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# Asymptotic behaviour of Verblunsky coefficients

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**Abstract**

Let $V(z)=\prod_{j=1}^{m}(z-\zeta_j)$, $\zeta_h\neq\zeta_k$, $h\neq k$ and $|\zeta_j|=1$, $j=1,\dots,m$, and consider the polynomials orthogonal with respect to $|V|^2\,d\mu$, $\varphi_n(|V|^2\,d\mu;z)$, where $\mu$ is a finite positive Borel measure on the unit circle with infinite points in its support, such that the reciprocal of its Szegő function has an analytic extension beyond $|z|<1$. In this paper we deduce the asymptotic behaviour of their Verblunsky coefficients. By means of this result, an asymptotic representation for these polynomials inside the unit circle is also obtained.


*Keywords:* Orthogonal polynomials; Verblunsky coefficients

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## 1. Introduction

Let $\Gamma = \{z\colon |z| = 1\}$ and let $\mathcal{M}$ be the family of the finite positive Borel measures on the unit circle with infinite points in its support. We denote by $\{\varphi_n(d\mu; z)\}_{n=0}^{\infty}$ the sequence of polynomials orthonormal with respect to $\mu \in \mathcal{M}$. So,

$$\varphi_n(d\mu; z) = \kappa_n(d\mu)z^n + \text{lower degree terms}, \quad \kappa_n(d\mu) > 0,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_n(d\mu; z)\overline{\varphi_m(d\mu; z)}\, d\mu(\theta) = \delta_{n,m}, \quad z = e^{i\theta},\ n, m = 0, 1, \ldots .$$

The corresponding monic orthogonal polynomials $\kappa_n(d\mu)^{-1}\varphi_n(d\mu; z)$ will be denoted by $\Phi_n(d\mu; z)$. As usual, $\Phi_n^*(z) = z^n\overline{\Phi_n(1/\bar{z})}$ denotes the reversed polynomial of $\Phi_n(z)$.

A rich theory has been developed since 1918 when Szegő established the notion of orthogonal polynomials on the unit circle (see [4,5,13] or [15]). This theory has links with many subjects such as the trigonometric moments problem, the analytic and harmonic functions theory, the spectral theory of operators, the prediction theory and even with the analysis of some physical systems (see, for example, [9] and references there in), being signal processing one of the most studied at the moment.

A basic tool in the theory is the recurrence formula

$$\Phi_0(d\mu; z) = 1, \qquad \Phi_{n+1}(d\mu; z) = z\Phi_n(d\mu; z) - \overline{\alpha_n(d\mu)}\Phi_n^*(d\mu; z), \quad n \geqslant 0, \tag{1}$$

where $\alpha_n(d\mu) = -\overline{\Phi_{n+1}(d\mu; 0)}$.

For $\mu \in \mathcal{M}$ with $\log \mu' \in L^1[0, 2\pi)$, the Szegő function is defined by

$$D(d\mu; z) = \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \mu'(\theta)\, d\theta\right), \quad |z| < 1,$$

and then

$$\lim_{n\to\infty} \kappa_n(d\mu) = \exp\left(-\frac{1}{4\pi}\int_0^{2\pi} \log \mu'(\theta)\, d\theta\right) = D(d\mu; 0)^{-1}. \tag{2}$$

The numbers $\{\alpha_n(d\mu)\}_{n\geqslant 0}$ are called Verblunsky coefficients. They play an important role in the theory of orthogonal polynomials as they completely determine the whole sequence $\{\Phi_n(d\mu; z)\}$.

Even though the connection between the Szegő function and the Verblunsky coefficients already appears in Geronimus works, the relationship between the analyticity of $D(d\mu; z)^{-1}$ and the asymptotic behaviour of the associated reflection coefficients $\Phi_n(d\mu; 0)$ was revealed by Nevai and Totik in [11]. In this article they proved that the condition

$$\limsup_{n\to\infty} \left|\Phi_n(d\mu; 0)\right|^{1/n} = \frac{1}{\rho} < 1, \tag{3}$$

guarantees the integrability of $\log \mu'$ and, moreover, the equivalence of (3) and the following assertion:

$$\sup\left\{r\colon\ D(d\mu; z)^{-1} \text{ is analytic for } |z| < r\right\} = \rho > 1.$$

As a consequence of that, if (3) holds the known asymptotic formulae for $\varphi_n(d\mu; z)$ and $\varphi_n^*(d\mu; z)$ are valid in a larger region than the one considered at first. Namely,

$$\lim_{n\to\infty} \frac{\varphi_n(d\mu; z)}{z^n} = \overline{D\left(d\mu; \frac{1}{\bar{z}}\right)^{-1}}, \quad |z| > \frac{1}{\rho}, \tag{4}$$

$$\lim_{n\to\infty} \varphi_n^*(d\mu; z) = D(d\mu; z)^{-1}, \quad |z| < \rho, \tag{5}$$

from which one can easily deduce

$$\lim_{n\to\infty} \frac{\varphi_n^{(j)}(d\mu; z)}{n^j z^{n-j}} = \overline{D\left(d\mu; \frac{1}{\bar{z}}\right)^{-1}}, \quad |z| > \frac{1}{\rho}, \tag{6}$$

$$\lim_{n\to\infty} \varphi_n^{*(j)}(d\mu; z) = \left(D(d\mu; z)^{-1}\right)^{(j)}, \quad |z| < \rho. \tag{7}$$

In (4)–(7) the convergence is uniform on each compact subset of the prescribed regions.

Furthermore, a strong connection between the algebraic singularities of the function $D(d\mu; z)^{-1}$ and the asymptotic behaviour of the reflection coefficients was revealed in [3] for the case of a weight function for which $D(d\mu; z)^{-1}$ has a unique singularity inside the disk $\{z\colon |z| < \rho\}$.

Some works of Nevai–Totik, Mhaskar–Saff, Barrios–López–Saff, Pan or Simon (see, for example, [11,10,3,12,13]) and other people show that Verblunsky coefficients converge to zero exponentially when the reciprocal of the Szegő function is analytic in $\{z\colon |z| < \rho\}$ with $\rho > 1$. For the coefficients associated to the generalized Jacobi weight functions on the unit circle (the simplest weight functions with a finite number of known algebraic singularities) Golinskii proved in [6] that the Verblunsky coefficients tend to zero with a potential rate. Our goal in this paper is to determine an asymptotic formula for the Verblunsky coefficients associated to the measure $|V(e^{i\theta})|^2\, d\mu$ defined in the following way:

(i) $V(z) = \prod_{j=1}^m (z - \zeta_j)$ with $|\zeta_j| = 1$, $j = 1, \ldots, m$, and $\zeta_h \neq \zeta_k$, $h \neq k$.
(ii) $\mu \in \mathcal{M}$ such that the function $D(d\mu; z)^{-1}$ has an analytic extension to $\{z\colon |z| < \rho\}$, $\rho > 1$.

The results that we state in the next section include accurate information about the asymptotic behaviour of $\alpha_n(|V|^2\, d\mu)$; they will be given in terms of the singularities of the reciprocal of the Szegő function. But we think that as important as the results themselves, are the foreseeable applications of them in the field of Fourier series. It is very well known that some problems of convergence of Fourier series are closely related to the behaviour of the reflection coefficients (see, for example, [7]).

We know that Verblunsky coefficients are the conjugate of the Fourier coefficients of the function $D(d\mu; z)^{-1}$. In the real case, the works by Guadalupe–Pérez, Badkov or Golinskii (see [8,2,6]) show that it is possible to obtain some properties of Fourier series from perturbations of the orthogonality measure. Because of this, we consider that an interesting future challenge is to study properties of Fourier series in terms of properties of Verblunsky coefficients.

The result about $\alpha_n(|V|^2\, d\mu)$ also allows us to deduce an asymptotic representation for the polynomials $\Phi_n(|V|^2\, d\mu; z)$ in $\{z\colon |z| < 1\}$. When $V(z) = \prod_{j=1}^m (z - \zeta_j)$ with $|\zeta_j| > 1$, $j = 1, \ldots, m$, $\zeta_i \neq \zeta_k$ if $i \neq k$, the asymptotic behaviour of $\varphi_n(|V|^2\, d\mu; z)$ for $1/\rho < |z| < \rho$ has been studied by Pan in [12].

Let us state some remarks concerning this result. It is well known that the asymptotics play an important role in applications of the theory of orthogonal polynomials. The problem of obtaining uniform asymptotic representations is more difficult when (as in our case) the measure has some singular points in the set where the polynomials are orthogonal. We want to point out some previous results in this direction. Badkov in [1] establishes a uniform asymptotic representation on the whole unit circle for a sequence of polynomials which are orthogonal on $\Gamma$ with respect to a weight function with a finite number of singularities of power type. The choice of the weight function justifies the asymptotic representation obtained which involves an interesting family of orthogonal polynomials. Also in [1], Badkov states an asymptotic formula, uniformly on several subsets of $|z| \geqslant 1$. Other uniform asymptotic formulae on closed arcs of $\Gamma$ can be seen in [14].

In our case, since $\mu$ is necessarily absolutely continuous the measure used, is in fact, $|V(e^{i\theta})|^2\mu'(\theta)\,d\theta$. We obtain a simple asymptotic representation uniformly for compacts subset of $\{z\colon |z|<1\}$.

## 2. Main results

Let the polynomial $V(z)=\prod_{j=1}^{m}(z-\zeta_j)$ with $\zeta_h \neq \zeta_k$, $h \neq k$ and $|\zeta_j|=1$, $j=1,\dots,m$. We consider a measure $\mu \in \mathcal{M}$, such that the function $D(d\mu;z)^{-1}$ has an analytic extension to $\{z\colon |z|<\rho\}$ with $\rho>1$.

The Szegő functions associated to the measures $d\mu$ and $|V|^2\,d\mu$ are linked by the formula

$$D\big(|V|^2\,d\mu;z\big)=\prod_{j=1}^{m}\left(1-\frac{z}{\zeta_j}\right)D(d\mu,z), \tag{8}$$

which can be easily obtained from [4, Lemma 3.1, p. 211]. Obviously, the singularities of $D(|V|^2\,d\mu;z)^{-1}$ in $\{z\colon |z|<\rho\}$ are on the zeros of the polynomial $V(z)$.

The asymptotic behaviour of the polynomials $\{\Phi_n(|V|^2\,d\mu;z)\}$ and their Verblunsky coefficients are stated in the following theorems.

**Theorem 1.** *Let $\{\zeta_j\}_{j=1}^{m}$ be distinct points on the unit circle and $V(z)=\prod_{j=1}^{m}(z-\zeta_j)$. Let $\mu \in \mathcal{M}$ such that $D(\mu,z)^{-1}$ has an analytic extension to a circle $\{z\colon |z|<\rho\}$, where $\rho>1$. Then*

$$\alpha_n\big(|V|^2\,d\mu\big)=\frac{1}{(n+1)\overline{V(0)}}\sum_{j=1}^{m}\frac{\overline{D(d\mu;\zeta_j)}}{D(d\mu;\zeta_j)}\overline{\zeta_j}^{\,m+n+1}+\mathrm{o}\left(\frac{1}{n}\right)\quad(n\to\infty). \tag{9}$$

From this result, we can obtain the asymptotic behaviour of the orthogonal polynomials in $\{z\colon |z|<1\}$:

**Theorem 2.** *Let $\{\zeta_j\}_{j=1}^{m}$ be distinct points on the unit circle and $V(z)=\prod_{j=1}^{m}(z-\zeta_j)$. Let $\mu \in \mathcal{M}$ be such that $D(d\mu;z)^{-1}$ has an analytic extension to $\{z\colon |z|<\rho\}$, $\rho>1$. Then*

$$V(z)^2\Phi_n\big(|V|^2\,d\mu;z\big)=\frac{D(d\mu;0)}{nD(d\mu;z)}P_m(z;n)+\mathrm{o}\left(\frac{1}{n}\right)\quad(n\to\infty), \tag{10}$$

*where $P_m(z;n)$ has degree $m-1$ and satisfies*

$$
\begin{aligned}
P_m(z;n) = {} & P_{m-1}(z;n+2) - \zeta_m P_{m-1}(z;n+1) \\
& + \left( \sum_{j=1}^{m} \frac{D(d\mu;\zeta_j)}{\overline{D(d\mu;\zeta_j)}} \zeta_j^{n+m+1} \right) V_{m-1}(z),
\end{aligned} \tag{11}
$$

*being* $V_{m-1}(z) = \prod_{j=1}^{m-1}(z-\zeta_j)$ *and* $P_1(z;n) = \frac{D(d\mu;\zeta_1)}{\overline{D(d\mu;\zeta_1)}}\zeta_1^{n+2}$.

*The convergence in* (10) *is uniform on each compact subset of* $\{z\colon |z|<1\}$.

## 3. Auxiliary results

It is well known that the reproducing kernel

$$
K_n(d\mu;z,y) \stackrel{\text{def}}{=} \sum_{j=0}^{n} \overline{\varphi_j(d\mu;y)}\varphi_j(d\mu;z),
$$

admits two different expressions, the so-called Christoffel–Darboux formulae (see [5, p. 8])

$$
\begin{aligned}
K_n(d\mu;z,y) &= \frac{\overline{\varphi_n^*(d\mu;y)}\varphi_n^*(d\mu;z) - \bar{y}z\overline{\varphi_n(d\mu;y)}\varphi_n(d\mu;z)}{1-\bar{y}z}, \\
K_n(d\mu;z,y) &= \frac{\overline{\varphi_{n+1}^*(d\mu;y)}\varphi_{n+1}^*(d\mu;z) - \overline{\varphi_{n+1}(d\mu;y)}\varphi_{n+1}(d\mu;z)}{1-\bar{y}z},
\end{aligned} \tag{12}
$$

with $\bar{y}z \neq 1$ in both cases.

From the orthogonality of the family $\{\varphi_j(d\mu;\cdot)\}_{j=0}^{n}$ the following reproducing property can be easily deduced:

**Lemma 1.** *Let $P$ be a polynomial of degree at most $n$. Then*

$$
P(z) = \frac{1}{2\pi}\int_0^{2\pi} K_n(d\mu;z,y)P(y)\,d\mu(\theta), \quad y = e^{i\theta}.
$$

The next lemma states the relationship between the orthogonal polynomials $\varphi_n(|V|^2\,d\mu;z)$ and $\varphi_n(d\mu;z)$.

**Lemma 2.** *Let $V(z) = \prod_{j=1}^{m}(z-\zeta_j)$, $|\zeta_j| = 1$, $j = 1,\dots,m$, and $\zeta_h \neq \zeta_k$, $h \neq k$. For $n \geqslant m$ we have*

$$
V(z)\varphi_{n-m}\big(|V|^2\,d\mu;z\big) = \frac{\kappa_{n-m}(|V|^2\,d\mu)}{\kappa_n(d\mu)}\varphi_n(d\mu;z) + \sum_{j=1}^{m} A_{nj}K_{n-1}(d\mu;z,\zeta_j), \tag{13}
$$

*where*

$$
A_{nj} = \frac{1}{2\pi}\int_0^{2\pi} V(z)\varphi_{n-m}\big(|V|^2\,d\mu;z\big)\overline{L_j(z)}\,d\mu(\theta), \quad z = e^{i\theta}, \tag{14}
$$

*and* $L_j(z) = \prod_{i\neq j}\frac{z-\zeta_i}{\zeta_j-\zeta_i}$, $j = 1,\dots,m$, *are the fundamental polynomials of Lagrange interpolation.*

**Proof.** (See [12].) Since $V(z)\varphi_{n-m}(|V|^2\,d\mu;z)$ is a polynomial of degree $n$ with leading coefficient $\kappa_{n-m}(|V|^2\,d\mu)$ we have

$$V(z)\varphi_{n-m}\big(|V|^2\,d\mu;z\big)=\frac{\kappa_{n-m}(|V|^2\,d\mu)}{\kappa_n(d\mu)}\varphi_n(d\mu;z)+\sum_{s=0}^{n-1}a_s\varphi_s(d\mu;z),\tag{15}$$

where

$$a_s=\frac{1}{2\pi}\int_0^{2\pi}V(z)\varphi_{n-m}\big(|V|^2\,d\mu;z\big)\overline{\varphi_s(d\mu;z)}\,d\mu(\theta),\qquad z=e^{i\theta}.$$

In order to compute $a_s$ we consider two cases:

(i) $0\leqslant s\leqslant m-1$; using $\varphi_s(d\mu;z)=\sum_{i=1}^m\varphi_s(d\mu;\zeta_i)L_i(z)$, we obtain directly

$$\begin{aligned}a_s&=\frac{1}{2\pi}\int_0^{2\pi}V(z)\varphi_{n-m}\big(|V|^2\,d\mu;z\big)\overline{\sum_{j=1}^m\varphi_s(d\mu;\zeta_j)L_j(z)}\,d\mu(\theta)\\&=\sum_{j=1}^m\overline{\varphi_s(d\mu;\zeta_j)}\frac{1}{2\pi}\int_0^{2\pi}V(z)\varphi_{n-m}\big(|V|^2\,d\mu;z\big)\overline{L_j(z)}\,d\mu(\theta)\\&=\sum_{j=1}^mA_{nj}\overline{\varphi_s(d\mu;\zeta_j)}.\end{aligned}$$

(ii) $m\leqslant s\leqslant n-1$; in this case $\Pi(z)=\varphi_s(d\mu;z)-\sum_{j=1}^m\varphi_s(d\mu;\zeta_j)L_j(z)$ is a polynomial of degree $s$. Since each zero $\zeta_j$ of $V(z)$ is also a zero of $\Pi(z)$, $\frac{\Pi(z)}{V(z)}$ is a polynomial of degree $s-m\leqslant n-m-1$. Therefore,

$$\begin{aligned}a_s&=\frac{1}{2\pi}\int_0^{2\pi}V(z)\varphi_{n-m}\big(|V|^2\,d\mu;z\big)\left(\overline{\Pi(z)}+\overline{\sum_{j=1}^m\varphi_s(d\mu;\zeta_j)L_j(z)}\right)d\mu(\theta)\\&=\frac{1}{2\pi}\int_0^{2\pi}\varphi_{n-m}\big(|V|^2\,d\mu;z\big)\frac{\overline{\Pi(z)}}{\overline{V(z)}}|V|^2\,d\mu(\theta)\\&\quad+\frac{1}{2\pi}\int_0^{2\pi}V(z)\varphi_{n-m}\big(|V|^2\,d\mu;z\big)\overline{\sum_{j=1}^m\varphi_s(d\mu;\zeta_j)L_j(z)}\,d\mu(\theta)\\&=\sum_{j=1}^m\overline{\varphi_s(d\mu;\zeta_j)}\frac{1}{2\pi}\int_0^{2\pi}V(z)\varphi_{n-m}\big(|V|^2\,d\mu;z\big)\overline{L_j(z)}\,d\mu(\theta)=\sum_{j=1}^mA_{nj}\overline{\varphi_s(d\mu;\zeta_j)}\end{aligned}$$

as in the former case. The proof is now completed by writing the expression of $a_s$ in (15). $\square$

We shall also need the following result (see [14]):

**Lemma 3.** *Let $\mu \in \mathcal{M}$. For $|\zeta| \geqslant 1$, we have*

$$(z-\zeta)\big(z-\zeta^*\big)\Phi_n\big(|z-\zeta|^2\,d\mu; z\big)$$
$$= \Phi_{n+2}(d\mu; z) + c_n(d\mu)\Phi_{n+1}(d\mu; z) + d_n(d\mu)\Phi_n^*(d\mu; z), \tag{16}$$

*where*

$$d_n(d\mu) = -\zeta^*\overline{\alpha_n\big(|z-\zeta|^2\,d\mu\big)}\frac{\kappa_n^2(d\mu)}{\kappa_n^2(|z-\zeta|^2\,d\mu)}, \tag{17}$$

$$c_n(d\mu) = -\frac{\Phi_{n+2}(d\mu;\zeta)}{\Phi_{n+1}(d\mu;\zeta)} - d_n(d\mu)\frac{\Phi_n^*(d\mu;\zeta)}{\Phi_{n+1}(d\mu;\zeta)}, \tag{18}$$

*and* $\zeta^* = \frac{1}{\bar{\zeta}}$.

**Proof.** Set $R_{n+2}(z) = (z-\zeta)(z-\zeta^*)\Phi_n(|z-\zeta|^2\,d\mu; z) - d_n(d\mu)\Phi_n^*(d\mu; z)$. It is easy to see that

$$\frac{1}{2\pi}\int_0^{2\pi} R_{n+2}(z)\,\overline{z^k}\,d\mu(\theta) = 0, \quad z = e^{i\theta},\ k = 0, 1, \dots, n. \tag{19}$$

In fact, it suffices to evaluate:

(i) $$\frac{1}{2\pi}\int_0^{2\pi}(z-\zeta)\big(z-\zeta^*\big)\Phi_n\big(|z-\zeta|^2\,d\mu; z\big)\overline{z^k}\,d\mu(\theta)$$
$$= -\frac{\zeta^*}{2\pi}\int_0^{2\pi}\Phi_n\big(|z-\zeta|^2\,d\mu; z\big)\overline{z^{k-1}}\,|z-\zeta|^2\,d\mu(\theta) = 0, \quad \text{if } k = 1, 2, \dots, n.$$

When $k = 0$, using (1) we obtain:

$$-\frac{\zeta^*}{2\pi}\int_0^{2\pi} z\Phi_n\big(|z-\zeta|^2\,d\mu; z\big)\,|z-\zeta|^2\,d\mu(\theta)$$
$$= -\overline{\alpha_n\big(|z-\zeta|^2\,d\mu\big)}\frac{\zeta^*}{2\pi}\int_0^{2\pi}\Phi_n^*\big(|z-\zeta|^2\,d\mu; z\big)|z-\zeta|^2\,d\mu(\theta) = -\zeta^*\frac{\overline{\alpha_n(|z-\zeta|^2\,d\mu)}}{\kappa_n^2(|z-\zeta|^2\,d\mu)}.$$

(ii) $$\frac{1}{2\pi}\int_0^{2\pi}\Phi_n^*(d\mu; z)\overline{z^k}\,d\mu(\theta) = \frac{1}{2\pi}\int_0^{2\pi} z^{n-k}\,\overline{\Phi_n(d\mu; z)}\,d\mu(\theta)$$
$$= \begin{cases} 0, & \text{if } k = 1, \dots, n; \\ \frac{1}{\kappa_n^2(d\mu)}, & \text{if } k = 0. \end{cases}$$

Now (19) immediately follows. From (19) we deduce that there exists $\tilde{c}_n \in \mathbb{C}$ such that the polynomial $R_{n+2}(z) = z^{n+2} + \cdots$ can be written

$$R_{n+2}(z) = \Phi_{n+2}(d\mu; z) + \tilde{c}_n\Phi_{n+1}(d\mu; z).$$

Setting $z = \zeta$ in this equality we obtain (16). $\square$

## 4. Proof of Theorems 1 and 2

**Proof of Theorem 1.** We use Lemma 2. Multiplying (13) by $\frac{1}{\kappa_{n-m}(|V|^2\,d\mu)}$ and setting $z=0$ we have

$$V(0)\Phi_{n-m}\big(|V|^2\,d\mu;0\big)=\Phi_n(d\mu;0)+\frac{1}{\kappa_{n-m}(|V|^2\,d\mu)}\sum_{j=1}^{m}A_{nj}K_{n-1}(d\mu;0,\zeta_j), \tag{20}$$

where

$$K_{n-1}(d\mu;0,\zeta_j)=\kappa_{n-1}(d\mu)\overline{\zeta_j}^{\,n-1}\varphi_{n-1}(d\mu;\zeta_j). \tag{21}$$

In order to get (9) from (20) it is enough to deduce the asymptotic behaviour of the $A_{nj}$.

Since the coefficients $A_{nj}$ are unique they can be obtained not only by (14) but also as the solution of the following system of linear equations:

$$\sum_{j=1}^{m}A_{nj}K_{n-1}(d\mu;\zeta_k,\zeta_j)=-\frac{\kappa_{n-m}(|V|^2\,d\mu)}{\kappa_n(d\mu)}\varphi_n(d\mu;\zeta_k),\quad k=1,\dots,m.$$

By using Cramer's rule we obtain $A_{nj}$, $j=1,\dots,m$, as the quotient $\Delta_j/\Delta$ where $\Delta=\det(K_{n-1}(d\mu;\zeta_i,\zeta_h))_{i,h=1,\dots,m}$ and $\Delta_j$ denotes the determinant obtained by substituting the $j$th column of $\Delta$ by the column $(-\frac{\kappa_{n-m}(|V|^2\,d\mu)}{\kappa_n(d\mu)}\varphi_n(d\mu;\zeta_i))_{i=1,\dots,m}$. In any case, for every $j$ $A_{nj}$ can be expressed in terms of $\varphi_n(d\mu;\zeta_i)$, $K_{n-1}(d\mu;\zeta_i,\zeta_h)$ with $i\neq h$ and $K_{n-1}(d\mu;\zeta_i,\zeta_i)$. So, in order to study how $A_{nj}$ behaves when $n\to\infty$, we first need to know how $K_{n-1}(d\mu;\zeta_i,\zeta_h)$ behaves for $i\neq h$ and $i=h$.

- Asymptotic behaviour of $K_{n-1}(d\mu;\zeta_i,\zeta_h)$.

If $|\zeta|=1$, we have from (12)

$$K_n(d\mu;z,\zeta)=\zeta\,\frac{\overline{\varphi_{n+1}(d\mu;\zeta)}\varphi_{n+1}(d\mu;z)-\overline{\varphi^*_{n+1}(d\mu;\zeta)}\varphi^*_{n+1}(d\mu;z)}{z-\zeta},$$

where the numerator vanishes at $z=\zeta$. Therefore, letting $z\to\zeta$ we obtain

$$K_n(d\mu;\zeta,\zeta)=\zeta\big(\overline{\varphi_{n+1}(d\mu;\zeta)}\varphi'_{n+1}(d\mu;\zeta)-\overline{\varphi^*_{n+1}(d\mu;\zeta)}\varphi^{*\prime}_{n+1}(d\mu;\zeta)\big).$$

Taking into account (4)–(7) we easily deduce

$$K_n(d\mu;\zeta,\zeta)=\frac{n}{|D(d\mu;\zeta)|^2}+\mathrm{o}(n)\quad(n\to\infty). \tag{22}$$

Let now $\eta\neq\zeta$, $|\eta|=1$. In the same way as before we get

$$K_n(d\mu;\zeta,\eta)=\mathrm{O}(1)\quad(n\to\infty). \tag{23}$$

- Asymptotic behaviour of $A_{nj}$.

For $j=1,\dots,m$ we develop both determinants in $A_{nj}=\Delta_j/\Delta$ and consider the different type of terms appearing in them. Taking into account (22) and (23) is not difficult to get

$$\Delta = \prod_{j=1}^{m} K_{n-1}(d\mu;\zeta_j,\zeta_j) + \mathrm{o}\big(n^m\big) \quad (n\to\infty),$$

$$\Delta_j = -\frac{\kappa_{n-m}(|V|^2\,d\mu)}{\kappa_n(d\mu)}\varphi_n(d\mu;\zeta_j)\prod_{h\neq j} K_{n-1}(d\mu;\zeta_h,\zeta_h) + \mathrm{o}\big(n^{m-1}\big) \quad (n\to\infty).$$

Consequently,

$$A_{nj} = -\frac{\kappa_{n-m}(|V|^2\,d\mu)}{\kappa_n(d\mu)}\varphi_n(d\mu;\zeta_j)\frac{1}{K_{n-1}(d\mu;\zeta_j,\zeta_j)} + \mathrm{o}\left(\frac{1}{n}\right),$$

where $\lim_{n\to\infty}\frac{\kappa_{n-m}(|V|^2\,d\mu)}{\kappa_n(d\mu)} = 1$ from (2) and (8). The behaviour of $\varphi_n(d\mu;\zeta_j)$ and $K_{n-1}(d\mu;\zeta_j,\zeta_j)$ when $n\to\infty$, is given by (4) and (22), respectively. Finally,

$$A_{nj} = -\zeta_j^n\frac{D(d\mu;\zeta_j)}{n} + \mathrm{o}\left(\frac{1}{n}\right) \quad (n\to\infty). \tag{24}$$

Now taking account (20), (21) and (24), we deduce

$$A_{nj}\overline{\zeta_j}^{n-1}\varphi_{n-1}(d\mu;\zeta_j) = -\zeta_j^n\frac{1}{n}\frac{D(d\mu;\zeta_j)}{\overline{D(d\mu;\zeta_j)}} + \mathrm{o}\left(\frac{1}{n}\right),$$

$$\Phi_{n-m}\big(|V|^2\,d\mu;0\big) = \frac{(-1)^{m+1}}{n\prod_{h=1}^m\zeta_h}\sum_{j=1}^m\zeta_j^n\frac{D(d\mu;\zeta_j)}{\overline{D(d\mu;\zeta_j)}} + \mathrm{o}\left(\frac{1}{n}\right),$$

and so (9) is obtained re-scaling and writing this result in terms of Verblunsky coefficients. □

In order to prove Theorem 2, we need to state two previous results:

**Lemma 4.** *Let $\mu\in\mathcal{M}$ be such that $D(d\mu;z)^{-1}$ has an analytic extension to $\{z\colon |z|<\rho\}$, $\rho>1$. Let $\zeta\in\mathbb{C}$ with $|\zeta|=1$. Then*

$$(z-\zeta)^2\Phi_n\big(|z-\zeta|^2\,d\mu;z\big) = \frac{1}{n}\frac{D(d\mu;\zeta)}{\overline{D(d\mu;\zeta)}}\frac{D(d\mu;0)}{D(d\mu;z)}\zeta^{n+2} + \mathrm{o}\left(\frac{1}{n}\right) \quad (n\to\infty), \tag{25}$$

*uniformly on each compact subset of $\{z\colon |z|<1\}$.*

**Proof.** Since $\zeta^* = \zeta$ for $|\zeta|=1$, using (16) we have

$$(z-\zeta)^2\Phi_n\big(|z-\zeta|^2\,d\mu;z\big) = \Phi_{n+2}(d\mu;z) + c_n(d\mu)\Phi_{n+1}(d\mu;z) + d_n(d\mu)\Phi_n^*(d\mu;z),$$

where $d_n(d\mu)$ and $c_n(d\mu)$ are as in (17) and (18). It is not difficult to see that:

- $\Phi_n(d\mu;z) = \mathrm{o}(\frac{1}{n})$ uniformly on each compact subset of $\{z\colon |z|<1\}$. In fact, we can state even more. Using (4) and the maximum principle, we get $\limsup_{n\to\infty}|\Phi_n(d\mu;z)|^{1/n}<1$ uniformly on compact subsets of $\{z\colon |z|<1\}$ and so, $\Phi_n(d\mu;0)$ tends to zero with an exponential rate when $n\to\infty$.
- $\lim_{n\to\infty}\frac{\kappa_n^2(d\mu)}{\kappa_n^2(|z-\zeta|^2\,d\mu)} = 1$, as we just pointed out in the proof of Theorem 1.
- $\lim_{n\to\infty}\Phi_n^*(d\mu;z) = \frac{D(d\mu;0)}{D(d\mu;z)}$ uniformly on each compact subset of $\{z\colon |z|\leqslant 1\}$, that follows directly from (2) and (5).
- $\alpha_n(|z-\zeta|^2\,d\mu) = -\frac{1}{n}\frac{\overline{D(d\mu;\zeta)}}{D(d\mu;\zeta)}\overline{\zeta}^{n+2} + \mathrm{o}(\frac{1}{n})$, from Theorem 1.

Therefore, we get

$$d_n(d\mu) = \frac{1}{n}\frac{D(d\mu;\zeta)}{\overline{D(d\mu;\zeta)}}\zeta^{n+2} + \mathrm{o}\left(\frac{1}{n}\right), \qquad c_n(d\mu) = -\zeta + \mathrm{O}\left(\frac{1}{n}\right),$$

and finally (25) follows. □

**Lemma 5.** *Let $\mu \in \mathcal{M}$ be such that $D(d\mu;z)^{-1}$ has an analytic extension to $\{|z| < \rho\}$, $\rho > 1$. Let $\zeta_1, \zeta_2$ be two different points on the unit circle. Then, as $n \to \infty$,*

$$\begin{aligned}&\big[(z-\zeta_1)(z-\zeta_2)\big]^2 \Phi_n\big(|z-\zeta_1|^2|z-\zeta_2|^2\,d\mu;z\big)\\ &\quad = \frac{1}{n}\frac{D(d\mu;0)}{D(d\mu;z)}\left[z\left(\zeta_1^{n+3}\frac{D(d\mu;\zeta_1)}{\overline{D(d\mu;\zeta_1)}} + \zeta_2^{n+3}\frac{D(d\mu;\zeta_2)}{\overline{D(d\mu;\zeta_2)}}\right)\right.\\ &\qquad \left. - \zeta_1\zeta_2\left(\zeta_1^{n+2}\frac{D(d\mu;\zeta_1)}{\overline{D(d\mu;\zeta_1)}} + \zeta_2^{n+2}\frac{D(d\mu;\zeta_2)}{\overline{D(d\mu;\zeta_2)}}\right)\right] + \mathrm{o}\left(\frac{1}{n}\right),\end{aligned} \tag{26}$$

*uniformly on each compact subset of* $\{z\colon |z| < 1\}$.

**Proof.** Let $d\mu_1 = |z-\zeta_1|^2\,d\mu$ and use again (16) after multiply it by $(z-\zeta_2)^2$, with $\mu = \mu_1$ and $\zeta = \zeta_2$.

By similar arguments to the ones used in the proof of Lemma 4 we get, when $n \to \infty$,

$$d_n(d\mu_1) = \frac{-1}{n\zeta_1}\sum_{j=1}^{2}\frac{D(d\mu;\zeta_j)}{\overline{D(d\mu;\zeta_j)}}\zeta_j^{n+3} + \mathrm{o}\left(\frac{1}{n}\right), \qquad c_n(d\mu_1) = -\zeta_2 + \mathrm{O}\left(\frac{1}{n}\right).$$

The asymptotics for $(z-\zeta_1)\Phi_{n+i}(d\mu_1;z)$, $i = 1, 2$, are known from Lemma 4 and

$$\Phi_n^*(d\mu_1;z) = \frac{-\zeta_1}{(z-\zeta_1)}\frac{D(d\mu;0)}{D(d\mu;z)} + \mathrm{o}(1),$$

holds uniformly on compact subsets of $\{z\colon |z| \leqslant 1\}$. Some simple calculations allow us to state (26). □

In fact, the results of Lemmas 4 and 5 lead us to state Theorem 2, that we now prove.

**Proof of Theorem 2.** Since we will prove Theorem 2 by induction in $m$, we write $V = V_m$ in order to clarify the induction step. For $m = 1$ Lemma 4 states that (10) holds, being

$$P_1(z;n) = \frac{D(d\mu;\zeta_1)}{\overline{D(d\mu;\zeta_1)}}\zeta_1^{n+2}.$$

From Lemma 5, (10) holds true if $m = 2$, being

$$\begin{aligned}P_2(z;n) &= z\left(\zeta_1^{n+3}\frac{D(d\mu;\zeta_1)}{\overline{D(d\mu;\zeta_1)}} + \zeta_2^{n+3}\frac{D(d\mu;\zeta_2)}{\overline{D(d\mu;\zeta_2)}}\right)\\ &\quad - \zeta_1\zeta_2\left(\zeta_1^{n+2}\frac{D(d\mu;\zeta_1)}{\overline{D(d\mu;\zeta_1)}} + \zeta_2^{n+2}\frac{D(d\mu;\zeta_2)}{\overline{D(d\mu;\zeta_2)}}\right),\end{aligned}$$

that satisfies (11). We suppose the theorem true for $m - 1 \in \mathbb{N}$.

Using (16) for $\zeta = \zeta_m$ and measure $|V_m|^2\,d\mu$, we have

$$\begin{aligned}
&V_{m-1}^2(z)(z-\zeta_m)^2\Phi_n\big(|V_m|^2\,d\mu;z\big)\\
&\quad= V_{m-1}^2(z)\Phi_{n+2}\big(|V_{m-1}|^2\,d\mu;z\big)+c_n\big(|V_{m-1}|^2\,d\mu\big)V_{m-1}^2(z)\Phi_{n+1}\big(|V_{m-1}|^2\,d\mu;z\big)\\
&\qquad+d_n\big(|V_{m-1}|^2\,d\mu\big)V_{m-1}^2(z)\Phi_n^*\big(|V_{m-1}|^2\,d\mu;z\big).
\end{aligned}\tag{27}$$

By Theorem 1 we know what happens with $\alpha_n(|V_{m-1}|^2\,d\mu)$ as $n\to\infty$ and so, as in the former particular cases, we obtain

$$c_n\big(|V_{m-1}|^2\,d\mu\big)=-\zeta_m+\mathrm{O}\left(\frac{1}{n}\right),$$

$$d_n\big(|V_{m-1}|^2\,d\mu\big)=\zeta_m\frac{(-1)^{m-1}}{nV_m(z)}\sum_{i=1}^{m}\frac{D(d\mu;\zeta_i)}{\overline{D(d\mu;\zeta_i)}}\zeta_i^{n+m+1}+\mathrm{o}\left(\frac{1}{n}\right).$$

The behaviour of $|V_{m-1}|^2\Phi_{n+2}(|V_m|^2\,d\mu;z)$ and $|V_{m-1}|^2\Phi_{n+1}(|V_{m-1}|^2\,d\mu;z)$ is given by the induction hypothesis and it is easy to verify that

$$\Phi_n^*\big(|V_{m-1}|^2\,d\mu;z\big)=\frac{(-1)^{m-1}}{V_{m-1}(z)}\frac{D(d\mu;0)}{D(d\mu;z)}\prod_{i=1}^{m-1}\zeta_i+\mathrm{o}(1),$$

uniformly on each compact subset of $\{z\colon |z|<1\}$.

Thus, (27) can be written

$$\begin{aligned}
V_m^2(z)\Phi_n\big(|V_m|^2\,d\mu;z\big)=\frac{1}{n}\frac{D(d\mu;0)}{D(d\mu;z)}\Bigg[&P_{m-1}(z;n+2)-\zeta_m P_{m-1}(z;n+1)\\
&+V_{m-1}(z)\sum_{i=1}^{m}\frac{D(d\mu;\zeta_i)}{\overline{D(d\mu;\zeta_i)}}\zeta_i^{n+m+1}\Bigg]+\mathrm{o}\left(\frac{1}{n}\right),
\end{aligned}$$

or

$$V_m^2(z)\Phi_n\big(|V_m|^2\,d\mu;z\big)=\frac{1}{n}\frac{D(d\mu;0)}{D(d\mu;z)}P_m(z;n)+\mathrm{o}\left(\frac{1}{n}\right),$$

where

$$\begin{aligned}
P_m(z;n)&=P_{m-1}(z;n+2)-\zeta_m P_{m-1}(z;n+1)\\
&\quad+V_{m-1}(z)\left(\sum_{i=1}^{m}\frac{D(d\mu;\zeta_i)}{\overline{D(d\mu;\zeta_i)}}\zeta_i^{n+m+1}\right).\qquad\Box
\end{aligned}$$

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