

Note

# A remark on Andrews–Askey integral<sup>☆</sup>

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## Abstract

In this paper, we use the Andrews–Askey integral and the  $q$ -Chu–Vandermonde formula to derive a more general integral formula. Applications of the new integral formula are also given, which include to derive the  $q$ -Pfaff–Saalschütz formula and the terminating Sears’s  ${}_3\phi_2$  transformation formula.

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## 1. Introduction and statement of result

The following is the Andrews–Askey integral [1] which can be derived from Ramanujan’s  ${}_1\psi_1$ :

$$\int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt; q)_\infty} d_q t = \frac{d(1-q)(q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty}, \quad (1.1)$$

provided that  $|q| < 1$  and there are no zero factors in the denominator of the integrals.

The Andrews–Askey integral is an important formula in basic hypergeometric series. In this paper, we derive a more general integral formula, which includes the  $q$ -Pfaff–Saalschütz formula and the terminating Sears’s  ${}_3\phi_2$  transformation formula as special cases. The main result is the following theorem:

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**Theorem 1.1.** If  $|q| < 1$  and there are no zero factors in the denominator of the integrals, then

$$\int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty P_n(\omega, c/a; q) P_m(\omega, d/b; q)}{(a\omega, b\omega; q)_\infty} d_q \omega \\ = \frac{t(1-q)(c; q)_n (d; q)_m (q, tq/s, s/t, abst; q)_\infty}{a^n b^m (as, at, bs, bt; q)_\infty} \sum_{k=0}^n \frac{(q^{-n}, as, at; q)_k q^k}{(q, c, abst; q)_k} {}_3\phi_2 \left( \begin{matrix} bs, bt, q^{-m} \\ d, abstq^k \end{matrix}; q, q \right), \quad (1.2)$$

where

$$P_0(a, b; q) = 1, \quad P_n(a, b; q) = (a-b)(a-bq) \cdots (a-bq^{n-1}), \quad n \geq 1.$$

## 2. Notations and known results

We first recall some definitions, notations and known results in [2,4] which will be used for the proof of Theorem 1.1. Throughout this paper, it is supposed that  $0 < |q| < 1$ . The  $q$ -shifted factorials are defined as

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k). \quad (2.1)$$

We also adopt the following compact notation for multiple  $q$ -shifted factorial:

$$(a_1, a_2, \dots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n, \quad (2.2)$$

where  $n$  is an integer or  $\infty$ .

In 1846, Heine introduced the  ${}_{r+1}\phi_r$  basic hypergeometric series, which is defined by

$${}_{r+1}\phi_r \left( \begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n x^n}{(q, b_1, b_2, \dots, b_r; q)_n}. \quad (2.3)$$

F.H. Jackson defined the  $q$ -integral by [5]

$$\int_0^d f(t) d_q t = d(1-q) \sum_{n=0}^{\infty} f(dq^n) q^n \quad (2.4)$$

and

$$\int_c^d f(t) d_q t = \int_0^d f(t) d_q t - \int_0^c f(t) d_q t. \quad (2.5)$$

In the context of this paper, convergence of series is no issue at all because they are the terminating series.

## 3. The proof of Theorem 1.1

In this section, we use the Andrews–Askey integral and the  $q$ -Chu–Vandermonde formula to prove Theorem 1.1.

**Proof.** Recall the  $q$ -Chu–Vandermonde convolution formula

$${}_2\phi_1 \left( \begin{matrix} q^{-n}, a \\ c \end{matrix}; q, q \right) = \frac{a^n (c/a; q)_n}{(c; q)_n}. \quad (3.1)$$

By the following relation

$$(a; q)_k = \frac{(a; q)_\infty}{(aq^k; q)_\infty}, \quad (3.2)$$

(3.1) can be written as

$$\sum_{k=0}^n \frac{(q^{-n}; q)_k q^k}{(q, c; q)_k} \cdot \frac{1}{(aq^k; q)_\infty} = \frac{a^n}{(c; q)_n} \cdot \frac{(c/a; q)_n}{(a; q)_\infty}. \quad (3.3)$$

If we let  $a = a\omega$  in (3.3) and multiply Eq. (3.3) by

$$\frac{(q\omega/s, q\omega/t; q)_\infty}{(b\omega; q)_\infty},$$

then we obtain

$$\sum_{k=0}^n \frac{(q^{-n}; q)_k q^k}{(q, c; q)_k} \cdot \frac{(q\omega/s, q\omega/t; q)_\infty}{(a\omega q^k, b\omega; q)_\infty} = \frac{a^n}{(c; q)_n} \cdot \frac{(q\omega/s, q\omega/t; q)_\infty P_n(\omega, c/a)}{(a\omega, b\omega; q)_\infty}. \quad (3.4)$$

Taking the  $q$ -integral on both sides of (3.4) with respect to variable  $\omega$ , we have

$$\sum_{k=0}^n \frac{(q^{-n}; q)_k q^k}{(q, c; q)_k} \cdot \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty}{(a\omega q^k, b\omega; q)_\infty} d_q \omega = \frac{a^n}{(c; q)_n} \cdot \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty P_n(\omega, c/a)}{(a\omega, b\omega; q)_\infty} d_q \omega. \quad (3.5)$$

Applying Andrews–Askey integral (1.1) to the integral on the left-hand side of (3.5), we have

$$\begin{aligned} & \sum_{k=0}^n \frac{(q^{-n}; q)_k q^k}{(q, c; q)_k} \cdot \frac{t(1-q)(q, tq/s, s/t, abstq^k; q)_\infty}{(asq^k, atq^k, bs, bt; q)_\infty} \\ &= \frac{a^n}{(c; q)_n} \cdot \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty P_n(\omega, c/a; q)}{(a\omega, b\omega; q)_\infty} d_q \omega, \end{aligned} \quad (3.6)$$

which can be rewritten as

$$\begin{aligned} & \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty P_n(\omega, c/a; q)}{(a\omega, b\omega; q)_\infty} d_q \omega \\ &= \frac{t(1-q)(c; q)_n (q, tq/s, s/t, abst; q)_\infty}{a^n (as, at, bs, bt; q)_\infty} {}_3\phi_2 \left( \begin{matrix} as, at, q^{-n} \\ c, abst \end{matrix}; q, q \right). \end{aligned} \quad (3.7)$$

On the other hand, if we multiply Eq. (3.4) by  $P_m(\omega, d/b; q)$ , we obtain

$$\begin{aligned} & \sum_{k=0}^n \frac{(q^{-n}; q)_k q^k}{(q, c; q)_k} \cdot \frac{(q\omega/s, q\omega/t; q)_\infty P_m(\omega, d/b; q)}{(a\omega q^k, b\omega; q)_\infty} \\ &= \frac{a^n}{(c; q)_n} \cdot \frac{(q\omega/s, q\omega/t; q)_\infty P_n(\omega, c/a; q) P_m(\omega, d/b; q)}{(a\omega, b\omega; q)_\infty}. \end{aligned} \quad (3.8)$$

We take the  $q$ -integral on both sides of (3.8) with respect to variable  $\omega$  and use (3.7) in the resulting equation with  $n = m$ ,  $c = da/b$ , and  $a = aq^k$ . After simple rearrangements, we have (1.2).  $\square$

Note that (3.7) is a special case of (1.2) when  $m = 0$ .

#### 4. Some applications

In this section, we give some applications of (1.2). First we point out that the  $q$ -Pfaff–Saalschütz formula [2,4] is a special case of (1.2).

**Theorem 4.1** (The  $q$ -Pfaff–Saalschütz formula). *We have*

$${}_3\phi_2 \left( \begin{matrix} a, b, q^{-n} \\ c, abc^{-1}q^{1-n} \end{matrix}; q, q \right) = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}. \quad (4.1)$$

**Proof.** Let  $m = 0$ ,  $b = aq^{1-n}/c$  in (1.2) to get

$$\begin{aligned} & \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty P_n(\omega, c/a; q)}{(a\omega, aq^{1-n}\omega/c; q)_\infty} d_q\omega \\ &= \frac{t(1-q)(c; q)_n (q, tq/s, s/t, a^2q^{1-n}st/c; q)_\infty}{a^n(as, at, asq^{1-n}/c, atq^{1-n}/c; q)_\infty} \cdot \sum_{k=0}^n \frac{(q^{-n}, as, at; q)_k q^k}{(q, c, a^2q^{1-n}st/c; q)_k}. \end{aligned} \quad (4.2)$$

Using the following relation

$$a^n(c/a; q)_n = (-c)^n q^{\binom{n}{2}} \frac{(aq^{1-n}/c; q)_\infty}{(aq/c; q)_\infty}, \quad (4.3)$$

we find that

$$a^n P_n(\omega, c/a; q) = (a\omega)^n (c/a\omega; q)_n = (-c)^n q^{\binom{n}{2}} \frac{(a\omega q^{1-n}/c; q)_\infty}{(a\omega q/c)_\infty}. \quad (4.4)$$

Using (4.4) in (4.2), we have

$$\begin{aligned} \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty}{(a\omega q/c, a\omega; q)_\infty} d_q\omega &= \frac{t(1-q)(c; q)_n}{(-c)^n q^{\binom{n}{2}}} \\ &\quad \times \frac{(q, tq/s, s/t, a^2q^{1-n}st/c; q)_\infty}{(as, at, asq^{1-n}/c, atq^{1-n}/c; q)_\infty} {}_3\phi_2 \left( \begin{matrix} as, at, q^{-n} \\ c, a^2q^{1-n}st/c \end{matrix}; q, q \right). \end{aligned} \quad (4.5)$$

Applying the Andrews–Askey integral (1.1) to (4.5), we obtain

$${}_3\phi_2 \left( \begin{matrix} as, at, q^{-n} \\ c, a^2q^{1-n}st/c \end{matrix}; q, q \right) = (-c)^n \frac{q^{\binom{n}{2}}}{(c; q)_n} \cdot \frac{(as/c, at/c; q^{-1})_n}{(a^2st/c; q^{-1})_n}. \quad (4.6)$$

After replacing  $as$  and  $at$  by  $a$  and  $b$ , respectively, and employing the following formula

$$(a; q)_n = (a^{-1}; q^{-1})_n (-a)^n q^{\binom{n}{2}},$$

we obtain the  $q$ -Pfaff–Saalschütz formula (4.1).  $\square$

We also have the following transformation formula, which includes the terminating Sears's  ${}_3\phi_2$  transformation formula.

**Theorem 4.2.** *We have*

$$\begin{aligned} & \sum_{k=0}^n \frac{(q^{-n}, as, at; q)_k q^k}{(q, c, abst; q)_k} {}_3\phi_2 \left( \begin{matrix} bs, bt, q^{-m} \\ d, abstq^k \end{matrix}; q, q \right) \\ &= \frac{(da/b; q)_m (cb/a; q)_n}{(c; q)_n (d; q)_m} \left( \frac{a}{b} \right)^n \left( \frac{b}{a} \right)^m \cdot \sum_{k=0}^m \frac{(q^{-m}, as, at; q)_k q^k}{(q, da/b, abst; q)_k} {}_3\phi_2 \left( \begin{matrix} bs, bt, q^{-n} \\ cb/a, abstq^k \end{matrix}; q, q \right). \end{aligned} \quad (4.7)$$

**Proof.** Using (1.2) gets

$$\begin{aligned} & \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty P_n(\omega, c/a; q) P_m(\omega, d/b; q)}{(a\omega, b\omega; q)_\infty} d_q\omega \\ &= \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty P_m(\omega, da/ba; q) P_n(\omega, cb/ab; q)}{(a\omega, b\omega; q)_\infty} d_q\omega \end{aligned}$$

$$\begin{aligned}
&= \frac{t(1-q)(da/b; q)_m (cb/a; q)_n (q, tq/s, s/t, abst; q)_\infty}{a^m b^n (as, at, bs, bt; q)_\infty} \\
&\quad \times \sum_{k=0}^m \frac{(q^{-m}, as, at; q)_k q^k}{(q, da/b, abst; q)_k} {}_3\phi_2 \left( \begin{matrix} bs, bt, q^{-n} \\ cb/a, abst q^k \end{matrix}; q, q \right). \quad (4.8)
\end{aligned}$$

Combining (1.2) and (4.8) yields (4.7).  $\square$

We want to point out that (4.7) have some important special cases. For example, we have

**Corollary 4.3** (The terminating Sears's  ${}_3\phi_2$  transformation formula).

$${}_3\phi_2 \left( \begin{matrix} a_1, a_2, q^{-m} \\ b_1, b_2 \end{matrix}; q, q \right) = \frac{(b_1 b_2 / a_1 a_2; q)_m}{(b_2; q)_m} \left( \frac{a_1 a_2}{b_1} \right)^m {}_3\phi_2 \left( \begin{matrix} b_1 / a_1, b_1 / a_2, q^{-m} \\ b_1, b_1 b_2 / a_1 a_2 \end{matrix}; q, q \right). \quad (4.9)$$

**Proof.** Setting  $n = 0$  in (4.7), we find that

$${}_3\phi_2 \left( \begin{matrix} bs, bt, q^{-m} \\ d, abst \end{matrix}; q, q \right) = \frac{(da/b; q)_m}{(d; q)_m} \left( \frac{b}{a} \right)^m {}_3\phi_2 \left( \begin{matrix} as, at, q^{-m} \\ da/b, abst \end{matrix}; q, q \right). \quad (4.10)$$

Replacing  $(bs, bt, abst, d)$  by  $(a_1, a_2, b_1, b_2)$ , we obtain (4.9).  $\square$

Identity (4.9) can be found in [6, p. 125, Eq. 2.18], which is used there to prove the Sears's  ${}_4\phi_3$  transformation formula [6, 7].

**Theorem 4.4.** For any integer  $i$ , such that  $0 \leq i \leq n$ , we have

$$\sum_{k=0}^i \frac{(q^{-i}, as, at; q)_k q^k}{(q, abst; q)_k} {}_3\phi_2 \left( \begin{matrix} bs, bt, q^{i-n} \\ 0, abst q^k \end{matrix}; q, q \right) = \frac{a^i b^{n-i}}{a^n} \cdot {}_3\phi_2 \left( \begin{matrix} as, at, q^{-n} \\ 0, abst \end{matrix}; q, q \right). \quad (4.11)$$

**Proof.** Let  $n = i$ ,  $m = n - i$  and  $c = d = 0$  in (1.2) to get

$$\begin{aligned}
\int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty \omega^i \omega^{n-i}}{(a\omega, b\omega; q)_\infty} d_q \omega &= \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty \omega^n}{(a\omega, b\omega; q)_\infty} d_q \omega = \frac{t(1-q)(q, tq/s, s/t, abst; q)_\infty}{a^i b^{n-i} (as, at, bs, bt; q)_\infty} \\
&\quad \times \sum_{k=0}^i \frac{(q^{-i}, as, at; q)_k q^k}{(q, abst; q)_k} {}_3\phi_2 \left( \begin{matrix} bs, bt, q^{i-n} \\ 0, abst q^k \end{matrix}; q, q \right). \quad (4.12)
\end{aligned}$$

On the other hand, by letting  $c = 0$  in (3.7), we have

$$\int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty \omega^n}{(a\omega, b\omega; q)_\infty} d_q \omega = \frac{t(1-q)(q, tq/s, s/t, abst; q)_\infty}{a^n (as, at, bs, bt; q)_\infty} {}_3\phi_2 \left( \begin{matrix} as, at, q^{-n} \\ 0, abst \end{matrix}; q, q \right). \quad (4.13)$$

Combining (4.12) and (4.13) gives (4.11).  $\square$

The following is the terminating  $q$ -binomial theorem [4]

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} x^k = (x; q)_n. \quad (4.14)$$

Using (1.2), we give an extension of it.

**Theorem 4.5.** For any integer  $n$ , we have

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} \left( -\frac{1}{a} \right)^k {}_3\phi_2 \left( \begin{matrix} as, at, q^{-k} \\ 0, ast \end{matrix}; q, q \right) = \frac{(s, t; q)_n}{(ast; q)_n}. \quad (4.15)$$

**Proof.** If we let  $x = \omega$  in (4.14) and multiply Eq. (4.14) by

$$\frac{(q\omega/s, q\omega/t; q)_\infty}{(a\omega, \omega; q)_\infty}$$

then we obtain

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} \frac{(q\omega/s, q\omega/t; q)_\infty \omega^k}{(a\omega, \omega; q)_\infty} = \frac{(q\omega/s, q\omega/t; q)_\infty (\omega; q)_n}{(a\omega, \omega; q)_\infty}. \quad (4.16)$$

Taking the  $q$ -integral on both sides of the above identity with respect to variable  $\omega$ , we have

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty \omega^k}{(a\omega, \omega; q)_\infty} d_q \omega = \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty (\omega; q)_n}{(a\omega, \omega; q)_\infty} d_q \omega. \quad (4.17)$$

By (3.7) with  $n = k$ ,  $b = 1$ , and  $c = 0$ , we have

$$\int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty \omega^k}{(a\omega, \omega; q)_\infty} d_q \omega = \frac{t(1-q)(q, tq/s, s/t, ast; q)_\infty}{a^k (as, at, s, t; q)_\infty} {}_3\phi_2 \left( \begin{matrix} as, at, q^{-k} \\ 0, ast \end{matrix}; q, q \right). \quad (4.18)$$

Note that

$$(\omega; q)_n = (-1)^n q^{\binom{n}{2}} P_n(\omega, 1/q^{n-1}; q), \quad (4.19)$$

$$(-1)^n q^{\binom{n}{2}} \frac{(a/q^{n-1}; q)_n}{a^n} = (1/a; q)_n. \quad (4.20)$$

Utilizing (4.19), (4.20), and (4.1) below in the order, we have

$$\begin{aligned} \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty (\omega; q)_n}{(a\omega, \omega; q)_\infty} d_q \omega &= (-1)^n q^{\binom{n}{2}} \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty P_n(\omega, 1/q^{n-1}; q)}{(a\omega, \omega; q)_\infty} d_q \omega \\ &= \frac{t(1-q)(1/a; q)_n (q, tq/s, s/t, ast; q)_\infty}{(as, at, s, t; q)_\infty} {}_3\phi_2 \left( \begin{matrix} as, at, q^{-n} \\ a/q^{n-1}, ast \end{matrix}; q, q \right) \\ &= \frac{t(1-q)(q, tq/s, s/t, ast; q)_\infty}{(as, at, s, t; q)_\infty} \cdot \frac{(s, t; q)_n}{(ast; q)_n}. \end{aligned} \quad (4.21)$$

Substituting (4.18) and (4.21) into (4.17) gives (4.15).  $\square$

It is obvious that, (4.15) leads to (4.14) when  $a = 1/s$ .

**Theorem 4.6.** We have

$$\begin{aligned} \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty P_n(\omega^2, c^2; q^2)}{(a\omega, b\omega; q)_\infty} d_q \omega &= \frac{t(1-q)(ac; q)_n (-bc; q)_n (q, tq/s, s/t, abst; q)_\infty}{a^n b^n (as, at, bs, bt; q)_\infty} \\ &\quad \times \sum_{k=0}^n \frac{(q^{-n}, as, at; q)_k q^k}{(q, ac, abst; q)_k} {}_3\phi_2 \left( \begin{matrix} bs, bt, q^{-n} \\ -bc, abst q^k \end{matrix}; q, q \right), \end{aligned} \quad (4.22)$$

provided that no zero factors in the denominator of the integrals.

**Proof.** Since,

$$\begin{aligned} P_n(\omega^2, c^2; q^2) &= (\omega^2 - c^2)(\omega^2 - c^2 q^2) \cdots (\omega^2 - c^2 q^{2n-2}) \\ &= (\omega - c)(\omega - cq) \cdots (\omega - cq^{n-1}) \cdot (\omega + c)(\omega + cq) \cdots (\omega + cq^{n-1}) \\ &= P_n(\omega, c; q) P_n(\omega, -c; q). \end{aligned} \quad (4.23)$$

Using (2.1), we have

$$\begin{aligned}
 & \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty P_n(\omega^2, c^2; q^2)}{(a\omega, b\omega; q)_\infty} d_q\omega \\
 &= \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty P_n(\omega, c; q) P_n(\omega, -c; q)}{(a\omega, b\omega; q)_\infty} d_q\omega \\
 &= \frac{t(1-q)(ac; q)_n(-bc; q)_n(q, tq/s, s/t, abst; q)_\infty}{a^n b^n (as, at, bs, bt; q)_\infty} \\
 &\quad \times \sum_{k=0}^n \frac{(q^{-n}, as, at; q)_k q^k}{(q, ac, abst; q)_k} {}_3\phi_2 \left( \begin{matrix} bs, bt, q^{-n} \\ -bc, abst q^k \end{matrix}; q, q \right),
 \end{aligned} \tag{4.24}$$

as desired.  $\square$

Carlitz discovered the following transformation formula [3]

$$\sum_{k=0}^n \frac{(a, b; q)_k}{(q; q)_k} (-ab)^{n-k} q^{(n-k)(n+k-1)/2} = (a; q)_{n+1} \sum_{k=0}^n \frac{(-b)^k q^{\binom{k}{2}}}{(q; q)_k (q; q)_{n-k} (1 - aq^{n-k})}. \tag{4.25}$$

Using (1.2), we give an extension of it.

**Theorem 4.7.** *We have*

$$\begin{aligned}
 & \sum_{k=0}^n \sum_{i=0}^k \frac{(a, 1/u; q)_k (q^{-k}, us, ut; q)_i}{(q; q)_k (q, u/q^{k-1}, uvst; q)_i} (-a/v)^{n-k} \cdot q^{i+(n-k)(n+k-1)/2} {}_3\phi_2 \left( \begin{matrix} vs, vt, q^{k-n} \\ 0, uvst q^i \end{matrix}; q, q \right) \\
 &= \frac{(a; q)_{n+1}}{(q; q)_n} \sum_{k=0}^n \left[ \begin{matrix} n \\ k \end{matrix} \right] \frac{(-1/u)^k q^{\binom{k}{2}}}{1 - aq^{n-k}} {}_3\phi_2 \left( \begin{matrix} us, ut, q^{-k} \\ 0, uvst \end{matrix}; q, q \right).
 \end{aligned} \tag{4.26}$$

**Proof.** If we let  $b = \omega$  in (4.25) and multiply Eq. (4.15) by

$$\frac{(q\omega/s, q\omega/t; q)_\infty}{(u\omega, v\omega; q)_\infty}$$

then we can obtain

$$\begin{aligned}
 & \sum_{k=0}^n \frac{(a; q)_k}{(q; q)_k} (-a)^{n-k} q^{(n-k)(n+k-1)/2} \cdot \frac{(q\omega/s, q\omega/t; q)_\infty (\omega; q)_k \omega^{n-k}}{(u\omega, v\omega; q)_\infty} \\
 &= \frac{(a; q)_{n+1}}{(q; q)_n} \sum_{k=0}^n \left[ \begin{matrix} n \\ k \end{matrix} \right] \frac{(-1)^k q^{\binom{k}{2}}}{1 - aq^{n-k}} \cdot \frac{(q\omega/s, q\omega/t; q)_\infty \omega^k}{(u\omega, v\omega; q)_\infty}.
 \end{aligned} \tag{4.27}$$

Taking the  $q$ -integral on both sides of the above identity with respect to variable  $\omega$ , we have

$$\begin{aligned}
 & \sum_{k=0}^n \frac{(a; q)_k}{(q; q)_k} (-a)^{n-k} q^{(n-k)(n+k-1)/2} \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty (\omega; q)_k \omega^{n-k}}{(u\omega, v\omega; q)_\infty} d_q\omega \\
 &= \frac{(a; q)_{n+1}}{(q; q)_n} \sum_{k=0}^n \left[ \begin{matrix} n \\ k \end{matrix} \right] \frac{(-1)^k q^{\binom{k}{2}}}{1 - aq^{n-k}} \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty \omega^k}{(u\omega, v\omega; q)_\infty} d_q\omega.
 \end{aligned} \tag{4.28}$$

Using (3.7) gets

$$\int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty \omega^k}{(u\omega, v\omega; q)_\infty} d_q \omega = \frac{t(1-q)(q, tq/s, s/t, uvst; q)_\infty}{u^k(us, ut, vs, vt; q)_\infty} {}_3\phi_2 \left( \begin{matrix} us, ut, q^{-k} \\ 0, uvst \end{matrix}; q, q \right). \quad (4.29)$$

Employing the following relations

$$(\omega; q)_k = (-1)^k q^{\binom{k}{2}} P_k(\omega, 1/q^{k-1}; q)$$

and

$$\omega^{n-k} = P_{n-k}(\omega, 0; q),$$

we have

$$\begin{aligned} \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty (\omega; q)_k \omega^{n-k}}{(u\omega, v\omega; q)_\infty} d_q \omega &= (-1)^k q^{\binom{k}{2}} \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty P_k(\omega, 1/q^{k-1}; q) P_{n-k}(\omega, 0; q)}{(u\omega, v\omega; q)_\infty} d_q \omega \\ &= (-1)^k q^{\binom{k}{2}} \cdot \frac{t(1-q)(u/q^{k-1}; q)_k (q, tq/s, s/t, uvst; q)_\infty}{u^k v^{n-k} (us, ut, vs, vt; q)_\infty} \\ &\quad \times \sum_{i=0}^k \frac{(q^{-k}, us, ut; q)_i q^i}{(q, u/q^{k-1}, uvst; q)_i} {}_3\phi_2 \left( \begin{matrix} vs, vt, q^{k-n} \\ 0, uvst q^i \end{matrix}; q, q \right). \end{aligned} \quad (4.30)$$

Using the following relation

$$(-1)^k q^{\binom{k}{2}} \cdot \frac{(u/q^{k-1}; q)_k}{u^k} = (1/u; q)_k,$$

(4.30) can be written as

$$\begin{aligned} \int_s^t \frac{(q\omega/s, q\omega/t; q)_\infty (\omega; q)_k \omega^{n-k}}{(u\omega, v\omega; q)_\infty} d_q \omega &= \frac{t(1-q)(1/u; q)_k (q, tq/s, s/t, uvst; q)_\infty}{v^{n-k} (us, ut, vs, vt; q)_\infty} \\ &\quad \times \sum_{i=0}^k \frac{(q^{-k}, us, ut; q)_i q^i}{(q, u/q^{k-1}, uvst; q)_i} {}_3\phi_2 \left( \begin{matrix} vs, vt, q^{k-n} \\ 0, uvst q^i \end{matrix}; q, q \right). \end{aligned} \quad (4.31)$$

Substituting (4.29) and (4.31) into (4.28) gives (4.26).  $\square$

It is obvious that, (4.26) leads to (4.25) when  $u = v = 1/s$ .

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