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Symmetrically localized frames and the removal of subsets of positive density

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Abstract

The theorems of Balan, Casazza, Heil, and Landau concerning the removal of sets of positive density from frames with positive excess are extended using a more general, symmetric concept of localization of frames.

Keywords: Frames; Localized frames; Density; Excess; Overcompleteness; Modulation spaces; Gabor systems; Weyl–Heisenberg systems

1. Introduction

The concept of localization of frames was recently introduced independently by Gröchenig [18] and the group consisting of Balan, Casazza, Heil, and Landau (BCHL) [4,5]. Frames having this new localization property are interesting in a number of ways; Gröchenig proved that a frame localized with respect to a Riesz basis is automatically a Banach frame for an often important family of Banach spaces associated to the Riesz basis. This was further generalized in [12–14, 17]. Further background and examples can be found in [1,7,11]. BCHL proved that the excess of an overcomplete localized frame has a certain degree of uniformity, and were able to give conditions under which excess of positive density could be removed from an overcomplete localized frame. Previous work done in this direction can be found in [2,3]. As often happens when a concept is introduced independently by several parties, the definitions found in [4,5,18] are

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different enough so that one definition is not a special case of the other, and are specific to their respective purposes. A more general definition which encompasses both Gröchenig's and BCHL's definitions in the most useful cases is introduced in [15]. This definition allows for a useful and natural equivalence class structure when dealing with l^1-self-localized frames, and extends the results of Gröchenig [15]. In this paper, we focus on extending the results of BCHL involving removing excess of positive density. In Section 2, we define symmetric localization. In Section 3, we provide the necessary definitions from [2]. In Section 4 we extend the Density-Relative Measure Theorem and the theorem concerning the removal of sets of positive density.

2. Symmetric localization

Before introducing the new definition, we fix basic notation. We recommend [6,8–10,16] for additional background.

Let $\mathcal{F} = \{f_x\}_{x\in X}$ be a *frame* for a separable Hilbert space $\mathcal{H}$ with frame bounds A, B. The *analysis operator* will be denoted $C : \mathcal{H} \to l^2(X)$, $C(f) = \{\langle f, f_x\rangle\}_{x\in X}$. The *synthesis operator* denoted $D : l^2(X) \to \mathcal{H}$, $D(\{c_x\}_{x\in X}) = \sum c_x f_x$ is the adjoint of C, $D = C^*$. The *frame operator* denoted $S = DC : \mathcal{H} \to \mathcal{H}$, $Sf = \sum_{x\in X}\langle f, f_x\rangle f_x$ is a positive, invertible operator such that $A \cdot I \leqslant S \leqslant B \cdot I$. The *canonical dual frame* of $\mathcal{F}$ will be denoted $\tilde{\mathcal{F}} = \{\tilde{f}_x\}_{x\in X} := \{S^{-1} f_x\}_{x\in X}$. A *frame sequence* $\mathcal{F} = \{f_x\}_{x\in X}$ is a frame for the closure of its span.

In the following, let G be a group of the form $\prod_{i=1}^{d} a_i\mathbb{Z} \times \prod_{j=1}^{e} \mathbb{Z}_{b_j}$. For every $g = (a_1 n_1, a_2 n_2, \ldots, a_d n_d, m_1, m_2, \ldots, m_e) \in G$, let

$$|g| = \sup\bigl\{|a_1 n_1|, |a_2 n_2|, \ldots, |a_d n_d|, \delta(m_1), \delta(m_2), \ldots, \delta(m_e)\bigr\}$$

where

$$\delta(m_j) = \begin{cases} 0 & \text{if } m_j = 0; \\ 1 & \text{otherwise.} \end{cases}$$

Define a metric on G by $d(g,h) = |g - h|$ for $g, h \in G$. Let $S_n(j)$ denote the ball of radius n centered at j in G and $|S_n(j)| := \#[S_n(j)]$, the cardinality of $S_n(j)$.

Definition 1 *(Symmetric localization).* Let $\mathcal{F} = \{f_x\}_{x\in X}$ and $\mathcal{E} = \{e_y\}_{y\in Y}$ be sequences in a Hilbert space $\mathcal{H}$, X and Y arbitrary index sets.

(1) $(\mathcal{F}, \mathcal{E})$ is *symmetrically l^p-localized* if there exist maps $a_X : X \to G$, $a_Y : Y \to G$ such that $\sup_{j\in G} |a_X^{-1}(j)|, \sup_{j\in G} |a_Y^{-1}(j)| \leqslant K < \infty$, and $r \in l^p(G)$ such that for all $x \in X$, $y \in Y$,

$$\bigl|\langle f_x, e_y\rangle\bigr| \leqslant r_{a_X(x) - a_Y(y)}.$$

(2) $(\mathcal{F}, \mathcal{E})$ has *uniform l^p-column decay* if for every $\epsilon > 0$ there is $N_\epsilon > 0$ such that for all $y \in Y$,

$$\sum_{x\in X\setminus a_X^{-1}(S_{N_\epsilon}(a_Y(y)))} \bigl|\langle f_x, e_y\rangle\bigr|^p < \epsilon.$$

(3) $(\mathcal{F}, \mathcal{E})$ has *uniform l^p-row decay* if for every $\epsilon > 0$ there is $N_\epsilon > 0$ such that for all $x \in X$,

$$\sum_{y\in Y\setminus a_Y^{-1}(S_{N_\epsilon}(a_X(x)))} \bigl|\langle f_x, e_y\rangle\bigr|^p < \epsilon.$$

Remark 2. The terms column and row decay come from considering the cross-Grammian matrix $[\langle f_x, e_y\rangle]_{X,Y}$.

If we let $Y = G$ and $a_Y = \mathrm{id}$, then we have the definition of BCHL provided $\sup_{j\in G} |a_X^{-1}(j)| \leqslant K < \infty$. Bounded point inverses are not only desired in applications but also used in nearly all of the theorems of BCHL so it is not a restrictive condition. This definition does not extend the definition of Gröchenig, however, every frame localized in the sense of Gröchenig is symmetrically localized. This is proved in [15].

Example 3. Let

$$\mathcal{F} = \left\{ f_k := \frac{\sin[\pi(x+k)]}{\pi(x+k)} \right\}_{k\in\frac{1}{2}\mathbb{Z}} \quad \text{and} \quad \mathcal{E} = \left\{ e_n := \frac{\sin[\pi(x+n)]}{\pi(x+n)} \right\}_{n\in\mathbb{Z}}$$

be contained in $L^2(\mathbb{R})$. $\mathcal{F}$ is a frame with frame bounds $A = B = 2$ and $\mathcal{E}$ is an orthonormal basis. Let $a_{\mathcal{E}} : \mathbb{Z} \to \mathbb{Z}$ be the identity function and $a_{\mathcal{F}} : \frac{1}{2}\mathbb{Z} \to \mathbb{Z}$ defined

$$a_{\mathcal{F}}(k) = \begin{cases} k & \text{if } k \in \mathbb{Z}; \\ k - 1/2 & \text{if } k \in \mathbb{Z}^+; \\ k + 1/2 & \text{if } k \in \mathbb{Z}^-. \end{cases}$$

Then $\mathcal{F}$ is l^p localized with respect to $\mathcal{E}$, for any $p > 1$, as

$$\big|\langle f_k, e_n\rangle\big| = \begin{cases} 1 & \text{if } k = n; \\ 0 & \text{if } k \in \mathbb{Z},\ k \neq n; \\ \frac{1}{|k-n|\pi} & \text{if } k \notin \mathbb{Z}. \end{cases}$$

$\frac{1}{|k-n|\pi} \leqslant \frac{1}{(|a(k)-n|)\pi}$ for $|a(k) - n| \neq 0$ so we have $|\langle f_k, e_n\rangle| \leqslant r_{a(k)-n}$, where

$$r_g = \begin{cases} 1 & \text{if } g = 0; \\ \frac{1}{|g|\pi} & \text{if } g \neq 0, \end{cases}$$

$r \in l^p(\mathbb{Z})$, $p > 1$.

3. Definitions of density, measure, and excess

Throughout this section, let $\mathcal{F} = \{f_x\}_{x\in X}$ and $\mathcal{E} = \{e_y\}_{y\in Y}$ be frame sequences for Hilbert space $\mathcal{H}$, X and Y arbitrary index sets. Let $a_X : X \to G$, $a_Y : Y \to G$ be associated maps such that $|a_X^{-1}(j)|, |a_Y^{-1}(j)| \leqslant K < \infty$ for all $j \in G$. We will also use free ultrafilters for more flexibility in convergence; free ultrafilters are discussed in Appendix A.

Definition 4. The *lower and upper densities of $\mathcal{F} = \{f_x\}_{x\in X}$ with respect to a_X* are respectively

$$D^-(a_X) = \liminf_{n\to\infty} \inf_{j\in G} \frac{|a_X^{-1} S_n(j)|}{|S_n(j)|}$$

and

$$D^+(a_X) = \liminf_{n\to\infty} \inf_{j\in G} \frac{|a_X^{-1} S_n(j)|}{|S_n(j)|}.$$

Let $c = \{c_N\}_{N\in\mathbb{N}}$ be any sequence, p a free ultrafilter. The *density with respect to a_X, p, c* is

$$D(p, c) := D(p, c; a_X) = p\text{-}\lim_{N\in\mathbb{N}} \frac{|a_X^{-1} S_N(c_N)|}{|S_N(c_N)|}.$$

If any of these expressions do not exist, then the respective density is ∞.

Definition 5. The *relative measure of $\mathcal{F}$ with respect to $\mathcal{E}$, p, c* is

$$M_{\mathcal{E}}(\mathcal{F}, p, c) = p\text{-}\lim_{N\in\mathbb{N}} \frac{\sum_{x\in a_X^{-1}S_N(c_N)} \langle P_{\mathcal{E}} f_x, \tilde{f}_x\rangle}{|a_X^{-1}S_N(c_N)|}.$$

The *relative measure of $\mathcal{E}$ with respect to $\mathcal{F}$, p, c* is

$$M_{\mathcal{F}}(\mathcal{E}, p, c) = p\text{-}\lim_{N\in\mathbb{N}} \frac{\sum_{y\in a_Y^{-1}S_N(c_N)} \langle P_{\mathcal{F}} e_y, \tilde{e}_y\rangle}{|a_Y^{-1}S_N(c_N)|}.$$

In the case that $\overline{\operatorname{span}}(\mathcal{F}) \subseteq \overline{\operatorname{span}}(\mathcal{E})$, $P_{\mathcal{E}}$ is the identity map so we can define the following:

The *measure of $\mathcal{F}$ with respect to p, c* is

$$M(\mathcal{F}, p, c) = p\text{-}\lim_{N\in\mathbb{N}} \frac{\sum_{x\in a_X^{-1}S_N(c_N)} \langle f_x, \tilde{f}_x\rangle}{|a_X^{-1}S_N(c_N)|}.$$

The *lower and upper measures of $\mathcal{F}$* are respectively

$$M^-(\mathcal{F}) = \liminf_{n\to\infty} \inf_{j\in G} \frac{\sum_{x\in a_X^{-1}S_n(j)} \langle f_x, \tilde{f}_x\rangle}{|a_X^{-1}S_n(j)|}$$

and

$$M^+(\mathcal{F}) = \limsup_{n\to\infty} \sup_{j\in G} \frac{\sum_{x\in a_X^{-1}S_n(j)} \langle f_x, \tilde{f}_x\rangle}{|a_X^{-1}S_n(j)|}.$$

If $\overline{\operatorname{span}}(\mathcal{E}) \subseteq \overline{\operatorname{span}}(\mathcal{F})$, we can define $M^-(\mathcal{E})$, $M^+(\mathcal{E})$, $M(\mathcal{E}, p, c)$ analogously.

Notice, the measure gives a kind of average over X of the diagonals of the Grammian matrix $[\langle P_{\mathcal{E}} f_x, f_{x'}\rangle]_{x,x'\in X}$.

Definition 6. The *excess* of a frame is the greatest integer n such that n elements can be deleted from a frame and still leave a complete set. The excess is infinite if there is no such upper bound.

Example 7. Let

$$\mathcal{F} = \left\{ f_k := \frac{\sin[\pi(x+k)]}{\pi(x+k)} \right\}_{k\in\frac{1}{2}\mathbb{Z}} \quad \text{and} \quad \mathcal{E} = \left\{ e_n := \frac{\sin[\pi(x+n)]}{\pi(x+n)} \right\}_{n\in\mathbb{Z}},$$

and suppose $a_{\frac{1}{2}\mathbb{Z}} : \frac{1}{2}\mathbb{Z} \to \mathbb{Z}$ and $a_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ are defined as in the previous example. Then

$$D^-(a_{\frac{1}{2}\mathbb{Z}}) = \liminf_{n\to\infty} \inf_{j\in G} \frac{|a^{-1}S_n(j)|}{|S_n(j)|} = \frac{2(2n+1)}{2n+1} = 2 = D^+(a_{\frac{1}{2}\mathbb{Z}}),$$

$$M^-(\mathcal{F}) = \liminf_{n\to\infty} \inf_{j\in G} \frac{\sum_{k\in a^{-1}S_n(j)} \langle f_k, S^{-1}f_k\rangle}{|a^{-1}S_n(j)|}$$

$$= \liminf_{n\to\infty} \inf_{j\in G} \frac{\sum_{k\in a^{-1}S_n(j)} \|S^{-1/2}f_k\|^2}{2(2n+1)} = \frac{1}{2}.$$

Likewise, $M^+(\mathcal{F}) = 1/2$. By a similar proof, $D^-(a_{\mathbb{Z}}) = D^+(a_{\mathbb{Z}}) = 1$ and $M^-(\mathcal{E}) = M^+(\mathcal{E}) = 1$.

4. Extending the results

4.1. Density-relative measure theorem

A major result in [4] relates density and relative measure such that, in particular, for $\mathcal{E}$ a Riesz basis, the relative measure and density are reciprocals of each other. This in turn allows one to quantify the redundancy of a frame by the reciprocal of the relative measure. We extend their theorem using the new definition. The proof is close to that of BCHL.

For the following theorem, let

$$d_X := \frac{|a_X^{-1} S_N(c_N)|}{|S_N(c_N)|} \quad \text{and} \quad m_X := \frac{\sum_{x\in a_X^{-1} S_N(c_N)} \langle f_x, \tilde{f}_x\rangle}{|a_X^{-1} S_N(c_N)|}.$$

Theorem 8 *(Density-Relative Measure Theorem). Let $\mathcal{F} = \{f_x\}_{x\in X}$ and $\mathcal{E} = \{e_y\}_{y\in Y}$ be frame sequences for Hilbert space $\mathcal{H}$, X and Y arbitrary index sets. Denote by $\tilde{\mathcal{F}} = \{\tilde{f}_x\}_{x\in X}$ and $\tilde{\mathcal{E}} = \{\tilde{e}_y\}_{y\in Y}$ the duals of $\mathcal{F}$ and $\mathcal{E}$, respectively. Let $a_X : X \to G$, $a_Y : Y \to G$ be maps. If $D^+(a_X)$, $D^+(a_Y) < \infty$, and $(\mathcal{F}, \mathcal{E})$ has l^2-column decay and l^2-row decay, then the following statements hold*:

(a) *For every sequence $c = \{c_N\}_{N\in\mathbb{N}} \subseteq G$,*

$$\lim_{N\to\infty} [d_Y m_Y - d_X m_X] = 0.$$

(b) *For every sequence $c = \{c_N\}_{N\in\mathbb{N}} \subseteq G$ and free ultrafilter p,*

$$D(p, c; a_Y) M_{\mathcal{F}}(\mathcal{E}, p, c) = D(p, c; a_X) M_{\mathcal{E}}(\mathcal{F}, p, c).$$

Proof. (a) Fix some sequence $c = \{c_N\}_{N\in\mathbb{N}} \subseteq G$. We must show that $|d_Y m_Y - d_X m_X| \to 0$. First, we make some preliminary observations and introduce some notation. Let A and B, A' and B' denote frame bounds for $\mathcal{F}$ and $\mathcal{E}$, respectively. Then $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{E}}$ have frame bounds $\frac{1}{B}$ and $\frac{1}{A}$, $\frac{1}{B'}$ and $\frac{1}{A'}$, respectively. Consequently, for $x \in X$ and $y \in Y$,

$$\|f_x\|^2 \leqslant B, \qquad \|\tilde{f}_x\|^2 \leqslant \frac{1}{A}, \qquad \|e_y\|^2 \leqslant B', \qquad \|\tilde{e}_y\|^2 \leqslant \frac{1}{A'}.$$

Fix any $\epsilon > 0$. Since $(\mathcal{F}, a_X, a_Y, \mathcal{E})$ has both l^2-row decay and l^2-column decay, there exists an integer $N_\epsilon > 0$ such that for all $y \in Y$,

$$\sum_{x\in X\setminus a_X^{-1}(S_{N_\epsilon}(a_Y(y)))} \big|\langle f_x, e_y\rangle\big|^2 \leqslant \epsilon$$

and for all $x \in X$,

$$\sum_{y\in Y\setminus a_Y^{-1}(S_{N_\epsilon}(a_X(x)))} \big|\langle f_x, e_y\rangle\big|^2 < \epsilon.$$

Also, since $D^+(a_X)$, $D^+(a_Y) < \infty$, we have

$$K = \max\Big\{\sup_{j\in G} \big|a_X^{-1}(j)\big|, \sup_{j\in G} \big|a_Y^{-1}(j)\big|\Big\} < \infty,$$

and for any set Γ contained in G, we have $|a_X^{-1}(\Gamma)| \leqslant K|\Gamma|$, $|a_Y^{-1}(\Gamma)| < K|\Gamma|$.

Let $P_{\mathcal{F}}$, $P_{\mathcal{E}}$ denote the orthogonal projections onto $\overline{\operatorname{span}}(\mathcal{F})$, $\overline{\operatorname{span}}(\mathcal{E})$, respectively. These projections can be realized in the following way:

$$P_{\mathcal{F}} f = \sum_{x\in X} \langle f, f_x\rangle \tilde{f}_x \quad \text{for } f \in \mathcal{H} \qquad \text{and} \qquad P_{\mathcal{E}} f = \sum_{y\in Y} \langle f, e_y\rangle \tilde{e}_y \quad \text{for } f \in \mathcal{H}.$$

Therefore,

$$\begin{aligned}
&\big|S_N(c_N)\big|(d_Y m_Y - d_X m_X)\\
&\quad = \sum_{y\in a_Y^{-1}(S_N(c_N))} \langle \tilde{e}_y, P_{\mathcal{F}} e_y\rangle - \sum_{x\in a_X^{-1}(S_N(c_N))} \langle P_{\mathcal{E}} f_x, \tilde{f}_x\rangle\\
&\quad = \sum_{y\in a_Y^{-1}(S_N(c_N))} \sum_{x\in X} \langle f_x, e_y\rangle\langle \tilde{e}_y, \tilde{f}_x\rangle - \sum_{x\in a_X^{-1}(S_N(c_N))} \sum_{y\in Y} \langle f_x, e_y\rangle\langle \tilde{e}_y, \tilde{f}_x\rangle.
\end{aligned}$$

We can rearrange the series as we like, since $a_Y^{-1}(S_N(c_N))$ and $a_X^{-1}(S_N(c_N))$ are finite sets and the infinite series over x or y converges absolutely by basic frame properties.

In particular, we rewrite the equation for $N > N_\epsilon$ as

$$\begin{aligned}
\big|S_N(c_N)\big|(d_Y m_Y - d_X m_X) = &\sum_{y\in a_Y^{-1}(S_N(c_N))} \sum_{x\in X\setminus a_X^{-1}(S_N(c_N))} \langle f_x, e_y\rangle\langle \tilde{e}_y, \tilde{f}_x\rangle\\
&- \sum_{x\in a_X^{-1}(S_N(c_N))} \sum_{y\in Y\setminus a_Y^{-1}(S_N(c_N))} \langle f_x, e_y\rangle\langle \tilde{e}_y, \tilde{f}_x\rangle.
\end{aligned}$$

We can go further, and rewrite the above equality as

$$\big|S_N(c_N)\big|(d_Y m_Y - d_X m_X) = T_1 + T_2 - T_3 - T_4,$$

where

$$\begin{aligned}
T_1 &= \sum_{y\in a_Y^{-1}(S_N(c_N))} \sum_{x\in X\setminus a_X^{-1}(S_{N+N_\epsilon}(c_N))} \langle f_x, e_y\rangle\langle \tilde{e}_y, \tilde{f}_x\rangle,\\
T_2 &= \sum_{y\in a_Y^{-1}(S_N(c_N))} \sum_{x\in a_X^{-1}(S_{N+N_\epsilon}(c_N))\setminus a_X^{-1}(S_N(c_N))} \langle f_x, e_y\rangle\langle \tilde{e}_y, \tilde{f}_x\rangle,\\
T_3 &= \sum_{x\in a_X^{-1}(S_{N-N_\epsilon}(c_N))} \sum_{y\in Y\setminus a_Y^{-1}(S_N(c_N))} \langle f_x, e_y\rangle\langle \tilde{e}_y, \tilde{f}_x\rangle,\\
T_4 &= \sum_{x\in a_X^{-1}(S_N(c_N))\setminus a_X^{-1}(S_{N-N_\epsilon}(c_N))} \sum_{y\in Y\setminus a_Y^{-1}(S_N(c_N))} \langle f_x, e_y\rangle\langle \tilde{e}_y, \tilde{f}_x\rangle.
\end{aligned}$$

We will estimate each of these quantities in turn.

Estimate T_1: If $y \in a_Y^{-1}(S_N(c_N))$, then $a_Y(y) \in S_N(c_N)$. So

$$a_X^{-1}\big(S_{N_\epsilon}\big(a_Y(y)\big)\big) \subseteq a_X^{-1}\big(S_{N+N_\epsilon}(c_N)\big).$$

Then by l^2 row decay, we have

$$\sum_{x\in X\setminus a_X^{-1}(S_{N+N_\epsilon}(c_N))} \big|\langle f_x, e_y\rangle\big|^2 \leqslant \sum_{x\in X\setminus a_X^{-1}(S_{N_\epsilon}(a_Y(y)))} \big|\langle f_x, e_y\rangle\big|^2 < \epsilon.$$

Using this and the fact that $\{\tilde{f}_x\}$ is a frame sequence, we estimate that

$$\begin{aligned}|T_1| &\leqslant \sum_{y\in a_Y^{-1}(S_N(c_N))}\Bigg(\sum_{x\in X\setminus a_X^{-1}(S_{N+N_\epsilon}(c_N))}\big|\langle f_x, e_y\rangle\big|^2\Bigg)^{1/2}\Bigg(\sum_{x\in X\setminus a_X^{-1}(S_{N+N_\epsilon}(c_N))}\big|\langle \tilde{e}_y, \tilde{f}_x\rangle\big|^2\Bigg)^{1/2}\\ &\leqslant \sum_{y\in a_Y^{-1}(S_N(c_N))}\epsilon^{1/2}\Big(\frac{1}{A}\|\tilde{e}_y\|^2\Big)^{1/2} \leqslant K\big|S_N(c_N)\big|\Big(\frac{\epsilon}{AA'}\Big)^{1/2}.\end{aligned}$$

Estimate T_2: We have

$$\big|a_X^{-1}\big(S_{N+N_\epsilon}(c_N)\big)\setminus a_X^{-1}\big(S_N(c_N)\big)\big| \leqslant K\big(\big|S_{N+N_\epsilon}(c_N)\big| - \big|S_N(c_N)\big|\big).$$

Since we have frame sequences $\{\tilde{e}_y\}_{y\in Y}$ and $\{e_y\}_{y\in Y}$, we have

$$\begin{aligned}|T_2| &\leqslant \sum_{x\in a_X^{-1}(S_{N+N_\epsilon}(c_N))\setminus a_X^{-1}(S_N(c_N))}\Bigg(\sum_{y\in Y}\big|\langle f_x, e_y\rangle\big|^2\Bigg)^{1/2}\Bigg(\sum_{y\in Y}\big|\langle \tilde{e}_y, \tilde{f}_x\rangle\big|^2\Bigg)^{1/2}\\ &\leqslant \sum_{x\in a_X^{-1}(S_{N+N_\epsilon}(c_N))\setminus a_X^{-1}(S_N(c_N))}\big(B'\|f_x\|^2\big)^{1/2}\Big(\frac{1}{A'}\|\tilde{f}_x\|^2\Big)^{1/2}\\ &\leqslant K\big(\big|S_{N+N_\epsilon}(c_N)\big| - \big|S_N(c_N)\big|\big)\Big(\frac{B'B}{AA'}\Big)^{1/2}.\end{aligned}$$

Estimate T_3: This estimate is similar to the one for T_1:

$$|T_3| \leqslant K\big|S_{N-N_\epsilon}(c_N)\big|\Big(\frac{\epsilon}{AA'}\Big)^{1/2}.$$

Estimate T_4: This estimate is similar to the one for T_2:

$$|T_4| \leqslant K\big(\big|S_N(c_N)\big| - \big|S_{N-N_\epsilon}(c_N)\big|\big)\Big(\frac{B'B}{AA'}\Big)^{1/2}.$$

Final estimate: Applying the above estimates, we find that if $N > N_\epsilon$, then

$$\begin{aligned}|d_Y m_Y - d_X m_X| &\leqslant \frac{|T_1|+|T_2|+|T_3|+|T_4|}{|S_N(c_N)|}\\ &\leqslant \frac{K|S_N(c_N)|}{|S_N(c_N)|}\Big(\frac{\epsilon}{AA'}\Big)^{1/2} + \frac{K(|S_{N+N_\epsilon}(c_N)| - |S_N(c_N)|)}{|S_N(c_N)|}\Big(\frac{B'B}{AA'}\Big)^{1/2}\\ &\quad + \frac{K|S_{N-N_\epsilon}(c_N)|}{|S_N(c_N)|}\Big(\frac{\epsilon}{AA'}\Big)^{1/2}\\ &\quad + \frac{K(|S_N(c_N)| - |S_{N-N_\epsilon}(c_N)|)}{|S_N(c_N)|}\Big(\frac{B'B}{AA'}\Big)^{1/2}.\end{aligned}$$

$|S_N(j)|$ is independent of $j \in G$, so we have $\lim_{N\to\infty}\frac{|S_N(c_N)|}{N^d} = C$ for some constant $C > 0$. Thus

$$\lim_{N\to\infty}|d_Y m_Y - d_X m_X| = K\Big(\frac{\epsilon}{AA'}\Big)^{1/2} + 0 + K\Big(\frac{\epsilon}{AA'}\Big)^{1/2} + 0.$$

Since ϵ was arbitrary, this implies $\lim_{N\to\infty}(d_Y m_Y - d_X m_X) = 0$ as required.

(b) Since ultrafilter limits exist for any bounded sequence and furthermore are linear and respect products, we have

$$\begin{aligned}0 &= \underset{N\in\mathbb{N}}{p\text{-lim}}(d_Y m_Y - d_X m_X)\\ &= \Big(\underset{N\in\mathbb{N}}{p\text{-lim}}(d_Y)\Big)\Big(\underset{N\in\mathbb{N}}{p\text{-lim}}(m_Y)\Big) - \Big(\underset{N\in\mathbb{N}}{p\text{-lim}}(d_X)\Big)\Big(\underset{N\in\mathbb{N}}{p\text{-lim}}(m_X)\Big)\\ &= D(p,c;a_Y)M_F(\mathcal{E},p,c) - D(p,c;a_X)M_E(\mathcal{F},p,c). \qquad \square\end{aligned}$$

Consequently, we have the following result:

Theorem 9. *Let $\mathcal{F} = \{f_x\}_{x\in X}$ and $\mathcal{E} = \{e_y\}_{y\in Y}$ be frames for $\mathcal{H}$, and let $a_X : X \to G$ and $a_Y : Y \to G$ be associated maps such that $D^+(a_X) < \infty$ and $D^+(a_Y) < \infty$. If $(\mathcal{F},\mathcal{E})$ has both l^2-column and l^2-row decay, then the following statements hold*:

(a) *For each free ultrafilter p and sequence c, we have*

$$\begin{aligned}&D^-(a_Y)M^-(\mathcal{E}) \leqslant D^-(a_X)M^+(\mathcal{F}) \leqslant D^+(a_Y)M^+(\mathcal{E}),\\ &D^-(a_Y)M^-(\mathcal{E}) \leqslant D^+(a_X)M^-(\mathcal{F}) \leqslant D^+(a_Y)M^+(\mathcal{E}),\\ &D^-(a_X)M^-(\mathcal{F}) \leqslant D^-(a_Y)M^+(\mathcal{E}) \leqslant D^+(a_X)M^+(\mathcal{F}),\\ &D^-(a_X)M^-(\mathcal{F}) \leqslant D^+(a_Y)M^-(\mathcal{E}) \leqslant D^+(a_X)M^+(\mathcal{F}).\end{aligned}$$

(b) *If $M^+(\mathcal{E}) < \frac{D^+(a_X)}{D^+(a_Y)}$, then there exists an infinite set $I \subset X$ such that $\{f_x\}_{x\in X\setminus I}$ is still a frame for $\mathcal{H}$.*

Proof. (a) Since the closed span of $\mathcal{F}$ and $\mathcal{E}$ is all of $\mathcal{H}$, we have

$$D(p,c;a_Y)M(\mathcal{E},p,c) = D(p,c;a_X)M(\mathcal{F},p,c)$$

for all p and c, by Theorem 8. By Lemma 2.5 found in [4], we have that there exist a free ultrafilter p and a sequence c which satisfy $D^-(a_X) = D(p,c;a_X)$. Hence

$$D^-(a_Y)M^-(\mathcal{E}) \leqslant D(p,c;a_Y)M(\mathcal{E},p,c) = D(p,c;a_X)M(\mathcal{F},p,c) \leqslant D^-(a_X)M^+(\mathcal{F}).$$

Similarly, we have that there exist a free ultrafilter p' and a sequence c' which satisfy $M^+(\mathcal{F}) = M(\mathcal{E},p,c)$. Hence

$$\begin{aligned}D^-(a_X)M^+(\mathcal{F}) &\leqslant D(p',c';a_X)M(\mathcal{F},p',c')\\ &= D(p',c';a_Y)M(\mathcal{E},p',c') \leqslant D^+(a_Y)M^+(\mathcal{E}).\end{aligned}$$

Hence

$$D^-(a_Y)M^-(\mathcal{E}) \leqslant D^-(a_X)M^+(\mathcal{F}) \leqslant D^+(a_Y)M^+(\mathcal{E}).$$

The other inequalities follow similarly.

(b) Suppose $M^+(\mathcal{E}) < \frac{D^+(a_X)}{D^+(a_Y)}$. By (a), $D^+(a_X)M^-(\mathcal{F}) \leqslant D^+(a_Y)M^+(\mathcal{E})$. So we have

$$M^-(\mathcal{F}) \leqslant \frac{D^+(a_Y)M^+(\mathcal{E})}{D^+(a_X)} < \frac{D^+(a_Y)\frac{D^+(a_X)}{D^+(a_Y)}}{D^+(a_X)} = 1.$$

Hence, by Proposition 1 from [4], we have that there exists an infinite set $I \subset X$ such that $\{f_x\}_{x\in X\setminus I}$ is still a frame for $\mathcal{H}$. $\square$

4.2. Removing sets of positive density

By posing a stronger form of localization, BCHL in [4] gave a stronger result concerning the removal of sets of positive density.

Theorem 10. (BCHL '05 [4]) *Let $\mathcal{F} = \{f_x\}_{x\in X}$ be a frame sequence with frame bounds A, B and associated map $a : X \to G$ such that*

(a) $0 < D^-(a) \leqslant D^+(a) < \infty$,
(b) $M^+(\mathcal{F}) < 1$, *and*
(c) $\mathcal{F}$ *is* l^1 *localized with respect to its dual.*

Then if we fix α such that $M^+(\mathcal{F}) < \alpha < 1$, for each $0 < \epsilon < 1 - \alpha$ there exists a subset $J \subset I_\alpha = \{x \in X\colon \langle f_x, \tilde{f}_x\rangle \leqslant \alpha\}$ such that $D^-(J, a) = D^+(J, a) > 0$ and $\mathcal{F}\backslash\{f_{x_j}\}_{j\in J}$ is a frame for its closed linear span with frame bounds $A(1 - \alpha - \epsilon), B$.

Using this theorem, we are able to prove a more general statement about the removal of subsets with positive density.

Theorem 11. *Let $\mathcal{F} = \{f_x\}_{x\in X}$ and $\mathcal{E} = \{e_y\}_{y\in Y}$ be frames for $\mathcal{H}$, with $a_X : X \to G$, $a_Y : Y \to G$ the associated maps. Assume the following*:

(a) $0 < D^-(a_X) \leqslant D^+(a_X) < \infty$,
(b) $0 < D^-(a_Y) \leqslant D^+(a_Y) < \infty$,
(c) $M^+(\mathcal{E}) \leqslant \frac{D^-(a_X)}{D^+(a_Y)}$,
(d) $\mathcal{F}$ *is* l^1 *localized with respect to its dual,*
(e) $(\mathcal{F}, \mathcal{E})$ *has both* l^2 *column and row decay.*

Then $M^+(\mathcal{F}) < 1$. Furthermore, there exists a subset $J \subset X$ with positive density such that $\{f_x\}_{x\in X\backslash J}$ forms a frame for its closed linear span.

Proof. By Theorem 9, we have

$$M^+(\mathcal{F}) \leqslant \frac{M^+(\mathcal{E})D^+(a_Y)}{D^-(a_X)} < 1.$$

The result follows from applying Theorem 10. □

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Appendix A. Free ultrafilters

In order to understand density and measure as defined in the next section, we will have to introduce the notion of free ultrafilters to define convergence for arbitrary sequences. Although

we can define a free ultrafilter for any nonempty set, we only define it here for $\mathbb{N}$. We can understand a free ultrafilter as an element of $\beta\mathbb{N}/\mathbb{N}$ or equivalently, as the following definition.

Definition A.1. A collection p of subsets of $\mathbb{N}$ is a *filter* if

(a) $\emptyset \notin p$,
(b) if $A, B \in p$ then $A \cap B \in p$,
(c) if $A \in p$ and $A \subseteq B \subseteq \mathbb{N}$ then $B \in p$.

A filter p is an *ultrafilter* if it is maximal in the sense that

(d) if p' is a filter on $\mathbb{N}$ such that $p \subseteq p'$, then $p = p'$

or equivalently,

(d′) for any $A \subseteq \mathbb{N}$ either $A \in p$ or its complement, $\mathbb{N}\backslash A \in p$.

An ultrafilter is a *free ultrafilter* if

(e) p contains no finite sets.

Definition A.2. Let p be an ultrafilter. Then we say that a sequence $\{c_N\}_{N\in\mathbb{N}}$ in $\mathbb{C}$ *converges to* $c \in \mathbb{C}$ *with respect to* p if for every $\epsilon > 0$ there exists a set $A \in p$ such that $|c_N - c| < \epsilon$ for all $N \in A$. In this case, we write

$$p\text{-}\lim_{N\in\mathbb{N}} c_N = c.$$

We have the following results concerning convergence with respect to free ultrafilters:

Proposition A.3. *Let p be a free ultrafilter, $\{c_N\}_{N\in\mathbb{N}}$ a sequence.*

(a) *If $\lim_{N\to\infty} c_N = c$, then $p\text{-}\lim_{N\in\mathbb{N}} c_N = c$.*
(b) *Every bounded sequence converges with respect to p to an accumulation point of that sequence.*
(c) *If c is an accumulation point of a bounded sequence $\{c_N\}_{N\in\mathbb{N}}$ then there exists a free ultrafilter p such that $p\text{-}\lim_{N\in\mathbb{N}} c_N = c$. In particular, there exists an ultrafilter p such that $p\text{-}\lim_{N\in\mathbb{N}} c_N = \liminf c_N$ and an ultrafilter q such that $q\text{-}\lim_{N\in\mathbb{N}} c_N = \limsup c_N$.*
(d) *p-limits are unique.*
(e) *p-limits are linear.*
(f) *p-limits respect products.*

Example A.4. We show how p-limits extend the concept of ordinary convergence. Consider the sequence $\{c_N\}_{N\in\mathbb{N}}$ where

$$c_N = \begin{cases} 1 & \text{if } N \text{ is even;} \\ 0 & \text{if } N \text{ is odd.} \end{cases}$$

This sequence does not converge in the ordinary sense. However, let p be a free ultrafilter. By (d′), we have that either the set of all positive even numbers is in p, or its complement, the set of all positive odd numbers is in p. If the first case is true, then $p\text{-}\lim_{N\in\mathbb{N}} c_N = 1$. If the second case is true, then $p\text{-}\lim_{N\in\mathbb{N}} c_N = 0$.

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