Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 332 (2007) 1323–1334
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

# Region of variability of two subclasses of univalent functions

S. Ponnusamy *, A. Vasudevarao

*Department of Mathematics, Indian Institute of Technology Madras, Chennai-600 036, India*

Received 20 July 2006
Available online 8 December 2006
Submitted by B.C. Berndt

**Abstract**

Let $\mathcal{F}_1$ ($\mathcal{F}_2$ respectively) denote the class of analytic functions $f$ in the unit disk $|z| < 1$ with $f(0) = 0 = f'(0) - 1$ satisfying the condition $\operatorname{Re} P_f(z) < 3/2$ ($\operatorname{Re} P_f(z) > -1/2$ respectively) in $|z| < 1$, where $P_f(z) = 1 + zf''(z)/f'(z)$. For any fixed $z_0$ in the unit disk and $\lambda \in [0, 1)$, we shall determine the region of variability for $\log f'(z_0)$ when $f$ ranges over the class $\{f \in \mathcal{F}_1\colon\ f''(0) = -\lambda\}$ and $\{f \in \mathcal{F}_2\colon\ f''(0) = 3\lambda\}$, respectively.


*Keywords:* Analytic; Univalent; Close-to-convex; Starlike functions; Variability region

## 1. Introduction and preliminaries

Let $\mathbb{D} := \{z\colon\ |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $\mathcal{H}$ denote the space of all analytic functions on $\mathbb{D}$. Here we think of $\mathcal{H}$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. Further, let $\mathcal{A} := \{f \in \mathcal{H}\colon\ f(0) = f'(0) - 1 = 0\}$ and $\mathcal{S}$ denote the class of *univalent* functions in $\mathcal{A}$.

A function $f \in \mathcal{A}$ is called *starlike* if $f(\mathbb{D})$ is a starlike domain with respect to the origin, and the class of univalent starlike functions is denoted by $\mathcal{S}^*$. It is called *convex* if $f(\mathbb{D})$ is a convex domain. Finally, it is called *close-to-convex* if there exists a convex (univalent) function $g$ and a number $\phi \in \mathbb{R}$ such that $\operatorname{Re}(e^{i\phi} f'(z)/g'(z)) > 0$ for $z \in \mathbb{D}$. Each univalent starlike function $f$ is

* Corresponding author.
*E-mail addresses:* samy@iitm.ac.in (S. Ponnusamy), alluvasu@iitm.ac.in (A. Vasudevarao).

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2006.11.019

characterized by the analytic condition $\mathrm{Re}(zf'(z)/f(z)) > 0$ in $\mathbb{D}$. Also, it is known that $zf'$ is starlike if and only if $f$ is convex, and that every close-to-convex function is univalent in $\mathbb{D}$. For a general reference about these definitions we refer [3,4].

Let $\mathcal{F}_1$ ($\mathcal{F}_2$ respectively) denote the subclass of locally univalent normalized functions $f \in \mathcal{A}$ such that

$$\mathrm{Re}\, P_f(z) < \frac{3}{2} \quad \left(\mathrm{Re}\, P_f(z) > -\frac{1}{2} \text{ respectively}\right), \quad z \in \mathbb{D},$$

where

$$P_f(z) = 1 + \frac{zf''(z)}{f'(z)}, \quad z \in \mathbb{D}. \tag{1.1}$$

It is known that (see [7, Eq. (16)] and [8]) $\mathcal{F}_1 \subset \mathcal{S}^*$ and $\mathcal{F}_2 \subset \mathcal{K}$. Here $\mathcal{K}$ denotes the class of all close-to-convex functions. For $f \in \mathcal{F}_j$ $(j = 1, 2)$, we denote by $\log f'$ the single-valued branch of the logarithm of $f'$ with $\log f'(0) = 0$. Using the well-known Herglotz representation for analytic function with positive real part in $\mathbb{D}$, we can write that if $f \in \mathcal{F}_1$, then there exists a unique positive unit measure $\mu$ on $(-\pi, \pi]$ such that

$$3 - 2\left(1 + \frac{zf''(z)}{f'(z)}\right) = \int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, d\mu(t).$$

This easily gives

$$\log f'(z) = \int_{-\pi}^{\pi} \log\left(1 - ze^{-it}\right) d\mu(t).$$

It follows that for each fixed $z_0 \in \mathbb{D}$ the region of variability

$$V_1(z_0) = \left\{\log f'(z_0)\colon\ f \in \mathcal{F}_1\right\}$$

coincides with the set $\{\log(1 - z)\colon\ |z| \leqslant |z_0|\}$. Similarly if $f \in \mathcal{F}_2$ then applying the Herglotz formula we may write

$$1 + \frac{zf''(z)}{f'(z)} = -\frac{1}{2} + \frac{3}{2} \int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, d\mu(t)$$

from which we can easily deduce that

$$\log f'(z) = 3 \int_{-\pi}^{\pi} \log\left(\frac{1}{1 - ze^{-it}}\right) d\mu(t)$$

and so for each fixed $z_0 \in \mathbb{D}$ the region of variability $V_2(z_0) = \{\log f'(z_0)\colon\ f \in \mathcal{F}_2\}$ coincides with the set $\{-3\log(1 - z)\colon\ |z| \leqslant |z_0|\}$.

For our discussion, we need alternate representations for functions in $\mathcal{F}_j$ $(j = 1, 2)$. Let $\mathcal{B}_0$ be the class of analytic functions $\omega$ in $\mathbb{D}$ such that $|\omega(z)| \leqslant 1$ in $\mathbb{D}$ and $\omega(0) = 0$. It is a simple exercise to see that each $f$ in $\mathcal{F}_1$ ($\mathcal{F}_2$ respectively) has the representation $P_f(0) = 1$ and

$$\omega_f(z) = \frac{P_f(z) - 1}{P_f(z) - 2} \quad \left(\frac{P_f(z) - 1}{P_f(z) + 2} \text{ respectively}\right), \quad z \in \mathbb{D}, \tag{1.2}$$

for some $\omega_f(z) \in \mathcal{B}_0$, and conversely. A simple application of the Schwarz lemma shows that if $f \in \mathcal{F}_1$, then $|f''(0)| = |-w'_f(0)| \leqslant 1$, whereas if $f \in \mathcal{F}_2$, then one has $|f''(0)| = |3w'_f(0)| \leqslant 3$.

One might question the significance of the classes $\mathcal{F}_1$ and $\mathcal{F}_2$ but on the positive side, we give a precise description of the region of variability of $\log f'(z_0)$ which always is a nice feature. To make this point precise, for $\lambda \in \overline{\mathbb{D}} = \{z \in \mathbb{C}: |z| \leqslant 1\}$ and for $z_0 \in \mathbb{D}$ fixed, we define

$$\begin{aligned} \mathcal{C}_1(\lambda) &= \bigl\{f \in \mathcal{F}_1: f''(0) = -\lambda\bigr\}, \\ \mathcal{C}_2(\lambda) &= \bigl\{f \in \mathcal{F}_2: f''(0) = 3\lambda\bigr\}, \\ V_j(z_0, \lambda) &= \bigl\{\log f'(z_0): f \in \mathcal{C}_j(\lambda)\bigr\}, \quad \text{for } j = 1, 2. \end{aligned}$$

Recently, the region of variability for functions of bounded derivative and of positive real part has been discussed in [9]. Also, the region of variability of $\log f'(z_0)$ when $f$ ranges over the class of convex functions $f$ with $f''(0) = 2\lambda$ has been investigated in [10]. See also [3, Exercises 10, 11 and 13 in Chapter 2].

In the present paper we wish to determine explicitly the region of variability $V_j(z_0, \lambda)$ of $\log f'(z_0)$ when $f$ ranges over the class $\mathcal{C}_j(\lambda)$, $j = 1, 2$.

We need some more preparation before we proceed to achieve our goal. For a positive integer $p$, let

$$(\mathcal{S}^*)^p = \bigl\{f = f_0^p: f_0 \in \mathcal{S}^*\bigr\}$$

and recall the following special result.

**Lemma 1.3.** *Let $f$ be an analytic function in $\mathbb{D}$ with $f(z) = z^p + \cdots$. If*

$$\operatorname{Re}\left(z\frac{f''(z)}{f'(z)}\right) > -1, \quad z \in \mathbb{D},$$

*then $f \in (\mathcal{S}^*)^p$.*

Although we could not find any historical reference for a proof of Lemma 1.3, it might be well known (see [4,5]), and we refer to [9] for an analytic proof of the lemma.

## 2. Basic properties of $V_1$, $V_2$, and main results

We now begin our discussion with a number of basic properties of the set $V_j(z_0, \lambda)$, $j = 1, 2$.

(1) For each $j = 1, 2$, $V_j(z_0, \lambda)$ is compact. For each $j = 1, 2$, since $\mathcal{C}_j(\lambda)$ is compact subset of $\mathcal{A}$, it follows that $V_j(z_0, \lambda)$ is compact.

(2) For each $j = 1, 2$, $V_j(z_0, \lambda)$ is convex. Indeed, if $f_0, f_1 \in \mathcal{C}_j(\lambda)$ $(j = 1, 2)$ and $0 \leqslant t \leqslant 1$, then the function

$$f_t(z) = \int_0^z \exp\bigl\{(1-t)\log f_0'(\zeta) + t\log f_1'(\zeta)\bigr\}\,d\zeta = \int_0^z \bigl(f_0'(\zeta)\bigr)^{1-t}\bigl(f_1'(\zeta)\bigr)^t\,d\zeta$$

also belongs to $\mathcal{C}_j(\lambda)$. Since $\log f_t'(z_0) = (1-t)\log f_0'(z_0) + t\log f_1'(z_0)$, the convexity of $V_j(z_0, \lambda)$ is evident.

(3) If $|\lambda| = 1$ or $z_0 = 0$, then

$$V_1(z_0, \lambda) = \bigl\{\log(1 - \lambda z_0)\bigr\} \quad \text{and} \quad V_2(z_0, \lambda) = \bigl\{-3\log(1 - \lambda z_0)\bigr\}.$$

For $|\lambda|<1$ and $z_0\neq 0$, the set $V_1(z_0,\lambda)$ has $\log(1-\lambda z_0)$ as an interior point, whereas the set $V_2(z_0,\lambda)$ has $-3\log(1-\lambda z_0)$ as an interior point.

Indeed, if $f\in\mathcal{F}_1$ and if $|\lambda|=|\omega_f'(0)|=1$, then it follows from the Schwarz lemma that $\omega_f(z)=\lambda z$, which implies that

$$P_f(z)=\frac{1-2\lambda z}{1-\lambda z}\quad\text{and}\quad \log f'(z)=\log(1-\lambda z)$$

showing that $V_1(z_0,\lambda)=\{\log(1-\lambda z_0)\}$. This also trivially holds for $z_0=0$. Now, for $\lambda\in\mathbb{D}$ and $a\in\overline{\mathbb{D}}$, we introduce

$$\delta(z,\lambda)=\frac{z+\lambda}{1+\bar{\lambda}z},\quad z\in\mathbb{D},$$

$$F_{a,\lambda}(z)=\int_0^z\exp\left\{\int_0^{\zeta_2}\frac{\delta(a\zeta_1,\lambda)}{\zeta_1\delta(a\zeta_1,\lambda)-1}\,d\zeta_1\right\}d\zeta_2,\quad z\in\mathbb{D},\tag{2.1}$$

$$H_{a,\lambda}(z)=\int_0^z\exp\left\{\int_0^{\zeta_2}\frac{3\delta(a\zeta_1,\lambda)}{1-\zeta_1\delta(a\zeta_1,\lambda)}\,d\zeta_1\right\}d\zeta_2,\quad z\in\mathbb{D}.\tag{2.2}$$

Then it is a simple exercise to see that

$$1+\frac{zF''_{a,\lambda}(z)}{F'_{a,\lambda}(z)}=1-\frac{z\delta(az,\lambda)}{1-z\delta(az,\lambda)}=\frac{1-2z\delta(az,\lambda)}{1-z\delta(az,\lambda)}$$

from which we can easily conclude that $F_{a,\lambda}\in\mathcal{C}_1(\lambda)$ and

$$\omega_{F_{a,\lambda}}(z)=z\delta(az,\lambda).\tag{2.3}$$

For a fixed $\lambda\in\mathbb{D}$ and $z_0\in\mathbb{D}\setminus\{0\}$ the function

$$\mathbb{D}\ni a\mapsto\log F'_{a,\lambda}(z_0)=\int_0^{z_0}\frac{\delta(a\zeta,\lambda)}{\zeta\delta(a\zeta,\lambda)-1}\,d\zeta$$

is a non-constant analytic function of $a\in\mathbb{D}$, and hence is an open mapping. To show that $\mathbb{D}\ni a\mapsto\log F'_{a,\lambda}(z_0)$ is non-constant, we let

$$h(z)=\frac{-2}{(1-\lambda^2)}\frac{\partial}{\partial a}\left\{\log F'_{a,\lambda}(z)\right\}\Big|_{a=0}.$$

We see that

$$h(z)=2\int_0^z\frac{\zeta}{(1-\lambda\zeta)^2}\,d\zeta=z^2+\cdots$$

so that

$$1+\frac{zh''(z)}{h'(z)}=\frac{2}{1-\lambda z}$$

and therefore,

$$\operatorname{Re}\left\{1+\frac{zh''(z)}{h'(z)}\right\}>\frac{2}{1+|\lambda|}>0,\quad z\in\mathbb{D}.$$

By Lemma 1.3 there exists $h_0 \in \mathcal{S}^*$ with $h = h_0^2$. The univalence of $h_0$ and $h_0(0) = 0$ implies that $h(z_0) \neq 0$ for all $z_0 \in \mathbb{D} \setminus \{0\}$. Consequently the mapping $\mathbb{D} \ni a \mapsto \log F'_{a,\lambda}(z_0)$ is a non-constant analytic function of $a$ and hence it is an open mapping. It follows now that $\log F'_{0,\lambda}(z_0) = \log(1 - \lambda z_0)$ is an interior point of $\{\log F'_{a,\lambda}(z_0)\colon a \in \mathbb{D}\} \subset V_1(z_0, \lambda)$.

Similarly, we find that

$$1 + \frac{zH''_{a,\lambda}(z)}{H'_{a,\lambda}(z)} = 1 + \frac{3z\delta(az,\lambda)}{1 - z\delta(az,\lambda)} = \frac{1 + 2z\delta(az,\lambda)}{1 - z\delta(az,\lambda)}$$

which implies that $H_{a,\lambda} \in \mathcal{C}_2(\lambda)$ and so, we obtain

$$\omega_{H_{a,\lambda}}(z) = z\delta(az,\lambda).$$

Now, by defining

$$g(z) = \frac{2}{3(1-\lambda^2)} \frac{\partial}{\partial a}\left\{\log H'_{a,\lambda}(z)\right\}\bigg|_{a=0} = 2\int_0^z \frac{\zeta}{(1-\lambda\zeta)^2}\, d\zeta$$

we easily see that for a fixed $\lambda \in \mathbb{D}$ and $z_0 \in \mathbb{D} \setminus \{0\}$ the function

$$\mathbb{D} \ni a \mapsto \log H'_{a,\lambda}(z_0) = \int_0^{z_0} \frac{3\delta(a\zeta,\lambda)}{1 - \zeta\delta(a\zeta,\lambda)}\, d\zeta$$

is a non-constant analytic function of $a \in \mathbb{D}$, and hence is an open mapping. Consequently, $\log H'_{0,\lambda}(z_0) = -3\log(1 - \lambda z_0)$ is an interior point of $\{\log H'_{a,\lambda}(z_0)\colon a \in \mathbb{D}\} \subset V_2(z_0, \lambda)$.

(4) For each $j = 1, 2$, we easily obtain that $V_j(e^{i\theta}z_0, \lambda) = V_j(z_0, e^{i\theta}\lambda)$ for $\theta \in \mathbb{R}$. This is a consequence of the fact that $e^{-i\theta} f(e^{i\theta}z) \in \mathcal{C}_j(e^{i\theta}\lambda)$ if and only if $f \in \mathcal{C}_j(\lambda)$.

In view of the last property it is sufficient to determine $V_j(z_0, \lambda)$ for $0 \leqslant \lambda < 1$ and $z_0 \in \mathbb{D} \setminus \{0\}$. Hence from now onwards, it suffices to consider the classes

$$\mathcal{C}_1(\lambda) = \left\{f \in \mathcal{F}_1\colon\ f''(0) = -\lambda\right\} \tag{2.4}$$

and

$$\mathcal{C}_2(\lambda) = \left\{f \in \mathcal{F}_2\colon\ f''(0) = 3\lambda\right\} \tag{2.5}$$

only for $0 \leqslant \lambda < 1$. In each case, since $V_j(z_0, \lambda)$ is a compact convex subset of $\mathbb{C}$ and has non-empty interior, the boundary $\partial V_j(z_0, \lambda)$ is a Jordan curve and $V_j(z_0, \lambda)$ is the union of $\partial V_j(z_0, \lambda)$ and its inner domain.

Our main results can now be stated.

**Theorem 2.6.** *For $0 \leqslant \lambda < 1$ and $z_0 \in \mathbb{D} \setminus \{0\}$, the boundary $\partial V_1(z_0, \lambda)$ is the Jordan curve given by*

$$(-\pi, \pi] \ni \theta \mapsto \log F'_{e^{i\theta},\lambda}(z_0) = \int_0^{z_0} \frac{\delta(e^{i\theta}z,\lambda)}{z\delta(e^{i\theta}z,\lambda) - 1}\, dz, \quad z \in \mathbb{D}. \tag{2.7}$$

*If $\log f'(z_0) = \log F'_{e^{i\theta},\lambda}(z_0)$ for some $f \in \mathcal{C}_1(\lambda)$ and $\theta \in (-\pi, \pi]$, then $f(z) = F_{e^{i\theta},\lambda}(z)$. Here $F_{e^{i\theta},\lambda}(z)$ is given by* (2.1).

**Theorem 2.8.** *For $0 \leqslant \lambda < 1$ and $z_0 \in \mathbb{D} \setminus \{0\}$, the boundary $\partial V_2(z_0, \lambda)$ is the Jordan curve given by*

$$(-\pi, \pi] \ni \theta \mapsto \log H'_{e^{i\theta},\lambda}(z_0) = \int_0^{z_0} \frac{3\delta(e^{i\theta}z, \lambda)}{1 - z\delta(e^{i\theta}z, \lambda)}\, dz, \quad z \in \mathbb{D}. \tag{2.9}$$

*If $\log f'(z_0) = \log H'_{e^{i\theta},\lambda}(z_0)$ for some $f \in \mathcal{C}_2(\lambda)$ and $\theta \in (-\pi, \pi]$, then $f(z) = H_{e^{i\theta},\lambda}(z)$. Here $H_{e^{i\theta},\lambda}(z)$ is given by* (2.2).

The proof of Theorem 2.6 will be given in Section 3. Since Theorem 2.8 is similar, we omit its proof although we present the necessary details in Section 4.

## 3. Region of variability of $V_1(z_0, \lambda)$

**Proposition 3.1.** *For $f \in \mathcal{C}_1(\lambda)$ we have*

$$\left| \frac{f''(z)}{f'(z)} - c(z, \lambda) \right| \leqslant r(z, \lambda), \quad z \in \mathbb{D}, \tag{3.2}$$

*where*

$$c(z, \lambda) = -\frac{\lambda(1 - |z|^2) + \bar{z}(|z|^2 - \lambda^2)}{(1 - |z|^2)(1 - 2\lambda \operatorname{Re} z + |z|^2)},$$
$$r(z, \lambda) = \frac{(1 - \lambda^2)|z|}{(1 - |z|^2)(1 - 2\lambda \operatorname{Re} z + |z|^2)}.$$

*For each $z \in \mathbb{D} \setminus \{0\}$, equality holds if and only if $f = F_{e^{i\theta},\lambda}$ for some $\theta \in \mathbb{R}$.*

**Proof.** Let $f \in \mathcal{C}_1(\lambda)$. Then, as pointed out in the introduction, there exists an $\omega_f \in \mathcal{B}_0$ such that

$$\omega_f(z) = \frac{P_f(z) - 1}{P_f(z) - 2}, \quad z \in \mathbb{D},$$

where $P_f(z)$ is defined by (1.1). Since $\omega_f \in \mathcal{B}_0$ satisfies $\omega'_f(0) = -f''(0) = \lambda$, it follows from the Schwarz lemma that

$$\left| \frac{\frac{\omega_f(z)}{z} - \lambda}{1 - \lambda \frac{\omega_f(z)}{z}} \right| \leqslant |z| \tag{3.3}$$

which by the definition of $P_f$ is equivalent to

$$\left| \frac{\frac{f''(z)}{f'(z)} - A(z, \lambda)}{\frac{f''(z)}{f'(z)} + B(z, \lambda)} \right| \leqslant |z| \left| \tau(z, \lambda) \right|, \tag{3.4}$$

where

$$A(z, \lambda) = -\frac{\lambda}{1 - \lambda z}, \qquad B(z, \lambda) = -\frac{1}{z - \lambda} \quad \text{and} \quad \tau(z, \lambda) = \frac{z - \lambda}{1 - \lambda z}. \tag{3.5}$$

A simple calculation shows that the inequality (3.4) is equivalent to

$$\left| \frac{f''(z)}{f'(z)} - \frac{A(z, \lambda) + |z|^2 |\tau(z, \lambda)|^2 B(z, \lambda)}{1 - |z|^2 |\tau(z, \lambda)|^2} \right| \leqslant \frac{|z||\tau(z, \lambda)||A(z, \lambda) + B(z, \lambda)|}{1 - |z|^2 |\tau(z, \lambda)|^2}. \tag{3.6}$$

Again, a computation shows that

$$1-|z|^2\big|\tau(z,\lambda)\big|^2=\frac{(1-|z|^2)(1-2\lambda\operatorname{Re} z+|z|^2)}{|1-\lambda z|^2}, \tag{3.7}$$

$$A(z,\lambda)+B(z,\lambda)=-\frac{1-\lambda^2}{(1-\lambda z)(z-\lambda)}, \tag{3.8}$$

$$A(z,\lambda)+|z|^2\big|\tau(z,\lambda)\big|^2B(z,\lambda)=-\frac{\lambda(1-|z|^2)+\bar{z}(|z|^2-\lambda^2)}{|1-\lambda z|^2}.$$

Using these, we easily have

$$\frac{A(z,\lambda)+|z|^2|\tau(z,\lambda)|^2B(z,\lambda)}{1-|z|^2|\tau(z,\lambda)|^2}:=c(z,\lambda)$$

and

$$\frac{|z||\tau(z,\lambda)||A(z,\lambda)+B(z,\lambda)|}{1-|z|^2|\tau(z,\lambda)|^2}:=r(z,\lambda).$$

Now the inequality (3.2) follows from the last two equalities and (3.6).

It is easy to see that the equality occurs for any $z\in\mathbb{D}$ in (3.2), when $f=F_{e^{i\theta},\lambda}$ for some $\theta\in\mathbb{R}$. Conversely if equality occurs for some $z\in\mathbb{D}\setminus\{0\}$ in (3.2), then equality must hold in (3.3). Thus from the Schwarz lemma there exists $\theta\in\mathbb{R}$ such that $\omega_f(z)=z\delta(e^{i\theta}z,\lambda)$ for all $z\in\mathbb{D}$. This gives that $f=F_{e^{i\theta},\lambda}$. □

For $\lambda=0$, we have the following interesting information which may be compared with the known estimate for the class of starlike functions in $\mathbb{D}$.

**Corollary 3.9.** *Let $f\in\mathcal{C}_1(0)$. Then we have*

$$\left|\frac{f''(z)}{f'(z)}+\frac{\bar{z}|z|^2}{1-|z|^4}\right|\leqslant\frac{|z|}{1-|z|^4},\quad z\in\mathbb{D}.$$

*In particular,*

$$\big(1-|z|^2\big)\left|\frac{f''(z)}{f'(z)}\right|\leqslant|z|,\quad z\in\mathbb{D}.$$

We define the norm for locally univalent functions $f$ by

$$\|f\|=\sup_{z\in\mathbb{D}}\big(1-|z|^2\big)\left|\frac{f''(z)}{f'(z)}\right|.$$

It is well known that $\|f\|\leqslant 6$ if $f$ is univalent in $\mathbb{D}$, and conversely if $\|f\|\leqslant 1$ then $f$ is univalent in $\mathbb{D}$, and these bounds are sharp (see [1]). On the other hand, Corollary 3.9 shows that $f\in\mathcal{C}_1(0)$, then $\|f\|\leqslant 1$. For recent investigations on the quantity $\|f\|$ when $f$ runs over various subclasses of locally univalent functions, we refer to [2,6].

**Corollary 3.10.** *Let $\gamma$: $z(t)$ $(0\leqslant t\leqslant 1)$ be a $C^1$-curve in $\mathbb{D}$ with $z(0)=0$ and $z(1)=z_0$. Then we have*

$$V_1(z_0,\lambda)\subset\overline{\mathbb{D}}\big(C(\lambda,\gamma),R(\lambda,\gamma)\big)=\big\{w\in\mathbb{C}\colon\ \big|w-C(\lambda,\gamma)\big|\leqslant R(\lambda,\gamma)\big\},$$

*where*

$$C(\lambda,\gamma)=\int_0^1 c\big(z(t),\lambda\big)z'(t)\,dt \quad \textit{and} \quad R(\lambda,\gamma)=\int_0^1 r\big(z(t),\lambda\big)\big|z'(t)\big|\,dt.$$

**Proof.** Let $f\in\mathcal{C}_1(\lambda)$. Then, we have

$$\int_0^1 \frac{f''(z(t))}{f'(z(t))}z'(t)\,dt=\log f'(z_0),$$

and therefore, from Proposition 3.1 we deduce that

$$\begin{aligned}
\big|\log f'(z_0)-C(\lambda,\gamma)\big| &= \left|\log f'(z_0)-\int_0^1 c\big(z(t),\lambda\big)z'(t)\,dt\right| \\
&= \left|\int_0^1 \left\{\frac{f''(z(t))}{f'(z(t))}-c\big(z(t),\lambda\big)\right\}z'(t)\,dt\right| \\
&\leqslant \int_0^1 r\big(z(t),\lambda\big)\big|z'(t)\big|\,dt=R(\lambda,\gamma).
\end{aligned}$$

Since $\log f'(z_0)\in V_1(z_0,\lambda)$ was arbitrary, the conclusion follows. □

**Lemma 3.11.** [10] *For $\theta\in\mathbb{R}$ and $0\leqslant\lambda<1$, the function*

$$G(z)=\int_0^z \frac{e^{i\theta}\zeta}{\{1+\lambda(e^{i\theta}-1)\zeta-e^{i\theta}\zeta^2\}^2}\,d\zeta, \quad z\in\mathbb{D},$$

*has a double zero at the origin and no zeros elsewhere in $\mathbb{D}$. Furthermore, there exists a starlike univalent function $G_0$ in $\mathbb{D}$ such that $G=2^{-1}e^{i\theta}G_0^2$ and $G_0(0)=G_0'(0)-1=0$.*

**Proposition 3.12.** *Let $z_0\in\mathbb{D}\setminus\{0\}$. Then for $\theta\in(-\pi,\pi]$ we have $\log F'_{e^{i\theta},\lambda}(z_0)\in\partial V_1(z_0,\lambda)$. Furthermore if $\log f'(z_0)=\log F'_{e^{i\theta},\lambda}(z_0)$ for some $f\in\mathcal{C}_1(\lambda)$ and $\theta\in(-\pi,\pi]$, then $f=F_{e^{i\theta},\lambda}$.*

**Proof.** Recall from (2.1)

$$\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)}=\frac{\delta(az,\lambda)}{z\delta(az,\lambda)-1}=\frac{az+\lambda}{az^2+\lambda(1-a)z-1}.$$

Thus we have from (3.5)

$$\begin{aligned}
\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)}-A(z,\lambda)&=\frac{(1-\lambda^2)az}{(1-\lambda z)(az^2+\lambda(1-a)z-1)},\\
\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)}+B(z,\lambda)&=\frac{1-\lambda^2}{(z-\lambda)(az^2+\lambda(1-a)z-1)}
\end{aligned}$$

and hence we may rewrite

$$\begin{aligned}
&\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} - c(z,\lambda) \\
&\quad = \frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} - \frac{A(z,\lambda) + |z|^2|\tau(z,\lambda)|^2 B(z,\lambda)}{1-|z|^2|\tau(z,\lambda)|^2} \\
&\quad = \frac{1}{1-|z|^2|\tau(z,\lambda)|^2}\left\{\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} - A(z,\lambda) - |z|^2\big|\tau(z,\lambda)\big|^2\left(\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} + B(z,\lambda)\right)\right\} \\
&\quad = \frac{(1-\lambda^2)\{a(1-\lambda\bar{z})z - |z|^2(\bar{z}-\lambda)\}}{(1-|z|^2)(1-2\lambda\operatorname{Re} z + |z|^2)(az^2+\lambda(1-a)z-1)}.
\end{aligned}$$

Now by substituting $a = e^{i\theta}$ we have

$$\begin{aligned}
\frac{F''_{e^{i\theta},\lambda}(z)}{F'_{e^{i\theta},\lambda}(z)} - c(z,\lambda) &= \frac{(1-\lambda^2)e^{i\theta}z\{\overline{1+\lambda(e^{i\theta}-1)z-e^{i\theta}z^2}\}}{(1-|z|^2)(1-2\lambda\operatorname{Re} z+|z|^2)(1+\lambda(e^{i\theta}-1)z-e^{i\theta}z^2)} \\
&= \frac{(1-\lambda^2)e^{i\theta}z|1+\lambda(e^{i\theta}-1)z-e^{i\theta}z^2|^2}{(1-|z|^2)(1-2\lambda\operatorname{Re} z+|z|^2)(1+\lambda(e^{i\theta}-1)z-e^{i\theta}z^2)^2} \\
&= r(z,\lambda)\frac{|1+\lambda(e^{i\theta}-1)z-e^{i\theta}z^2|^2 e^{i\theta}z}{(1+\lambda(e^{i\theta}-1)z-e^{i\theta}z^2)^2|z|}.
\end{aligned}$$

From Lemma 3.11, we can rewrite the last expression as

$$\frac{F''_{e^{i\theta},\lambda}(z)}{F'_{e^{i\theta},\lambda}(z)} - c(z,\lambda) = r(z,\lambda)\frac{G'(z)}{|G'(z)|}. \tag{3.13}$$

Since the function $G_0$ is starlike, for any $z_0 \in \mathbb{D}\setminus\{0\}$ the linear segment joining 0 and $G_0(z_0)$ lies entirely in $G_0(\mathbb{D})$. Define $\gamma_0$ by

$$\gamma_0\colon\ z(t) = G_0^{-1}\big(tG_0(z_0)\big), \quad 0 \leqslant t \leqslant 1. \tag{3.14}$$

Since $G(z(t)) = 2^{-1}e^{i\theta}G_0(z(t))^2 = 2^{-1}e^{i\theta}(tG_0(z_0))^2 = t^2G(z_0)$, we have

$$G'\big(z(t)\big)z'(t) = 2tG(z_0), \quad t \in [0,1]. \tag{3.15}$$

From this and (3.13) we have

$$\begin{aligned}
\log F'_{e^{i\theta},\lambda}(z_0) - C(\lambda,\gamma_0) &= \int_0^1 \left\{\frac{F''_{e^{i\theta},\lambda}(z(t))}{F'_{e^{i\theta},\lambda}(z(t))} - c\big(z(t),\lambda\big)\right\}z'(t)\,dt \\
&= \int_0^1 r\big(z(t),\lambda\big)\frac{G'(z(t))z'(t)}{|G'(z(t))z'(t)|}\big|z'(t)\big|\,dt \\
&= \frac{G(z_0)}{|G(z_0)|}\int_0^1 r\big(z(t),\lambda\big)\big|z'(t)\big|\,dt \\
&= \frac{G(z_0)}{|G(z_0)|}R(\lambda,\gamma_0).
\end{aligned} \tag{3.16}$$

Thus we have $\log F'_{e^{i\theta},\lambda}(z_0) \in \partial\overline{\mathbb{D}}(C(\lambda,\gamma_0), R(\lambda,\gamma_0))$. From Corollary 3.10 we have also $\log F'_{e^{i\theta},\lambda}(z_0) \in V_1(z_0,\lambda) \subset \overline{\mathbb{D}}(C(\lambda,\gamma_0), R(\lambda,\gamma_0))$. Hence we have $\log F'_{e^{i\theta},\lambda}(z_0) \in \partial V_1(z_0,\lambda)$.

Next, we deal with uniqueness. Suppose that $\log f'(z_0) = \log F'_{e^{i\theta},\lambda}(z_0)$ for some $f \in \mathcal{C}_1(\lambda)$ and $\theta \in (-\pi,\pi]$. Introduce

$$h(t) = \frac{\overline{G(z_0)}}{|G(z_0)|}\left\{\frac{f''(z(t))}{f'(z(t))} - c\big(z(t),\lambda\big)\right\} z'(t),$$

where $\gamma_0$: $z(t)$, $0 \leqslant t \leqslant 1$, as in (3.14). Then $h(t)$ is continuous function on $[0,1]$ and satisfies $|h(t)| \leqslant r(z(t),\lambda)|z'(t)|$ for $0 \leqslant t \leqslant 1$. Furthermore we have from (3.16)

$$\begin{aligned}\int_0^1 \operatorname{Re} h(t)\,dt &= \int_0^1 \operatorname{Re}\left\{\frac{\overline{G(z_0)}}{|G(z_0)|}\left\{\frac{f''(z(t))}{f'(z(t))} - c\big(z(t),\lambda\big)\right\} z'(t)\right\} dt \\ &= \operatorname{Re}\left\{\frac{\overline{G(z_0)}}{|G(z_0)|}\big\{\log f'(z_0) - C(\lambda,\gamma_0)\big\}\right\} \\ &= \operatorname{Re}\left\{\frac{\overline{G(z_0)}}{|G(z_0)|}\big\{\log F'_{e^{i\theta},\lambda}(z_0) - C(\lambda,\gamma_0)\big\}\right\} \\ &= \int_0^1 r\big(z(t),\lambda\big)\big|z'(t)\big|\,dt\end{aligned}$$

which gives that $h(t) = r(z(t),\lambda)|z'(t)|$ for all $t \in [0,1]$. From (3.13) and (3.15) this implies that

$$\frac{f''}{f'} = \frac{F''_{e^{i\theta},\lambda}}{F'_{e^{i\theta},\lambda}}$$

on the curve $\gamma_0$. Using the identity theorem for analytic functions, we conclude that the last equality holds in $\mathbb{D}$ and hence, by normalization, we obtain that $f = F_{e^{i\theta},\lambda}$ in $\mathbb{D}$. □

**Proof of Theorem 2.6.** We need to prove that the closed curve $(-\pi,\pi] \ni \theta \mapsto \log F'_{e^{i\theta},\lambda}(z_0)$ is simple. Suppose that $\log F'_{e^{i\theta_1},\lambda}(z_0) = \log F'_{e^{i\theta_2},\lambda}(z_0)$ for some $\theta_1,\theta_2 \in (-\pi,\pi]$ with $\theta_1 \neq \theta_2$. Then, from Proposition 3.12, we have $F_{e^{i\theta_1},\lambda} = F_{e^{i\theta_2},\lambda}$. From (2.3) this gives a contradiction

$$e^{i\theta_1}z = \tau\left(\frac{\omega_{F_{e^{i\theta_1},\lambda}}}{z},\lambda\right) = \tau\left(\frac{\omega_{F_{e^{i\theta_2},\lambda}}}{z},\lambda\right) = e^{i\theta_2}z.$$

Thus the curve is simple.

Since $V_1(z_0,\lambda)$ is a compact convex subset of $\mathbb{C}$ and has nonempty interior, the boundary $\partial V_1(z_0,\lambda)$ is a simple closed curve. From Proposition 3.1 the curve $\partial V_1(z_0,\lambda)$ contains the curve $(-\pi,\pi] \ni \theta \mapsto \log F'_{e^{i\theta},\lambda}(z_0)$. Note that a simple closed curve cannot contain any simple closed curve other than itself. Thus, $\partial V_1(z_0,\lambda)$ is given by $(-\pi,\pi] \ni \theta \mapsto \log F'_{e^{i\theta},\lambda}(z_0)$. □

## 4. Region of variability $V_2(z_0,\lambda)$

**Proposition 4.1.** *For $f \in \mathcal{C}_2(\lambda)$ we have*

$$\left|\frac{f''(z)}{f'(z)} + 3c(z,\lambda)\right| \leqslant 3r(z,\lambda), \quad z \in \mathbb{D},$$

*where* $c(z,\lambda)$ *and* $r(z,\lambda)$ *given in Proposition* 3.1. *For each* $z \in \mathbb{D} \setminus \{0\}$, *equality holds if and only if* $f = H_{e^{i\theta},\lambda}$ *for some* $\theta \in \mathbb{R}$.

**Proof.** Let $f \in \mathcal{C}_2(\lambda)$. Then, as stated in the introduction, there exists an $\omega_f \in \mathcal{B}_0$ such that

$$\omega_f(z) = \frac{P_f(z) - 1}{P_f(z) + 2}, \quad z \in \mathbb{D},$$

where $P_f(z)$ is defined by (1.1). Since $\omega'_f(0) = \lambda$, it follows from the Schwarz lemma that

$$\left| \frac{\frac{\omega_f(z)}{z} - \lambda}{1 - \lambda \frac{\omega_f(z)}{z}} \right| \leqslant |z|$$

which may be written equivalently as

$$\left| \frac{\frac{f''(z)}{f'(z)} + 3A(z,\lambda)}{\frac{f''(z)}{f'(z)} - 3B(z,\lambda)} \right| \leqslant |z| \left|\tau(z,\lambda)\right|,$$

where $A(z,\lambda)$, $B(z,\lambda)$ and $\tau(z,\lambda)$ are defined by (3.5). The remaining part of the proof follows exactly in the same lines of the proof of Proposition 3.1 and so, we omit the details. □

For $\lambda = 0$, we have the following interesting information which may be compared with the known estimate for the class of close-to-convex functions in $\mathbb{D}$.

**Corollary 4.2.** *Let* $f \in \mathcal{C}_2(0)$. *Then we have*

$$\left| \frac{f''(z)}{f'(z)} - \frac{3\bar{z}|z|^2}{1 - |z|^4} \right| \leqslant \frac{3|z|}{1 - |z|^4}, \quad z \in \mathbb{D}.$$

*In particular,*

$$\left(1 - |z|^2\right) \left| \frac{f''(z)}{f'(z)} \right| \leqslant 3|z|, \quad z \in \mathbb{D},$$

*and* $\|f\| \leqslant 3$.

**Corollary 4.3.** *Let* $\gamma\colon\ z(t)$, $0 \leqslant t \leqslant 1$, *be a* $C^1$*-curve in* $\mathbb{D}$ *with* $z(0) = 0$ *and* $z(1) = z_0$. *Then we have*

$$V_2(z_0,\lambda) \subset \overline{\mathbb{D}}\big(-3C(\lambda,\gamma), 3R(\lambda,\gamma)\big) = \big\{w \in \mathbb{C}\colon \left|w + 3C(\lambda,\gamma)\right| \leqslant 3R(\lambda,\gamma)\big\},$$

*where* $C(\lambda,\gamma)$ *and* $R(\lambda,\gamma)$ *are defined in Corollary* 3.10.

**Proof.** Proof exactly follows from Proposition 4.1 using the method of proof of Corollary 3.10. □

Finally, we state the following result which can proved with the help of the proof of Proposition 3.12.

**Proposition 4.4.** *Let* $z_0 \in \mathbb{D} \setminus \{0\}$. *Then for* $\theta \in (-\pi,\pi]$ *we have* $\log H'_{e^{i\theta},\lambda}(z_0) \in \partial V_2(z_0,\lambda)$. *Furthermore if* $\log f'(z_0) = \log H'_{e^{i\theta},\lambda}(z_0)$ *for some* $f \in \mathcal{C}_2(\lambda)$ *and* $\theta \in (-\pi,\pi]$, *then* $f = H_{e^{i\theta},\lambda}$.

## References


[1] J. Becker, Ch. Pommerenke, Schlichtheitskriterien und Jordangebiete, J. Reine Angew. Math. 354 (1984) 74–94.
[2] J.H. Choi, Y.C. Kim, S. Ponnusamy, T. Sugawa, Norm estimates for the Alexander transforms of convex functions of order alpha, J. Math. Anal. Appl. 303 (2005) 661–668.
[3] P.L. Duren, Univalent Functions, Grundlehren Math. Wiss., vol. 259, Springer, New York, 1983.
[4] A.W. Goodman, Univalent Functions, vols. I and II, Mariner Publishing Co., Tampa, Florida, 1983.
[5] D.J. Hallenbeck, A.E. Livingston, Applications of extreme point theory to classes of multivalent functions, Trans. Amer. Math. Soc. 221 (1976) 339–359.
[6] Y.C. Kim, S. Ponnusamy, T. Sugawa, Mapping properties of nonlinear integral operators and pre-Schwarzian derivatives, J. Math. Anal. Appl. 299 (2004) 433–447.
[7] S. Ponnusamy, S. Rajasekaran, New sufficient conditions for starlike and univalent functions, Soochow J. Math. 21 (1995) 193–201.
[8] S. Ponnusamy, V. Singh, Univalence of certain integral transforms, Glas. Mat. Ser. III 31 (51) (1996) 253–261.
[9] H. Yanagihara, Regions of variability for functions of bounded derivatives, Kodai Math. J. 28 (2005) 452–462.
[10] H. Yanagihara, Regions of variability for convex functions, Math. Nachr. 279 (2006) 1723–1730.