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Long-time asymptotics for a mixed nonlinear Schrödinger equation with the Schwartz initial data

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A B S T R A C T

In this paper, we define a general analytical domain and two reflection coefficients to study the long-time asymptotics for a defocusing mixed nonlinear Schrödinger equation with the Schwartz initial data. It is shown that the solution of the initial value problem for the mixed Schrödinger equation can be expressed in terms of the solution associated with a Riemann-Hilbert problem. The long-time asymptotic of the mixed Schrödinger equation is further obtained via the Deift-Zhou nonlinear steepest descent method. The long-time asymptotic for the classical Schrödinger equation, derivative Schrödinger equation and modified Schrödinger equation are obtained directly from our results as special cases.

1. Introduction

The nonlinear Schrödinger equation is one of the most celebrated soliton equations, which has been widely used for the description of various nonlinear phenomena, such as quantum field theory, weakly nonlinear dispersive water waves, and nonlinear optics [10,22,29]. To study the effect of higher-order perturbations, various modifications and generalizations of the NLS equations have been proposed and studied [23,31]. 1970s, Wadati et al. proposed a kind of mixed NLS equation

$$q_t + iq_{xx} - ia(|q|^2 q)_x - 2b^2|q|^2 q = 0, \tag{1.1}$$

which was further solved by the inverse scattering method [25,47]. The equation (1.1) can be used to describe Alfvén waves propagating along the magnetic field in cold plasmas and the deep-water gravity waves [38,43]. The term $i(|q|^2 q)_x$ in the equation (1.1) is called the self-steepening term, which causes an optical pulse to become asymmetric and steepen upward at the trailing edge [1,53]. The equation (1.1) also describes the short pulses propagate in a long optical fiber characterized by a nonlinear refractive index [39,46]. Brizhik

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et al. showed that the NLS equation (1.1), unlike the classical NLS equation, possesses static localized solutions when the effective nonlinearity parameter is larger than a certain critical value [7].

For $a = 0$, the equation (1.1) reduces the classical defocusing Schrödinger equation

$$iu_t + u_{xx} - 2b^2|u|^2u = 0, \tag{1.2}$$

which has important applications in a wide variety of fields such as nonlinear optics, deep water waves, plasma physics, etc. [4,14].

For $b = 0$, the equation (1.1) reduces to the Kaup-Newell equation [24]

$$iu_t + u_{xx} - ia(|u|^2u)_x = 0, \tag{1.3}$$

which has been shown to be gauge equivalent to the Chen-Lee-Liu (CLL) equation [8,16]

$$iu_t + u_{xx} + i|u|^2u_x = 0 \tag{1.4}$$

and the Gerdjikov-Ivanov equation [48]

$$iu_t + u_{xx} - iu^2u_x^* + \frac{1}{2}|u|^4u = 0. \tag{1.5}$$

For $a, b \neq 0$, the equation (1.1) is equivalent to the modified NLS equation

$$q_t + q_{xx} + ia(|q|^2q)_x - 2|q|^2q = 0, \tag{1.6}$$

which was also called the perturbation NLS equation [36].

In recent years, there are various works on mixed NLS equation (1.1) and the modified NLS equation (1.6). For example, Date constructed the periodic soliton solutions [42]. In 1991, Guo and Tian studied the unique existence and the decay behaviors of the smooth solutions to the initial value problem [19]. And then in 1994, Tian and Zhang proved that the Cauchy problem of initial value in Sobolev space has a unique weak solution [45]. The existence of global solution and blow-up for this equation were also investigated [44]. The bright, dark envelope solutions and Soliton behavior with N-fold Darboux transformation for mixed nonlinear Schrödinger equation were also discussed [21,33,34].

The IST procedure, as one of the most powerful tool to investigate solitons of nonlinear models, was first discovered by Gardner, Green, Kruskal and Miura [17]. The IST for the focusing NLS equation with zero boundary conditions was first developed by Zakharov and Shabat [55], later for the defocusing case with nonzero boundary conditions [56]. The next important step of the development of IST method is the Riemann-Hilbert (RH) method as the modern version of IST was established by Zakharov and Shabat [57], which involves the determination of a analytic function in given sectors of the complex plane, from the knowledge of the jumps of this function across the boundaries of the sectors. It has since become clear that the RH method is applicable to construction and asymptotic analysis of solutions for a wide class of integrable systems [2,3,20,30,40,41,50,58,59]. In 1993, based on the RH problem and by deforming contours in the spirit of the classical method of steepest descent, Deift and Zhou developed nonlinear steepest descent method to obtain the long-time asymptotics behavior of the solution rigorously for the MKdV equation [11,13]. It subsequently has become a powerful tool for the long-time asymptotics of the nonlinear evolution equations in complete integrable system, for example, the non-focusing NLS equation [9], the modified Schrodinger equation [26–28], the KdV equation [18], the Camassa-Holm equation [6], Fokas-Lenells equation [51], Kundu-Eckhaus equation [60] and Harry Dym equation [49].

In this paper, by defining a general domain (2.25) and its boundary (2.26) Σ, also using two reflection coefficients (3.11) (see next section 2), we apply Deift-Zhou nonlinear steepest descent method to uniformly study the long-time asymptotics for the defocusing mixed nonlinear Schrödinger equation (1.1) with Schwartz decaying initial value

$$q(t=0,x)=q_0(x)\in\mathcal{S}(R).$$

Our result could recover the long-time asymptotics for the classical defocusing Schrödinger equation (1.2), defocusing derivative Schrödinger equation (1.3) and defocusing modified Schrödinger equation (1.6) as special cases of results.

The structure of this manuscript is the following. In section 2, starting from the Lax pair of the mixed NLS equation, we construct Jost solutions and spectral matrix, and their analyticity symmetries are further analyzed. In section 3, based on the results obtained in section 2, we establish a RH problem on the jump contour Σ, which further is changed into a new RH problem on real axis R as the jump contour. In section 4, through the decomposition of the jump matrix, rational approximation of scattering data, scaling transformation and taking limit with respect to t, we change the RH problem into a model RH problem which can be solved via the Weber equation. In section 5, based on the reconstruction formula between solutions of the mixed NLS equation and the RH problem, we obtain the long-time asymptotics for the defocusing mixed NLS equation. In section 6, we give conclusion and remarks to discuss recent development on asymptotic analysis and future work.

2. Spectral analysis

2.1. Lax pair

The defocusing mixed NLS equation (1.1) admits the following Lax pair [47]

$$\psi_x=(-ai\sigma_3\lambda^2+Q_1\lambda+Q_0)\psi, \tag{2.1}$$

$$\psi_t=(-2a^2\sigma_3\lambda^4+V_3\lambda^3+V_2\lambda^2+V_1\lambda+V_0)\psi, \tag{2.2}$$

where

$$\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ Q_1=\begin{pmatrix}2ib&aq\\ar&-2ib\end{pmatrix},\ Q_0=\begin{pmatrix}0&-bq\\-br&0\end{pmatrix},$$
$$\psi=\begin{pmatrix}\phi\\\varphi\end{pmatrix},V_3=\begin{pmatrix}8iab&2a^2q\\2a^2r&-8iab\end{pmatrix},V_2=\begin{pmatrix}4ib-ia^2rq&-6abq\\-6abr&-(4ib-ia^2rq)\end{pmatrix},$$
$$V_1=\begin{pmatrix}2iabqr&-2bq+iaq_x+a^2rq^2\\-2br-iar_x+a^2qr^2&-2iabqr\end{pmatrix},$$
$$V_0=\begin{pmatrix}\frac{1}{2}ibqr&ib(-q_x+iar^2q)\\ib(r_x+iar^2q)&-\frac{1}{2}ibqr\end{pmatrix},$$

where $r=q^*$, while $\lambda\in C$ and ψ are called an eigenvalue and an eigenfunction, respectively.

To deal with the mixed NLS equation (1.1) with two arbitrary constants a and b in a simple and unified way, we rewrite the Lax pair (2.1)-(2.2) in the form

$$\psi_x+i\lambda(a\lambda-2b)\sigma_3\psi=P_1\psi, \tag{2.3}$$

$$\psi_t+2i\lambda^2(a\lambda-2b)^2\sigma_3\psi=P_2\psi, \tag{2.4}$$

where

$$\begin{aligned} &P_1 = (\lambda a - b)Q, \quad Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \\ &P_2 = -i(a^2\lambda^2 - 2ab\lambda + b^2)Q^2\sigma_3 + (2a^2\lambda^3 - 6ab\lambda^2 + 4b^2\lambda)Q \\ &+ i(\lambda a - b)Q_x\sigma_3 + (\lambda a^2 - ab)Q^3 \end{aligned} \tag{2.5}$$

with Q being a Hermitian matrix and P_1 satisfies $P_1^H(\bar{\lambda}) = P_1(\lambda)$.

2.2. Asymptotic

Due to $q_0(x) \in \mathcal{S}(R)$, so as $x \to \pm\infty$, the Lax pair (2.3)-(2.4) becomes

$$\begin{aligned} &\psi_x + i\lambda(a\lambda - 2b)\sigma_3\psi \sim 0, \\ &\psi_t + 2i\lambda^2(a\lambda - 2b)^2\sigma_3\psi \sim 0, \end{aligned}$$

which implies that the Lax pair (2.3)-(2.4) admits the Jost solutions with asymptotic

$$\psi \sim e^{-i\zeta(x,t,\lambda)\sigma_3}, \quad x \to \pm\infty, \tag{2.6}$$

where $\zeta(x,t,\lambda) = \lambda(a\lambda - 2b)x + 2\lambda^2(a\lambda - 2b)^2 t$. Therefore, making transformation

$$\Psi = \psi e^{i\zeta(x,t,\lambda)\sigma_3}, \tag{2.7}$$

then we have

$$\Psi \sim I, \ x \to \pm\infty, \tag{2.8}$$

and Ψ satisfies a new Lax pair

$$\Psi_x + i\lambda(a\lambda - 2b)[\sigma_3, \Psi] = P_1\Psi, \tag{2.9}$$

$$\Psi_t + 2i\lambda^2(a\lambda - 2b)^2[\sigma_3, \Psi] = P_2\Psi, \tag{2.10}$$

which can be written in full derivative form

$$d(e^{i\zeta(x,t,\lambda)\hat{\sigma}_3}\Psi) = e^{i\zeta(x,t,\lambda)\hat{\sigma}_3}(P_1 dx + P_2 dt)\Psi. \tag{2.11}$$

We consider asymptotic expansion

$$\Psi = \Psi_0 + \frac{\Psi_1}{\lambda} + \frac{\Psi_2}{\lambda^2} + O(\frac{1}{\lambda^3}), \qquad \lambda \to \infty, \tag{2.12}$$

where Ψ_0, Ψ_1, Ψ_2 are independent of λ. Substituting (2.12) into (2.9) and comparing the coefficients of λ, we obtain that Ψ_0 is a diagonal matrix and

$$\Psi_1 = \frac{i}{2}Q\Psi_0\sigma_3, \tag{2.13}$$

$$\Psi_{0x} + ia[\sigma_3, \Psi_2] = aQ\Psi_1 + bQ\Psi_0. \tag{2.14}$$

In the same way, substituting (2.12) into (2.10) and comparing the coefficients of λ in the same order leads to

$$\Psi_{0x} = \frac{i}{2} aqr\sigma_3\Psi_0. \tag{2.15}$$

$$\Psi_{0t} = [\frac{3}{4} ia^2r^2q^2 + \frac{1}{2}a(rq_x - r_xq)]\sigma_3\Psi_0. \tag{2.16}$$

Noting that the mixed NLS equation (1.1) admits the conservation law

$$\left(\frac{1}{2}arq\right)_t = \left[\frac{3}{4}ia^2r^2q^2 + \frac{1}{2}(rq_x - r_xq)\right]_x, \tag{2.17}$$

so two equations (2.15) and (2.16) are compatible if we define

$$\Psi_0(x,t) = e^{i\int_{(x,t)}^{(+\infty,t)} \Delta\sigma_3} = e^{\frac{1}{2}i\int_x^{+\infty} |q(x',t)|^2 dx'\sigma_3}, \tag{2.18}$$

where Δ is the closed real-valued one-form

$$\Delta(x,t) = \frac{1}{2}aqrdx + \left[\frac{3}{4}a^2r^2q^2 - \frac{i}{2}(rq_x - r_xq)\right]dt. \tag{2.19}$$

We introduce a new function ω by

$$\Psi(x,t,\lambda) = e^{-i\int_{(-\infty,t)}^{(x,t)} \Delta\hat{\sigma}_3}\omega(x,t,\lambda)\Psi_0(x,t), \tag{2.20}$$

then ω admits the asymptotic

$$\omega = \mathbb{I} + O(\frac{1}{\lambda}), \quad \lambda \to \infty, \tag{2.21}$$

and satisfies the Lax pair

$$\omega_x + i\lambda(a\lambda - 2b)[\sigma_3, \omega] = e^{-i\int_{(-\infty,t)}^{(x,t)} \Delta\hat{\sigma}_3}H_1\omega, \tag{2.22a}$$

$$\omega_t + 2i\lambda^2(a\lambda - 2b)^2[\sigma_3, \omega] = e^{-i\int_{(-\infty,t)}^{(x,t)} \Delta\hat{\sigma}_3}H_2\omega, \tag{2.22b}$$

where

$$H_1 = P_1 - \frac{1}{2}iarq\sigma_3, \quad H_2 = P_2 - i\left(\frac{3}{4}a^2r^2q^2 - \frac{i}{2}(rq_x - r_xq)\right)\sigma_3.$$

2.3. *Analyticity and symmetry*

The Lax pair (2.22a)-(2.22b) can be written into a full derivative form

$$\begin{aligned} d(e^{i\zeta(x,t,\lambda)\hat{\sigma}_3}\omega(x,t,\lambda)) &= e^{i\zeta(x,t,\lambda)\hat{\sigma}_3}e^{-i\int_{(-\infty,t)}^{(x,t)} \Delta\hat{\sigma}_3}(H_1dx + H_2dt)\omega \\ &= e^{i\zeta(x,t,\lambda)\hat{\sigma}_3}e^{-i\int_{(-\infty,t)}^{(x,t)} \Delta\hat{\sigma}_3}(P_1dx + P_2dt - i\Delta\sigma_3)\omega. \end{aligned} \tag{2.23}$$

Throughout this section we assume that $q(x,t)$ is sufficiently smooth. We choose two integrated contours $(\pm\infty, t) \to (x,t)$ in the Fig. 1 to define two solutions of equation (2.23) by

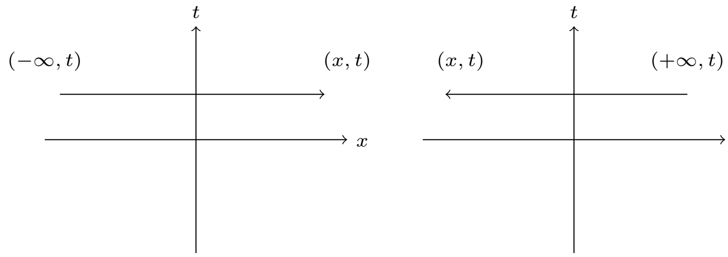

Fig. 1. Path integrals of the ω_1 and ω_2.

$$\omega_j(x,t)e^{i\int_{(-\infty,t)}^{(x,t)}\Delta\sigma_3} = e^{i\int_{(-\infty,t)}^{(x,t)}\Delta\sigma_3} + \int_{\pm\infty}^{x} e^{-i\lambda(a\lambda-2b)(x-y)\hat{\sigma}_3}H_1(y,t,\lambda)$$
$$\times\, e^{-i\int_{(-\infty,t)}^{(x,t)}\Delta\sigma_3}\omega_j(y,t)dy, \quad j=1,2.$$

Let $\lambda = \xi + i\eta$, then

$$\lambda(a\lambda - 2b) = a(\xi^2-\eta^2) - 2b\xi + 2i(a\xi - b)\eta = 0, \tag{2.24}$$

which gives

$$\mathrm{Im}[\lambda(a\lambda-2b)] = 2(a\mathrm{Re}\lambda - b)\mathrm{Im}\lambda.$$

We define two domains by

$$D^+ = \{\lambda|(a\mathrm{Re}\lambda - b)\ \mathrm{Im}\lambda > 0\}, \quad D^- = \{\lambda|(a\mathrm{Re}\lambda - b)\ \mathrm{Im}\lambda < 0\}, \tag{2.25}$$

then boundary of the domains D^+ and D^- is given by

$$\Sigma = \{\lambda|(a\mathrm{Re}\lambda - b)\ \mathrm{Im}\lambda = 0\}. \tag{2.26}$$

Remark 2.1. Let's discuss the above domains and boundary for different a and b.

(i) When $a = 0, \quad b \neq 0$. The domains (2.25) reduce to

$$D^+ = \{\lambda|\ \mathrm{Im}\lambda > 0\}, \quad D^- = \{\lambda|\ \mathrm{Im}\lambda < 0\}, \tag{2.27}$$

which denote upper half complex plane (yellow) and lower half complex plane (white), respectively. And the boundary (2.26) reduces to real axis

$$\Sigma = \{\lambda|\ \mathrm{Im}\lambda = 0\} = R. \tag{2.28}$$

This case corresponds to the domains and the boundary of classical NLS equation, see Fig. 2.

(ii) When $a \neq 0, b = 0$. The domains (2.25) reduce to

$$D^+ = \{\lambda|\mathrm{Re}\lambda\ \mathrm{Im}\lambda > 0\}, \quad D^- = \{\lambda|\mathrm{Re}\lambda\ \mathrm{Im}\lambda < 0\},$$

which denote yellow part and white part in the Fig. 3. The boundary (2.26) reduces to real axis and imaginary axis

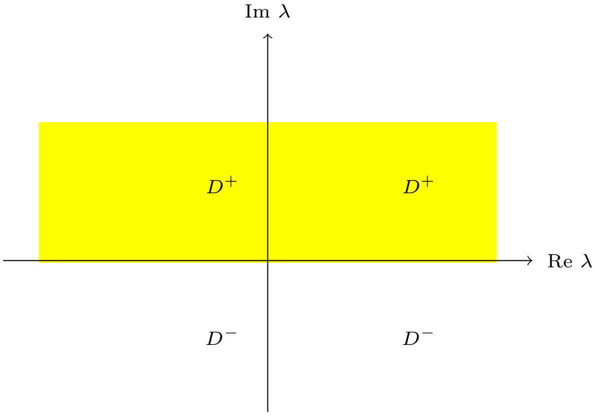

Fig. 2. Analytical domains D^+, D^- and boundary Σ corresponding to the classical NLS equation. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

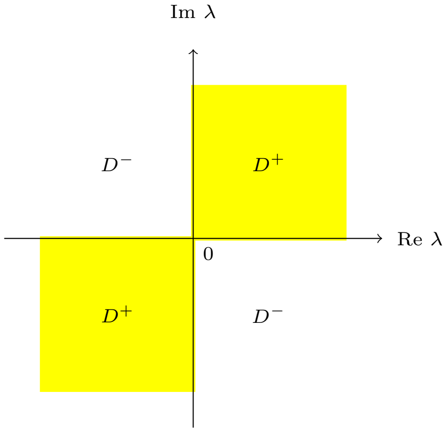

Fig. 3. Analytical domains D^+, D^- and boundary Σ corresponding to the derivative NLS equation.

$$\Sigma = iR \cup R. \tag{2.29}$$

This case corresponds to the domains and boundary of the derivative NLS equation.

(iii) When $a, b \neq 0$. The domains (2.25) means that D^+ is yellow part and D^- is white part in the Fig. 4. And the boundary (2.26) is real axis R and a line $\mathrm{Re}\lambda = b/a$, that is,

$$\Sigma = R \cup \{\lambda |\ Re\lambda = b/a\}. \tag{2.30}$$

This case corresponds to the domains and boundary of the modified NLS equation.

We denote the first and second columns of $\omega_j(x,t,\lambda)$ by $\omega_j^{(1)}$ and $\omega_j^{(2)}$ respectively, then we can show that

Proposition 2.1. *The matrices $\omega_1(x,t,\lambda)$ and $\omega_2(x,t,\lambda)$ have the following properties*

(i) $\det\omega_1 = \det\omega_2 = 1$.

(ii) $\omega_1^{(1)}$ is analytic in D^+ and

$$\omega_1^{(1)} = e_1 + O(\frac{1}{\lambda}), \quad as\ \lambda \to \infty. \tag{2.31}$$

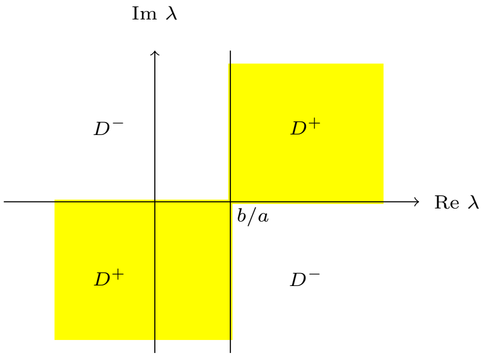

Fig. 4. Analytical domains D^+, D^- and boundary Σ corresponding to the modified NLS equation.

(iii) $\omega_1^{(2)}$ *is analytic in* D^- *and*

$$\omega_1^{(2)} = e_2 + O(\frac{1}{\lambda}), \quad \text{as } \lambda \to \infty. \tag{2.32}$$

(iv) $\omega_2^{(1)}$ *is analytic in* D^- *and*

$$\omega_2^{(1)} = e_1 + O(\frac{1}{\lambda}), \quad \text{as } \lambda \to \infty. \tag{2.33}$$

(v) $\omega_2^{(2)}$ *is analytic in* D^+ *and*

$$\omega_2^{(2)} = e_2 + O(\frac{1}{\lambda}), \quad \text{as } \lambda \to \infty, \tag{2.34}$$

where e_1 *and* e_2 *are the first and second column of the identity matrix.*

Since the eigenfunctions $\psi_1(x,t,\lambda)$ and $\psi_2(x,t,\lambda)$ are the matrix solution of Lax pair (2.3)-(2.4), there exists a matrix $S(\lambda)$ such that

$$\psi_1 = \psi_2 S(\lambda), \tag{2.35}$$

where

$$S(\lambda) = \begin{pmatrix} s_{11}(\lambda) & s_{12}(\lambda) \\ s_{21}(\lambda) & s_{22}(\lambda) \end{pmatrix}.$$

Again by using transformations (2.7) and (2.20), we have

$$\omega_1(x,t,\lambda) = \omega_2(x,t,\lambda) e^{i\int_{(-\infty,t)}^{(+\infty,t)} \Delta\hat{\sigma}_3} e^{-i\zeta(x,t,\lambda)\hat{\sigma}_3} S(\lambda). \tag{2.36}$$

Further we can show that

Proposition 2.2. *The eigenfunctions* $\omega_j,\ j=1,2$ *and scattering matrix* $S(\lambda)$ *satisfy two kinds of symmetry relations*

$$\omega_j(x,t,\lambda) = \sigma_1 \overline{\omega_j^*(x,t,\overline{\lambda})} \sigma_1, \quad S(\lambda) = \sigma_1 \overline{S(\overline{\lambda})} \sigma_1, \tag{2.37}$$

and

$$\omega_j(x,t,\lambda)=\sigma_*\overline{\omega_j\left(x,t,2b/a-\overline{\lambda}\right)}\sigma_*,\quad S(\lambda)=-\sigma_*\overline{S\left(2b/a-\overline{\lambda}\right)}\sigma_*, \tag{2.38}$$

where

$$\sigma_*=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\ \sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

3. The RH problem

3.1. The original RH problem

We rewrite equation (2.36) into a matrix form

$$\left(\frac{\omega_1^{(1)}}{s_{11}(\lambda)},\omega_2^{(2)}\right)e^{i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\sigma_3}=\left(\omega_2^{(1)},\frac{\omega_1^{(2)}}{s_{22}}\right)e^{i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\sigma_3}V(x,t,\lambda), \tag{3.1}$$

where

$$V(x,t,\lambda)=e^{-i\zeta\hat{\sigma}_3}\begin{pmatrix}\frac{1}{s_{11}s_{22}} & -\frac{s_{12}}{s_{22}}\\ \frac{s_{21}}{s_{11}} & 1\end{pmatrix}. \tag{3.2}$$

Defining

$$M(x,t,\lambda)=\begin{cases}(\frac{\omega_1^{(1)}}{s_{11}(\lambda)},\omega_2^{(2)}), & \lambda\in D^+,\\ (\omega_2^{(1)},\frac{\omega_1^{(2)}}{s_{22}}), & \lambda\in D^-,\end{cases} \tag{3.3}$$

and the scattering coefficient $s_{11}(z)$ satisfies $\inf_{\mathrm{Re}z=b/a}|s_{11}(z)|>1/2$, then $M(x,t,\lambda)$ satisfies the following RH problem:
(i) $M(x,t,\lambda)$ is analytic in $C\backslash\Sigma$,
(ii) The boundary value $M_\pm(x,t,\lambda)$ at Σ satisfies the jump condition

$$\left[M(x,t,\lambda)e^{i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\sigma_3}\right]_+=\left[M(x,t,\lambda)e^{i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\sigma_3}\right]_-V(x,t,\lambda),\quad \lambda\in\Sigma, \tag{3.4}$$

where the jump matrix $V(\lambda)$ is given by

$$V(x,t,\lambda)=\begin{pmatrix}1-r(\lambda)\overline{r(\bar{\lambda})} & -\overline{r(\bar{\lambda})}e^{-2it\theta(\lambda)}\\ r(\lambda)e^{2it\theta(\lambda)} & 1\end{pmatrix},\quad r(\lambda)=\frac{s_{21}(\lambda)}{s_{11}(\lambda)}, \tag{3.5}$$

and $|r(\lambda)|<1$,

$$\theta(\lambda)=\zeta(\lambda)/t=2\lambda^2(a\lambda-2b)^2+\frac{x}{t}\lambda(a\lambda-2b). \tag{3.6}$$

(iii) Asymptotic behavior

$$M(x,t,\lambda)=\mathbb{I}+O(\frac{1}{\lambda}),\qquad as\quad \lambda\to\infty. \tag{3.7}$$

Combining (2.13) and (2.18) implies that the potential $q(x,t)$ is given by

$$q(x,t) = 2ie^{-2i\int_{(-\infty,t)}^{(x,t)}\Delta} \lim_{\lambda\to\infty} [\lambda M(x,t,\lambda)]_{12}. \tag{3.8}$$

3.2. The new RH problem

In this section, we translate the RH problem on Σ in λ-plane to a real axis in z-plane which is similar to the type of nonlinear Schrödinger equation. The advantage of this is overcoming the contradiction in the plane wave region.

Let $z = \lambda(a\lambda - 2b)$, then the boundary $\Sigma = \{\lambda|\mathrm{Im}[\lambda(a\lambda - 2b)] = 0\}$ is changed to real axis R. Again define

$$N(x,t,\lambda) = (a\lambda - 2b)^{-\frac{1}{2}\hat{\sigma}_3} M(x,t,\lambda), \tag{3.9}$$

then $N(x,t,z)$ satisfies a new RH problem:

(i) $N(x,t,z)$ is analytic in $C\backslash R$,

(ii) The boundary value $N_\pm(x,t,z)$ at Σ satisfies the jump condition

$$\left[N(x,t,\lambda)e^{i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\sigma_3}\right]_+ = \left[N(x,t,\lambda)e^{i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\sigma_3}\right]_- e^{-i\zeta(x,t,\lambda)\hat{\sigma}_3} V_N, \quad z\in R, \tag{3.10}$$

where

$$V_N = \begin{pmatrix} 1 - z\rho_1(z)\rho_2(z) & -\rho_1(z) \\ z\rho_2(z) & 1 \end{pmatrix},$$

and two reflection coefficients are given by

$$\rho_1(z) = \frac{\overline{r(\bar{\lambda})}}{a\lambda - 2b}, \quad \rho_2(z) = \frac{r(\lambda)}{\lambda}. \tag{3.11}$$

(iii) Asymptotic behavior

$$N(x,t,z) = \mathbb{I} + O(\frac{1}{z}), \quad as \quad z\to\infty. \tag{3.12}$$

The solution of the mixed NLS equation (1.1) given by

$$q(x,t) = 2ie^{-2i\int_{(-\infty,t)}^{(x,t)}\Delta} \lim_{z\to\infty} [zN(x,t,z)]_{12}. \tag{3.13}$$

Next, based on the above RH problem, we analyze the asymptotic behavior of the mixed NLS equation when $t\to\infty$ by using the Deift-Zhou nonlinear steepest descent method.

4. Deformation of RH problem

4.1. Stationary point and steepest descent lines

Under the transformation $z = \lambda(a\lambda - 2b)$, (3.6) can be written as

$$\theta(z) = \frac{x}{t}z + 2z^2, \tag{4.1}$$

and we set

$$\varphi(z) = i\theta(z) = i(\frac{x}{t}z + 2z^2), \tag{4.2}$$

which admits the first order stationary point

$$z_0 = -\frac{x}{4t}. \tag{4.3}$$

Correspondingly, the two steepest descent lines are

$$L = \{z = z_0 + ue^{\frac{i\pi}{4}}, \quad u \geq 0\} \cup \{z = z_0 + ue^{\frac{5i\pi}{4}}, \quad u \geq 0\}, \tag{4.4}$$

$$\bar{L} = \{z = z_0 + ue^{-\frac{i\pi}{4}}, \quad u \geq 0\} \cup \{z = z_0 + ue^{\frac{3i\pi}{4}}, \quad u \geq 0\}. \tag{4.5}$$

4.2. Decomposition of jump matrix

We deform the RH problem with one jump matrix into the one with two jump matrices on R by triangulating the jump matrix up and down. It can be shown that the jump matrix V_N admits the following triangular factorizations:

$$V_N = \begin{pmatrix} 1 & -\rho_1(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z\rho_2(z) & 1 \end{pmatrix}, \quad z > z_0.$$

$$V_N = \begin{pmatrix} 1 & 0 \\ \frac{z\rho_2(z)}{1 - z\rho_1(z)\rho_2(z)} & \end{pmatrix} \begin{pmatrix} 1 - z\rho_1(z)\rho_2(z) & 0 \\ 0 & \frac{1}{1 - z\rho_1(z)\rho_2(z)} \end{pmatrix} \begin{pmatrix} 1 & \frac{-\rho_1(z)}{1 - z\rho_1(z)\rho_2(z)} \\ 0 & 1 \end{pmatrix}, \quad z < z_0.$$

According to matrix decomposition, we search for a transformation to remove the diagonal matrix in the middle for the second decomposition.

Consider a scalar RH problem

$$\begin{cases} \delta_+(z) = \delta_-(z)(1 - z\rho_1(z)\rho_2(z)), & z < z_0, \\ \delta_+(z) = \delta_-(z), & z > z_0, \\ \delta(z) \to 1, & as\ z \to \infty, \end{cases} \tag{4.6}$$

which admits a solution

$$\delta(z) = \exp\left[\frac{1}{2\pi i}\int_{-\infty}^{z_0} \frac{\ln(1 - z\rho_1(z)\rho_2(z))}{\xi - z} d\xi\right]. \tag{4.7}$$

Theorem 4.1. $\delta(z)$ and $\delta(z)^{-1}$ are uniformly bounded in z and for $|z_0| \leq M$, namely

$$(1 - \|z\rho_1(z)\rho_2(z)\|^2_{L^\infty})^{\frac{1}{2}} \leq |\delta(z)| \leq (1 - \|z\rho_1(z)\rho_2(z)\|^2_{L^\infty})^{-\frac{1}{2}} \tag{4.8}$$

Noticing that $|z\rho_1(z)\rho_2(z)| = |r(\lambda)| < 1$, then the proof process can be referred to [11].

Defining a transformation

$$N^{(1)}(x,t,z) = N(x,t,z)\delta^{-\sigma_3}, \tag{4.9}$$

then $N^{(1)}$ satisfies a new RH problem

(i) $N^{(1)}(x,t,z)$ is analytic in $C \setminus \Sigma^{(1)}$;

(ii) $N^{(1)}(x,t,z)$ satisfies the jump condition

$$\left[N^{(1)}(x,t,z)e^{i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\sigma_3}\right]_+ = \left[N^{(1)}(x,t,z)e^{i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\sigma_3}\right]_- V_N^{(1)}(x,t,z), \quad z\in\Sigma^{(1)}, \tag{4.10}$$

where $\Sigma^{(1)} = R$, and the jump matrix $V_N^{(1)}(z)$ is defined by

$$V_N^{(1)} = \begin{cases} e^{-it\theta\hat{\sigma}_3}\begin{pmatrix} 1 & 0 \\ \delta_-^{-2}\frac{z\rho_2(z)}{1-z\rho_1(z)\rho_2(z)} & 1\end{pmatrix}\begin{pmatrix} 1 & \frac{-\rho_1(z)}{1-z\rho_1(z)\rho_2(z)}\delta_+^2 \\ 0 & 1\end{pmatrix}, & z<z_0, \\ e^{-it\theta\hat{\sigma}_3}\begin{pmatrix} 1 & -\rho_1(z)\delta_+^2 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ z\rho_2(z)\delta_-^{-2} & 1\end{pmatrix}, & z>z_0; \end{cases} \tag{4.11}$$

(iii) Asymptotic condition

$$N^{(1)}(x,t,z) = \mathbb{I} + O(z^{-1}), \qquad as \quad z\to\infty. \tag{4.12}$$

The solution of the mixed NLS equation (1.1) is given by

$$q(x,t) = 2ie^{-2i\int_{(-\infty,t)}^{(x,t)}\Delta}\lim_{z\to\infty}[zN^{(1)}(x,t,z)]_{12}. \tag{4.13}$$

4.3. Rational approximation of scattering data

We show that the scattering data

$$\frac{z\rho_2(z)}{1-z\rho_1(z)\rho_2(z)}, \quad \frac{\rho_1(z)}{1-z\rho_1(z)\rho_2(z)}, \quad \rho_1(z), \quad z\rho_2(z), \tag{4.14}$$

can be approximated by rational functions

$$[\frac{z\rho_2(z)}{1-z\rho_1(z)\rho_2(z)}], \quad [\frac{\rho_1(z)}{1-z\rho_1(z)\rho_2(z)}], \quad [\rho_1](z), \quad [z\rho_2](z), \tag{4.15}$$

respectively. By using the similar way to [11], we can establish the following estimates.

Theorem 4.2. Let

$$f(z) = \begin{cases} f_1 = \frac{\rho_1(z)}{1-z\rho_1(z)\rho_2(z)}, & z<z_0, \\ f_2 = \frac{z\rho_2(z)}{1-z\rho_1(z)\rho_2(z)}, & z<z_0, \\ f_3 = \rho_1(z), & z>z_0, \\ f_4 = z\rho_2(z), & z>z_0. \end{cases} \tag{4.16}$$

We decompose f into three parts

$$f = h_I + h_{II} + [f], \tag{4.17}$$

where the piecewise rational $[f]$ and h_{II} have an analytic continuation to the steepest descent line L and $\bar{L}$, but h_I can't. Moreover, we have

$$|e^{-2it\theta(z)}h_1| \le \frac{c}{(1+|z|^2)t^l}, \quad z\in R, \tag{4.18}$$

$$|e^{-2it\theta(z)}h_2| \le \frac{c}{(1+|z|^2)t^l}, \quad z\in L, \tag{4.19}$$

$$|e^{-2it\theta(z)}[f]| \leq Ce^{-4tz_0^2u^2} \leq Ce^{-4\varepsilon^2 t}, \quad z \in L, \tag{4.20}$$

where l is an arbitrary positive integer.

4.4. Analytic extension of jump matrix

The purpose of this section is to deform the RH problem on $\Sigma^{(1)} = R$ to the one on $\Sigma^{(2)} = R \cup L \cup \bar{L}$, see Fig. 5 We extend the jump matrix to the fast descent line L and $\bar{L}$. Thus, the jump matrix can be decomposed into

$$V_N^{(1)} = (b_-)^{-1}_{x,t,\delta}(b_+)_{x,t,\delta}, \tag{4.21}$$

where

$$(b_\pm)_{x,t,\delta} = \delta_\pm^{\hat{\sigma}_3} e^{-it\theta(z)\hat{\sigma}_3} b_\pm. \tag{4.22}$$

From the definition of f, we have

$$b_+ = \mathbb{I} + \omega_+ = \begin{cases} \begin{pmatrix} 1 & f_1 \\ 0 & 1 \end{pmatrix}, & z < z_0, \\ \begin{pmatrix} 1 & f_3 \\ 0 & 1 \end{pmatrix}, & z > z_0, \end{cases} \tag{4.23}$$

$$b_- = \mathbb{I} - \omega_- = \begin{cases} \begin{pmatrix} 1 & f_2 \\ 0 & 1 \end{pmatrix}, & z < z_0, \\ \begin{pmatrix} 1 & f_4 \\ 0 & 1 \end{pmatrix}, & z > z_0. \end{cases} \tag{4.24}$$

We further decompose b_+ and b_- as

$$b_+ = b_+^o b_+^a = (\mathbb{I} + \omega_+^o)(\mathbb{I} + \omega_+^a) = \begin{cases} \begin{pmatrix} 1 & (h_I)^1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (h_{II} + [f]_1)^1 \\ 0 & 1 \end{pmatrix}, & z < z_0, \\ \begin{pmatrix} 1 & (h_I)^3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (h_{II} + [f]_3)^3 \\ 0 & 1 \end{pmatrix}, & z > z_0. \end{cases} \tag{4.25}$$

$$b_- = b_+^o b_+^a = (\mathbb{I} - \omega_-^o)(\mathbb{I} - \omega_-^a) = \begin{cases} \begin{pmatrix} 1 & (h_I)^2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (h_{II} + [f]_2)^2 \\ 0 & 1 \end{pmatrix}, & z < z_0, \\ \begin{pmatrix} 1 & (h_I)^4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (h_{II} + [f]_4)^4 \\ 0 & 1 \end{pmatrix}, & z > z_0. \end{cases} \tag{4.26}$$

$(b_-^a)^{-1}$ and b_+^a can be analytically extended to L and $\bar{L}$ respectively. $(b_-^o)^{-1}b_+^o$ doesn't have the property of analytic continuation but decays fast about time. Therefore, we make the following transformation

$$N^{(2)} = N^{(1)}\phi, \tag{4.27}$$

where

$$\phi = \begin{cases} I, & z \in \Omega_2 \cap \Omega_5, \\ (b_-^a)^{-1}_{x,t,\delta}, & z \in \Omega_1 \cap \Omega_4, \\ (b_-^a)^{-1}_{x,t,\delta}, & z \in \Omega_3 \cap \Omega_6. \end{cases} \tag{4.28}$$

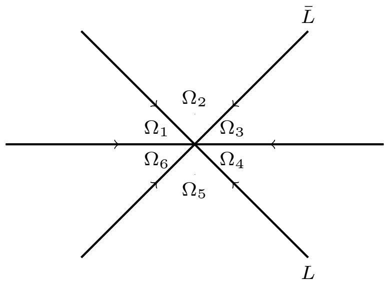

Fig. 5. The orient contour of $\Sigma^{(2)}$.

The description of the region is shown in Fig. 5.

The RH problem on $\Sigma^{(2)}$ is given by

(i) $N^{(2)}$ is analytic in $C \setminus \Sigma^{(2)}$,

(ii) $N^{(2)}(x,t,z)$ satisfies the jump condition

$$\left[N^{(2)}(x,t,z)e^{i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\sigma_3}\right]_+ = \left[N^{(2)}(x,t,z)e^{i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\sigma_3}\right]_- V_N^{(2)}(x,t,z), \quad z\in\Sigma^{(2)}, \tag{4.29}$$

(iii) Asymptotic

$$N^{(2)}(x,t,z)\to I, \qquad z\to\infty, \tag{4.30}$$

where the jump matrix is

$$V_N^{(2)}(x,t,z) = \begin{cases} (b_-^o)^{-1}_{x,t,\delta}(b_+^o)_{x,t,\delta}, & z\in R, \\ (b_+^a)_{x,t,\delta}, & z\in \bar{L}, \\ (b_-^a)^{-1}_{x,t,\delta}, & z\in L. \end{cases} \tag{4.31}$$

The solution of the mixed NLS equation is represented by

$$q(x,t) = 2ie^{-2i\int_{(-\infty,t)}^{(x,t)}\Delta}\lim_{z\to\infty}[zN^{(2)}(x,t,z)]_{12}. \tag{4.32}$$

Now we change $\Sigma^{(2)}$ into

$$\Sigma^{(3)} = L\cap\bar{L} = \Sigma^{(2)}\backslash R, \tag{4.33}$$

which can help us evaluate the contribution on the solution of RH problem by $h_I(z)$ and $h_{II}(z)$. We decompose the jump matrix into

$$V_N^{(2)} = (b_-^{(2)})^{-1}_{x,t,\delta}(b_+^{(2)})_{x,t,\delta}, \tag{4.34}$$

and define

$$(\mu_\pm^{(2)})_{x,t,\delta} = \pm((b_\pm^{(2)})_{x,t,\delta} - I), \tag{4.35}$$

$$(\mu^{(2)})_{x,t,\delta} = (\mu_+^{(2)})_{x,t,\delta} + (\mu_-^{(2)})_{x,t,\delta}. \tag{4.36}$$

The Cauchy operators are

$$(C_{\pm}f)(z) = \lim_{\substack{z' \to z \\ z' \in \pm side\ of\ \Sigma^{(2)}}} \frac{1}{2\pi i} \int_{\Sigma^{(2)}} \frac{f(\xi)}{\xi - z'} d\xi, \tag{4.37}$$

and

$$C_{(\mu^{(2)})_{x,t,\delta}} f = C_+(f(\mu_-^{(2)})_{x,t,\delta}) + C_-(f(\mu_+^{(2)})_{x,t,\delta}). \tag{4.38}$$

According to Beal-Cofiman theorem, we have

$$\gamma^{(2)} = (1 - C_{\mu^{(2)}})^{-1}\mathbb{I} = \mathbb{I} + (1 - C_{\mu^{(2)}})^{-1} C_{\mu^{(2)}} \mathbb{I}, \tag{4.39}$$

and the solution of RH problem is

$$N^{(2)} = \mathbb{I} + \frac{1}{2\pi i} \int_{\Sigma^{(2)}} \frac{\gamma^{(2)}(\xi)(\mu^{(2)})_{x,t,\delta}(\xi)}{\xi - z} d\xi. \tag{4.40}$$

From (4.32) and (4.39)-(4.40), we have

$$q(x,t) = -\frac{1}{\pi} e^{-2i \int_{(-\infty,t)}^{(x,t)} \Delta} \left(\int_{\Sigma^{(2)}} (1 - C_{\mu^{(2)}})^{-1} (\mu^{(2)})_{x,t,\delta}(\xi) d\xi \right)_{12}. \tag{4.41}$$

Using a method similar to that used for rational approximation of scattering data, we can prove that the contribution to RH problem is mainly from the rational part of the scattering data. Also, the contribution of h_I and h_{II} is an infinitely small quantity of t. Setting $\Sigma^{(3)} = \Sigma^{(2)} \backslash R$ and $\mu^{(3)} = [f]$, we have

$$q(x,t) = 2ie^{-2i \int_{(-\infty,t)}^{(x,t)} \Delta} [N_1^{(3)}(x,t,z)]_{12} + \mathcal{O}(t^{-1}), \tag{4.42}$$

where

$$N^{(3)} = I + \frac{N_1^{(3)}}{z} + \cdots \tag{4.43}$$

is the solution of following RH problem:

(i) $N^{(3)}$ is analytic in $C \backslash \Sigma^{(3)}$,

(ii) The boundary value $N^{(3)}(x,t,z)$ at $\Sigma^{(3)}$ satisfies the jump condition

$$\left[N^{(3)}(x,t,z) e^{i \int_{(-\infty,t)}^{(+\infty,t)} \Delta \sigma_3} \right]_+ = \left[N^{(3)}(x,t,z) e^{i \int_{(-\infty,t)}^{(+\infty,t)} \Delta \sigma_3} \right]_- V_N^{(3)}(x,t,z), \quad z \in \Sigma^{(3)}, \tag{4.44}$$

(iii) $N^{(3)}(x,t,z) \to I, \quad$ as $z \to \infty$.

Here, we show the jump matrix $V_N^{(3)}$ in Fig. 6.

4.5. Scaling transformation

Let $\Sigma^{(3)}$ deform to $\Sigma^{(4)} = \Sigma^{(3)} - z_0$ and we do a translational scaling

$$z = (8t)^{-\frac{1}{2}} \tilde{z} + z_0 \longleftrightarrow \tilde{z} = (8t)^{\frac{1}{2}} (z - z_0). \tag{4.45}$$

Next, we define the scaling operator

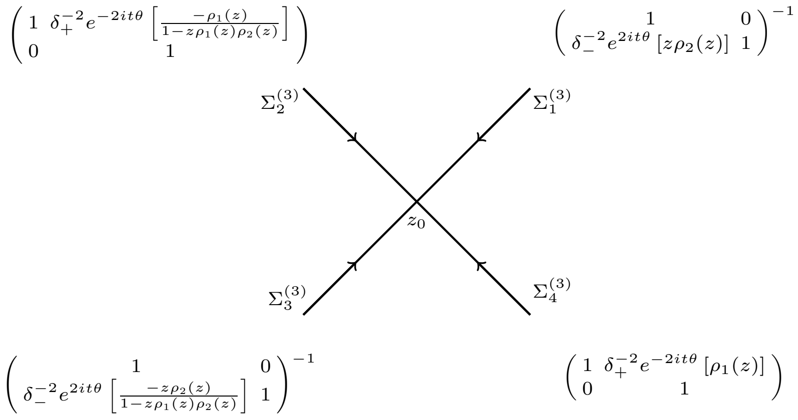

Fig. 6. $\Sigma^{(3)}$ and jump matrix.

$$\begin{aligned} &P : \Sigma^{(3)} \to \Sigma^{(4)}, \\ &f \to Pf(z) = f((8t)^{-\frac{1}{2}}\tilde{z} + z_0). \end{aligned} \tag{4.46}$$

The RH problem becomes

(i) $N^{(4)}(x,t,z)$ is analytic in $C \backslash \Sigma^{(4)}$,

(ii) The boundary value $N^{(4)}(x,t,z)$ at $\Sigma^{(4)}$ satisfies the jump condition

$$\left[N^{(4)}(x,t,z)e^{i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\sigma_3}\right]_+ = \left[N^{(4)}(x,t,z)e^{i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\sigma_3}\right]_- V_N^{(4)}(x,t,z), \quad z \in \Sigma^{(4)}, \tag{4.47}$$

(iii) Asymptotic condition

$$N^{(4)}(x,t,z) \to I, \quad z \to \infty, \tag{4.48}$$

and the solution of mixed Schrodinger equation is given by

$$q(x,t) = \frac{i}{\sqrt{2t}} e^{-2i\int_{(-\infty,t)}^{(x,t)}\Delta}[N_1^{(4)}(x,t,z)]_{12} + \mathcal{O}(t^{-1}), \tag{4.49}$$

where

$$N^{(4)} = PN^{(3)} = I + \frac{N_1^{(3)}}{(8t)^{-1/2}\tilde{z} + z_0} + \cdots = I + \frac{N_1^{(4)}}{z} + \cdots \tag{4.50}$$

For the sake of discussion, we isolate the oscillations in the jump matrix $V_N^{(3)}$,

$$V_N^{(3)} = \delta^{\hat{\sigma}_3} e^{-it\theta\hat{\sigma}_3} V^{(3)}, \tag{4.51}$$

$$V_N^{(4)} = PV_N^{(3)} = P(\eta^{\hat{\sigma}_3} e^{-it\theta\hat{\sigma}_3} V^{(3)}) = P(\eta e^{-it\theta})^{\hat{\sigma}_3} PV^{(3)}, \tag{4.52}$$

and $V_N^{(4)}$ is shown in Fig. 7. We have

$$P(\eta(z)e^{-it\theta(z)}) = \delta^0\delta^1, \tag{4.53}$$

where

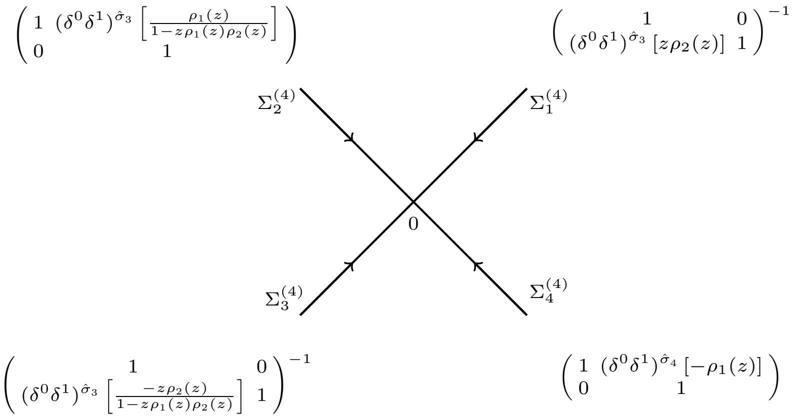

Fig. 7. $\Sigma^{(4)}$ and jump matrix.

$$\delta^0 = (8t)^{-\frac{i\nu}{2}} e^{\chi(z_0)} e^{2itz_0^2}, \quad \delta^1 = z^{i\nu} e^{\chi(\frac{z}{\sqrt{8t}}+z_0)-\chi(z_0)} e^{-\frac{iz^2}{4}}. \tag{4.54}$$

For

$$\eta(z) = (z-z_0)^{i\nu} e^{\chi(z)}, \tag{4.55}$$

where

$$\begin{aligned} \nu = \nu(z_0) &= -\frac{1}{2\pi}\log(1 - z_0\rho_1(z_0)\rho_2(z_0)) > 0, \\ \chi(z) &= -\frac{1}{2\pi i}\int_{-\infty}^{z_0} \log|z-s| \mathrm{d}\log(1 - z\rho_1(z)\rho_2(z)). \end{aligned} \tag{4.56}$$

Next, our main task is to remove the oscillating factor from the RH problem. Setting

$$N^{(N)} = (\delta^0)^{-\hat\sigma_3} N^{(4)}, \tag{4.57}$$

then we have

$$N_+^{(N)} = (\delta^0)^{-\sigma_3} N_-^{(4)} (\delta^0)^{\sigma_3} (\delta^0)^{-\sigma_3} V_N^{(4)} (\delta^0)^{\sigma_3} = M_-^N V_N^{(N)}. \tag{4.58}$$

From (4.57), we can obtain

$$N_1^{(4)} = (\delta^0)^{\hat\sigma_3} N_1^{(N)}. \tag{4.59}$$

Thus,

$$q(x,t) = \frac{i}{\sqrt{2t}} e^{-2i\int_{(-\infty,t)}^{(x,t)}\Delta} (\delta^0)^2 (N_1^{(N)})_{12} + \mathcal{O}(t^{-1}), \tag{4.60}$$

where $N^{(N)}$ is the solution of the following RH problem:

(i) $N^{(N)}(x,t,z)$ is analytic in $C\backslash\Sigma^{(N)}$,

(ii) The boundary value $N^{(N)}(x,t,z)$ at $\Sigma^{(N)} = \Sigma^{(4)}$ satisfies the jump condition

$$\left[N^{(N)}(x,t,z)) e^{i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\sigma_3}\right]_+ = \left[N^{(N)}(x,t,z)) e^{i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\sigma_3}\right]_- V_N^{(N)}, \quad z \in \Sigma^{(N)}, \tag{4.61}$$

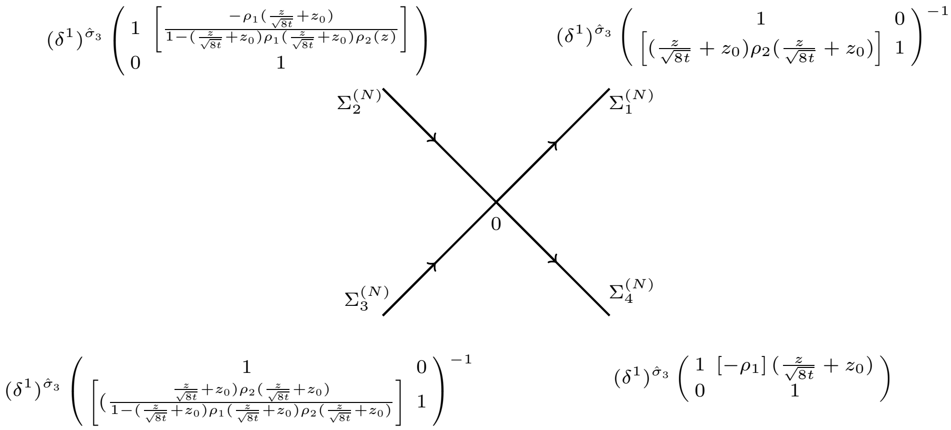

Fig. 8. $\Sigma^{(N)}$ and jump matrix.

(iii) Asymptotic condition

$$N^{(N)}(x,t,z) \to I, \quad z \to \infty. \tag{4.62}$$

$V_N^{(N)}$ can be seen clearly in Fig. 8.

We now deform the jump curve $\Sigma^{(N)}$ to Σ^∞ by taking the limit to attribute RH problem on the phase point z_0. When $t \to \infty$, it follows

$$(8t)^{-\frac{1}{2}}z + z_0 \to z_0, \quad \delta^1 \to z^{i\nu}e^{-\frac{iz^2}{4}}, \quad V_N^{(N)}(x,t,z) \to V_N^\infty(x,t,z), \tag{4.63}$$

where $V_N^\infty(x,t,z)$ is given in Fig. 9. Moreover, we find that

$$||V_N^{(N)} - V_N^\infty||_{L^1\cap L^\infty} \le ct^{-\frac{1}{2}}\log t, \tag{4.64}$$

and obtain the RH problem of $(V_N^\infty, \Sigma^\infty)$:

(i) $N^\infty(x,t,z)$ is analytic in $C\backslash\Sigma^\infty$,

(ii) The boundary value $N^\infty(x,t,z)$ at Σ^∞ satisfies the jump condition

$$\left[N^\infty(x,t,z))e^{i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\sigma_3}\right]_+ = \left[N^\infty(x,t,z))e^{i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\sigma_3}\right]_- V_N^\infty, \quad z \in R, \tag{4.65}$$

(iii) Asymptotic condition

$$N^\infty(x,t,z) \to I, \quad z \to \infty, \tag{4.66}$$

where the contour Σ^∞ and the jump matrix V_N^∞ are given by Fig. 9. The solution of mixed Schrödinger equation is given by

$$q(x,t) = \frac{i}{\sqrt{2t}}e^{-2i\int_{(-\infty,t)}^{(x,t)}\Delta}(\delta^0)^2[N_1^\infty(x,t,z)]_{12} + \mathcal{O}(t^{-1}\log t), \tag{4.67}$$

where

$$N^\infty = I + \frac{N_1^\infty}{z} + \cdots \tag{4.68}$$

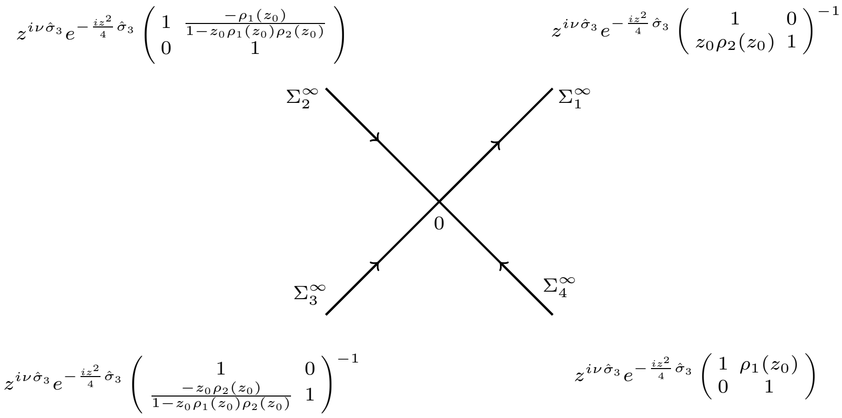

Fig. 9. Σ^∞ and jump matrix.

We define $\Sigma^e = \Sigma^\infty \cup R$ and six domains $\Omega_j, j = 1, \cdots, 6$, see Fig. 10. Define a function

$$\phi(z) = \begin{cases} z^{-i\nu\sigma_3}, & z \in \Omega_2 \cup \Omega_5, \\ z^{-i\nu\sigma_3}(b_+^e)^{-1}, & z \in \Omega_1 \cup \Omega_3, \\ z^{-i\nu\sigma_3}(b_-^e)^{-1}, & z \in \Omega_4 \cup \Omega_6. \end{cases} \tag{4.69}$$

Making a transformation

$$\vartheta = N^\infty \phi^{-1} e^{-\frac{iz^2}{4}\sigma_3},$$

we then get the following model RH problem

(i) ϑ is analytic in $C\backslash R$,

(ii) The boundary value $\vartheta(x, t, z)$ at R satisfies the jump condition

$$\left[\vartheta(x,t,z)e^{i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\sigma_3}\right]_+ = \left[\vartheta(x,t,z)e^{i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\sigma_3}\right]_- V(z_0), \quad z \in R, \tag{4.70}$$

(iii) Asymptotic condition

$$\vartheta e^{\frac{iz^2}{4}\sigma_3} z^{-i\nu\sigma_3} \to I, \quad z \to \infty. \tag{4.71}$$

This kind of RH problem can be solved by the Weber equation and parabolic-cylinder function.

5. Long-time asymptotics

By using (4.70) and the Liouville theorem, we can show that $(\frac{\mathrm{d}\vartheta}{\mathrm{d}z} + \frac{1}{2}iz\sigma_3\vartheta)\vartheta^{-1}$ is a constant matrix with respect to z, so there exist constants $\beta_{ij}, i, j = 1, 2$ such that

$$\left(\frac{\mathrm{d}\vartheta}{\mathrm{d}z} + \frac{1}{2}iz\sigma_3\vartheta\right)\vartheta^{-1} = \frac{1}{2}i[\sigma_3, N_1^\infty] = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}. \tag{5.1}$$

Comparing two sides of (5.1) leads to

$$(N_1^\infty)_{12} = -i\beta_{12}. \tag{5.2}$$

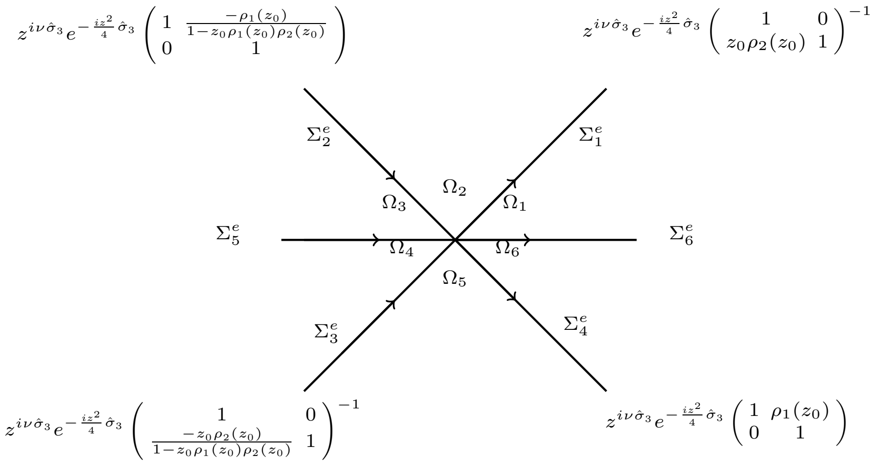

Fig. 10. Σ^e and domains Ω_j.

Substituting (5.2) into (4.67) gives

$$q(x,t) = \frac{1}{\sqrt{2t}} e^{-2i\int_{(-\infty,t)}^{(x,t)}\Delta} (\delta^0)^2 \beta_{12} + \mathcal{O}(t^{-1}\log t). \tag{5.3}$$

We can use parabolic-cylinder function to calculate β_{12}. Letting

$$\vartheta = \begin{pmatrix}\vartheta_{11} & \vartheta_{12}\\ \vartheta_{21} & \vartheta_{22}\end{pmatrix},$$

and starting from (5.1), we find that for Im$z > 0$,

$$\vartheta_{11}^{+} = e^{-\frac{3\pi\nu}{4}} D_a(e^{\frac{-3\pi i}{4}}z), \tag{5.4a}$$

$$\vartheta_{21}^{+} = e^{-\frac{3\pi\nu}{4}} \beta_{12}^{-1}[\partial_z D_a(e^{\frac{-3\pi i}{4}}z) + \frac{iz}{2} D_a(e^{\frac{-3\pi i}{4}}z)], \tag{5.4b}$$

$$\vartheta_{22}^{+} = e^{\frac{\pi\nu}{4}} D_{-a}(e^{\frac{-\pi i}{4}}z), \tag{5.4c}$$

$$\vartheta_{12}^{+} = e^{\frac{3\pi\nu}{4}} \beta_{21}^{-1}[\partial_z D_{-a}(e^{\frac{-\pi i}{4}}z) - \frac{iz}{2} D_{-a}(e^{\frac{-\pi i}{4}}z)]; \tag{5.4d}$$

for Im$z < 0$,

$$\vartheta_{11}^{-} = e^{\frac{\pi\nu}{4}} D_a(e^{\frac{\pi i}{4}}z), \tag{5.5a}$$

$$\vartheta_{21}^{-} = e^{\frac{\pi\nu}{4}} \beta_{12}^{-1}[\partial_z D_a(e^{\frac{\pi i}{4}}z) + \frac{iz}{2} D_a(e^{\frac{\pi i}{4}}z)], \tag{5.5b}$$

$$\vartheta_{22}^{-} = e^{\frac{-3\pi\nu}{4}} D_a(e^{\frac{3\pi i}{4}}z), \tag{5.5c}$$

$$\vartheta_{12}^{-} = e^{-\frac{3\pi\nu}{4}} \beta_{21}^{-1}[\partial_z D_{-a}(e^{\frac{3\pi i}{4}}z) - \frac{iz}{2} D_{-a}(e^{\frac{3\pi i}{4}}z)], \tag{5.5d}$$

where $a = -i\nu(z_0)$ and $D_a(\zeta) = D_a(e^{-\frac{3}{4}\pi i}z)$ is a standard parabolic cylinder function satisfying the Weber equation

$$\partial_\zeta^2 D_a(\zeta) + \left(\frac{1}{2} - \frac{\zeta^2}{4} + i\beta_{12}\beta_{21}\right) D_a(\zeta) = 0. \tag{5.6}$$

We rewrite the formulae (4.70) in the form

$$e^{-i\int_{(-\infty,t)}^{(+\infty,t)}\Delta\hat{\sigma}_3}\vartheta_-^{-1}\vartheta_+ = V(z_0) = \begin{pmatrix} 1 - z_0\rho_1(z_0)\rho_2(z_0) & -\rho_1(z_0) \\ z_0\rho_2(z_0) & 1 \end{pmatrix}, \tag{5.7}$$

then considering the $(2,1)$-element of the above matrix and using (5.4a)-(5.5d), we find that

$$\begin{aligned}(a\lambda_0 - 2b)r(\lambda_0) &= z_0\rho_2(z_0) = e^{2i\int_{(-\infty,t)}^{(+\infty,t)}\Delta}(\vartheta_{11}^-\vartheta_{21}^+ - \vartheta_{21}^-\vartheta_{11}^+) \\ &= e^{2i\int_{(-\infty,t)}^{(+\infty,t)}\Delta}\frac{(2\pi)^{\frac{1}{2}}e^{\frac{i\pi}{4}}e^{-\frac{\pi\nu}{2}}}{\beta_{12}\Gamma(-a)},\end{aligned} \tag{5.8}$$

where λ_0 satisfies $\lambda_0(a\lambda_0 - 2b) = z_0$. Thus,

$$\beta_{12} = e^{2i\int_{(-\infty,t)}^{(+\infty,t)}\Delta}\frac{(2\pi)^{\frac{1}{2}}e^{\frac{i\pi}{4}}e^{-\frac{\pi\nu}{2}}}{(a\lambda_0 - 2b)r(\lambda_0)\Gamma(-a)}. \tag{5.9}$$

Substituting (5.9) into (5.3) leads to the following result.

Theorem 5.1. *As $t \to \infty$, such that $|z_0| = |-\frac{x}{4t}| < M$,*

$$q(x,t) = t^{-1/2}\alpha(z_0)e^{2i\int_{(x,t)}^{(\infty,t)}\Delta}e^{i(4tz_0^2 - \nu(z_0)\log 8t)} + \mathcal{O}(t^{-1}\log t), \tag{5.10}$$

where

$$\alpha(z_0) = \frac{\pi^{\frac{1}{2}}e^{\frac{\pi\nu}{2}}e^{\frac{i\pi}{4}}e^{2\chi(z_0)}}{(a\lambda_0 - 2b)r(\lambda_0)\Gamma(-a)}, \tag{5.11}$$

whose modulus is

$$|\alpha(z_0)|^2 = -\frac{1}{4\pi}\ln(1 - |r(\lambda_0)|^2), \tag{5.12}$$

and angle is

$$\begin{aligned}\arg\alpha(z_0) = &\frac{1}{\pi}\int_{-\infty}^{z_0}\log|z_0 - \lambda(a\lambda - 2b)|\mathrm{d}\log(1 - |r(\lambda)|^2) \\ &+ \frac{\pi}{4} - \arg[(a\lambda_0 - 2b)r(\lambda_0)] + \arg\Gamma(a).\end{aligned} \tag{5.13}$$

Theorem 5.2. *As $t \to \infty$, we have*

$$e^{2i\int_{(x,t)}^{(\infty,t)}\Delta} \sim e^{-\frac{ia}{2\pi}\int_{-\infty}^{\lambda_0}\log(1-|r(\lambda)|^2)(a\lambda - b)\mathrm{d}\lambda}. \tag{5.14}$$

Proof. Noticing that

$$e^{2i\int_{(x,t)}^{(+\infty,t)}\Delta} = e^{ia\int_x^{+\infty}|q(x',t)|^2\mathrm{d}x'}. \tag{5.15}$$

From Theorem 5.1, we know that

$$|q(x,t)|^2 \sim \frac{\nu}{8t}, \quad t \to \infty,$$

by which, we obtain that

$$\int_x^\infty |q(x',t)|^2 \mathrm{d}x' \sim \int_x^\infty \frac{\nu}{8t}\mathrm{d}x' = -\frac{1}{2\pi}\int_{-\infty}^{\lambda_0} \log(1-|r(\lambda)|^2)(a\lambda-b)\mathrm{d}\lambda. \tag{5.16}$$

Therefore, we have

$$e^{2i\int_{(x,t)}^{(\infty,t)}\Delta} \sim e^{-\frac{ia}{2\pi}\int_{-\infty}^{\lambda_0}\log(1-|r(\lambda)|^2)(a\lambda-b)\mathrm{d}\lambda}. \tag{5.17}$$

Summarizing Theorem 5.1 and Theorem 5.2, we obtain the main results in this paper.

Theorem 5.3. *Suppose that the initial value $q_0(x)$ belongs to the Schwartz space. $z_0 = -\frac{x}{4t}$ is a stationary phase point and $|z_0| < M$ with $M > 1$ being a fixed constant, then long-time asymptotic for the mixed NLS equation (1.1) is given by*

$$q(x,t) = t^{-1/2}\alpha(z_0)e^{\frac{ix^2}{4t}-i\nu(z_0)\log 8t}e^{-\frac{ia}{2\pi}\int_{-\infty}^{\lambda_0}\log(1-|r(\lambda)|^2)(a\lambda-b)\mathrm{d}\lambda} + \mathcal{O}(t^{-1}\log t), \tag{5.18}$$

where $\alpha(z_0)$ is given by (5.11).

6. Conclusion and remarks

In this paper, we define a general analytical domain and two reflection coefficients to uniformly study the long-time asymptotics for a defocusing mixed nonlinear Schrödinger equation (1.1) with the Schwartz initial data. Our formula (5.18) is an unified asymptotic formula which can cover results on classical defocusing Schrödinger equation, derivative Schrödinger equation and modified Schrödinger equation. For $a = 0,\ b \neq 0$, the formula (5.18) gives the asymptotic of the defocusing Schrödinger equation [13]; For $a \neq 0,\ b = 0$, the formula (5.18) gives the asymptotic of derivative Schrödinger equation [52]; For $a, b \neq 0$, the formula (5.18) gives the asymptotic of the modified Schrödinger equation [26].

The key technique to gain the asymptotic (5.18) is classical Deift-Zhou steepest descent method, in which the Schwartz initial data $q_0(x) \in \mathcal{S}(R)$ is inquired. In 2003, for much weaker weighted Sobolev initial data $q_0(x) \in H^{1,1}(R)$, Deift and Zhou obtained asymptotics of defocusing NLS equation with decay error $\mathcal{O}(t^{-(\frac{1}{2}+k)}),\ 0 < k < 1/4$ by using Deift-Zhou steepest descent method and Perturbation theory [12].

In recent years, McLaughlin and Miller developed a $\bar\partial$-steepest descent method for obtaining asymptotic of RH problems based on $\bar\partial$-problems [37]. This method has been successfully adapted to study the NLS equation and derivative NLS equation [5,15,32]. We recently have applied this method to obtain asymptotic for the Kundu-Eckhaus equation and modified NLS equation with weighted Sobolev initial data [35,54]. The advantages of this method are that it can avoid delicate estimates involving L^p estimates of Cauchy projection operators [12], and it also can improve error estimates to be sharp one $\mathcal{O}(t^{-3/4})$ without additional restrictions on the initial data [15].

Motivated to the obtained results above on NLS equations and derivative NLS equations [5,15,32], for our mixed NLS equation (1.1), we naturally hope to extend the results [5,15,32] in an uniform way like in this paper by replacing the Schwartz initial data $q_0(x) \in \mathcal{S}(R)$ with certain much weaker weighted Sobolev initial data. Recently, for the case without soliton, we have successfully obtained asymptotic for the mixed NLS equation with much weaker weighted Sobolev initial data $q_0(x) \in H^{2,2}(R)$. For the case supports soliton solutions, there are still certain technique difficulties in the calculation, which will be considered in our future work.

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