L^{r'}(\mathbb{R}^2))} + t^{\frac{1}{r}}\|c_0\|_{W^{1,r}(\mathbb{R}^2)}\right) \\
&\le C\left(1+t^{\frac{1}{r}}+t^{\frac{1}{r}-\frac{\theta}{q}}+\|c_\varepsilon\|_{L^r(0,t;L^{r'}(\mathbb{R}^2))}\right).
\end{aligned}
\tag{4.64}
$$

More precisely, we decompose $w_\varepsilon := w_1 + w_2 + w_3$ as follows:

$$
\begin{aligned}
&\partial_t w_1 - \Delta w_1 = -\nabla \cdot (v_0 c_\varepsilon), && w_1(x,0) = 0,\\
&\partial_t w_2 - \Delta w_2 = n_\varepsilon - c_\varepsilon, && w_2(x,0) = 0,\\
&\partial_t w_3 - \Delta w_3 = 0, && w_3(x,0) = c_0(x).
\end{aligned}
$$

Due to the estimates (2.6), (2.10) and (2.14), we have

$$
\begin{aligned}
&\|\nabla w_1\|_{L^r(\mathbb{R}^2\times(0,t))} \le C\|v_0 c_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))},\\
&\|\nabla w_2\|_{L^r(\mathbb{R}^2\times(0,t))} \le C\left(\|n_\varepsilon\|_{L^r(0,t;L^{r'}(\mathbb{R}^2))} + \|c_\varepsilon\|_{L^r(0,t;L^{r'}(\mathbb{R}^2))}\right),\\
&\|\nabla w_3\|_{L^r(\mathbb{R}^2\times(0,t))} \le Ct^{\frac{1}{r}}\|c_0\|_{W^{1,r}(\mathbb{R}^2)}.
\end{aligned}
$$

Combining the above estimates, we obtain (4.64). Using (4.56) and (4.63), the interpolation inequality implies that for some $\ell_1 \in (1,q)$

$$
\|c_\varepsilon\|_{L^r(0,t;L^{r'}(\mathbb{R}^2))} \le \left(\int_0^t \|c_\varepsilon\|^{r\theta_3}_{L^{\ell_1}(\mathbb{R}^2)}\|c_\varepsilon\|^{r(1-\theta_3)}_{L^r(\mathbb{R}^2)}\right)^{\frac{1}{r}} \le C\|c_\varepsilon\|^{1-\theta_3}_{L^r(\mathbb{R}^2\times(0,t))}t^{\theta_3} \le Ct^{\theta_3}, \tag{4.65}
$$

where $\theta_3 = \frac{r\ell_1}{2(r-\ell_1)} \in (0,1)$. Therefore, we have

$$
\|\nabla w_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} \le C\left(1 + t^{\frac{1}{r}} + t^{\frac{1}{r}-\frac{\theta}{q}} + t^{\frac{r\ell_1}{2(r-\ell_1)}}\right).
$$

Next, we show that

$$
\|\nabla \tilde{w_\varepsilon}\|_{L^r(\mathbb{R}^2\times(0,t))} \le C, \tag{4.66}
$$

where C depends on T. Since the initial value of (4.58) is zero, the maximal estimate (2.6) for $\tilde{w_\varepsilon}$ shows that

$$
\|\nabla \tilde{w_\varepsilon}\|_{L^r(\mathbb{R}^2\times(0,t))} \le C\|v_\varepsilon c_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} \le C\left(\|v_\varepsilon w_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} + \|v_\varepsilon \tilde{w_\varepsilon}\|_{L^r(\mathbb{R}^2\times(0,t))}\right). \tag{4.67}
$$

Using (4.60), (4.61), Hölder's inequality and the Gagliardo-Nirenberg inequality (2.15), we can show the boundedness of the term $\|v_\varepsilon w_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))}$ in (4.67) such that

$$
\begin{aligned}
\|v_\varepsilon w_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} &= \left(\int_0^t \|v_\varepsilon w_\varepsilon\|^r_{L^r(\mathbb{R}^2)}\right)^{\frac{1}{r}}\\
&\le \left(\int_0^t \|v_\varepsilon\|^r_{L^4(\mathbb{R}^2)}\|w_\varepsilon\|^r_{L^{\bar{r}}(\mathbb{R}^2)}\right)^{\frac{1}{r}}\\
&\le \|v_\varepsilon\|_{L^\infty(0,t;L^4(\mathbb{R}^2))}\left(\int_0^t \|w_\varepsilon\|^r_{L^{\bar{r}}(\mathbb{R}^2)}\right)^{\frac{1}{r}}\\
&\le Ct^\sigma\left(\int_0^t \|\nabla w_\varepsilon\|^{\frac{r}{2}}_{L^r(\mathbb{R}^2)}\|w_\varepsilon\|^{\frac{r}{2}}_{L^r(\mathbb{R}^2)}\right)^{\frac{1}{r}}\\
&\le Ct^\sigma\left(\|\nabla w_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} + \|w_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))}\right),
\end{aligned} \tag{4.68}
$$

where $\frac{1}{\tilde{r}} = \frac{1}{r} - \frac{1}{4}$ and $\sigma = \frac{2q-3}{4(q-1)}$. Using the similar decomposition used for $\|\nabla w_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))}$, the maximal estimates (2.6), (2.10), (2.14) for w_ε and the Sobolev embedding imply

$$\begin{aligned}
&\|w_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))}\\
&\le C\|\nabla w_\varepsilon\|_{L^r(0,t;L^{r'}(\mathbb{R}^2))}\\
&\le C\left(\|v_0c_\varepsilon\|_{L^r(0,t;L^{r'}(\mathbb{R}^2))} + t^{\frac{1}{2}}\|n_\varepsilon\|_{L^r(0,t;L^{r'}(\mathbb{R}^2))} + t^{\frac{1}{2}}\|c_\varepsilon\|_{L^r(0,t;L^{r'}(\mathbb{R}^2))} + t^{\frac{1}{r}}\|c_0\|_{W^{1,r'}(\mathbb{R}^2)}\right)\\
&\le C\left(t^{\frac{1}{r}} + t^{\frac{1}{2}+\frac{1}{r}-\frac{\theta}{q}} + (1+t^{\frac{1}{2}})\|c_\varepsilon\|_{L^r(0,t;L^{r'}(\mathbb{R}^2))}\right)\\
&\le C\left(t^{\frac{1}{r}} + t^{\frac{1}{2}+\frac{1}{r}-\frac{\theta}{q}} + t^{\frac{r\ell_1}{2(r-\ell_1)}} + t^{\frac{r\ell_1+r-\ell_1}{2(r-\ell_1)}}\right),
\end{aligned} \tag{4.69}$$

where we used (4.65) with $\ell_1 \in (1,q)$. Combining (4.68) with (4.61) and (4.69), we have

$$\|v_\varepsilon w_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} \le C\left(1 + t^{\frac{1}{r}} + t^{\frac{1}{r}-\frac{\theta}{q}} + t^{\frac{1}{2}+\frac{1}{r}-\frac{\theta}{q}} + t^{\frac{r\ell_1}{2(r-\ell_1)}} + t^{\frac{r\ell_1+r-\ell_1}{2(r-\ell_1)}}\right). \tag{4.70}$$

As to the term $\|v_\varepsilon\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))}$ in (4.67), we compute as

$$\begin{aligned}
\|v_\varepsilon\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} &= \left(\int_0^t \|v_\varepsilon\tilde{w}_\varepsilon\|^r_{L^r(\mathbb{R}^2)}\right)^{\frac{1}{r}}\\
&\le \left(\int_0^t \|v_\varepsilon\|^r_{L^4(\mathbb{R}^2)}\|\tilde{w}_\varepsilon\|^r_{L^{\tilde{r}}(\mathbb{R}^2)}\right)^{\frac{1}{r}}\\
&\le \|v_\varepsilon\|_{L^\infty(0,t;L^4(\mathbb{R}^2))}\left(\int_0^t \|\tilde{w}_\varepsilon\|^r_{L^{\tilde{r}}((\mathbb{R}^2))}\right)^{\frac{1}{r}}\\
&\le Ct^\sigma\left(\int_0^t \|\nabla\tilde{w}_\varepsilon\|^{\frac{r}{2}}_{L^r(\mathbb{R}^2)}\|\tilde{w}_\varepsilon\|^{\frac{r}{2}}_{L^r(\mathbb{R}^2)}\right)^{\frac{1}{r}}\\
&\le Ct^\sigma\left(\|\nabla\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} + \|\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))}\right),
\end{aligned}$$

where $\frac{1}{\tilde{r}} = \frac{1}{r} - \frac{1}{4}$ and $\sigma = \frac{2q-3}{4(q-1)}$. Since $\tilde{w}_\varepsilon = c_\varepsilon - w_\varepsilon$ and $\|c_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} \le C$, we have

$$\|v_\varepsilon\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} \le Ct^\sigma\left(\|\nabla\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} + 1 + t^{\frac{1}{r}} + t^{\frac{1}{2}+\frac{1}{r}-\frac{\theta}{q}} + t^{\frac{r\ell_1}{2(r-\ell_1)}} + t^{\frac{r\ell_1+r-\ell_1}{2(r-\ell_1)}}\right). \tag{4.71}$$

Collecting (4.70) and (4.71), we end up with

$$\begin{aligned}
\|\nabla\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} &\le Ct^\sigma\left(\|\nabla\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} + 1 + t^{\frac{1}{r}} + t^{\frac{1}{r}-\frac{\theta}{q}} + t^{\frac{1}{2}+\frac{1}{r}-\frac{\theta}{q}} + t^{\frac{r\ell_1}{2(r-\ell_1)}} + t^{\frac{r\ell_1+r-\ell_1}{2(r-\ell_1)}}\right)\\
&=: Ct^\sigma\left(\|\nabla\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} + f(t)\right).
\end{aligned} \tag{4.72}$$

For sufficiently small $t_0 \in (0,T)$, we have

$$\|\nabla\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t_0))} \le C,$$

where the above C depends on t_0. We use an inductive argument to show the boundedness of $\|\nabla\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))}$ depending on T. Denote $I_j := [jt_0, (j+1)t_0]$ the closed time interval for nonnegative integer j. Suppose that there exists $C > 0$ depending on t_0 such that

$$\|\nabla\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^2\times I_j)} \le C. \tag{4.73}$$

Then for $t \in I_{j+1}$, we use decomposition with the maximal estimates (2.6), (2.10) for $\tilde{w}_\varepsilon$ to see that

$$\begin{aligned}\|\nabla\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^2\times I_{j+1})} &\le C\left(\|v_\varepsilon c_\varepsilon\|_{L^r(\mathbb{R}^2\times I_{j+1})} + (t-(j+1)t_0)^{\frac{1}{r}}\|w_\varepsilon(\cdot,(j+1)t_0)\|_{W^{1,r}(\mathbb{R}^2)}\right)\\ &\le C\left(\|v_\varepsilon c_\varepsilon\|_{L^r(\mathbb{R}^2\times I_{j+1})} + t_0^{\frac{1}{r}}\|w_\varepsilon(\cdot,(j+1)t_0)\|_{W^{1,r}(\mathbb{R}^2)}\right).\end{aligned} \tag{4.74}$$

Owing to (4.67)–(4.71) and (4.74), we repeat the similar procedure for $t \in I_{j+1}$ to obtain

$$\|\nabla\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^2\times I_{j+1})} \le C\left(t_0^\sigma\|\nabla\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^2\times I_{j+1})} + t_0^\sigma f(t_0) + t_0^{\frac{1}{r}}\|\nabla w_\varepsilon(\cdot,(j+1)t_0)\|_{L^r(\mathbb{R}^2)}\right),$$

which has a similar structure to (4.72). Thus, from (4.73), we have

$$\|\nabla\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^2\times I_{j+1})} \le C,$$

where C depends on t_0. Therefore, we obtain (4.66). Hence, we obtain

$$\|\nabla c_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} \le \|\nabla w_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} + \|\nabla\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^2\times(0,t))} \le C.$$

• (The case $d = 3$) We follow the similar procedure to the two dimensional case. From Young's inequality and (4.53), we have for $t \in (0,T)$

$$\begin{aligned}\|\nabla v_\varepsilon(\cdot,t)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} &\le \int_0^t \|\nabla S * n_\varepsilon(\cdot,s)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} ds\\ &\le \int_0^t \frac{C}{(t-s)^{\frac{1}{2}}}\|n_\varepsilon(\cdot,s)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} ds\\ &\le \int_0^t \frac{C}{(t-s)^{\frac{1}{2}}}\|n_\varepsilon(\cdot,s)\|_{L^1(\mathbb{R}^3)}^{\frac{2q-3}{3(q-1)}}\|n_\varepsilon(\cdot,s)\|_{L^q(\mathbb{R}^3)}^{\frac{q}{3(q-1)}} ds\\ &\le C\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}}\|n_\varepsilon(\cdot,s)\|_{L^q(\mathbb{R}^3)}^{\frac{q}{3(q-1)}} ds\\ &\le Ct^\sigma\|n_\varepsilon\|_{L^q(\mathbb{R}^3\times(0,T))}^{\frac{q}{3(q-1)}} \le Ct^\sigma,\end{aligned}$$

where $\sigma = \frac{3q-5}{6(q-1)} > 0$ and $q > \frac{5}{3}$. Therefore, by the Sobolev embedding, we have

$$\|v_\varepsilon\|_{L^\infty(0,t;L^3(\mathbb{R}^3))} \le C\|\nabla v_\varepsilon\|_{L^\infty(0,t;L^{\frac{3}{2}}(\mathbb{R}^3))} \le Ct^\sigma. \tag{4.75}$$

We recall (4.51) such that

$$\|n_\varepsilon\|_{L^r(0,T;L^{r'}(\mathbb{R}^3))} \le Ct^{\frac{1}{r}-\frac{\theta}{q}}, \tag{4.76}$$

where $r \in (\frac{3}{2}, \frac{3q}{2})$, $r' = \frac{3r}{3+r}$ and $\theta = \frac{qr'-q}{qr'-r'}$. Using the similar procedure to the two dimensional case, due to the Gagliardo-Nirenberg inequality (2.15) and (4.56), we observe that

$$\int_0^t \|c_\varepsilon^{\frac{\ell}{2}}\|_{L^{\frac{10}{3}}(\mathbb{R}^3)}^{\frac{10}{3}} \le C \int_0^t \|\nabla c_\varepsilon^{\frac{\ell}{2}}\|_{L^2(\mathbb{R}^3)}^2 \|c_\varepsilon^{\frac{\ell}{2}}\|_{L^2(\mathbb{R}^3)}^{\frac{4}{3}} \le C \quad \text{for any } \ 1 < \ell \le q.$$

Then, (4.55) implies

$$\|c_\varepsilon\|_{L^{\frac{5\ell}{3}}(\mathbb{R}^3\times(0,t))} + \|v_0 c_\varepsilon\|_{L^{\frac{5\ell}{3}}(\mathbb{R}^3\times(0,t))} \le C \quad \text{for any } \ 1 < \ell \le q. \tag{4.77}$$

According to (4.76) and (4.77), decomposition with the maximal estimates (2.6), (2.10), (2.14) for w_ε implies

$$\begin{aligned}
&\|\nabla w_\varepsilon\|_{L^r(\mathbb{R}^3\times(0,t))} \\
&\le C\left(\|v_0 c_\varepsilon\|_{L^r(\mathbb{R}^3\times(0,t))} + \|n_\varepsilon\|_{L^r(0,t;L^{r'}(\mathbb{R}^3))} + \|c_\varepsilon\|_{L^r(0,t;L^{r'}(\mathbb{R}^3))} + t^{\frac{1}{r}}\|c_0\|_{W^{1,r}(\mathbb{R}^3)}\right) \\
&\le C\left(1 + t^{\frac{1}{r}} + t^{\frac{1}{r}-\frac{\theta}{q}} + \|c_\varepsilon\|_{L^r(0,t;L^{r'}(\mathbb{R}^3))}\right),
\end{aligned} \tag{4.78}$$

where $r \in [\frac{5}{3}, \frac{3q}{2})$, $r' = \frac{3r}{3+r}$ and $\theta = \frac{qr'-q}{qr'-r'}$. Similarly to (4.65), we have for some $\ell_2 \in (1, q)$

$$\|c_\varepsilon\|_{L^r(0,t;L^{r'}(\mathbb{R}^3))} \le \left(\int_0^t \|c_\varepsilon\|_{L^{\ell_2}(\mathbb{R}^3)}^{r\theta_4} \|c_\varepsilon\|_{L^r(\mathbb{R}^3)}^{r(1-\theta_4)}\right)^{\frac{1}{r}} \le C\|c_\varepsilon\|_{L^r(\mathbb{R}^3\times(0,t))}^{1-\theta_4} t^{\theta_4} \le Ct^{\theta_4},$$

where $\theta_4 = \frac{1/r'-1/r}{1/\ell_2-1/r} = \frac{r\ell_2}{3(r-\ell_2)} \in (0,1)$. Therefore, (4.78) becomes

$$\|\nabla w_\varepsilon\|_{L^r(\mathbb{R}^3\times(0,t))} \le C\left(1 + t^{\frac{1}{r}} + t^{\frac{1}{r}-\frac{\theta}{q}} + t^{\frac{r\ell_2}{3(r-\ell_2)}}\right). \tag{4.79}$$

Using (4.75), the estimate of $\tilde{w}_\varepsilon$ is also verified as follows:

$$\begin{aligned}
\|v_\varepsilon \tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^3\times(0,t))} &= \left(\int_0^t \|v_\varepsilon \tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^3)}^r\right)^{\frac{1}{r}} \\
&\le \left(\int_0^t \|v_\varepsilon\|_{L^3(\mathbb{R}^3)}^r \|\tilde{w}_\varepsilon\|_{L^{r''}(\mathbb{R}^3)}^r\right)^{\frac{1}{r}} \\
&\le \|v_\varepsilon\|_{L^\infty(0,t;L^3(\mathbb{R}^3))} \|\tilde{w}_\varepsilon\|_{L^r(0,t;L^{r''}(\mathbb{R}^3))} \\
&\le C\|v_\varepsilon\|_{L^\infty(0,t;L^3(\mathbb{R}^3))} \|\nabla\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^3\times(0,t))} \\
&\le Ct^\sigma \|\nabla\tilde{w}_\varepsilon\|_{L^r(\mathbb{R}^3\times(0,t))},
\end{aligned}$$

where $r'' = \frac{3r}{3-r}$. Moreover, due to (4.75) and (4.79), we see that

$$\begin{aligned}
\|v_\varepsilon w_\varepsilon\|_{L^r(\mathbb{R}^3\times(0,t))} &\le C\|v_\varepsilon\|_{L^\infty(0,T;L^3(\mathbb{R}^3))} \|\nabla w_\varepsilon\|_{L^r(\mathbb{R}^3\times(0,t))} \\
&\le Ct^\sigma\left(1 + t^{\frac{1}{r}} + t^{\frac{1}{r}-\frac{\theta}{q}} + t^{\frac{r\ell_2}{3(r-\ell_2)}}\right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|\nabla \tilde{w_\varepsilon}\|_{L^r(\mathbb{R}^3\times(0,t))} &\le \|v_\varepsilon c_\varepsilon\|_{L^r(\mathbb{R}^3\times(0,t))}\\ &\le \|v_\varepsilon \tilde{w_\varepsilon}\|_{L^r(\mathbb{R}^3\times(0,t))} + \|v_\varepsilon w_\varepsilon\|_{L^r(\mathbb{R}^3\times(0,t))}\\ &\le Ct^\sigma\left(\|\nabla \tilde{w_\varepsilon}\|_{L^r(\mathbb{R}^3\times(0,t))} + 1 + t^{\frac{1}{r}} + t^{\frac{1}{r}-\frac{\theta}{q}} + t^{\frac{r\ell_2}{3(r-\ell_2)}}\right).\end{aligned}$$

For sufficiently small $t_0 \in (0,T)$, we have

$$\|\nabla \tilde{w_\varepsilon}\|_{L^r(\mathbb{R}^3\times(0,t))} \le C \quad \text{for} \quad t < t_0,$$

where the above C depends on t_0. Repeating the same argument as in the two dimensional case, we have

$$\|\nabla \tilde{w_\varepsilon}\|_{L^r(\mathbb{R}^3\times(0,t))} \le C,$$

where the above C depends on T. Finally, we obtain

$$\|\nabla c_\varepsilon\|_{L^r(\mathbb{R}^3\times(0,t))} \le \|\nabla w_\varepsilon\|_{L^r(\mathbb{R}^3\times(0,t))} + \|\nabla \tilde{w_\varepsilon}\|_{L^r(\mathbb{R}^3\times(0,t))} \le C,$$

where $C = C(\|n_0\|_{L^1(\mathbb{R}^3)}, \|c_0\|_{(L^{\frac{3}{5}r}\cap W^{1,r})(\mathbb{R}^3)}, \|u_0\|_{L^\infty(\mathbb{R}^3)}, T) > 0$. This completes the proof. □

Lemma 4.3. *Let $d = 2, 3$, $T > 0$ and q satisfy (4.47). Suppose that $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ be a regular solution of (4.46) in $\mathbb{R}^d \times (0,T)$. Then, there exists $C = C(\|n_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^d(\mathbb{R}^d)}, \|u_0\|_{L^{\tilde{q}}(\mathbb{R}^d)}, T) > 0$ independent of ε such that for all $t \in (0,T)$*

$$\sup_{t\in(0,T)} \int_{\mathbb{R}^d} |u_\varepsilon(\cdot,t)|^d \le C, \tag{4.80}$$

$$\int_0^t \int_{\mathbb{R}^d} |u_\varepsilon|^{\tilde{q}} \le C, \tag{4.81}$$

where $\frac{1}{\tilde{q}} = \frac{1}{q} - \frac{2}{d+2}$.

Proof. Applying the gradient estimate (4.53) for Stokes operator to the representation formula (4.54), we compute

$$\|u_\varepsilon(\cdot,t)\|_{L^d(\mathbb{R}^d)} \le C\|u_0\|_{L^d(\mathbb{R}^d)} + C\int_0^t (t-s)^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{d})}\|n_\varepsilon(\cdot,s)\|_{L^q(\mathbb{R}^d)}ds \quad \text{for} \quad t \in (0,T).$$

Using Hölder's inequality and (4.49), we obtain for $t \in (0,T)$

$$\begin{aligned}\|u_\varepsilon\|_{L^\infty(0,t;L^d(\mathbb{R}^d))} &\le C\|u_0\|_{L^d(\mathbb{R}^d)} + Ct^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{d})+1-\frac{1}{q}}\|n_\varepsilon\|_{L^q(\mathbb{R}^d\times(0,t))}\\ &\le C\left(\|u_0\|_{L^d(\mathbb{R}^d)} + T^\eta\right),\end{aligned}$$

where $\eta = \frac{3}{2} - \frac{d+2}{2q} > 0$ since $q > 2 - \frac{1}{d}$. This completes the proof of (4.80). As to (4.81), we let $\tilde{q}$ satisfy $\frac{1}{\tilde{q}} = \frac{1}{q} - \frac{2}{d+2}$ for $d = 2, 3$. Again using (4.53) for (4.54), we compute for $t \in (0,T)$

$$\begin{aligned}\|u_\varepsilon(\cdot,t)\|_{L^{\tilde{q}}(\mathbb{R}^d)} &\le C\|u_0\|_{L^{\tilde{q}}(\mathbb{R}^d)} + C\int_0^t (t-s)^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})}\|n_\varepsilon(\cdot,s)\|_{L^q(\mathbb{R}^d)}ds\\ &\le C\|u_0\|_{L^{\tilde{q}}(\mathbb{R}^d)} + Ct^{-\frac{d}{d+2}} * \|n_\varepsilon(\cdot,t)\|_{L^q(\mathbb{R}^d)},\end{aligned}$$

which implies

$$\begin{aligned}\|u_\varepsilon\|_{L^{\tilde{q}}(\mathbb{R}^d\times(0,t))} &\le CT^{\frac{1}{\tilde{q}}}\|u_0\|_{L^{\tilde{q}}(\mathbb{R}^d)} + C[s^{-\frac{d}{d+2}}]_{L^{\frac{d+2}{d}}(0,t)}\|n_\varepsilon\|_{L^q(\mathbb{R}^d\times(0,t))}\\ &\le CT^{\frac{1}{\tilde{q}}}\|u_0\|_{L^{\tilde{q}}(\mathbb{R}^d)} + C\|n_\varepsilon\|_{L^q(\mathbb{R}^d\times(0,t))},\end{aligned}$$

where we used the weak version of Young's convolution inequality (2.16). Thus, by (4.49), we prove (4.81). $\square$

Lemma 4.4. *Let $d = 2, 3$, $T > 0$, and q satisfy (4.47). Suppose that $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ be a regular solution of (4.46) in $\mathbb{R}^d \times (0, T)$. Then, there exists $C > 0$ independent of ε such that for all $t \in (0, T)$*

$$\int_0^t \int_{\mathbb{R}^d} |\Delta c_\varepsilon|^q dxds \le C, \tag{4.82}$$

where C depends on $\|c_0\|_{W^{2,q}(\mathbb{R}^d)}$ and the conditions in Lemmas 4.2, 4.3.

Proof. Let $\frac{1}{\tilde{q}} = \frac{1}{q} - \frac{1}{r}$ where $r > \frac{d+2}{2}$ for $d = 2, 3$ satisfies the conditions in Lemma 4.2. Invoking (4.52) and (4.81), Hölder's inequality implies that

$$\|u_\varepsilon \cdot \nabla c_\varepsilon\|_{L^q(\mathbb{R}^d\times(0,t))} \le C\|u_\varepsilon\|_{L^{\tilde{q}}(\mathbb{R}^d\times(0,t))}\|\nabla c_\varepsilon\|_{L^r(\mathbb{R}^d\times(0,t))} \le C. \tag{4.83}$$

In view of the maximal estimate (2.9), we obtain

$$\begin{aligned}&\|\partial_t c_\varepsilon\|_{L^q(\mathbb{R}^d\times(0,t))} + \|\Delta c_\varepsilon\|_{L^q(\mathbb{R}^d\times(0,t))}\\ &\le C\left(\|n_\varepsilon\|_{L^q(\mathbb{R}^d\times(0,t))} + \|c_\varepsilon\|_{L^q(\mathbb{R}^d\times(0,t))} + \|u_\varepsilon \cdot \nabla c_\varepsilon\|_{L^q(\mathbb{R}^d\times(0,t))} + \|c_0\|_{W^{2,q}(\mathbb{R}^d)}\right).\end{aligned}$$

Thus, due to (4.49), (4.63), (4.77) and (4.83), we deduce that

$$\|\Delta c_\varepsilon\|_{L^q(\mathbb{R}^d\times(0,t))} \le C,$$

where C depends on T. This completes the proof. $\square$

To obtain some boundedness of n_ε for super-solution, we introduce the following two lemmas.

Lemma 4.5. *Let $T > 0$, Ω be any bounded domain in $\mathbb{R}^d$, $d = 2, 3$, and q satisfy (4.47). Suppose that $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ is a regular solution of (4.46) in $\mathbb{R}^d \times (0, T)$. Then, for any $0 < \gamma \le q - 1$ there exists $C > 0$ such that for all $t \in (0, T)$*

$$\int_0^t \int_\Omega n_\varepsilon^{\gamma-2}|\nabla n_\varepsilon|^2 dxds \le C, \tag{4.84}$$

where C depends on the conditions in Lemmas 4.2, 4.3.

Proof. For a given bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, we consider a test function $\varphi \in C_0^\infty(\mathbb{R}^d)$ which is non-increasing in $|x|$ and satisfies

$$0 \le \varphi \le 1 \quad \text{and} \quad \varphi(x) = \begin{cases} 1 & \text{for } x \in \Omega \\ 0 & \text{for } x \in \mathbb{R}^d \setminus B_R \end{cases},$$

where $B_R \subset \mathbb{R}^d$ is a ball centered in 0 with radius $R > 0$ and $\Omega \subset\subset B_{\frac{R}{2}}$. Testing the first equation of (4.46) by $n_\varepsilon^{\gamma-1}\varphi$ with $\gamma \le q-1$ and integrating over B_R, integration by parts implies

$$\begin{aligned}\frac{1}{\gamma}\frac{d}{dt}\int_{B_R} n_\varepsilon^\gamma \varphi + \frac{1}{\gamma}\int_{B_R} u_\varepsilon \cdot \nabla(n_\varepsilon^\gamma)\varphi =&(1-\gamma)\int_{B_R} n_\varepsilon^{\gamma-2}|\nabla n_\varepsilon|^2\varphi - \int_{B_R} n_\varepsilon^{\gamma-1}\nabla n_\varepsilon \cdot \nabla\varphi \\ &+ \int_{B_R} n_\varepsilon \nabla c_\varepsilon \cdot \nabla(n_\varepsilon^{\gamma-1}\varphi) - \mu\int_{B_R} n_\varepsilon^{q+\gamma-1}\varphi - \varepsilon\int_{B_R} n_\varepsilon^{\beta+\gamma-1}\varphi. \end{aligned} \tag{4.85}$$

We integrate (4.85) over $(0,t)$ to obtain

$$\begin{aligned}(1-\gamma)\int_0^t\int_\Omega n_\varepsilon^{\gamma-2}|\nabla n_\varepsilon|^2 &\le (1-\gamma)\int_0^t\int_{B_R} n_\varepsilon^{\gamma-2}|\nabla n_\varepsilon|^2\varphi \\ &= \frac{1}{\gamma}\int_{B_R} n_\varepsilon^\gamma(\cdot,t)\varphi - \frac{1}{\gamma}\int_{B_R} n_0^\gamma\varphi - \frac{1}{\gamma}\int_0^t\int_{B_R} n_\varepsilon^\gamma u_\varepsilon\cdot\nabla\varphi - \frac{1}{\gamma}\int_0^t\int_{B_R} n_\varepsilon^\gamma\Delta\varphi \\ &\quad - \int_0^t\int_{B_R} n_\varepsilon\nabla c_\varepsilon\cdot\nabla(n_\varepsilon^{\gamma-1}\varphi) + \mu\int_0^t\int_{B_R} n_\varepsilon^{q+\gamma-1}\varphi + \varepsilon\int_0^t\int_{B_R} n_\varepsilon^{\beta+\gamma-1}\varphi. \end{aligned} \tag{4.86}$$

Using Young's inequality and Hölder's inequality, we can easily see that

$$\frac{1}{\gamma}\int_{B_R} n_\varepsilon^\gamma(\cdot,t)\varphi \le \frac{|B_R|^{1-\gamma}}{\gamma}\left(\int_{B_R} n_\varepsilon(x,t)\right)^\gamma \le \frac{|B_R|^{1-\gamma}}{\gamma}\left(\int_{\mathbb{R}^d} n_0(x)\right)^\gamma \le C, \tag{4.87}$$

$$-\frac{1}{\gamma}\int_0^t\int_{B_R} n_\varepsilon^\gamma\Delta\varphi \le \frac{1}{q}\int_0^t\int_{\mathbb{R}^d} n_\varepsilon^q + \frac{q-\gamma}{\gamma q}\int_0^t\int_{B_R}|\Delta\varphi|^{\frac{q}{q-\gamma}} \le C, \tag{4.88}$$

$$\mu\int_0^t\int_{B_R} n_\varepsilon^{q+\gamma-1}\varphi \le \frac{\mu(q+\gamma-1)}{q}\int_0^t\int_{\mathbb{R}^d} n_\varepsilon^q + \frac{\mu(1-\gamma)}{q}|B_R|t \le C, \tag{4.89}$$

$$\varepsilon\int_0^T\int_{B_R} n_\varepsilon^{\beta+\gamma-1}\varphi \le \frac{\varepsilon(\beta+\gamma-1)}{\beta}\int_0^t\int_{\mathbb{R}^d} n_\varepsilon^\beta + \frac{\varepsilon(1-\gamma)}{\beta}|B_R|t \le C. \tag{4.90}$$

Moreover, since $\frac{\gamma}{q} + (\frac{1}{q} - \frac{2}{d+2}) \le 1 - \frac{2}{d+2} < 1$, we observe that

$$-\frac{1}{\gamma}\int_0^t\int_{B_R} n_\varepsilon^\gamma u_\varepsilon\cdot\nabla\varphi \le C\|n_\varepsilon\|^\gamma_{L^q(\mathbb{R}^d\times(0,t))}\|u_\varepsilon\|_{L^{\tilde{q}}(\mathbb{R}^d\times(0,t))} \le C, \tag{4.91}$$

where $\frac{1}{\tilde{q}} = \frac{1}{q} - \frac{2}{d+2}$, and we used (4.49) and (4.81). On the other hand, by integration by parts, we have

$$
\begin{aligned}
-\int_0^t \int_{B_R} n_\varepsilon \nabla c_\varepsilon \cdot \nabla(n_\varepsilon^{\gamma-1}\varphi) &= \frac{1}{\gamma}\int_0^t \int_{B_R} \nabla(n_\varepsilon^\gamma)\cdot \nabla c_\varepsilon \varphi + \int_0^t \int_{B_R} n_\varepsilon^\gamma \Delta c_\varepsilon \varphi \\
&= \left(1-\frac{1}{\gamma}\right)\int_0^t \int_{B_R} n_\varepsilon^\gamma \Delta c_\varepsilon \varphi - \frac{1}{\gamma}\int_0^t \int_{B_R} n_\varepsilon^\gamma \nabla c_\varepsilon \cdot \nabla\varphi. \qquad (4.92)
\end{aligned}
$$

By virtue of Hölder's inequality, we estimate the first term to the right side of (4.92) such that

$$
\left(1-\frac{1}{\gamma}\right)\int_0^t \int_{B_R} n_\varepsilon^\gamma \Delta c_\varepsilon \varphi \le C\left(\int_0^t \int_{B_R} n_\varepsilon^q\right)^{\frac{\gamma}{q}}\left(\int_0^t \int_{B_R} |\Delta c_\varepsilon|^q\right)^{\frac{1}{q}}(|B_R|t)^{\frac{q-\gamma-1}{q}}. \qquad (4.93)
$$

Due to (4.49) and (4.82), the boundedness of $-\frac{1}{\gamma}\int_0^t \int_{B_R} n_\varepsilon^\gamma \Delta c_\varepsilon \varphi$ is obtained. For some $r > q$ satisfying the conditions in Lemma 4.2, we show the boundedness of the second term to the right side of (4.92) as

$$
-\frac{1}{\gamma}\int_0^t \int_{B_R} n_\varepsilon^\gamma \nabla c_\varepsilon \cdot \nabla\varphi \le C\left(\int_0^t \int_{B_R} n_\varepsilon^q\right)^{\frac{\gamma}{q}}\left(\int_0^t \int_{B_R} |\nabla c_\varepsilon|^r\right)^{\frac{1}{r}}(|B_R|t)^{1-\frac{\gamma}{q}-\frac{1}{r}} \le C,
$$

where we used (4.49) and (4.52). Thus, we have

$$
-\int_0^t \int_{B_R} n_\varepsilon \nabla c_\varepsilon \cdot \nabla(n_\varepsilon^{\gamma-1}\varphi) \le C. \qquad (4.94)
$$

We collect the estimates (4.86)–(4.94) to complete the proof. □

Lemma 4.6. *Let $T > 0$, Ω be any bounded domain in $\mathbb{R}^d$, $d = 2, 3$, and q satisfy (4.47). Suppose that $(n_\varepsilon, c_\varepsilon, u_\varepsilon)$ is a regular solution of (4.46) in $\mathbb{R}^d \times (0, T)$. Then for any $\gamma \in (0, 1)$ satisfying $\gamma \le q - 1$, there exist $k \in \mathbb{N}$ and $C > 0$ such that*

$$
\|\partial_t(1+n_\varepsilon)^{\frac{\gamma}{2}}\|_{L^1(0,T;(W_0^{k,2}(\Omega))^*)} \le C,
$$

where C depends on the conditions in Lemmas 4.2, 4.3.

Proof. We choose $k \in \mathbb{N}$ sufficiently large such that $k > \frac{d+2}{2}$, which enables us to have the following embedding properties:

$$
W_0^{k,2}(\Omega) \hookrightarrow L^\infty(\Omega) \quad \text{and} \quad W_0^{k,2}(\Omega) \hookrightarrow W^{1,p}(\Omega), \qquad (4.95)
$$

where $p := \max\{2, \frac{2q}{2q-\gamma-2}\}$. For any test function $\psi \in W_0^{k,2}(\Omega)$, multiplying (4.46) by $(1+n_\varepsilon)^{\frac{\gamma-2}{2}}\psi$ and integrating by parts over Ω, we obtain for $t \in (0, T)$

$$
\begin{aligned}
\frac{2}{\gamma}\int_\Omega \partial_t(1+n_\varepsilon)^{\frac{\gamma}{2}}\psi =& \int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}\partial_t n_\varepsilon\psi \\
=& \int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}\Delta n_\varepsilon\psi - \int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}\nabla\cdot(n_\varepsilon\nabla c_\varepsilon)\psi - \int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}u_\varepsilon\cdot\nabla n_\varepsilon\psi \\
& - \mu\int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}n_\varepsilon^q\psi - \varepsilon\int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}n_\varepsilon^\beta\psi \\
=& \frac{2-\gamma}{2}\int_\Omega (1+n_\varepsilon)^{\frac{\gamma-4}{2}}|\nabla n_\varepsilon|^2\psi - \int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}\nabla n_\varepsilon\cdot\nabla\psi - \int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}\nabla\cdot(n_\varepsilon\nabla c_\varepsilon)\psi \\
& - \int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}u_\varepsilon\cdot\nabla n_\varepsilon\psi - \mu\int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}n_\varepsilon^q\psi - \varepsilon\int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}n_\varepsilon^\beta\psi.
\end{aligned}
$$

As in Lemma 1.3 of [27], we have

$$
\begin{aligned}
\left|\int_\Omega (1+n_\varepsilon)^{\frac{\gamma-4}{2}}|\nabla n_\varepsilon|^2\psi\right| &\le \left(\int_\Omega (1+n_\varepsilon)^{\gamma-2}|\nabla n_\varepsilon|^2\right)\|\psi\|_{L^\infty(\Omega)} \\
&\le \left(\int_\Omega n_\varepsilon^{\gamma-2}|\nabla n_\varepsilon|^2\right)\|\psi\|_{L^\infty(\Omega)},
\end{aligned} \tag{4.96}
$$

$$
\begin{aligned}
\left|\int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}\nabla n_\varepsilon\cdot\nabla\psi\right| &\le \left(\int_\Omega (1+n_\varepsilon)^{\gamma-2}|\nabla n_\varepsilon|^2\right)^{\frac{1}{2}}\cdot\left(\int_\Omega |\nabla\psi|^2\right)^{\frac{1}{2}} \\
&\le \frac{1}{2}\left(1+\int_\Omega n_\varepsilon^{\gamma-2}|\nabla n_\varepsilon|^2\right)\|\nabla\psi\|_{L^2(\Omega)},
\end{aligned} \tag{4.97}
$$

$$
\begin{aligned}
\left|\mu\int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}n_\varepsilon^q\psi\right| &\le \mu\int_\Omega (1+n_\varepsilon)^{\frac{2q+\gamma-2}{2}}|\psi| \\
&\le \mu\left(\int_\Omega (1+n_\varepsilon)^q\right)^{\frac{2q+\gamma-2}{2q}}\cdot\left(\int_\Omega |\psi|^{\frac{2q}{2-\gamma}}\right)^{\frac{2-\gamma}{2q}} \\
&\le C\left(1+\int_\Omega (1+n_\varepsilon)^q\right)\cdot\|\psi\|_{L^{\frac{2q}{2-\gamma}}(\Omega)},
\end{aligned} \tag{4.98}
$$

$$
\begin{aligned}
\left|-\varepsilon\int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}n_\varepsilon^\beta\psi\right| &\le \varepsilon\int_\Omega n_\varepsilon^{\frac{2\beta+\gamma-2}{2}}|\psi| \\
&\le \varepsilon\left(\int_\Omega n_\varepsilon^\beta\right)^{\frac{2\beta+\gamma-2}{2\beta}}\cdot\left(\int_\Omega |\psi|^{\frac{2\beta}{2-\gamma}}\right)^{\frac{2-\gamma}{2\beta}} \\
&\le \varepsilon C\left(1+\int_\Omega n_\varepsilon^\beta\right)\cdot\|\psi\|_{L^{\frac{2\beta}{2-\gamma}}(\Omega)}.
\end{aligned} \tag{4.99}
$$

As to the chemotaxis term $\int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}\nabla\cdot(n_\varepsilon\nabla c_\varepsilon)\psi$, we compute

$$
\begin{aligned}
&\left|\int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}\nabla\cdot(n_\varepsilon\nabla c_\varepsilon)\psi\right|\\
&=\left|-\int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}\nabla n_\varepsilon\cdot\nabla c_\varepsilon\psi-\int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}n_\varepsilon\Delta c_\varepsilon\psi\right|\\
&=\left|-\frac{2}{\gamma}\int_\Omega \nabla(1+n_\varepsilon)^{\frac{\gamma}{2}}\cdot\nabla c_\varepsilon\psi-\int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}n_\varepsilon\Delta c_\varepsilon\psi\right|\\
&=\left|\frac{2}{\gamma}\int_\Omega (1+n_\varepsilon)^{\frac{\gamma}{2}}\Delta c_\varepsilon\psi+\frac{2}{\gamma}\int_\Omega (1+n_\varepsilon)^{\frac{\gamma}{2}}\nabla c_\varepsilon\cdot\nabla\psi-\int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}n_\varepsilon\Delta c_\varepsilon\psi\right|\\
&\le\frac{2}{\gamma}\int_\Omega (1+n_\varepsilon)^{\frac{\gamma}{2}}|\Delta c_\varepsilon||\psi|+\frac{2}{\gamma}\int_\Omega (1+n_\varepsilon)^{\frac{\gamma}{2}}|\nabla c_\varepsilon\cdot\nabla\psi|+\int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}n_\varepsilon|\Delta c_\varepsilon||\psi|\\
&\le(\frac{2}{\gamma}+1)\left(\int_\Omega (1+n_\varepsilon)^q\right)^{\frac{\gamma}{2q}}\left(\int_\Omega|\Delta c_\varepsilon|^q\right)^{\frac{1}{q}}\left(\int_\Omega|\psi|^{\frac{2q}{2q-\gamma-2}}\right)^{\frac{2q-\gamma-2}{2q}}\\
&+\left(\int_\Omega (1+n_\varepsilon)^q\right)^{\frac{\gamma}{2q}}\left(\int_\Omega|\nabla c_\varepsilon|^q\right)^{\frac{1}{q}}\left(\int_\Omega|\nabla\psi|^{\frac{2q}{2q-\gamma-2}}\right)^{\frac{2q-\gamma-2}{2q}}\\
&\le C\left(1+\int_\Omega (1+n_\varepsilon)^q+\int_\Omega|\Delta c_\varepsilon|^q+\int_\Omega|\nabla c_\varepsilon|^q\right)\|\psi\|_{W^{1,\frac{2q}{2q-\gamma-2}}(\Omega)}.
\end{aligned}
\tag{4.100}
$$

On the other hand, as to the convection term $\int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}u_\varepsilon\cdot\nabla n_\varepsilon\psi$, we find

$$
\begin{aligned}
\left|\int_\Omega (1+n_\varepsilon)^{\frac{\gamma-2}{2}}u_\varepsilon\cdot\nabla n_\varepsilon\psi\right|&=\left|\frac{2}{\gamma}\int_\Omega \nabla(1+n_\varepsilon)^{\frac{\gamma}{2}}\cdot u_\varepsilon\psi\right|\\
&\le\frac{2}{\gamma}\left(\int_\Omega (1+n_\varepsilon)^{\frac{\gamma}{2}}|\nabla u_\varepsilon||\psi|+\int_\Omega (1+n_\varepsilon)^{\frac{\gamma}{2}}|u_\varepsilon||\nabla\psi|\right)\\
&\le\frac{2}{\gamma}\left(\int_\Omega (1+n_\varepsilon)^q\right)^{\frac{\gamma}{2q}}\left(\int_\Omega|\nabla u_\varepsilon|^q\right)^{\frac{1}{q}}\left(\int_\Omega|\psi|^{\frac{2q}{2q-\gamma-2}}\right)^{\frac{2q-\gamma-2}{2q}}\\
&+\frac{2}{\gamma}\left(\int_\Omega (1+n_\varepsilon)^q\right)^{\frac{\gamma}{2q}}\left(\int_\Omega|u_\varepsilon|^q\right)^{\frac{1}{q}}\left(\int_\Omega|\nabla\psi|^{\frac{2q}{2q-\gamma-2}}\right)^{\frac{2q-\gamma-2}{2q}}\\
&\le C\left(1+\int_\Omega (1+n_\varepsilon)^q+\int_\Omega|\nabla u_\varepsilon|^q+\int_\Omega|u_\varepsilon|^q\right)\|\psi\|_{W^{1,\frac{2q}{2q-\gamma-2}}(\Omega)}.
\end{aligned}
\tag{4.101}
$$

Together with (4.96)–(4.101), using the embedding properties (4.95), we obtain

$$\begin{aligned}&\left|\frac{2}{\gamma}\int_\Omega \partial_t(1+n_\varepsilon)^{\frac{\gamma}{2}}\psi\right|\\&\le C\left(1+\int_\Omega (n_\varepsilon^{\gamma-2}|\nabla n_\varepsilon|^2+n_\varepsilon^q+\varepsilon n_\varepsilon^\beta+|\Delta c_\varepsilon|^q+|\nabla c_\varepsilon|^q+|\nabla u_\varepsilon|^q+|u_\varepsilon|^q)\right)\\&\times\left(\|\psi\|_{W^{1,p}(\Omega)}+\|\psi\|_{L^\infty(\Omega)}\right)\\&\le C\left(1+\int_\Omega (n_\varepsilon^{\gamma-2}|\nabla n_\varepsilon|^2+n_\varepsilon^q+\varepsilon n_\varepsilon^\beta+|\Delta c_\varepsilon|^q+|\nabla c_\varepsilon|^q+|\nabla u_\varepsilon|^q+|u_\varepsilon|^q)\right)\|\psi\|_{W_0^{k,2}(\Omega)}.\end{aligned}$$

Integration over $(0,T)$ implies that

$$\begin{aligned}&\|\partial_t(1+n_\varepsilon)^{\frac{\gamma}{2}}\|_{L^1(0,T;(W_0^{k,2}(\Omega))^*)}\\&\le C\left(T+\int_0^T\int_\Omega (n_\varepsilon^{\gamma-2}|\nabla n_\varepsilon|^2+n_\varepsilon^q+\varepsilon n_\varepsilon^\beta+|\Delta c_\varepsilon|^q+|\nabla c_\varepsilon|^q+|\nabla u_\varepsilon|^q+|u_\varepsilon|^q)\right),\end{aligned}$$

which is bounded, in virtue of (4.48), (4.50), (4.82), (4.84) and the Sobolev embeddings on $\Omega\times(0,T)$. This completes the proof. □

We are now in position to prove the compactness property of n_ε and c_ε.

Lemma 4.7. *Let $T>0$, $q>2-\frac{1}{d}$, Ω be any bounded domain in $\mathbb{R}^d$, $d=2,3$. Suppose that $(n_\varepsilon,c_\varepsilon,u_\varepsilon)$ is a regular solution of (4.46) in $\mathbb{R}^d\times(0,T)$. Then, we have*

$$n_\varepsilon\to n \ \ \text{a.e. in } \ \Omega\times(0,T).$$

Moreover, for any $1\le p<q$

$$(n_\varepsilon)_{\varepsilon\in(0,1)} \text{ is strongly precompact in } L^p(\Omega\times(0,T)).$$

Proof. The proof can be found in Lemma 1.4 in [27]. □

Lemma 4.8. *Let $T>0$, Ω be any bounded domain in $\mathbb{R}^d$, $d=2,3$, and q satisfy (4.47). Suppose that $(n_\varepsilon,c_\varepsilon,u_\varepsilon)$ is a regular solution of (4.46) in $\mathbb{R}^d\times(0,T)$. Then, for $2<r<2q$ if $d=2$ or $\frac{5}{3}\le r<\frac{3q}{2}$ if $d=3$*

$$(c_\varepsilon)_{\varepsilon\in(0,1)} \text{ is strongly precompact in } L^r(0,T;W^{1,\hat r}(\Omega)),$$

where $\hat r\in[\frac{dr}{d+r},r)$.

Proof. Let $r'=\frac{dr}{d+r}$ for $d=2,3$. Using (4.52) and (4.80), Hölder's inequality implies

$$\|u_\varepsilon\cdot\nabla c_\varepsilon\|_{L^r(0,T;L^{r'}(\Omega))}\le\|u_\varepsilon\|_{L^\infty(0,T;L^d(\Omega))}\|\nabla c_\varepsilon\|_{L^r(\Omega\times(0,T))}. \tag{4.102}$$

Since $r'<r$, (4.56) implies that

$$c_\varepsilon\in L^r(0,T;L^{r'}(\Omega)). \tag{4.103}$$

In view of the maximal estimate (2.9), collecting (4.51), (4.102) and (4.103), we infer that

$$\partial_t c_\varepsilon,\ \Delta c_\varepsilon \in L^r(0,T;L^{r'}(\Omega)).$$

By the Rellich-Kondrachov theorem, the Sobolev space $W^{2,r'}(\Omega)$ is compactly embedded in $W^{1,\hat{r}}(\Omega)$ for any $\hat{r}\in[1,r)$. Again using the Rellich-Kondrachov theorem, we see that $W^{1,\hat{r}}(\Omega)$ is continuously embedded in $L^r(\Omega)$ for any $\hat{r}\in[r',r)$. Now, the Aubin-Lions lemma (Theorem 2.3 in [23]) can be applied and completes the proof. □

Proof of Theorem 1.2. From Lemmas 4.1–4.8 and (4.49), we can pick a sequence $\varepsilon=\varepsilon_j$ such that for any bounded $\Omega\subset\mathbb{R}^d$ and $T>0$, there exists (n,c,u) satisfying the following convergences

$$n_\varepsilon \to n \ \text{ a.e. in } \ \Omega\times(0,T), \tag{4.104}$$
$$n_\varepsilon^{\frac{\gamma}{2}} \rightharpoonup n^{\frac{\gamma}{2}} \ \text{ in } \ L^2(0,T;W^{1,2}_{loc}(\mathbb{R}^d)), \tag{4.105}$$
$$n_\varepsilon \rightharpoonup n \ \text{ in } \ L^q_{loc}(\mathbb{R}^d\times(0,T)), \tag{4.106}$$
$$n_\varepsilon \to n \ \text{ in } \ L^p_{loc}(\mathbb{R}^d\times(0,T)), \tag{4.107}$$
$$c_\varepsilon \rightharpoonup c \ \text{ in } \ L^r(0,T;W^{1,r}_{loc}(\mathbb{R}^d)), \tag{4.108}$$
$$c_\varepsilon \to c \ \text{ in } \ L^r(0,T;W^{1,\hat{r}}_{loc}(\mathbb{R}^d)), \tag{4.109}$$
$$u_\varepsilon \rightharpoonup u \ \text{ in } \ L^r(0,T;W^{1,r}_{loc}(\mathbb{R}^d)), \tag{4.110}$$

as $\varepsilon=\varepsilon_j\searrow 0$ for any $\hat{r}\in[\frac{dr}{d+r},r)$, $p\in(0,q)$, $2<r<2q$ if $d=2$ or $\frac{5}{3}\le r<\frac{3q}{2}$ if $d=3$, and for any $\gamma\in(0,1)$ satisfying $\gamma\le q-1$. To show the convergence to a γ-very weak solution, let us choose a nonnegative test function $\varphi\in C_0^\infty(\Omega\times[0,T))$ such that $\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega\times(0,T)$ and $\operatorname{supp}(\varphi)\subset\Omega\times[0,T)$ for any bounded $\Omega\subset\mathbb{R}^d$. Multiplying (4.46) by φ and integrating by parts, we obtain

$$\begin{aligned}
-\int_0^T\!\!\int_{\mathbb{R}^d} n_\varepsilon\partial_t\varphi - \int_{\mathbb{R}^d} n_0\varphi(\cdot,0) &= \int_0^T\!\!\int_{\mathbb{R}^d} n_\varepsilon\Delta\varphi + \int_0^T\!\!\int_{\mathbb{R}^d} n_\varepsilon\nabla c_\varepsilon\cdot\nabla\varphi - \int_0^T\!\!\int_{\mathbb{R}^d}\mu {n_\varepsilon}^q\varphi \\
&\quad - \varepsilon\int_0^T\!\!\int_{\mathbb{R}^d}\mu {n_\varepsilon}^\beta\varphi + \int_0^T\!\!\int_{\mathbb{R}^d} n_\varepsilon u_\varepsilon\cdot\nabla\varphi,
\end{aligned}$$

$$-\int_0^T\!\!\int_{\mathbb{R}^d} c_\varepsilon\partial_t\varphi - \int_{\mathbb{R}^d} c_0\varphi(\cdot,0) = -\int_0^T\!\!\int_{\mathbb{R}^d}\nabla c_\varepsilon\cdot\nabla\varphi - \int_0^T\!\!\int_{\mathbb{R}^d} c_\varepsilon\varphi + \int_0^T\!\!\int_{\mathbb{R}^d} n_\varepsilon\varphi + \int_0^T\!\!\int_{\mathbb{R}^d} c_\varepsilon u_\varepsilon\cdot\nabla\varphi,$$

$$-\int_0^T\!\!\int_{\mathbb{R}^d} u_\varepsilon\cdot\partial_t\varphi - \int_{\mathbb{R}^d} u_0\cdot\varphi(\cdot,0) = -\int_0^T\!\!\int_{\mathbb{R}^d}\nabla u_\varepsilon\cdot\nabla\varphi - \int_0^T\!\!\int_{\mathbb{R}^d} n_\varepsilon\nabla\phi\cdot\varphi.$$

By (4.106) and the properties of φ, we have

$$-\int_0^T\!\!\int_{\mathbb{R}^d} n_\varepsilon\varphi_t \to -\int_0^T\!\!\int_{\mathbb{R}^d} n\varphi_t, \tag{4.111}$$

$$\int_0^T\!\!\int_{\mathbb{R}^d} n_\varepsilon\Delta\varphi \to \int_0^T\!\!\int_{\mathbb{R}^d} n\Delta\varphi, \tag{4.112}$$

$$\int_0^T \int_{\mathbb{R}^d} n_\varepsilon \varphi \to \int_0^T \int_{\mathbb{R}^d} n\varphi. \tag{4.113}$$

From (4.107) and (4.108), we choose $p < q$ and $r < \frac{dq}{d-1}$ sufficiently close to q and $\frac{dq}{d-1}$, respectively. Then, we infer that

$$\int_0^T \int_{\mathbb{R}^d} n_\varepsilon \nabla c_\varepsilon \cdot \nabla\varphi \to \int_0^T \int_{\mathbb{R}^d} n\nabla c \cdot \nabla\varphi. \tag{4.114}$$

We apply Fatou's lemma to the degradation term such that

$$-\int_0^T \int_{\mathbb{R}^d} \mu n^q \varphi \geq -\limsup_{\varepsilon=\varepsilon_j \searrow 0} \int_0^T \int_{\mathbb{R}^d} \mu {n_\varepsilon}^q \varphi. \tag{4.115}$$

The nonnegativity of n_ε implies

$$-\varepsilon \int_0^T \int_{\mathbb{R}^d} \mu {n_\varepsilon}^\beta \varphi \leq 0. \tag{4.116}$$

Moreover, due to (4.107) and (4.110), we choose $p < q$ and $r < \frac{dq}{d-1}$ sufficiently close to q and $\frac{dq}{d-1}$, respectively, to obtain

$$\int_0^T \int_{\mathbb{R}^d} n_\varepsilon u_\varepsilon \cdot \nabla\varphi \to \int_0^T \int_{\mathbb{R}^d} nu \cdot \nabla\varphi. \tag{4.117}$$

Similarly to (4.111)–(4.113), we have

$$-\int_0^T \int_{\mathbb{R}^d} c_\varepsilon \partial_t \varphi \to -\int_0^T \int_{\mathbb{R}^d} c\partial_t\varphi, \tag{4.118}$$

$$-\int_0^T \int_{\mathbb{R}^d} \nabla c_\varepsilon \cdot \nabla\varphi \to -\int_0^T \int_{\mathbb{R}^d} \nabla c \cdot \nabla\varphi, \tag{4.119}$$

$$-\int_0^T \int_{\mathbb{R}^d} c_\varepsilon \varphi \to -\int_0^T \int_{\mathbb{R}^d} c\varphi. \tag{4.120}$$

From (4.109) and (4.110), we can choose $r, \hat{r} > 2$ so as to satisfy

$$\int_0^T \int_{\mathbb{R}^d} c_\varepsilon u_\varepsilon \cdot \nabla\varphi \to \int_0^T \int_{\mathbb{R}^d} cu \cdot \nabla\varphi. \tag{4.121}$$

Furthermore, (4.110) implies

$$-\int_0^T \int_{\mathbb{R}^d} u_\varepsilon \cdot \partial_t \varphi \to -\int_0^T \int_{\mathbb{R}^d} u \cdot \partial_t \varphi, \tag{4.122}$$

$$-\int_0^T \int_{\mathbb{R}^d} \nabla u_\varepsilon \cdot \nabla \varphi \to -\int_0^T \int_{\mathbb{R}^d} \nabla u \cdot \nabla \varphi. \tag{4.123}$$

The convergence of the last term follows from (4.107) such that

$$-\int_0^T \int_{\mathbb{R}^d} n_\varepsilon \nabla \phi \cdot \varphi \to -\int_0^T \int_{\mathbb{R}^d} n \nabla \phi \cdot \varphi. \tag{4.124}$$

Thus, from (4.111)–(4.124), (n, c, u) is a very weak sub-solution of (1.1) in $\mathbb{R}^d \times (0, T)$.
Now, we show that (n, c, u) is a γ-entropy super-solution. Multiplying the first equation of (1.1) by $\gamma n_\varepsilon^{\gamma-1}\varphi$ and integrating by parts, we obtain

$$\begin{aligned}
&-\int_0^T \int_{\mathbb{R}^d} n_\varepsilon{}^\gamma \partial_t \varphi - \int_{\mathbb{R}^d} n_0^\gamma \varphi(\cdot, 0) = \gamma(1-\gamma) \int_0^T \int_{\mathbb{R}^d} n_\varepsilon{}^{\gamma-2} |\nabla n_\varepsilon|^2 \varphi + \int_0^T \int_{\mathbb{R}^d} n_\varepsilon{}^\gamma \Delta \varphi + \gamma \int_0^T \int_{\mathbb{R}^d} n_\varepsilon{}^\gamma \nabla c_\varepsilon \cdot \nabla \varphi \\
&+ \gamma(\gamma-1) \int_0^T \int_{\mathbb{R}^d} \varphi n_\varepsilon{}^{\gamma-1} \nabla n_\varepsilon \cdot \nabla c_\varepsilon - \gamma \int_0^T \int_{\mathbb{R}^d} n_\varepsilon{}^{q+\gamma-1} \varphi - \gamma\varepsilon \int_0^T \int_{\mathbb{R}^d} \mu n_\varepsilon{}^{\beta+\gamma-1} \varphi + \int_0^T \int_{\mathbb{R}^d} n_\varepsilon{}^\gamma u_\varepsilon \cdot \nabla \varphi.
\end{aligned}$$

By (4.104), (4.107) and the properties of φ, we have for $0 < \gamma < 1$

$$-\int_0^T \int_{\mathbb{R}^d} n_\varepsilon{}^\gamma \partial_t \varphi \to -\int_0^T \int_{\mathbb{R}^d} n^\gamma \partial_t \varphi, \tag{4.125}$$

$$\int_0^T \int_{\mathbb{R}^d} n_\varepsilon{}^\gamma \Delta \varphi \to \int_0^T \int_{\mathbb{R}^d} n^\gamma \Delta \varphi. \tag{4.126}$$

Combining (4.104), (4.107) and inequality $|n_\varepsilon^\gamma - n^\gamma| < C_\gamma |n_\varepsilon - n|^\gamma$ for $0 < \gamma < 1$, we have the strong convergence such that $n_\varepsilon^\gamma \to n^\gamma$ in $L^p_{loc}(\mathbb{R}^d \times (0, T))$ so that

$$\gamma \int_0^T \int_{\mathbb{R}^d} n_\varepsilon{}^\gamma \nabla c_\varepsilon \cdot \nabla \varphi \to \gamma \int_0^T \int_{\mathbb{R}^d} n^\gamma \nabla c \cdot \nabla \varphi \tag{4.127}$$

holds. As to the convergence of $\gamma(\gamma-1) \int_0^T \int_{\mathbb{R}^d} \varphi n_\varepsilon{}^{\gamma-1} \nabla n_\varepsilon \cdot \nabla c_\varepsilon$, we write

$$\int_0^T \int_{\mathbb{R}^d} \varphi n_\varepsilon{}^{\gamma-1} \nabla n_\varepsilon \cdot \nabla c_\varepsilon = \frac{2}{\gamma} \int_0^T \int_{\mathbb{R}^d} \varphi n_\varepsilon^{\frac{\gamma}{2}} \nabla n_\varepsilon^{\frac{\gamma}{2}} \cdot \nabla c_\varepsilon. \tag{4.128}$$

We observe that $\gamma < q - \frac{2(d-1)}{d}$ is equivalent to

$$\frac{1}{2}+\frac{\gamma}{2q}+\frac{d-1}{dq}<1.$$

We choose $p<q$ and $r<\frac{dq}{d-1}$ close to q and $\frac{dq}{d-1}$, respectively, such that

$$\frac{1}{2}+\frac{\gamma}{2p}+\frac{1}{r}<1.$$

Again we choose $\hat{r}<r$ close to r such that

$$\frac{1}{2}+\frac{\gamma}{2p}+\frac{1}{\hat{r}}<1.$$

By making use of (4.105), (4.107) and (4.109), we infer from (4.128) that

$$\gamma(\gamma-1)\int_0^T\int_{\mathbb{R}^d}\varphi {n_\varepsilon}^{\gamma-1}\nabla n_\varepsilon\cdot\nabla c_\varepsilon\to\gamma(\gamma-1)\int_0^T\int_{\mathbb{R}^d}\varphi n^{\gamma-1}\nabla n\cdot\nabla c, \tag{4.129}$$

where $\gamma\in\left(0,q-\frac{2(d-1)}{d}\right)$. Further, as in (1.61) of [27], we have the following convergence for the degradation terms:

$$-\gamma\int_0^T\int_{\mathbb{R}^d}{n_\varepsilon}^{q+\gamma-1}\varphi\to-\gamma\int_0^T\int_{\mathbb{R}^d}n^{q+\gamma-1}\varphi, \tag{4.130}$$

$$\gamma\varepsilon\int_0^T\int_{\mathbb{R}^d}\mu {n_\varepsilon}^{\beta+\gamma-1}\varphi\to 0. \tag{4.131}$$

Again as in (1.66) of [27], we have

$$\gamma(1-\gamma)\int_0^T\int_{\mathbb{R}^d}n^{\gamma-2}|\nabla n|^2\varphi\le\gamma(1-\gamma)\liminf_{\varepsilon=\varepsilon_j\searrow 0}\int_0^T\int_{\mathbb{R}^d}n_\varepsilon^{\gamma-2}|\nabla n_\varepsilon|^2\varphi. \tag{4.132}$$

Similarly to (4.127), we finally obtain

$$\int_0^T\int_{\mathbb{R}^d}{n_\varepsilon}^{\gamma}u_\varepsilon\cdot\nabla\varphi\to\int_0^T\int_{\mathbb{R}^d}n^\gamma u\cdot\nabla\varphi. \tag{4.133}$$

Therefore, from (4.125)–(4.127) and (4.129)–(4.133), (n,c,u) is a γ-entropy super-solution of (1.1) in $\mathbb{R}^d\times(0,T)$. Note that Assumption 1.2 satisfies the initial conditions for Lemmas 4.2–4.8. This completes the proof. □

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Appendix A

We present the proof of blow-up criterion.

Proof of Theorem 2.2. We suppose to the contrary that there exists $C > 0$ such that

$$\limsup_{t \nearrow T_{\max}} \int_0^t \|\nabla c\|_{L^{d+2}}^{d+2} \le C. \tag{A.134}$$

Then, we show that the maximal time of existence can not be finite, which leads to a contradiction. Let $0 < t < T_{\max}$. For the estimate of (n, c, u) in $L^2(\mathbb{R}^d) \times H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$, we multiply n to both sides of the first equation of (1.1) and integrate over $\mathbb{R}^d$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|n\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla n\|_{L^2(\mathbb{R}^d)}^2 \le C\|\nabla c\|_{L^{d+2}(\mathbb{R}^d)}^{d+2}\|n\|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{2}\|\nabla n\|_{L^2(\mathbb{R}^d)}^2.$$

By Grönwall's inequality, we have

$$n \in L^\infty(0, t; L^2(\mathbb{R}^d)), \quad \nabla n \in L^2(\mathbb{R}^d \times (0, t)). \tag{A.135}$$

Moreover, we observe that for $2 < p < \infty$

$$\begin{aligned}
\frac{d}{dt}\|n\|_{L^p(\mathbb{R}^d)}^p + \|\nabla n^{\frac{p}{2}}\|_{L^2(\mathbb{R}^d)}^2 &\le C\int_{\mathbb{R}^d} |n\nabla c\nabla n^{p-1}| \\
&\le C\|\nabla c\|_{L^{d+2}(\mathbb{R}^d)}\|\nabla n^{\frac{p}{2}}\|_{L^2(\mathbb{R}^d)}\|n^{\frac{p}{2}}\|_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^d)} \\
&\le C\|\nabla c\|_{L^{d+2}(\mathbb{R}^d)}\|\nabla n^{\frac{p}{2}}\|_{L^2(\mathbb{R}^d)}^{\frac{2(d+1)}{d+2}}\|n^{\frac{p}{2}}\|_{L^2(\mathbb{R}^d)}^{\frac{2}{d+2}} \\
&\le C\|\nabla c\|_{L^{d+2}(\mathbb{R}^d)}^{d+2}\|n\|_{L^p(\mathbb{R}^d)}^p + \frac{1}{2}\|\nabla n^{\frac{p}{2}}\|_{L^2(\mathbb{R}^d)}^2,
\end{aligned}$$

where C is independent of p. Therefore, by (A.134), we have $n \in L^\infty(0, t; L^p(\mathbb{R}^d))$ and $\nabla n^{\frac{p}{2}} \in L^2(\mathbb{R}^d \times (0, t))$. Since $\|n\|_{L^p(\mathbb{R}^d)} \le C$, where C is independent of p, we have

$$n \in L^\infty(\mathbb{R}^d \times (0, t)). \tag{A.136}$$

For the estimate of u, multiplying u to both sides of the third equation of (1.1) and integrating over $\mathbb{R}^d$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 \le C\|n\|_{L^2(\mathbb{R}^d)}\|u\|_{L^2(\mathbb{R}^d)} \le C\|u\|_{L^2(\mathbb{R}^d)}^2 + C\|n\|_{L^2(\mathbb{R}^d)}^2.$$

Then, due to Grönwall's inequality, we observe that

$$u \in L^\infty(0, T; L^2(\mathbb{R}^d)), \quad \nabla u \in L^2(\mathbb{R}^d \times (0, T)). \tag{A.137}$$

Moreover, multiplying $-\Delta u$ to both sides of the third equation of (1.1) and integrating over $\mathbb{R}^d$, we have for $d = 2$

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2(\mathbb{R}^2)}^2 + \|\Delta u\|_{L^2(\mathbb{R}^2)}^2 \le C\|\nabla u\|_{L^\infty(\mathbb{R}^2)}\|\nabla u\|_{L^2(\mathbb{R}^2)}^2 + C\|n\|_{L^2(\mathbb{R}^2)}\|\Delta u\|_{L^2(\mathbb{R}^2)},$$

and for $d = 3$

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2_{L^2(\mathbb{R}^3)} + \|\Delta u\|^2_{L^2(\mathbb{R}^3)} \le C\|n\|_{L^2(\mathbb{R}^3)}\|\Delta u\|_{L^2(\mathbb{R}^3)} \le C\|n\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{2}\|\Delta u\|^2_{L^2(\mathbb{R}^3)}.$$

Therefore, we have

$$\nabla u \in L^\infty(0,T;L^2(\mathbb{R}^d)), \quad \Delta u \in L^2(\mathbb{R}^d \times (0,T)). \tag{A.138}$$

For the estimate of c in $H^1(\mathbb{R}^d)$, we need to show that $\int_0^t \|\nabla u\|_{L^\infty(\mathbb{R}^d)} \le C$. We recall the vorticity equation such that

$$\partial_t w - \Delta w + \kappa u \nabla w = -\nabla \times (n\nabla\phi),$$

where $w = \nabla \times u$. Multiplying $-\Delta w$ to both sides of the above equation and integrating over $\mathbb{R}^d$, we have for $d = 2$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla w\|^2_{L^2(\mathbb{R}^2)} + \|\Delta w\|^2_{L^2(\mathbb{R}^2)} \\
&\le \|u\|_{L^4(\mathbb{R}^2)}\|\nabla w\|_{L^4(\mathbb{R}^2)}\|\Delta w\|_{L^2(\mathbb{R}^2)} + C\|\nabla n\|_{L^2(\mathbb{R}^2)}\|\Delta w\|_{L^2(\mathbb{R}^2)} \\
&\le C\|u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\nabla u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\nabla w\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\Delta w\|^{\frac{3}{2}}_{L^2(\mathbb{R}^2)} + C\|\nabla n\|_{L^2(\mathbb{R}^2)}\|\Delta w\|_{L^2(\mathbb{R}^2)},
\end{aligned}$$

and for $d = 3$

$$\frac{1}{2}\frac{d}{dt}\|\nabla w\|^2_{L^2(\mathbb{R}^3)} + \|\Delta w\|^2_{L^2(\mathbb{R}^3)} \le C\|\nabla n\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{2}\|\Delta w\|^2_{L^2(\mathbb{R}^3)}.$$

Hence, in view of (A.135), (A.137) and (A.138), Grönwall's inequality implies

$$\nabla w \in L^\infty(0,t;L^2(\mathbb{R}^d)), \quad \Delta w \in L^2(\mathbb{R}^d \times (0,t)). \tag{A.139}$$

Therefore, via the Sobolev embedding, we have

$$\int_0^t \|\nabla u\|_{L^\infty(\mathbb{R}^d)} \le \int_0^t \|\nabla u\|_{H^2(\mathbb{R}^d)} \le C\int_0^t \|w\|_{H^2(\mathbb{R}^d)} \le C \tag{A.140}$$

as desired. Now, multiplying $-\Delta c$ to both sides of above equation and integrating over $\mathbb{R}^d$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla c\|^2_{L^2(\mathbb{R}^d)} + \|\Delta c\|^2_{L^2(\mathbb{R}^d)} + \|\nabla c\|^2_{L^2(\mathbb{R}^d)} \\
&\le C\|\nabla u\|_{L^\infty(\mathbb{R}^d)}\|\nabla c\|^2_{L^2(\mathbb{R}^d)} + \|n\|_{L^2(\mathbb{R}^d)}\|\Delta c\|_{L^2(\mathbb{R}^d)} \\
&\le C\|\nabla u\|_{L^\infty(\mathbb{R}^d)}\|\nabla c\|^2_{L^2(\mathbb{R}^d)} + C\|n\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\|\Delta c\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

By (A.135) and (A.140), we see that

$$c \in L^\infty(0,t;H^1(\mathbb{R}^d)) \cap L^2(0,t;H^2(\mathbb{R}^d)). \tag{A.141}$$

It is direct from (A.139) and (A.140) that

$$u \in L^\infty(0,t;H^1(\mathbb{R}^d)) \cap L^2(0,t;H^2(\mathbb{R}^d)). \tag{A.142}$$

Next, we consider the estimate of (n,c,u) in $H^1(\mathbb{R}^d)\times H^2(\mathbb{R}^d)\times H^2(\mathbb{R}^d)$. Multiplying $-\Delta n$ to both sides of the first equation of (1.1) and integrating over $\mathbb{R}^d$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla n\|^2_{L^2(\mathbb{R}^d)} + \|\Delta n\|^2_{L^2(\mathbb{R}^d)}\\
&\le C\|\nabla u\|_{L^\infty(\mathbb{R}^d)}\|\nabla n\|^2_{L^2(\mathbb{R}^d)} + C\|\nabla n\|_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^d)}\|\nabla c\|_{L^{d+2}(\mathbb{R}^d)}\|\Delta n\|_{L^2(\mathbb{R}^d)}\\
&+ C\|n\|_{L^\infty(\mathbb{R}^d)}\|\Delta c\|_{L^2(\mathbb{R}^d)}\|\Delta n\|_{L^2(\mathbb{R}^d)} + C\|n^q\|_{L^2(\mathbb{R}^d)}\|\Delta n\|_{L^2(\mathbb{R}^d)}\\
&\le C\|\nabla u\|_{L^\infty(\mathbb{R}^d)}\|\nabla n\|^2_{L^2(\mathbb{R}^d)} + C\|\nabla n\|^2_{L^2(\mathbb{R}^d)}\|\nabla c\|^{d+2}_{L^{d+2}(\mathbb{R}^d)}\\
&+ C\|n\|^2_{L^\infty(\mathbb{R}^d)}\|\Delta c\|^2_{L^2(\mathbb{R}^d)} + C\|n^q\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\|\Delta n\|^2_{L^2(\mathbb{R}^d)}\\
&\le C\left(\|\nabla u\|_{L^\infty(\mathbb{R}^d)} + \|\nabla c\|^{d+2}_{L^{d+2}(\mathbb{R}^d)}\right)\|\nabla n\|^2_{L^2(\mathbb{R}^d)} + C\|n\|^2_{L^\infty(\mathbb{R}^d)}\|\Delta c\|^2_{L^2(\mathbb{R}^d)} + C\|n^q\|^2_{L^2(\mathbb{R}^d)}\\
&\le C\left(\|\nabla u\|_{L^\infty(\mathbb{R}^d)} + \|\nabla c\|^{d+2}_{L^{d+2}(\mathbb{R}^d)}\right)\|\nabla n\|^2_{L^2(\mathbb{R}^d)} + C\|n\|^2_{L^\infty(\mathbb{R}^d)}\|\Delta c\|^2_{L^2(\mathbb{R}^d)} + C\|n\|^{2q-1}_{L^\infty(\mathbb{R}^d)}\|n\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

Therefore, we see that

$$\nabla n \in L^\infty(0,t;L^2(\mathbb{R}^d)),\ \ \Delta n \in L^2(\mathbb{R}^d\times(0,t)).$$

It follows that

$$n \in L^\infty(0,t;H^1(\mathbb{R}^d)) \cap L^2(0,t;H^2(\mathbb{R}^d)). \tag{A.143}$$

For the estimate of c in $H^2(\mathbb{R}^d)$, multiplying $\Delta^2 c$ to both sides of the second equation of (1.1) and integrating over $\mathbb{R}^d$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta c\|^2_{L^2(\mathbb{R}^d)} + \|\nabla\Delta c\|^2_{L^2(\mathbb{R}^d)} + \|\Delta c\|^2_{L^2(\mathbb{R}^d)}\\
&\le C\|\nabla u\|_{L^\infty(\mathbb{R}^d)}\|\Delta c\|^2_{L^2(\mathbb{R}^d)} + C\|\Delta u\|_{L^2(\mathbb{R}^d)}\|c\|_{L^\infty(\mathbb{R}^d)}\|\nabla\Delta c\|_{L^2(\mathbb{R}^d)} + \|\nabla n\|_{L^2(\mathbb{R}^d)}\|\nabla\Delta c\|_{L^2(\mathbb{R}^d)}\\
&\le C\|\nabla u\|_{L^\infty(\mathbb{R}^d)}\|\Delta c\|^2_{L^2(\mathbb{R}^d)} + C\|\Delta u\|^2_{L^2(\mathbb{R}^d)}\|c\|^2_{L^\infty(\mathbb{R}^d)} + C\|\nabla n\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\|\nabla\Delta c\|^2_{L^2(\mathbb{R}^d)},
\end{aligned}$$

which, together with (A.141)–(A.143), implies

$$\Delta c \in L^\infty(0,t;L^2(\mathbb{R}^d)),\ \ \nabla\Delta c \in L^2(\mathbb{R}^d\times(0,t)).$$

Hence, we obtain

$$c \in L^\infty(0,t;H^2(\mathbb{R}^d)) \cap L^2(0,t;H^3(\mathbb{R}^d)). \tag{A.144}$$

Similarly, we compute for $d=2$

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2_{L^2(\mathbb{R}^2)} + \|\nabla\Delta u\|^2_{L^2(\mathbb{R}^2)} \le C\|\Delta n\|_{L^2(\mathbb{R}^2)}\|\Delta u\|_{L^2(\mathbb{R}^2)} + C\|\nabla u\|_{L^\infty(\mathbb{R}^2)}\|\Delta u\|^2_{L^2(\mathbb{R}^2)},$$

and for $d=3$

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2_{L^2(\mathbb{R}^3)} + \|\nabla\Delta u\|^2_{L^2(\mathbb{R}^3)} \le C\|\Delta n\|_{L^2(\mathbb{R}^3)}\|\Delta u\|_{L^2(\mathbb{R}^3)}.$$

Therefore, we have

$$\Delta u \in L^\infty(0,t;L^2(\mathbb{R}^d)), \;\; \nabla\Delta u \in L^2(\mathbb{R}^d \times (0,t)),$$

and thereby we have

$$u \in L^\infty(0,t;H^2(\mathbb{R}^d)) \cap L^2(0,t;H^3(\mathbb{R}^d)). \tag{A.145}$$

Then, we consider the estimate of (n,c,u) in $H^2(\mathbb{R}^d)\times H^3(\mathbb{R}^d)\times H^3(\mathbb{R}^d)$. For the $H^2(\mathbb{R}^d)$ estimate of n, we observe that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|n\|^2_{H^2(\mathbb{R}^d)} + \|\nabla n\|^2_{H^2(\mathbb{R}^d)}\\
&\le C\|u\|_{L^\infty(\mathbb{R}^d)}\|n\|_{H^2(\mathbb{R}^d)}\|\nabla n\|_{H^2(\mathbb{R}^d)} + C\|\nabla u\|_{L^\infty(\mathbb{R}^d)}\|n\|_{H^1(\mathbb{R}^d)}\|\nabla n\|_{H^2(\mathbb{R}^d)}\\
&+ C\|n\|_{H^2(\mathbb{R}^d)}\|\nabla c\|_{H^2(\mathbb{R}^d)}\|\nabla n\|_{H^2(\mathbb{R}^d)} + C\|n^q\|_{H^2(\mathbb{R}^d)}\|n\|_{H^2(\mathbb{R}^d)}\\
&\le C\|u\|^2_{L^\infty(\mathbb{R}^d)}\|n\|^2_{H^2(\mathbb{R}^d)} + C\|\nabla u\|^2_{L^\infty(\mathbb{R}^d)}\|n\|^2_{H^1(\mathbb{R}^d)} + C\|n\|^2_{H^2(\mathbb{R}^d)}\|\nabla c\|^2_{H^2(\mathbb{R}^d)}\\
&+ C\|n\|^{q-1}_{H^2(\mathbb{R}^d)}\|n\|^2_{H^2(\mathbb{R}^d)} + \frac{1}{2}\|\nabla n\|^2_{H^2(\mathbb{R}^d)}.
\end{aligned}$$

Then, with the aid of (A.143)–(A.145), we have

$$n \in L^\infty(0,t;H^2(\mathbb{R}^d)) \cap L^2(0,t;H^3(\mathbb{R}^d)). \tag{A.146}$$

Similarly, we observe that for $d=2$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|^2_{H^3(\mathbb{R}^2)} + \|\nabla u\|^2_{H^3(\mathbb{R}^2)} &\le C\|\nabla u\|_{L^\infty(\mathbb{R}^2)}\|u\|_{H^3(\mathbb{R}^2)}\|\nabla u\|_{H^3(\mathbb{R}^2)}\\
&+ \frac{1}{4}\|\nabla u\|^2_{H^3(\mathbb{R}^2)} + C\|n\|^2_{H^2(\mathbb{R}^2)},
\end{aligned}$$

and for $d=3$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|^2_{H^3(\mathbb{R}^3)} + \|\nabla u\|^2_{H^3(\mathbb{R}^3)} &\le C\|n\|_{H^2(\mathbb{R}^3)}\|\nabla u\|_{H^3(\mathbb{R}^3)}\\
&\le C\|n\|^2_{H^2(\mathbb{R}^3)} + \frac{1}{2}\|\nabla u\|^2_{H^3(\mathbb{R}^3)}.
\end{aligned}$$

By (A.143), we have

$$u \in L^\infty(0,t;H^3(\mathbb{R}^d)) \cap L^2(0,t;H^4(\mathbb{R}^d)). \tag{A.147}$$

For the sake of c, we compute such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|c\|^2_{H^3(\mathbb{R}^d)} + \|\nabla c\|^2_{H^3(\mathbb{R}^d)} + \|c\|^2_{H^3(\mathbb{R}^d)}\\
&\le C\|\nabla u\|_{L^\infty(\mathbb{R}^d)}\|c\|_{H^3(\mathbb{R}^d)}\|\nabla c\|_{H^3(\mathbb{R}^d)} + C\|u\|_{H^3(\mathbb{R}^d)}\|\nabla c\|_{L^\infty(\mathbb{R}^d)}\|\nabla c\|_{H^3(\mathbb{R}^d)} + \|n\|_{H^2(\mathbb{R}^d)}\|\nabla c\|_{H^3(\mathbb{R}^d)}\\
&\le C\|\nabla u\|^2_{L^\infty(\mathbb{R}^d)}\|c\|^2_{H^3(\mathbb{R}^d)} + C\|u\|^2_{H^3(\mathbb{R}^d)}\|\nabla c\|^2_{L^\infty(\mathbb{R}^d)} + C\|n\|^2_{H^2(\mathbb{R}^d)} + \frac{1}{2}\|\nabla c\|^2_{H^3(\mathbb{R}^d)}.
\end{aligned}$$

From (A.136), (A.140), (A.143), (A.144) and (A.147), we obtain

$$c \in L^\infty(0,t;H^3(\mathbb{R}^d)) \cap L^2(0,t;H^4(\mathbb{R}^d)). \tag{A.148}$$

Thus, by collecting (A.146)–(A.148), we have that for all $t < T_{\max}$

$$(n,c,u) \in L^\infty(0,t;H^2(\mathbb{R}^d) \times H^3(\mathbb{R}^d) \times H^3(\mathbb{R}^d)).$$

For $m \geq 4$, proceeding similarly to the above, we obtain

$$(n,c,u) \in L^\infty(0,t;H^{m-1}(\mathbb{R}^d) \times H^m(\mathbb{R}^d) \times H^m(\mathbb{R}^d)),$$

which contradicts the blow-up criterion since $t < T_{\max}$ is arbitrary. This completes the proof. □

References

[1] Ken Abe, Yoshikazu Giga, The L^∞-Stokes semigroup in exterior domains, J. Evol. Equ. 14 (1) (2014) 1–28.
[2] Thierry Cazenave, Semilinear Schrodinger Equations, American Mathematical Soc., 2003.
[3] Myeongju Chae, Kyungkeun Kang, Jihoon Lee, Existence of smooth solutions to coupled chemotaxis-fluid equations, Discrete Contin. Dyn. Syst. Ser. A 33 (6) (2013) 2271–2297.
[4] Myeongju Chae, Kyungkeun Kang, Jihoon Lee, Global existence and temporal decay in Keller-Segel models coupled to fluid equations, Comm. Partial Differential Equations 39 (7) (2014) 1205–1235.
[5] Myeongju Chae, Kyungkeun Kang, Jihoon Lee, Ki-Ahm Lee, A regularity condition and temporal asymptotics for chemotaxis-fluid equations, Nonlinearity 31 (2) (2018) 351.
[6] J.C. Coll, B.F. Bowden, G.V. Meehan, G.M. Konig, A.R. Carroll, D.M. Tapiolas, P.M. Alino, A. Heaton, R. De Nys, P.A. Leone, et al., Chemical aspects of mass spawning in corals. I. Sperm-attractant molecules in the eggs of the scleractinian coral montipora digitata, Mar. Biol. 118 (2) (1994) 177–182.
[7] Elio Espejo, Takashi Suzuki, Reaction terms avoiding aggregation in slow fluids, Nonlinear Anal. Real World Appl. 21 (2015) 110–126.
[8] Gerald B. Folland, Real Analysis: Modern Techniques and Their Applications, John Wiley & Sons, 2013.
[9] Yoshikazu Giga, Hermann Sohr, Abstract L^p estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains, J. Funct. Anal. 102 (1) (1991) 72–94.
[10] Kyungkeun Kang, Jae-Myoung Kim, Regularity criteria of the magnetohydrodynamic equations in bounded domains or a half space, J. Differential Equations 253 (2) (2012) 764–794.
[11] Evelyn F. Keller, Lee A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret. Biol. 26 (3) (1970) 399–415.
[12] Alexander Kiselev, Lenya Ryzhik, Biomixing by chemotaxis and enhancement of biological reactions, Comm. Partial Differential Equations 37 (2) (2012) 298–318.
[13] Olga Aleksandrovna Ladyzhenskaia, Vsevolod Alekseevich Solonnikov, Nina N. Ural'ceva, Linear and Quasi-Linear Equations of Parabolic Type, vol. 23, American Mathematical Soc., 1968.
[14] Johannes Lankeit, Chemotaxis can prevent thresholds on population density, arXiv preprint, arXiv:1403.1837, 2014.
[15] Gary M. Lieberman, Second Order Parabolic Differential Equations, World Scientific, 1996.
[16] Ji Liu, Yifu Wang, Boundedness and decay property in a three-dimensional Keller–Segel–Stokes system involving tensor-valued sensitivity with saturation, J. Differential Equations 261 (2) (2016) 967–999.
[17] Alexander Lorz, A coupled Keller–Segel–Stokes model: global existence for small initial data and blow-up delay, Commun. Math. Sci. 10 (2) (2012) 555–574.
[18] Richard L. Miller, Demonstration of sperm chemotaxis in echinodermata: Asteroidea, Holothuroidea, Ophiuroidea, J. Exp. Zool. Part A: Ecol. Genet. Physiol. 234 (3) (1985) 383–414.
[19] Roberta Dal Passo, Harald Garcke, Günther Grün, On a fourth-order degenerate parabolic equation: global entropy estimates, existence, and qualitative behavior of solutions, SIAM J. Math. Anal. 29 (2) (1998) 321–342.
[20] Youshan Tao, Michael Winkler, Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system, Z. Angew. Math. Phys. 66 (5) (2015) 2555–2573.
[21] Youshan Tao, Michael Winkler, Blow-up prevention by quadratic degradation in a two-dimensional Keller–Segel–Navier–Stokes system, Z. Angew. Math. Phys. 67 (6) (2016) 138.
[22] J. Ignacio Tello, Michael Winkler, A chemotaxis system with logistic source, Comm. Partial Differential Equations 32 (6) (2007) 849–877.
[23] Roger Temam, Navier-Stokes Equations: Theory and Numerical Analysis, vol. 343, American Mathematical Soc., 2001.
[24] Idan Tuval, Luis Cisneros, Christopher Dombrowski, Charles W. Wolgemuth, John O. Kessler, Raymond E. Goldstein, Bacterial swimming and oxygen transport near contact lines, Proc. Natl. Acad. Sci. USA 102 (7) (2005) 2277–2282.
[25] Giuseppe Viglialoro, Very weak global solutions to a parabolic–parabolic chemotaxis-system with logistic source, J. Math. Anal. Appl. 439 (1) (2016) 197–212.
[26] Yulan Wang, Zhaoyin Xiang, Global existence and boundedness in a Keller–Segel–Stokes system involving a tensor-valued sensitivity with saturation: the 3d case, J. Differential Equations 261 (9) (2016) 4944–4973.
[27] Michael Winkler, Chemotaxis with logistic source: very weak global solutions and their boundedness properties, J. Math. Anal. Appl. 348 (2) (2008) 708–729.

[28] Michael Winkler, Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction, J. Math. Anal. Appl. 2 (384) (2011) 261–272.
[29] Michael Winkler, How far can chemotactic cross-diffusion enforce exceeding carrying capacities?, J. Nonlinear Sci. 24 (5) (2014) 809–855.