



A unique continuation property for the wave equation in a time-dependent domain



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ABSTRACT

We prove a unique continuation property for the wave equation in a time-dependent domain $\Omega(t)$, $0 \leq t \leq T$. That is, if $u(t)$ is a finite energy solution of the equation $u_{tt} - \Delta u = 0$, $x \in \Omega(t)$, with $u(x, t)|_{\partial\Omega(t)} = 0$, $0 \leq t \leq T$, satisfying $u(t) = 0$ on some neighborhood $\omega(t)$ of a portion of the boundary $\partial\Omega(t)$, $0 \leq t \leq T$, then we have $u(t) = 0$ on $\Omega(t)$, $0 \leq t \leq T$, under the assumptions that T is sufficiently large and $\Omega(t)$ does not move so rapidly.

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1. Introduction

We consider the initial-boundary value problem for the wave equation

$$u_{tt} - \Delta u = 0, x \in \Omega(t), 0 \leq t \leq T, \quad (1.1)$$

with

$$u(x, 0) = u_0(x) \text{ and } u_t(x, 0) = u_1(x) \text{ for } x \in \Omega(0) \text{ and } u(x, t)|_{\partial\Omega(t)} = 0, 0 \leq t \leq T, \quad (1.2)$$

where $\Omega(t)$ is a bounded domain in R^N for each t with the boundary $\partial\Omega(t)$. We set

$$Q(t_1, t_2) = \cup_{t_1 \leq t \leq t_2} \Omega(t) \times \{t\} \text{ and } \Gamma(t_1, t_2) = \cup_{t_1 \leq t \leq t_2} \partial\Omega(t) \times \{t\}.$$

We assume that $\Gamma(0, T)$ is sufficiently smooth, say, to be of C^2 class. Concerning the movement of $\Omega(t)$ we first assume that $\Omega(t)$ is time-like, that is, letting $\mathbf{n} = (n_x, n_t)$ be the outward unit normal at $(x, t) \in \Gamma(0, T)$ we assume

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Hyp.A

$$|n_t| < |n_x| \text{ at each } (x, t) \in \Gamma(0, T). \quad (1.3)$$

For a function $u(x, t)$ defined on $Q(0, T)$ we say $u(t) \in C([0, T]; H_0^1(\Omega(t)))$ if $u(t) \in H_0^1(\Omega(t))$ for each $t, 0 \leq t \leq T$, and $u(x, t)$ has an extension $\tilde{u}(x, t)$ such that $\tilde{u}(t) \in C([0, T]; H^1(\mathbb{R}^N))$ and $\tilde{u}(x, t) = u(x, t)$ if $(x, t) \in Q(0, T)$. Such a way of definition is applied to other spaces $C([0, T]; L^2(\Omega(t)))$ and $C([0, T]; H^2(\Omega(t)))$. We denote $L^2(\Omega(t))$ norm and innerproduct in $L^2(\Omega(t))$ by $\|u\|_{\Omega(t)}$ and $(u, v)_{\Omega(t)}$, respectively.

The following existence and uniqueness result of strong solution is essentially included in Inoue [3], where C^∞ solutions for the equation with nonlinear absorbing term u^3 in $\cup_{0 \leq t < \infty} \Omega(t) \times \{t\} \subset \mathbb{R}^3 \times \mathbb{R}^+$ are considered.

Proposition 1.1. *Let $(u_0, u_1) \in C_0^2(\Omega(0)) \times C_0^1(\Omega(0))$. Then, under **Hyp.A**, the problem (1.1)–(1.2) admits a unique solution $u(x, t)$ such that $u(t) \in C([0, T]; H^2(\Omega(t)) \cap H_0^1(\Omega(t)))$, $u_t(t) \in C([0, T]; H^1(\Omega(t)))$ and $u_{tt}(t) \in C([0, T]; L^2(\Omega(t)))$.*

Combining a density argument and an energy estimate (see (4.8)) we can show the existence and uniqueness of finite energy solution $u(x, t)$ for each $(u_0, u_1) \in H_0^1(\Omega(0)) \times L^2(\Omega(0))$ such that $u(t) \in C([0, T]; H_0^1(\Omega(t)))$ and $u_t(t) \in C([0, T]; L^2(\Omega(t)))$. These results are essentially included in Cooper [2], where the case $N = 3$ and $\Gamma(0, T)$ is of C^∞ class is considered. (See also [9].)

The object of this note is to show a unique continuation property for finite energy solution of the problem (1.1)–(1.2) which asserts that if $u(t) = 0$ near a certain portion of the boundary $\Gamma(0, T)$ with a large $T > 0$, then $u(t) = 0$ on $Q(0, T)$. To be more precise we set for $x_0 \in \mathbb{R}^N$,

$$\Gamma^+(0, T)(x_0) = \{(x, t) \in \Gamma(0, T) | (x - x_0) \cdot n_x \geq 0\} \quad (1.4)$$

and we take a relatively open set $\omega(0, T) = \cup_{0 \leq t \leq T} \omega(t) \times \{t\} \subset \overline{Q(0, T)}$ such that $\Gamma^+(0, T)(x_0) \subset \omega(0, T)$. We show that if a finite energy solution $u(x, t)$ satisfies $u(x, t) = 0$ on $\omega(0, T)$, then $u(x, t) = 0$ on $Q(0, T)$ provided that T is sufficiently large and $\Omega(t)$ does not move so rapidly with respect to t . Our result follows from an observability estimate (see (5.5)).

Such a unique continuation result is well known when $\Omega(t) = \Omega$, independent of t , and has been applied to control theory and decay estimates (cf. Lions [5], Ruiz [10], Zuazua [12] and [8] etc.). Related problems are considered by Bardos and Chen [1] when $\Omega(t)$ is expanding and by Miranda [6] when $Q(0, T)$ is changed to a cylindrical domain $\Omega \times [0, T]$ by changing variables. Recently very general results on boundary observability estimates for more general wave equations have been established by Shao [11] by use of Carleman estimate for sufficiently smooth solutions, where it is assumed that $\Omega(t)$ is time-like and further, $\Omega(t_1)$ and $\Omega(t_2)$ are diffeomorphic for any t_1, t_2 . In this case the existence and uniqueness of finite energy solution is rather standard, while in this note we do not assume such a diffeomorphism. We also note that our observability estimate (5.5) holds for finite energy solutions and it is not included in [11].

We would expect any applications of our result here to control theory and decay properties to the wave equations with some localized dissipations. For the proof of our result we use only elementary multiplier method which combines the ideas used in Lee [4], Zuazua [12] and Nakao [7–9] etc.

2. Preliminaries

For the outward normal $\mathbf{n} = (n_x, n_t)$ on $\Gamma(0, T)$ we set

$$\beta(x, t) = -n_t n_x / |n_x|^2. \quad (2.1)$$

It is easy to see that β can be extended as a C^1 class map $\beta : R^N \times R \rightarrow R^N$. We may assume $|\beta(x, t)| < 1$ under **Hyp.A**. We denote Jacobian of the map $\beta(\cdot, t) : R^N \rightarrow R^N$ for each t by $\beta'(t)$. We set

$$r^+(T) = \sup_{(x,t) \in Q(0,T)} (|\beta| + |\nabla \beta| + |\beta_t| + |\beta'|). \quad (2.2)$$

We note that $\beta(x, t)$ can be taken as $\beta(x, t) = 0$ and $r^+(T) = 0$ if $\Omega(t) = \Omega$, independent of t , because $n_t = 0$ in this case.

For the portion $\Gamma^+(0, T)(x_0)$ of the boundary $\Gamma(0, T)$ defined by (1.4) we take a relatively open set $\omega(0, T)$ in $\overline{Q(0, T)}$ such that $\Gamma^+(0, T)(x_0) \subset \omega(0, T)$. We can take a relatively open set $\tilde{\omega}(0, T) = \cup_{0 \leq t \leq T} \tilde{\omega}(t) \times \{t\}$ such that $\Gamma^+(0, T)(x_0) \subset \tilde{\omega}(0, T)$ and $\tilde{\omega}(0, T) \subset \omega(0, T)$. Then there exists a C^1 nonnegative function $\eta(x, t)$ defined on $\overline{Q(0, T)}$ such that $\eta(x, t) = 1$ on $\tilde{\omega}(0, T)$, $\eta(x, t) = 0$ on $\omega(0, T)^c \equiv \cup_{0 \leq t \leq T} [\omega(t)^c \cap \overline{\Omega(t)} \times \{t\}]$, $0 \leq \eta(x, t) \leq 1$ on $Q(0, T)$ and $|\nabla \eta(x, t)|^2 / \eta(x, t) \leq C(T) < \infty$ for $(x, t) \in Q(0, T)$ with $\eta(x, t) \neq 0$. We see that $\sup_{(x,t) \in Q(0,T)} (|\nabla \eta(x, t)| + |\eta_t(x, t)|) \leq C(T) < \infty$. But, we make an assumption:

Hyp.B₁ There exists a function $\eta(x, t)$ as above satisfying

$$\sup_{\{(x,t) \in Q(0,T) | \eta(x,t) \neq 0\}} (|\nabla \eta(x, t)| + |\eta_t(x, t)| + |\nabla \eta|^2 / \eta(x, t)) \leq C < \infty \quad (2.3)$$

with a constant C independent of T .

Similarly we can take a C^1 class vector valued function $\phi(x, t)$ defined on $\overline{Q(0, T)}$ such that $\phi(x, t) = n_x$ on $\Gamma^+(0, T)(x_0)$, $\phi(x, t) \cdot n_x \geq 0$ on $\Gamma(0, T)$ and $\phi(x, t) = 0$ on $\tilde{\omega}(0, T)^c$. Further we have $\sup_{(x,t) \in Q(0,T)} (|\phi| + |\nabla \phi| + |\phi_t| + |\phi'|) \leq C(T) < \infty$, where we note that $\phi'(t)$ denotes the Jacobian of the map $\phi(\cdot, t) : R^N \rightarrow R^N$. But we make an assumption:

Hyp.B₂

There exists a function $\phi(x, t)$ as above satisfying

$$\sup_{(x,t) \in Q(0,T)} (|\phi| + |\nabla \phi| + |\phi_t| + |\phi'|) \leq C < \infty \quad (2.4)$$

with a constant C independent of T .

Further, to apply Poincare's inequality we assume:

Hyp.B₃ There exists a ball $B_0(t) \subset R^N$ centered at some $x_0(t) \in R^N$ for each $t, 0 \leq t \leq T$, with the radius $d_0 > 0$ independent of t such that $\Omega(t) \subset B_0(t)$.

Now we state our result.

Theorem 2.1. *Let $(u_0, u_1) \in H_0^1(\Omega(0)) \times L^2(\Omega(0))$. Then, under Hyp.A, there exists a unique finite energy solution $u(x, t)$ of the problem (1.1)–(1.2).*

Further, under Hyp.B_i, $i = 1, 2, 3$, there exist $\delta_0, 0 < \delta_0 < 1$, and $T_0 > 0$ such that if $T > T_0$, $r^+(T) < \delta_0$ and $u(x, t) = 0$ on $\omega(0, T)$, then we have $u(x, t) = 0$ on $Q(0, T)$

3. Some integral identities for strong solutions

Let $u(x, t)$ be a strong solution as in Proposition 1.1. We derive some basic integral identities for this solution.

First, multiplying the equation by u_t and integrating it on $Q(0, T)$ we have (cf. [4,7,9])

$$\int_{Q(0,T)} (u_{tt} - \Delta u) u_t dV$$

$$\begin{aligned}
&= E(T) - E(0) + \frac{1}{2} \int_{\Gamma(0,T)} (n_t |u_t|^2 + |\nabla u|^2 n_t - 2n_x \cdot \nabla u u_t) dS \\
&= E(T) - E(0) + \frac{1}{2} \int_{\Gamma(0,T)} (|\frac{\partial u}{\partial \mathbf{n}}|^2 n_t (n_t^2 - |n_x|^2) dS
\end{aligned} \tag{3.1}$$

where we set

$$E(t) = \frac{1}{2} (\|u_t(t)\|_{\Omega(t)}^2 + \|\nabla u(t)\|_{\Omega(t)}^2).$$

Next, taking a function $\eta \in C^1(\overline{Q(0,T)})$ independent of the function in **Hyp.B₁** we multiply the equation by ηu and integrating it we have

$$\begin{aligned}
&\int_{Q(0,T)} (u_{tt} - \Delta u) \eta u dV \\
&= \int_{Q(0,T)} \{ (u_t \eta u)_t - \eta_t u_t u - \eta |u_t|^2 - \nabla \cdot (\eta u \nabla u) + \nabla u \cdot (u \nabla \eta + \eta \nabla u) \} dV \\
&\quad = (u_t(T), \eta(T) u(T))_{\Omega(T)} - (u_t(0), \eta(0) u(0))_{\Omega(0)} \\
&\quad + \int_{Q(0,T)} \{ (-\eta_t u_t + \nabla u \cdot \nabla \eta) u + \eta (|\nabla u|^2 - u_t^2) \} dV
\end{aligned} \tag{3.2}$$

Further, taking a vector valued function $\phi \in C^1(\overline{Q(0,T)})$ independent of the function in **Hyp.B₂**, multiplying the equation by $\phi \cdot \nabla u$ and integrating it we see

$$\begin{aligned}
&\int_{Q(0,T)} (u_{tt} - \Delta u) \phi \cdot \nabla u dV \\
&= \int_{Q(0,T)} \{ (u_t \phi \cdot \nabla u)_t - u_t \phi_t \cdot \nabla u - \frac{1}{2} \phi \cdot \nabla |u_t|^2 \} dV \\
&+ \int_{Q(0,T)} \{ -\nabla \cdot (\phi \cdot \nabla u \nabla u) + \nabla u \cdot \phi' \nabla u + \frac{1}{2} \nabla \cdot (\phi |\nabla u|^2) - \frac{1}{2} (\nabla \cdot \phi) |\nabla u|^2 \} dV \\
&\quad = (u_t(T), \phi(T) \cdot \nabla u(T))_{\Omega(T)} - (u_t(0), \phi(t_1) \cdot \nabla u(0))_{\Omega(0)} \\
&\quad + \int_{Q(0,T)} \{ \nabla u \cdot \phi' \nabla u - u_t \phi_t \cdot \nabla u \} dV + \frac{1}{2} \int_{Q(0,T)} (u_t^2 - |\nabla u|^2) \nabla \cdot \phi dV \\
&\quad + \frac{1}{2} \int_{\Gamma(0,T)} \left(\frac{\partial u}{\partial \mathbf{n}} \right)^2 \phi \cdot n_x (n_t^2 - |n_x|^2) dS.
\end{aligned} \tag{3.3}$$

In particular, by (3.1) and (3.3) with $\phi = \beta$ ($= -n_t n_x / |n_x|^2$ on $\Gamma(0,T)$) we see

$$\begin{aligned}
&\int_{Q(0,T)} (u_{tt} - \Delta u) (u_t + \beta \cdot \nabla u) dV \\
&= \tilde{E}(T) - \tilde{E}(0) + \frac{1}{2} \int_{\Gamma(0,T)} \left(\frac{\partial u}{\partial \mathbf{n}} \right)^2 (n_t + \beta \cdot n_x) (n_t^2 - |n_x|^2) dS
\end{aligned}$$

$$+ \int_{Q(0,T)} \{ \nabla u \cdot \beta' \nabla u - u_t \beta_t \cdot \nabla u + \frac{1}{2} (u_t^2 - |\nabla u|^2) \nabla \cdot \beta \} dV$$

where we set

$$\tilde{E}(t) = E(t) + (u_t(t), \beta(t) \cdot \nabla u(t))_{\Omega(t)}.$$

The above identity implies by the relation $n_t + \beta \cdot n_x = 0$ on $\Gamma(0, T)$,

$$\tilde{E}(T) = \tilde{E}(0) - \int_{Q(0,T)} \{ \nabla u \cdot \beta' \nabla u - u_t \beta_t \cdot \nabla u + \frac{1}{2} (u_t^2 - |\nabla u|^2) \nabla \cdot \beta \} dV. \quad (3.4)$$

We note that $\tilde{E}(t)$ and $E(t)$ are mutually equivalent if $|\beta(t)| = |n_t|/|n_x| \leq \delta < 1$ with some constant $\delta > 0$.

4. A basic integral inequality and existence of finite energy solution

We first derive a basic integral inequality concerning the energy $\tilde{E}(t)$ or $E(t)$ for strong solution $u(t)$. Next we confirm through a density argument based on an energy inequality that there exists a unique finite energy solution for each $(u_0, u_1) \in H_0^1(\Omega(0)) \times L^2(\Omega(0))$ and the basic inequality holds for finite energy solutions.

We take $\eta = 1$ in (3.2). For a fixed $x_0 \in R^N$ we take $\phi = x - x_0$ in (3.3). Combining resulted two identities we have

$$\begin{aligned} & \left(\frac{N}{2} - \alpha \right) \int_{Q(0,T)} |u_t|^2 dV + \left(1 + \alpha - \frac{N}{2} \right) \int_{Q(0,T)} |\nabla u|^2 dV \\ &= -(u_t(T), (x - x_0) \cdot \nabla u(T))_{\Omega(T)} - \alpha (u_t(T), u(T))_{\Omega(T)} + (u_t(0), (x - x_0) \cdot \nabla u(0))_{\Omega(0)} \\ & \quad + \alpha (u_t(0), u(0))_{\Omega(0)} + \frac{1}{2} \int_{\Gamma(0,T)} \left(\frac{\partial u}{\partial \mathbf{n}} \right)^2 (|n_x|^2 - n_t^2) (x - x_0) \cdot n_x dS \\ & \leq C(x_0, T) (E(0) + E(T)) + \frac{1}{2} \int_{\Gamma^+(0,T)(x_0)} \left(\frac{\partial u}{\partial \mathbf{n}} \right)^2 (|n_x|^2 - n_t^2) (x - x_0) \cdot n_x dS, \end{aligned} \quad (4.1)$$

with a constant $C(x_0, T)$ depending on x_0 and T , where we have used the assumption $|n_t| < |n_x|$ and Poincaré's inequality $\|u(t)\|_{\Omega(t)} \leq d(T) \|\nabla u(t)\|_{\Omega(t)}$, $0 \leq t \leq T$. Note that under **Hyp.B₃**, both of $d(T) = d$ and $C(x_0, T) = C(x_0)$ are independent of T .

We can take positive constants ϵ_0 and α such that

$$\epsilon_0 < \frac{1}{2} \text{ and } \frac{N}{2} - 1 + \epsilon_0 \leq \alpha \leq \frac{N}{2} - \epsilon_0.$$

Then (4.1) yields

$$\epsilon_0 \int_0^T E(s) ds \leq C(x_0) \left(E(0) + E(T) + \frac{1}{2} \int_{\Gamma^+(0,T)(x_0)} \left(\frac{\partial u}{\partial \mathbf{n}} \right)^2 (|n_x|^2 - n_t^2) dS \right), \quad (4.2)$$

under **Hyp.B₃**.

We proceed to the estimation of the boundary integral in (4.2) under **Hyp.B₁** and **Hyp.B₂**. We take a function $\eta(x, t)$ as in **Hyp.B₁**. Then by (3.2) we see

$$\int_{Q(0,T)} \eta |\nabla u|^2 dV \leq C(E(0) + E(T)) + \frac{1}{2} \int_{Q(0,T)} \eta |\nabla u|^2 dV + C \int_{\omega(0,T)} (|u_t|^2 + |u|^2) dV$$

and hence,

$$\int_{\tilde{\omega}(0,T)} |\nabla u|^2 dV \leq C(E(0) + E(T)) + C \int_{\omega(0,T)} (|u_t|^2 + |u|^2) dV. \quad (4.3)$$

Next, we take a vector valued function $\phi(x, t)$ as in **Hyp.B₂**. Then we see from (3.3) that

$$\begin{aligned} & \int_{\Gamma^+(0,T)(x_0)} \left(\frac{\partial u}{\partial \mathbf{n}} \right)^2 (|n_x|^2 - n_t^2) dS \\ & \leq C \left(\int_{\tilde{\omega}(0,T)} (|u_t|^2 + |\nabla u|^2) dV + E(0) + E(T) \right). \end{aligned} \quad (4.4)$$

It follows from (4.3) and (4.4) that

$$\begin{aligned} & \int_{\Gamma^+(0,T)} \left(\frac{\partial u}{\partial \mathbf{n}} \right)^2 (|n_x|^2 - n_t^2) dS \\ & \leq C \left(E(0) + E(T) + \int_{\omega(0,T)} (|u_t|^2 + |u|^2) dV \right). \end{aligned} \quad (4.5)$$

Therefore we obtain from (4.2) and (4.5)

$$\int_0^T E(s) ds \leq C(x_0) \left(E(0) + E(T) + \int_{\omega(0,T)} (|u_t|^2 + |u|^2) dV \right). \quad (4.6)$$

This is the basic integral inequality for strong solution.

To consider finite energy solution we can use the arguments in Cooper [2]. Indeed, we return to the identity (3.4) with T replaced by t , $0 \leq t \leq T$, which implies the energy inequality

$$\tilde{E}(t) \leq \tilde{E}(0) + \delta_0(T) \int_0^t E(s) ds, \quad 0 \leq t \leq T. \quad (4.7)$$

Since $\tilde{E}(t)$ and $E(t)$ are mutually equivalent on $[0, T]$, that is, there exist constants $k_0(T), k_1(T) > 0$ such that $k_0 E(t) \leq \tilde{E}(t) \leq k_1 E(t)$, $0 \leq t \leq T$, we have from (4.7)

$$E(t) \leq C(T) E(0) e^{\delta_0(T)t}, \quad 0 \leq t \leq T. \quad (4.8)$$

Let $(u_0, u_1) \in H_0^1(\Omega(0)) \times L^2(\Omega(0))$ and take a sequence $(u_{0,n}, u_{1,n}) \in C_0^2(\Omega(0)) \times C_0^1(\Omega(0))$ such that $(u_{0,n}, u_{1,n}) \rightarrow (u_0, u_1)$ in $(H_0^1(\Omega(0)) \times L^2(\Omega(0)))$. The problem has a unique strong solution $u_n(x, t)$ with

$u_n(0) = u_{0,n}$ and $u_{n,t}(0) = u_{1,n}$ (see Proposition 1.1). By the energy inequality (4.8) with u replaced by $u_n(t) - u_m(t)$ we see that there exists a function $u(x, t)$ on $Q(0, T)$ with $u(t) \in C([0, T]; H_0^1(\Omega(t)))$ and $u_t(t) \in C([0, T]; L^2(\Omega(t)))$ such that $u_n(t)$ and $u_{n,t}(t)$ are uniformly convergent to $u(t)$ and $u_t(t)$ in $H_0^1(\Omega(t))$ and $L^2(\Omega(t))$ on $[0, T]$, respectively. (We extend $u_n(t)(x, t)$ and $u_{n,t}(x, t)$ as $\tilde{u}_n(t)(x, t) = 0$, and $\tilde{u}_{n,t}(x, t) = 0$ for $x \notin \Omega(t)$.) Thus the function $u(x, t)$ is finite energy solution on $(0, T)$ with $u(0) = u_0$ and $u_t(0) = u_1$. The uniqueness of such a solution is assured by Cooper [2] by use of mollifier argument.

By the density argument it is clear that the basic inequality (4.6) and the energy identity (3.4) are valid for the finite energy solutions.

5. Completion of the proof of Theorem 2.1

It is clear that the identity (3.4) holds with 0 and T replaced by any t_1 and t_2 , respectively, such that $0 \leq t_1 < t_2 \leq T$. Then, under **Hyp.** B_i , $i = 1, 2, 3$, we have

$$\sup_{0 \leq t \leq T} \tilde{E}(t) \leq \inf_{0 \leq t \leq T} \tilde{E}(t) + \delta_0 \int_{Q(0, T)} (|\nabla u|^2 + |u_t|^2) dV, \quad (5.1)$$

where we assume that there exists $\delta_0 > 0$ independent of T such that $r^+(T) \leq \delta_0 < 1$. It follows from (4.6) and (5.1) that

$$\int_0^T E(t) dt \leq C(x_0) \left\{ \inf_{0 \leq t \leq T} E(t) + \delta_0 \int_0^T E(t) dt + \int_{\omega(0, T)} (|u_t|^2 + |u|^2) dV \right\}. \quad (5.2)$$

We assume that $C(x_0)\delta_0 < 1/2$. Then (5.2) implies

$$(T \inf_{0 \leq t \leq T} E(t) \leq) \int_0^T E(t) dt \leq C(x_0) \left\{ \inf_{0 \leq t \leq T} E(t) + \int_{\omega(0, T)} (|u_t|^2 + |u|^2) dV \right\}. \quad (5.3)$$

Hence, there exists $T_0 > 0$ such that if $T > T_0$, then

$$\inf_{0 \leq t \leq T} E(t) \leq C(x_0) \int_{\omega(0, T)} (|u_t|^2 + |u|^2) dV. \quad (5.4)$$

By (5.3) and (5.4) we arrive at the desired observability inequality

$$\int_0^T E(t) dt \leq C(x_0) \int_{\omega(0, T)} (|u_t|^2 + |u|^2) dV, \quad (5.5)$$

which shows that if $u(x, t) = 0$ on $\omega(0, T)$, then $E(t) = 0$ on $Q(0, T)$, that is, $u(x, t) = 0$ on $Q(0, T)$. The proof of Theorem is completed.

6. An open problem

It is an interesting question whether in Theorem 2.1 we can replace the condition $u(x, t) = 0$ on $\omega(0, T)$ by

$$u_t(x, t) = 0 \text{ on } \omega(0, T) \quad (6.1)$$

or not. When $\Omega(t) = \Omega$, independent of t , it is easy to show that the assertion is valid. Indeed we set $u_\epsilon(x, t) = u(x, \cdot) * \rho_\epsilon(t)$, where $\rho_\epsilon(t)$, $0 < \epsilon \ll 1$, is a mollifier with respect to t and we set $u(x, t) = 0$ for $t < 0$. Then $v_\epsilon(x, t) \equiv \partial u_\epsilon(x, t)/\partial t = u_t(x, \cdot) * \rho_\epsilon(t)$ is a solution of the equation (1.1) on $[\epsilon, T - \epsilon]$ with $v_\epsilon(x, t) = 0$ on $\omega(\epsilon, T - \epsilon) = \omega \times [\epsilon, T - \epsilon]$. Therefore we can apply Theorem 2.1 to see that $v_\epsilon(x, t) = 0$ on $\Omega \times (\epsilon, T - \epsilon)$ for $T > T_0$ and for any sufficiently small $\epsilon > 0$. Hence we have $u_t(x, t) = 0$ on $\Omega \times [0, T]$. This implies $u(x, t) = u(x)$, independent of t , on $[0, T]$ and $-\Delta u = 0$ on Ω . By the boundary condition $u|_{\partial\Omega} = 0$ we conclude that $u(x, t) = 0$ on $\Omega \times [0, T]$. But when $\Omega(t)$ is time-dependent the problem is open at this time.

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