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Strong convergence theorems for nonexpansive semigroup in Banach spaces

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Abstract

Let K be a nonempty closed convex subset of a reflexive and strictly convex Banach space E with a uniformly Gâteaux differentiable norm, and $\mathcal{F} = \{T(t)\colon t > 0\}$ a nonexpansive self-mappings semigroup of K, and $f : K \to K$ a fixed contractive mapping. The strongly convergent theorems of the following implicit and explicit viscosity iterative schemes $\{x_n\}$ are proved.

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n,$$

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n.$$

And the cluster point of $\{x_n\}$ is the unique solution to some co-variational inequality.

Keywords: Nonexpansive semigroup; Viscosity approximation methods; Reflexive and strictly convex Banach space; Chebyshev set

1. Introduction

Let E be a Banach space and let K be a nonempty closed convex subset of E. A (one-parameter) nonexpansive semigroup is a family $\mathcal{F} = \{T(t)\colon t > 0\}$ of self-mappings of K such that

(i) $T(0)x = x$ for $x \in K$;
(ii) $T(t + s)x = T(t)T(s)x$ for $t, s > 0$ and $x \in K$;
(iii) $\lim_{t\to 0} T(t)x = x$ for $x \in K$;
(iv) for each $t > 0$, $T(t)$ is nonexpansive, that is,

$$\|T(t)x - T(t)y\| \leqslant \|x - y\|, \quad \forall x, y \in K.$$

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We shall denote by F the common fixed point set of $\mathcal{F}$, that is,

$$F := Fix(\mathcal{F}) = \{x \in K:\ T(t)x = x,\ t > 0\} = \bigcap_{t>0} Fix(T(t)).$$

(Here $Fix(T) = \{x \in C:\ Tx = x\}$ is the set of fixed points of a mapping T.)

Let $T : K \to K$ be a nonexpansive mapping (that is, $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in K$). Assume that the fixed point set $Fix(T)$ of T is nonempty. One classical method to study nonexpansive mappings is to use contractions to approximate nonexpansive mappings. More precisely, for a fixed point $u \in K$, define for each $0 < t < 1$, a contraction T_t by $T_t x = tu + (1 - t)Tx$, $x \in K$. Let x_t be the fixed point of T_t obtained by Banach contraction mapping principle. Thus,

$$x_t = tu + (1 - t)Tx_t. \tag{1.1}$$

Browder [4] (Reich [9], respectively) proves that as $t \to 0$, x_t converges strongly to a fixed point of T in a Hilbert space (uniformly smooth Banach space, respectively). Halpern [6] firstly introduced the following explicit iterative scheme (1.2) in Hilbert space,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n. \tag{1.2}$$

He pointed out that the control conditions (C1) and (C2) are necessary for the convergence of the iteration scheme (1.2) to a fixed point of T.

(C1) $\lim_{n\to\infty} \alpha_n = 0$,
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

In 1992, Wittmann [24], still in Hilbert space, obtained a strong convergence result [24, Theorem 2] for the iteration scheme (1.2) under the control conditions (C1), (C2) and

(C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$.

Shioji and Takahashi [11] extended Wittmann's results to a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. In 2004, for $T : K \to K$ a nonexpansive mapping with $F(T) \neq \emptyset$, and $f : K \to K$ a fixed contractive mapping, H.K. Xu [20] proposed the following viscosity iterative process $\{x_n\}$:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \tag{1.3}$$

and prove that $\{x_n\}$ converges to a fixed point p of T in a uniformly smooth Banach space. (Related results can be found in [7,12–15].)

It is an interesting problem to extend above (Browder's, Halpern's and so on) results to the nonexpansive semigroup case. However, only partial answers have been obtained. In [10], Shioji and Takahashi introduced the implicit iteration (1.4) in a Hilbert space,

$$x_n = \alpha_n u + (1 - \alpha_n)\sigma_{t_n}(x_n), \quad n \geqslant 1, \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{t_n\}$ is a sequence of positive real numbers divergent to ∞, and for each $t > 0$ and $x \in C$, $\sigma_t(x)$ is the average given by

$$\sigma_t(x) = \frac{1}{t}\int_0^t T(s)x\,ds.$$

Under certain restrictions to the sequence $\{\alpha_n\}$, Shioji and Takahashi [11] prove the strong convergence of $\{x_n\}$ to a member of F. (See also Xu [22].) Recently, Chen and Song [5] introduced the following implicit and explicit viscosity iteration processes defined by (1.5) and (1.6) to nonexpansive semigroup case,

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)\sigma_{t_n}(x_n), \quad n \geqslant 1, \tag{1.5}$$

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\sigma_{t_n}(x_n), \quad n \geqslant 1. \tag{1.6}$$

And proved that $\{x_n\}$ converges to a same point of F in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm.

Note however that their iterate x_n at step n is constructed through the average of the semigroup over the interval $(0,t)$. Suzuki [16] is the first to introduce again in a Hilbert space the following implicit iteration process:

$$x_n = \alpha_n u + (1-\alpha_n)T(t_n)x_n, \quad n \geqslant 1, \tag{1.7}$$

for the nonexpansive semigroup case.

In 2002, Dominguez Benavides, López Acedo and Xu [3] in a uniformly smooth Banach space, showed that if $\mathcal{F}$ satisfies an asymptotic regularity condition and α_n fulfills the control conditions (C1) and (C2) and

(C4) $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$,

then both the implicit iteration process (1.7) and the explicit iteration process (1.8) converge to a same point of F (cf. the discussion in [1,2]).

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T(t_n)x_n, \quad n \geqslant 1. \tag{1.8}$$

Recently, Xu [21] studied the strong convergence of the implicit iteration process (1.4) and (1.7) in a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping.

In this paper, under the framework of a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, we will study the convergence of the following implicit and explicit viscosity iterative schemes:

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad n \geqslant 1, \tag{1.9}$$

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad n \geqslant 1. \tag{1.10}$$

Our work improves and generalizes some of the results obtained in the above paper. In particular, our results extend the main results of Chen and Song [5] to a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. At the same time, the main conclusions of Dominguez Benavides, López Acedo and Xu [3], Aleyner and Censor [1, Theorem 20], Aleyner and Reich [2, Theorem 3.1] are not only proved in more generalized Banach space, but the control condition (C4) or (C3) for the iterative coefficient α_n is removed also.

2. Preliminaries

Throughout this paper, let J denote the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^*,\ \langle x, f\rangle = \|x\|\|f\|,\ \|x\| = \|f\|\}, \quad \forall x \in E,$$

where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. In the sequel, we shall denote the single-valued duality mapping by j. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (respectively $x_n \rightharpoonup x$, $x_n \rightharpoondown x$) will denote strong (respectively weak, weak*) convergence of the sequence $\{x_n\}$ to x.

Recall that the norm of Banach space E is said to be *Gâteaux differentiable* (or E is said to be *smooth*), if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{$*$}$$

exists for each x, y on the unit sphere $S(E)$ of E. Moreover, if for each y in $S(E)$ the limit defined by $(*)$ is uniformly attained for x in $S(E)$, we say that the norm of E is *uniformly Gâteaux differentiable*. The norm of E is said to be *Fréchet differentiable*, if for each $x \in S(E)$, the limit $(*)$ is attained uniformly for $y \in S(E)$. The norm of E is said to be *uniformly Fréchet differentiable* (or E is said to be *uniformly smooth*), the limit $(*)$ is attained uniformly for $(x,y) \in S(E) \times S(E)$. A Banach space E is said to *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for $\|x\| = \|y\| = 1$, $x \neq y$; *uniformly convex* if for all $\varepsilon \in [0,2]$, $\exists \delta_\varepsilon > 0$ such that $\frac{\|x+y\|}{2} < 1 - \delta_\varepsilon$ for $\|x\| = \|y\| = 1$ and $\|x-y\| \geqslant \varepsilon$. It is well known that each uniformly convex Banach space E is reflexive and strictly convex [18, Theorems 4.1.6, 4.1.2], and every uniformly smooth Banach space E is a reflexive Banach space with uniformly Gâteaux differentiable norm [18, Theorems 4.3.7, 4.1.6] (also see [8]).

Lemma 2.1. *(See [18, Theorems 4.3.1, 4.3.2].) E is a smooth Banach space if and only if the normal duality mapping J in E is single valued. Moreover, for $x, y \in E$,*

$$\langle y, J(x)\rangle = \lim_{t\to 0} \frac{\|x+ty\|^2 - \|x\|^2}{2t}.$$

Now, we present the concept of uniformly asymptotically regular semigroup (also see [1–3]). Let K be a nonempty closed convex subset of a Banach space E, $\mathcal{F} = \{T(t)\colon\ t > 0\}$ a continuous operator semigroup on K. Then $\mathcal{F}$ is said to be *uniformly asymptotically regular* (in short, u.a.r.) on K if for all $h \geqslant 0$ and any bounded subset C of K,

$$\lim_{t\to\infty} \sup_{x\in C} \big\| T(h)\big(T(t)x\big) - T(t)x\big\| = 0.$$

The nonexpansive semigroup $\{\sigma_t\colon\ t > 0\}$ defined by the following lemma is an example of u.a.r. operator semigroup. Other examples of u.a.r. operator semigroup can be found in [1, Examples 17, 18].

Lemma 2.2. *(See [5, Lemma 2.7].) Let K be a nonempty closed convex subset of a uniformly convex Banach space E, and D a bounded closed convex subset of K, and $\mathcal{F} = \{T(t)\colon\ t > 0\}$ a nonexpansive semigroup on K such that $F := \bigcap_{t>0} Fix(T(t))$ is nonempty. For each $h > 0$, set $\sigma_t(x) = \frac{1}{t}\int_0^t T(s)x\,ds$, then*

$$\lim_{t\to\infty} \sup_{x\in D} \big\|\sigma_t(x) - T(h)\sigma_t(x)\big\| = 0.$$

Example. The set $\{\sigma_t\colon\ t > 0\}$ defined by Lemma 2.2 is an u.a.r. nonexpansive semigroup. In fact, it is obvious that $\{\sigma_t\colon\ t > 0\}$ is a nonexpansive semigroup. For each fixed $h > 0$, we have

$$\big\|\sigma_t(x) - \sigma_h\sigma_t(x)\big\| = \left\| \frac{1}{h}\int_0^h \big(\sigma_t(x) - T(s)\sigma_t(x)\big)\,ds \right\| \leqslant \frac{1}{h}\int_0^h \big\|\sigma_t(x) - T(s)\sigma_t(x)\big\|\,ds.$$

Therefore, using Lemma 2.2,

$$\lim_{t\to\infty} \sup_{x\in D} \big\|\sigma_t(x) - \sigma_h\sigma_t(x)\big\| \leqslant \frac{1}{h}\int_0^h \lim_{t\to\infty} \sup_{x\in D} \big\|\sigma_t(x) - T(s)\sigma_t(x)\big\|\,ds = 0.$$

Finally, we also need the following definitions and results [17,18]. Let μ be a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on N if and only if

$$\inf\{a_n; n \in N\} \leqslant \mu(a) \leqslant \sup\{a_n; n \in N\}$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$. Occasionally, we shall use $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on N is called a *Banach limit* if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$. Using the Hahn–Banach theorem, or the Tychonoff fixed point theorem, we can prove the existence of a Banach limit. We know that if μ is a Banach limit, then

$$\liminf_{n\to\infty} a_n \leqslant \mu_n(a_n) \leqslant \limsup_{n\to\infty} a_n$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$. So, if $a = (a_1, a_2, \ldots)$, $b = (b_1, b_2, \ldots) \in l^\infty$ and $a_n \to c$ (respectively, $a_n - b_n \to 0$), as $n \to \infty$, we have

$$\mu_n(a_n) = \mu(a) = c \quad \big(\text{respectively, } \mu_n(a_n) = \mu_n(b_n)\big).$$

Subsequently, the following result was showed in Refs. [17, Lemma 1] and [18, Lemma 4.5.4].

Lemma 2.3. *(See [17, Lemma 1].) Let K be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm, and $\{x_n\}$ a bounded sequence of E. If $z_0 \in K$, then*

$$\mu_n\|x_n - z_0\|^2 = \min_{y\in K} \mu_n\|x_n - y\|^2$$

if and only if

$$\mu_n\langle y - z_0, J(x_n - z_0)\rangle \leqslant 0, \quad \forall y \in K.$$

3. Implicit iteration scheme

In order to prove the strong convergence of the iterative process (1.9), we first apply the property of Chebyshev set to show the following proposition.

Let (M, d) a metric space. A subset A of M is called a Chebyshev set, if for each $x \in M$, there exists an unique element $y \in A$ such that $d(x, y) = d(x, A)$, where $d(x, A) = \inf_{y\in A} d(x, y)$.

Day–James Theorem. *(See [8, Theorem 5.1.18, Corollary 5.1.19].) E is a reflexive strictly convex Banach space if and only if every nonempty closed convex subset of E is a Chebyshev set.*

Proposition 3.1. *Let E be a reflexive strictly convex Banach space with a uniformly Gâteaux differentiable norm, and K a nonempty closed convex subset of E. Suppose x_n is a bounded sequence in K such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, an approximate fixed point of nonexpansive self-mapping T on K. Define the set*

$$K^* = \left\{x \in K:\ \mu_n\|x_n - x\|^2 = \inf_{y\in K} \mu_n\|x_n - y\|^2\right\}.$$

If $Fix(T) \neq \emptyset$, then $K^* \cap Fix(T) \neq \emptyset$.

Proof. Set $g(y) = \mu_n\|x_n - y\|^2$, $\forall y \in K$, then $g(y)$ is a convex and continuous function, and $g(y) \to \infty$ as $\|y\| \to \infty$. Using [18, Theorem 1.3.11], there exists $x \in K$ such that $g(x) = \inf_{y\in K} g(y)$ by the reflexivity of E, that is, K^* is nonempty. Clearly, K^* is closed convex by the convexity and continuity of $g(y)$.

Since $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, for $\forall x \in K^*$, we get that

$$g(Tx) = \mu_n\|x_n - Tx\|^2 = \mu_n\|Tx_n - Tx\|^2 \leqslant \mu_n\|x_n - x\|^2 = g(x).$$

Hence, $Tx \in K^*$. As x is arbitrary, then $T(K^*) \subset K^*$.

Let $p \in Fix(T)$. It follows from Day–James's theorem that there exists an unique $v \in K^*$ such that

$$\|p - v\| = \inf_{x\in K^*} \|p - x\|.$$

Since $p = Tp$ and $Tv \in K^*$,

$$\|p - Tv\| = \|Tp - Tv\| \leqslant \|p - v\|.$$

Hence $v = Tv$ by the uniqueness of $v \in K^*$. Thus $v \in K^* \cap Fix(T)$. This completes the proof. □

Theorem 3.2. *Let E be a real reflexive strictly convex Banach space with a uniformly Gâteaux differentiable norm, and K a nonempty closed convex subset of E, and $\{T(t)\}$ a u.a.r. nonexpansive semigroup from K into itself such that $F := Fix(\mathcal{F}) = \bigcap_{t>0} Fix(T(t)) \neq \emptyset$, and $f : K \to K$ a fixed contractive mapping with contractive coefficient $\beta \in (0, 1)$. Suppose $\lim_{n\to\infty} t_n = \infty$ and $\alpha_n \in (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. If $\{x_n\}$ is defined by*

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)T(t_n)x_n, \quad n \geqslant 1.$$

Then as $n \to \infty$, $\{x_n\}$ converges strongly to some common fixed point p of $\mathcal{F}$ such that p is the unique solution in F to the following co-variational inequality:

$$\left\langle f(p) - p, J(y - p)\right\rangle \leqslant 0 \quad \text{for all } y \in F. \tag{3.1}$$

Proof. We first show that the uniqueness of solution to the variational inequality (3.1) in F. In fact, suppose $p, q \in F$ satisfy (3.1), we have that

$$\left\langle f(p) - p, J(q - p)\right\rangle \leqslant 0, \tag{3.2}$$

$$\left\langle f(q) - q, J(p - q)\right\rangle \leqslant 0. \tag{3.3}$$

Combining (3.2) and (3.3), it follows that

$$(1-\beta)\|p-q\|^2 \leqslant \big\langle (p-q)-\big(f(p)-f(q)\big),\ J(p-q)\big\rangle \leqslant 0.$$

We must have $p=q$ and the uniqueness is proved.

Now we show the boundedness of $\{x_n\}$. Indeed, for any fixed $y \in F$,

$$\begin{aligned}
&\|x_n-y\|^2\\
&\quad= \big\langle \alpha_n\big(f(x_n)-y\big)+(1-\alpha_n)\big(T(t_n)x_n-y\big), J(x_n-y)\big\rangle\\
&\quad= \alpha_n\big\langle f(x_n)-f(y)+f(y)-y, J(x_n-y)\big\rangle+(1-\alpha_n)\big\langle T(t_n)x_n-T(t_n)y, J(x_n-y)\big\rangle\\
&\quad\leqslant \alpha_n\big\|f(x_n)-f(y)\big\|\big\|J(x_n-y)\big\|+\alpha_n\big\langle f(y)-y, J(x_n-y)\big\rangle+(1-\alpha_n)\big\|T(t_n)x_n-T(t_n)y\big\|\big\|J(x_n-y)\big\|\\
&\quad\leqslant \big(1-(1-\beta)\alpha_n\big)\|x_n-y\|^2+\alpha_n\big\langle f(y)-y, J(x_n-y)\big\rangle.
\end{aligned}$$

Therefore,

$$\|x_n-y\|^2 \leqslant \frac{1}{1-\beta}\big\langle f(y)-y, J(x_n-y)\big\rangle \leqslant \frac{1}{1-\beta}\big\|f(y)-y\big\|\|x_n-y\|. \tag{3.4}$$

Furthermore,

$$\|x_n-y\| \leqslant \frac{1}{1-\beta}\big\|f(y)-y\big\|.$$

Thus $\{x_n\}$ is bounded, and so are $\{T(t_n)x_n\}$ and $\{f(x_n)\}$. This implies that

$$\lim_{n\to\infty}\big\|x_n-T(t_n)x_n\big\| = \lim_{n\to\infty}\alpha_n\big\|T(t_n)x_n-f(x_n)\big\| = 0.$$

Since $\{T(t)\}$ is u.a.r. nonexpansive semigroup and $\lim_{n\to\infty} t_n=\infty$, then for all $h>0$,

$$\lim_{n\to\infty}\big\|T(h)\big(T(t_n)x_n\big)-T(t_n)x_n\big\| \leqslant \lim_{n\to\infty}\sup_{x\in C}\big\|T(h)\big(T(t_n)x\big)-T(t_n)x\big\| = 0,$$

where C is any bounded subset of K containing $\{x_n\}$. Hence,

$$\begin{aligned}
\big\|x_n-T(h)x_n\big\| &\leqslant \big\|x_n-T(t_n)x_n\big\|+\big\|T(t_n)x_n-T(h)\big(T(t_n)x_n\big)\big\|+\big\|T(h)\big(T(t_n)x_n\big)-T(h)x_n\big\|\\
&\leqslant 2\big\|x_n-T(t_n)x_n\big\|+\big\|T(h)\big(T(t_n)x_n\big)-T(t_n)x_n\big\| \to 0.
\end{aligned}$$

That is, for all $h>0$,

$$\lim_{n\to\infty}\big\|x_n-T(h)x_n\big\| = 0. \tag{3.5}$$

We claim that the set $\{x_n\}$ is sequentially compact. Indeed, define the set

$$K^* = \Big\{x\in K\colon\ \mu_n\|x_n-x\| = \inf_{y\in K}\mu_n\|x_n-y\|\Big\}.$$

By Proposition 3.1, we can found $p\in K^*$ such that $p=T(h)p$. Since h is arbitrary, it follows that $p\in F$. Using Lemma 2.3 together with $p\in K^*$, we get that

$$\mu_n\big\langle y-p, J(x_n-p)\big\rangle \leqslant 0, \quad \forall y\in K.$$

From (3.4), we have

$$\mu_n\|x_n-p\|^2 \leqslant \frac{1}{1-\beta}\mu_n\big\langle f(p)-p, J(x_n-p)\big\rangle \leqslant 0,$$

i.e.

$$\mu_n\|x_n-p\| = 0.$$

Hence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which strongly converges to $p\in F$ as $k\to\infty$.

Next we show that p is a solution in F to the variational inequality (3.1). In fact, for any fixed $y\in F$, there exists a constant $M>0$ such that $\|x_n-y\|\leqslant M$, then

$$\begin{aligned}\|x_n - y\|^2 &= \alpha_n\big\langle f(x_n) - f(p) + p - x_n, J(x_n - y)\big\rangle + \alpha_n\big\langle f(v) - p, J(x_n - y)\big\rangle + \alpha_n\big\langle x_n - y, J(x_n - y)\big\rangle \\ &\quad + (1-\alpha_n)\big\langle T(t_n)x_n - T(t_n)y, J(x_n - y)\big\rangle \\ &\leqslant (1+\beta)\alpha_n M\|x_n - v\| + \alpha_n\big\langle f(p) - p, J(x_n - y)\big\rangle + \|x_n - y\|^2.\end{aligned}$$

Therefore,

$$\big\langle f(p) - p, J(y - x_n)\big\rangle \leqslant (1+\beta)M\|x_n - p\|. \tag{3.6}$$

Since the duality mapping J is single-valued and norm topology to weak* topology uniformly continuous on any bounded subset of a Banach space E with a uniformly Gâteaux differentiable norm, we have

$$\big\langle f(p) - p, J(y - x_{n_k})\big\rangle \to \big\langle f(p) - p, J(y - v)\big\rangle.$$

Taking limit as $n_k \to \infty$ in two sides of (3.6), we get

$$\big\langle f(p) - p, J(y - p)\big\rangle \leqslant 0 \quad \forall y \in F.$$

This is, $p \in F$ is a solution of the variational inequality (3.1). From this we conclude that $p \in F$ is the unique solution of the variational inequality (3.1). In a similar way it can be show that each cluster point of the sequence $\{x_n\}$ is equal to p. Therefore, the entire sequence $\{x_n\}$ converges to p and the proof is complete. □

Corollary 3.3. *Let E be an uniformly convex Banach space with a uniformly Gâteaux differentiable norm, and K, f, t_n, α_n be as Theorem* 3.2. *Assumed $\{T(t)\}$ a nonexpansive semigroup from K into itself such that $F := Fix(\mathcal{F}) = \bigcap_{t>0} Fix(T(t)) \neq \emptyset$, and $\{x_n\}$ given by*

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x\,ds.$$

Then as $n \to \infty$, $\{x_n\}$ converges strongly to some common fixed point p of $\mathcal{F}$ such that p is the unique solution in F to the co-variational inequality (3.1).

Remark 3.4. The conclusion of Theorem 3.2 still holds if E is assumed to have the fixed point property for nonexpansive self-mappings instead of to be a strictly convex space. In fact, the same proof works (remains valid) disregarding of Proposition 3.1. In particular, when E is an uniformly smooth Banach space and therefore, when $f(x) \equiv u$ for all $x \in K$, our result contains Theorem 3.1 in [3].

4. Explicit iterative scheme

In order to prove our main result we will need the following numerical lemma (see, e.g., [10–14,19–21,23]).

Lemma 4.1. *(See [23, Lemma 2.5].) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leqslant (1-\gamma_n)a_n + \beta_n, \quad n \geqslant 0,$$

where $\{\gamma_n\} \subset (0,1)$ and $\{\beta_n\}$ is real number sequence such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\beta_n}{\gamma_n} \leqslant 0$.

Then $\{a_n\}$ converges to zero, as $n \to \infty$.

Theorem 4.2. *Let E be a real reflexive strictly convex Banach space with a uniformly Gâteaux differentiable norm, and K a nonempty closed convex subset of E, and $\{T(t)\}$ a u.a.r. nonexpansive semigroup from K into itself such that $F := Fix(\mathcal{F}) \neq \emptyset$, and $f : K \to K$ a fixed contractive mapping with contractive coefficient $\beta \in (0,1)$. Suppose $\lim_{n\to\infty} t_n = \infty$ and $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $\{x_n\}$ is given by the following equation*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad n \geqslant 1. \tag{4.1}$$

Then as $n \to \infty$, $\{x_n\}$ converges strongly to some common fixed point p of $\mathcal{F}$ such that p is the unique solution in F to the co-variational inequality (3.1).

Proof. Firstly, we show that $\{x_n\}$ is bounded. Take $u \in F$. It follows that

$$\begin{aligned}
\|x_{n+1} - u\| &\leqslant (1-\alpha_n)\big\|T(t_n)x_n - u\big\| + \alpha_n\big\|f(x_n) - u\big\| \\
&\leqslant (1-\alpha_n)\|x_n - u\| + \alpha_n\big(\beta\|x_n - u\| + \big\|f(u) - u\big\|\big) \\
&= \big(1 - (1-\beta)\alpha_n\big)\|x_n - u\| + \alpha_n\big\|f(u) - u\big\| \\
&\leqslant \max\left\{\|x_n - u\|, \frac{1}{1-\beta}\big\|f(u) - u\big\|\right\} \\
&\ \ \vdots \\
&\leqslant \max\left\{\|x_1 - u\|, \frac{1}{1-\beta}\big\|f(u) - u\big\|\right\}.
\end{aligned}$$

Thus $\{x_n\}$ is bounded, which leads to the boundedness of $\{f(x_n)\}$ and $\{T(t_n)x_n\}$. Using the assumption that $\lim_{n\to\infty} \alpha_n = 0$, we get that

$$\big\|x_{n+1} - T(t_n)x_n\big\| = \alpha_n\big\|f(x_n) - T(t_n)x_n\big\|. \tag{4.2}$$

Since $\{T(t)\}$ is u.a.r. nonexpansive semigroup, then for $h > 0$,

$$\lim_{n\to\infty}\big\|T(h)\big(T(t_n)x_n\big) - T(t_n)x_n\big\| \leqslant \lim_{n\to\infty}\sup_{x\in C}\big\|T(h)\big(T(t_n)x\big) - T(t_n)x\big\| = 0, \tag{4.3}$$

where C is any bounded subset of K containing $\{x_n\}$.

Hence, for all $h > 0$,

$$\begin{aligned}
\big\|x_{n+1} - T(h)x_{n+1}\big\| &\leqslant \big\|x_{n+1} - T(t_n)x_n\big\| + \big\|T(t_n)x_n - T(h)\big(T(t_n)x_n\big)\big\| + \big\|T(h)\big(T(t_n)x_n\big) - T(h)x_{n+1}\big\| \\
&\leqslant 2\big\|x_{n+1} - T(t_n)x_n\big\| + \big\|T(t_n)x_n - T(h)\big(T(t_n)x_n\big)\big\|.
\end{aligned}$$

Combining (4.2) and (4.3), we get that for all $h > 0$,

$$\lim_{n\to\infty}\big\|x_{n+1} - T(h)x_{n+1}\big\| = 0. \tag{4.4}$$

From Theorem 3.2, there exists the unique solution $p \in F$ to the variational inequality (3.1). Since $p = T(t)p$, for all $t > 0$, we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \alpha_n\big\langle f(x_n) - p, J(x_{n+1} - p)\big\rangle + (1-\alpha_n)\big\langle T(t_n)x_n - p, J(x_{n+1} - p)\big\rangle \\
&\quad \leqslant \alpha_n\big\langle f(p) - p, J(x_{n+1} - p)\big\rangle + \alpha_n\big\langle f(x_n) - f(p), J(x_{n+1} - p)\big\rangle + (1-\alpha_n)\big\|T(t_n)x_n - p\big\|\|x_{n+1} - p\| \\
&\quad \leqslant \alpha_n\big\langle f(p) - p, J(x_{n+1} - p)\big\rangle + \alpha_n\big\|f(x_n) - f(p)\big\|\|x_{n+1} - p\| + (1-\alpha_n)\|x_n - p\|\|x_{n+1} - p\| \\
&\quad \leqslant \alpha_n\big\langle f(p) - p, J(x_{n+1} - p)\big\rangle + \alpha_n\frac{\beta^2\|x_n - p\|^2 + \|x_{n+1} - p\|^2}{2} + (1-\alpha_n)\frac{\|x_n - p\|^2 + \|x_{n+1} - p\|^2}{2}.
\end{aligned}$$

And thus,

$$\|x_{n+1} - p\|^2 \leqslant \big(1 - \alpha_n\big(1 - \beta^2\big)\big)\|x_n - p\|^2 + 2\alpha_n\big\langle f(p) - p, J(x_{n+1} - p)\big\rangle,$$

that is

$$\|x_{n+1} - p\|^2 = (1-\gamma_n)\|x_n - p\|^2 + \gamma_n\lambda_n, \tag{4.5}$$

where $\gamma_n = \alpha_n(1-\beta^2)$ and $\lambda_n = \frac{2}{1-\beta^2}\langle f(p) - p, J(x_{n+1} - p)\rangle$.

In order to prove that $x_n \to p$ as $n \to \infty$, we apply Lemma 4.1 to (4.5). Indeed, since by assumption $\sum_{n=1}^{\infty} \gamma_n = \infty$, we only need to show that $\limsup_{n\to\infty} \lambda_n \leqslant 0$ to conclude $\lim_n \|x_n - p\| = 0$. We claim that

$$\limsup_{n\to\infty} \bigl\langle f(p) - p, J(x_{n+1} - p)\bigr\rangle \leqslant 0. \tag{4.6}$$

Let $z_m = \alpha_m f(z_m) + (1 - \alpha_m)T(t_m)z_m$, where t_m and α_m satisfies the condition of Theorem 3.2. Then it follows from Theorem 3.2 that $p = \lim_{m\to\infty} z_m$.

Since

$$\begin{aligned}
\|z_m - x_{n+1}\|^2 &= (1-\alpha_m)\bigl\langle T(t_m)z_m - x_{n+1}, J(z_m - x_{n+1})\bigr\rangle + \alpha_m\bigl\langle f(z_m) - x_{n+1}, J(z_m - x_{n+1})\bigr\rangle \\
&= (1-\alpha_m)\bigl(\bigl\langle T(t_m)z_m - T(t_m)x_{n+1}, J(z_m - x_{n+1})\bigr\rangle + \bigl\langle T(t_m)x_{n+1} - x_{n+1}, J(z_m - x_{n+1})\bigr\rangle\bigr) \\
&\quad + \alpha_m\bigl\langle f(z_m) - z_m - \bigl(f(p) - p\bigr), J(z_m - x_{n+1})\bigr\rangle + \alpha_m\bigl\langle f(p) - p, J(z_m - x_{n+1})\bigr\rangle \\
&\quad + \alpha_m\bigl\langle z_m - x_{n+1}, J(z_m - x_{n+1})\bigr\rangle \\
&\leqslant \|x_{n+1} - z_m\|^2 + \bigl\|T(t_m)x_{n+1} - x_{n+1}\bigr\|M + \alpha_m\bigl\langle f(p) - p, J(z_m - x_{n+1})\bigr\rangle \\
&\quad + M\bigl(\bigl\|f(z_m) - f(p)\bigr\| + \|z_m - p\|\bigr),
\end{aligned}$$

and hence

$$\bigl\langle f(p) - p, J(x_{n+1} - z_m)\bigr\rangle \leqslant \frac{\|x_{n+1} - T(t_m)x_{n+1}\|}{\alpha_m} M + (1+\beta)M\|z_m - p\|, \tag{4.7}$$

where M is a constant such that $M \geqslant \|x_{n+1} - z_m\|$. Therefore, taking upper limit as $n \to \infty$ firstly, and then as $m \to \infty$ in (4.7), (using (4.4))

$$\limsup_{m\to\infty}\limsup_{n\to\infty} \bigl\langle f(p) - p, J(x_{n+1} - z_m)\bigr\rangle \leqslant 0. \tag{4.8}$$

On the other hand, since $\lim_{m\to\infty} z_m = p$ due to the fact the duality map J is single-valued and norm topology to weak* topology uniformly continuous on bounded sets of E, we obtain $\lim_{m\to\infty}(x_{n+1} - z_m) = x_{n+1} - p$, therefore

$$\bigl\langle f(p) - p, J(x_{n+1} - z_m)\bigr\rangle \to \bigl\langle f(p) - p, J(x_{n+1} - p)\bigr\rangle \quad \text{uniformly}.$$

Thus given $\epsilon > 0$, there exists $N \geqslant 1$ such that if $m > N$, for all n we have

$$\bigl\langle f(p) - p, J(x_{n+1} - p)\bigr\rangle < \bigl\langle f(p) - p, J(x_{n+1} - z_m)\bigr\rangle + \varepsilon. \tag{4.9}$$

Hence, by taking upper limit as $n \to \infty$ firstly, and then as $m \to \infty$ in two sides of (4.9),

$$\limsup_{n\to\infty} \bigl\langle f(p) - p, J(x_{n+1} - p)\bigr\rangle \leqslant \limsup_{m\to\infty}\limsup_{n\to\infty} \bigl\langle f(p) - p, J(x_{n+1} - z_m)\bigr\rangle + \varepsilon \leqslant \varepsilon.$$

Since ε is arbitrary, (4.6) is proved. Finally, we apply Lemma 4.1 to conclude that $x_n \to p$. □

Similar to the discussion of Theorem 3.3, the following result is clearly gained.

Corollary 4.3. *(See [5, Theorem 3.2].) Let E be an uniformly convex Banach space with an uniformly Gâteaux differentiable norm, and K, f, t_n, α_n be as Theorem 4.2. Assumed $\{T(t)\}$ a nonexpansive semigroup from K into itself such that $F := Fix(\mathcal{F}) = \bigcap_{t>0} Fix(T(t)) \neq \emptyset$, and $\{x_n\}$ given by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x\,ds.$$

Then as $n \to \infty$, $\{x_n\}$ converges strongly to some common fixed point p of $\mathcal{F}$ such that p is the unique solution in F to the co-variational inequality (3.1).

Remark 4.4. (i) The conclusion of Theorem 4.2 still holds if E is an uniformly smooth Banach space and therefore, when $f(x) \equiv u$ for all $x \in K$, our result contains [3, Theorem 3.2] and [1, Theorem 20], and the control conditions $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$ in [3, Theorem 3.2] and $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$, $\sum_{n=0}^{\infty} |r_n - r_{n+1}| < \infty$ in [1, Theorem 20] can be respectively removed.

(ii) The method of proof in Theorem 4.2 carries over to a reflexive Banach space with a uniformly Gâteaux differentiable norm which has the fixed point property for nonexpansive self-mappings. Therefore, the condition $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ in [3, Theorem 3.1] ($f(x) \equiv u$) can be dropped.

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