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Characterizations of the harmonic Bergman space on the ball ☆

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ABSTRACT

In the harmonic Bergman space with the normal weight, we prove norm equivalences in terms of radial, gradient and invariant gradient norms. Using this, we give new characterizations in terms of Lipschitz type conditions with Euclidean, pseudo-hyperbolic and hyperbolic metrics on the ball.

1. Introduction

For a fixed integer $n \geqslant 2$, let $B = B_n$ denote the open unit ball in $\mathbf{R}^n$. Given $\alpha > -1$ real and $1 \leqslant p < \infty$, we let $L^p_\alpha = L^p(V_\alpha)$ denote the weighted Lebesgue spaces on B where dV_α denotes the weighted measure defined by

$$dV_\alpha(x) = \left(1 - |x|^2\right)^\alpha dV(x).$$

Here V denotes the Lebesgue volume measure on B. For $\alpha = 0$, we have $L^p = L^p_0 = L^p(V)$. For simplicity, we use the notation $dy = dV(y)$, etc.

For $1 \leqslant p < \infty$ and $\alpha > -1$ real, we let

$$b^p_\alpha = L^p_\alpha \cap h(B)$$

denote the weighted harmonic Bergman space, where $h(B)$ is the space of all harmonic functions in B. In case $\alpha = 0$, we have $b^p = b^p_0$. As is well known, b^2 is a closed subspace of L^2 and hence is a Hilbert space. Since each point evaluation in B is a bounded linear functional on the Hilbert space b^2_α, for each $x \in B$ there exists a unique function $R^\alpha_x = R^\alpha(x, \cdot)$ in b^2_α such that it satisfies the following reproducing formula:

$$f(x) = \int_B f(y) R^\alpha_x(y)\, dV_\alpha(y), \quad f \in b^2_\alpha.$$

The function $R^\alpha_x(y)$, $x, y \in B$ is called the reproducing kernel of b^2_α. One can write $R^\alpha_x(y)$ for a series involving extended zornal harmonics; see [6]. Note that the kernel $R^\alpha_x(y)$ is real and symmetric. We also have the following estimate of these kernels:

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$$\begin{aligned} &\left|R_x^\alpha(x)\right| \approx \left(1-|x|^2\right)^{-(n+\alpha)} \quad \text{and} \quad \left|R_x^\alpha(y)\right| \lesssim [x,y]^{-(n+\alpha)}, \\ &\left|\nabla_y R_x^\alpha(y)\right| \lesssim [x,y]^{-(n+\alpha+1)} \end{aligned} \tag{1.1}$$

for all $x, y \in B$. From now on, ∇_y denotes the gradient with respect to y-variable and

$$[x,y] = \sqrt{1 - 2x\cdot y + |x|^2|y|^2}.$$

Since the function $R^\alpha(x,y)$ is bounded in y whenever x is fixed, we can consider the following integral operator

$$Q_\alpha(f)(x) = \int_B f(y) R^\alpha(x,y)\, dV_\alpha(y), \quad f \in L^1_\alpha.$$

It is easy to show that the operator Q_α maps L^2_α boundedly onto the harmonic Bergman space b^2_α.

To state our main result, we write ρ and β for the pseudo-hyperbolic and hyperbolic metrics on B, respectively; see Section 2 for the definitions. The following theorem is our main result in this paper.

Theorem 1.1. *Let $1 \leqslant p < \infty$ and $\alpha > -1$. Then the following conditions are equivalent*:

(a) $f \in b^p_\alpha$.
(b) *There exists a continuous function $g \in L^p_\alpha$ such that*

$$\left|f(x) - f(y)\right| \leqslant \rho(x,y)\left[g(x) + g(y)\right]$$

for all $x, y \in B$.
(c) *There exists a continuous function $g \in L^p_\alpha$ such that*

$$\left|f(x) - f(y)\right| \leqslant \beta(x,y)\left[g(x) + g(y)\right]$$

for all $x, y \in B$.
(d) *There exists a continuous function $g \in L^p_{\alpha+p}$ such that*

$$\left|f(x) - f(y)\right| \leqslant |x-y|\left[g(x) + g(y)\right]$$

for all $x, y \in B$.

Analogous result for Bergman spaces of holomorphic functions are proved in [7].

In case $p = \infty$ and $\alpha = 0$, we can replace the harmonic Bergman space b^p_α by the harmonic Bloch space.

The harmonic Bloch space $\mathcal{B}$ is the space of harmonic functions f on B such that the function $(1-|x|^2)|\nabla f(x)|$ is bounded on B. Note that it is a Banach space equipped with norm

$$\|f\|_{\mathcal{B}} = \left|f(0)\right| + \sup_{x\in B}\left(1-|x|^2\right)\left|\nabla f(x)\right|.$$

The harmonic little Bloch space $\mathcal{B}_0$ is the space of harmonic functions $f \in \mathcal{B}$ for which $(1-|x|^2)|\nabla f(x)|$ is vanishing on ∂B. For each $\alpha > -1$, one can see that $Q_\alpha : L^\infty \to \mathcal{B}$ is bounded by Lemmas 4 and 5 of [5].

As a companion result for the harmonic Bloch space, we have the following. Here, we denote $C_0 = C_0(B)$.

Theorem 1.2. *The following conditions are equivalent*:

(a) $f \in \mathcal{B}$ ($\mathcal{B}_0$ *resp.*).
(b) *There exists a continuous function $g \in L^\infty$ (C_0 resp.) such that*

$$\left|f(x) - f(y)\right| \leqslant \rho(x,y)\left[g(x) + g(y)\right]$$

for all $x, y \in B$.
(c) *There exists a continuous function $g \in L^\infty$ (C_0 resp.) such that*

$$\left|f(x) - f(y)\right| \leqslant \beta(x,y)\left[g(x) + g(y)\right]$$

for all $x, y \in B$.
(d) *There exists a continuous function $(1-|x|^2)g(x)$ in L^∞ (C_0 resp.) such that*

$$\left|f(x) - f(y)\right| \leqslant |x-y|\left[g(x) + g(y)\right]$$

for all $x, y \in B$.

In Section 2 we review Möbius transformations and some metrics on B. Also some properties of them are stated and proved. In Section 3 we prove norm equivalences in terms of radial, gradient and invariant gradient norms. In the last section, our main theorems are proved.

2. Preliminaries

We first recall Möbius transformations on B. All relevant details can be found in [1, pp. 17–30] or [3]. Let $a \in B$. The canonical Möbius transformation ϕ_a that exchanges a and 0 is given by

$$\phi_a(x) = a + \left(1 - |a|^2\right)(a - x^*)^*$$

for $x \in B$ (note $\phi_a = -T_a$ in the notation of [1]). Here $x^* = x/|x|^2$ denotes the inversion of x relative to the sphere ∂B. Avoiding x^* notation, we have

$$\phi_a(x) = \frac{(1 - |a|^2)(a - x) + |a - x|^2 a}{[x, a]^2}.$$

Here, as elsewhere, we write $x \cdot y$ for the dot product of $x, y \in \mathbf{R}^n$. The map ϕ_a is an involution of B, i.e., $\phi_a^{-1} = \phi_a$. The following results are well-known identities:

$$\left|\phi_a(x)\right| = \frac{|x - a|}{[x, a]},$$

$$1 - \left|\phi_a(x)\right|^2 = \frac{(1 - |x|^2)(1 - |a|^2)}{[x, a]^2}, \tag{2.1}$$

$$\left[\phi_a(x), a\right] = \frac{1 - |a|^2}{[x, a]}, \tag{2.2}$$

$$J\phi_a(x) = \left(\frac{1 - |a|^2}{[x, a]^2}\right)^n, \tag{2.3}$$

where $J\phi_a$ denotes the Jacobian of ϕ_a.

We now recall the pseudo-hyperbolic and hyperbolic metrics on B. Let ρ be the pseudo-hyperbolic metric on B defined by

$$\rho(x, y) = \left|\phi_y(x)\right| = \frac{|x - y|}{[x, y]},$$

where

$$\phi_x(y) = \frac{(1 - |x|^2)(x - y) + |x - y|^2 x}{[y, x]^2}$$

for $x, y \in B$. Also, we write β for the hyperbolic metric on B defined by

$$\beta(x, y) = \frac{1}{2} \log \frac{1 + \rho(x, y)}{1 - \rho(x, y)}$$

for $x, y \in B$.

For $a \in B$ and $r \in (0, 1)$, let $E_r(a)$ denote the pseudo-hyperbolic ball with radius r and center a. A straightforward calculation shows that $E_r(a)$ is a Euclidean ball with

$$(\text{center}) = \frac{1 - r^2}{1 - r^2|a|^2} a \quad \text{and} \quad (\text{radius}) = \frac{1 - |a|^2}{1 - r^2|a|^2} r. \tag{2.4}$$

This leads to the next lemma.

Lemma 2.1. *The inequalities*

$$\frac{1 - \rho(x, y)}{1 + \rho(x, y)} \leqslant \frac{1 - |x|^2}{1 - |y|^2} \leqslant \frac{1 + \rho(x, y)}{1 - \rho(x, y)}$$

and

$$\frac{1 - \rho(x, y)}{1 + \rho(x, y)} \leqslant \frac{[x, a]}{[y, a]} \leqslant \frac{1 + \rho(x, y)}{1 - \rho(x, y)}$$

hold for $x, y, a \in B$.

Proof. See Lemmas 2.1 and 2.2 of [3]. □

Note that (2.4) implies that

$$\left|E_r(a)\right| = |B|\left(\frac{1-|a|^2}{1-r^2|a|^2}r\right)^n. \tag{2.5}$$

Here, $|E| = V(E)$ for Borel sets $E \subset B$. Consequently, we obtain by Lemma 2.1 an estimate on size of pseudo-hyperbolic balls: Given $\delta, t \in (0,1)$, there is a positive constant $C = C(\delta, t, n)$ such that

$$C^{-1} \leqslant \frac{|E_{r_1}(x)|}{|E_{r_2}(y)|} \leqslant C, \quad y \in E_{r_3}(x),$$

whenever $r_1, r_2, r_3 < \delta$ and $t < r_1/r_2 < t^{-1}$.

The following relations can be proved by a simple calculation.

Lemma 2.2. *Let x, $y \in B$ and $y = tx$ where t is a scalar. Then we have*

$$\lim_{y\to x}\frac{\rho(y,x)}{|x-y|} = \lim_{y\to x}\frac{\beta(x,y)}{|x-y|} = \frac{1}{1-|x|^2}.$$

Lemma 2.3. *Let $1 \leqslant p < \infty$ and $r \in (0,1)$. Then there exists a positive constant $C = C(r)$ such that*

$$\left|f(x)\right|^p \leqslant \frac{C}{(1-|x|^2)^n}\int_{E_r(x)}\left|f(y)\right|^p\,dy$$

for $x \in B$ and $f \in h(B)$.

Proof. Note that we see by (2.4) that $E_r(0)$ is a Euclidean ball with center 0 and radius r. Using Jensen's inequality and (2.5), we obtain for $f \in h(B)$

$$\left|f(0)\right|^p \leqslant \frac{1}{|E_r(0)|}\int_{E_r(0)}\left|f(y)\right|^p\,dy = \frac{1}{r^n|B|}\int_{E_r(0)}\left|f(y)\right|^p\,dy.$$

Since $[x,y] \geqslant |x-y| \geqslant 1-|x|$, we have after replacing f by $f\circ\phi_x$ and making a change of variable,

$$\left|f(0)\right|^p \leqslant \frac{1}{r^n|B|}\int_{E_r(x)}\left|f(y)\right|^p\left(\frac{1-|x|^2}{[y,x]^2}\right)^n dy \leqslant \frac{C}{(1-|x|^2)^n}\int_{E_r(x)}\left|f(y)\right|^p\,dy$$

for $x \in B$. We have the desired result. □

3. Norm equivalences

The following is change of variable formula for weighted Lebesgue spaces.

Proposition 3.1. *Suppose $\alpha > -1$ is real and f is in L^1_α. Then*

$$\int_B f\circ\phi\,dV_\alpha = \int_B f(x)\frac{(1-|a|^2)^{n+\alpha}}{[x,a]^{2(n+\alpha)}}\,dV_\alpha(x),$$

where ϕ is any Möbius transformation of B and $a = \phi(0)$.

Proof. Note that

$$\int_B f(Ux)\,dV_\alpha(x) = \int_B f(x)\,dV_\alpha(x) \tag{3.1}$$

for any orthogonal transformation U on B.

Let $a = \phi(0)$. Then there exists a orthogonal transformation U such that $\phi = \phi_a U$. From (3.1), we may assume that $\phi = \phi_a$. Eqs. (2.1) and (2.3) imply that

$$\begin{aligned}\int_B f\circ\phi(x)\,dV_\alpha(x) &= \int_B f\circ\phi(x)\big(1-|x|^2\big)^\alpha\,dx\\ &= \int_B f(x)\big(1-\big|\phi_a(x)\big|^2\big)^\alpha J\phi_a(x)\,dx\\ &= \int_B f(x)\frac{(1-|a|^2)^{n+\alpha}}{[x,a]^{2(n+\alpha)}}\,dV_\alpha(x),\end{aligned}$$

which completes the proof. □

For $\alpha > -1$ and c real, we define $I_{\alpha,c}$ as follows:

$$I_{\alpha,c}(x) = \int_B \frac{(1-|y|^2)^\alpha}{[x,y]^{n+\alpha+c}}\,dy, \quad x\in B.$$

The following estimate for $I_{\alpha,c}$ is taken from Lemma 2.5 of [3].

Lemma 3.2. *Let $\alpha > -1$ and c real. Then*

$$I_{\alpha,c}(x) \approx \begin{cases} 1 & \text{if } c<0,\\ 1-\log(1-|x|^2) & \text{if } c=0,\\ (1-|x|^2)^{-c} & \text{if } c>0\end{cases}$$

for $x\in B$. The constants suppressed above are independent of x.

We use the notation $\mathcal{D}$ for the radial differentiation defined by

$$\mathcal{D}f(x) = \sum_{j=1}^n x_j D_j f(x), \quad x\in B$$

for functions $f\in C^1(B)$. Here, D_j denotes the partial differentiation with respect to the jth component. It is easy to see that $\mathcal{D}f$ is harmonic if f is.

For f, a twice differentiable function in B, we define the invariant Laplacian $\widetilde{\Delta} f$ by

$$(\widetilde{\Delta} f)(x) = \Delta(f\circ\phi_x)(0), \quad x\in B,$$

where ϕ_x is a Möbius transformation that exchanges a and 0. The following result explains why the operator $\widetilde{\Delta}$ is called invariant Laplacian.

Proposition 3.3. *Let f be twice differentiable in B. Then $\widetilde{\Delta}(f\circ\phi) = (\widetilde{\Delta} f)\circ\phi$ for all Möbius transformations ϕ.*

Proof. For fixed $a\in B$, let $a=\phi(x)$. Then we have

$$\phi\circ\phi_x = \phi_a\circ U$$

for some orthogonal transformation U. Note that the Laplacian commutes with orthogonal transformations, namely,

$$\Delta(f\circ U) = (\Delta f)\circ U$$

for $f\in C^2(B)$; see [2] for details.

Thus we have

$$\begin{aligned}\widetilde{\Delta}(f\circ\phi)(x) &= \Delta(f\circ\phi\circ\phi_x)(0) = \Delta(f\circ\phi_a\circ U)(0)\\ &= \Delta(f\circ\phi_a)(0) = \widetilde{\Delta} f(a)\\ &= (\widetilde{\Delta} f)\circ\phi(x).\end{aligned}$$

The proof is complete. □

From a simple computation, the invariant Laplacian can be described in terms of Laplacian and radial derivative as follows:

Proposition 3.4. *Let f be a twice differentiable in B. Then*

$$\widetilde{\Delta} f(x) = \left(1-|x|^2\right)^2 (\Delta f)(x) + 2(n-2)\left(1-|x|^2\right)\mathcal{D} f(x)$$

for $x \in B$.

Proof. Recall that

$$\phi_x(y) = \frac{(1-|x|^2)(x-y) + |x-y|^2 x}{[y,x]^2}.$$

For fixed $x \in B$, we write $\phi_x(y) = (\phi_1(y), \ldots, \phi_n(y))$ where

$$\phi_i(y) = \frac{(1-|x|^2)(x_i-y_i) + |x-y|^2 x_i}{[y,x]^2}.$$

Direct calculation implies that $D_j\phi_i(0) = (|x|^2-1)\delta_{ij}$ and

$$D_j^2\phi_i(0) = \begin{cases} 2x_i(|x|^2-1) & \text{if } i=j, \\ 2x_i(1-|x|^2) & \text{if } i \neq j. \end{cases}$$

Here and elsewhere δ_{ij} is a Kronecker delta. Thus we have by the chain rule

$$\begin{aligned}
\Delta(f\circ\phi_x)(0) &= \left(1-|x|^2\right)^2 \sum_{i,j=1}^n \left[(D_jD_i f)(x)\delta_{ij}\sum_{k=1}^n \delta_{jk}\right] + \sum_{i=1}^n (D_i f)(x)\sum_{j=1}^n \left(D_j^2\phi_i\right)(0) \\
&= \left(1-|x|^2\right)^2 \sum_{i=1}^n \left(D_i^2 f\right)(x) + \sum_{i=1}^n (D_i f)(x)\left(D_i^2\phi_i\right)(0) + \sum_{i=1}^n (D_i f)(x)\sum_{i\neq j}\left(D_j^2\phi_i\right)(0) \\
&= \left(1-|x|^2\right)^2(\Delta f)(x) + 2\left(|x|^2-1\right)\sum_{i=1}^n x_i(D_i f)(x) + 2(n-1)\left(1-|x|^2\right)\sum_{i=1}^n x_i(D_i f)(x) \\
&= \left(1-|x|^2\right)^2(\Delta f)(x) + 2(n-2)\left(1-|x|^2\right)\mathcal{D} f(x).
\end{aligned}$$

The proof is complete. $\square$

Given f harmonic on B, we define $\widetilde{\nabla} f(x)$ of f at x by

$$\widetilde{\nabla} f(x) = \nabla(f\circ\phi_x)(0)$$

for $x \in B$. We call $|\widetilde{\nabla} f|$ the invariant gradient of f at x by the following proposition.

Proposition 3.5. *Let $f \in h(B)$. Then $|\widetilde{\nabla} f|$ is Möbius invariant, namely,*

$$\left|\widetilde{\nabla}(f\circ\phi)(x)\right| = \left|(\widetilde{\nabla} f)\circ\phi(x)\right|$$

for any Möbius transformation ϕ.

Proof. Note that

$$\begin{aligned}
\Delta\left(|f|^2\right)(x) &= \sum_{j=1}^n D_j^2\left(|f|^2\right)(x) = 2\sum_{j=1}^n \left\{\left[D_j f(x)\right]^2 + f(x)\left(D_j^2 f\right)(x)\right\} \\
&= 2\left\{\left|\nabla f(x)\right|^2 + 2f(x)\Delta f(x)\right\} \qquad (3.2)
\end{aligned}$$

for a twice differentiable function f in B.

Let $f \in h(B)$. Then we have by (3.2)

$$\widetilde{\Delta}\left(|f|^2\right)(0) = \Delta\left(|f|^2\right)(0) = 2\left|\nabla f(0)\right|^2 = 2\left|\widetilde{\nabla} f(0)\right|^2.$$

From this, we have

$$2\left|\widetilde{\nabla} f(x)\right|^2 = 2\left|\nabla(f\circ\phi_x)(0)\right|^2 = \widetilde{\Delta}\left(|f\circ\phi_x|^2\right)(0) = \widetilde{\Delta}\left(|f|^2\right)(x). \qquad (3.3)$$

Combining (3.3) with Proposition 3.3, we obtain the desired result. $\square$

Lemma 3.6. *For $f \in h(B)$,*

$$\left(1-|x|^2\right)\left|\mathcal{D}f(x)\right| \leqslant \left(1-|x|^2\right)\left|\nabla f(x)\right| = \left|\widetilde{\nabla} f(x)\right|$$

holds for all $x \in B$.

Proof. By the Cauchy–Schwarz inequality for $\mathbf{R}^n$, we have the first inequality. To prove the second inequality, we fix $x \in B$ and write

$$\phi_x(y) = \left(\phi_1(y), \dots, \phi_n(y)\right),$$

where

$$\phi_i(y) = \frac{(1-|x|^2)(x_i - y_i) + |x-y|^2 x_i}{[y,x]^2}.$$

Note that

$$(D_j\phi_i)(0) = \left(|x|^2 - 1\right)\delta_{ij},$$

where δ_{ij} is a Kronecker delta. Also, the chain rule implies

$$D_j(f \circ \phi_x)(y) = \sum_{i=1}^{n} (D_i f)\left(\phi_x(y)\right)(D_j\phi_i)(y).$$

From these facts, we have

$$\begin{aligned}\left|\widetilde{\nabla} f(x)\right|^2 &= \left|\nabla(f\circ\phi_x)(0)\right|^2 = \sum_{j=1}^{n}\left|\sum_{i=1}^{n}(D_i f)(x)(D_j\phi_i)(0)\right|^2 = \sum_{j=1}^{n}\left|\sum_{i=1}^{n}(D_i f)(x)\left(|x|^2-1\right)\delta_{ij}\right|^2 \\ &= \left(1-|x|^2\right)^2\sum_{j=1}^{n}\left|D_j f(x)\right|^2 = \left(1-|x|^2\right)^2\left|\nabla f(x)\right|^2.\end{aligned}$$

Thus, we have the desired result. □

Theorem 3.7. *Let $1 \leqslant p < \infty$ and $\alpha > -1$. Then the following conditions are equivalent*:

(a) $f \in b^p_\alpha$.
(b) $(1-|x|^2)\mathcal{D}f(x) \in L^p_\alpha$.
(c) $(1-|x|^2)|\nabla f(x)| \in L^p_\alpha$.
(d) $|\widetilde{\nabla} f(x)| \in L^p_\alpha$.

Remark. In case $\alpha = 0$, one can obtain the equivalences of (a)–(c) by some application of Theorem 1.3 of [4].

Proof. Lemma 3.6 shows that (d) is equivalent to (c), and (c) implies (b).

(b) ⇒ (a): Assume (b). Taking $m = 1$ and $r = \alpha > -1$ in the proof of the part $\|f\| \lesssim \|f\|_{p,m,1}$ in Theorem 5.1 of [4], we get

$$\int_B \left|f(x)\right|^p\left(1-|x|^2\right)^\alpha dx \lesssim \int_B \left|\mathcal{D}f(x)\right|^p\left(1-|x|^2\right)^{p+\alpha} dx + \left|f(0)\right|^p$$

so that

$$\int_B \left|f(x)\right|^p dV_\alpha(x) \lesssim \int_B \left|\left(1-|x|^2\right)\mathcal{D}f(x)\right|^p dV_\alpha(x).$$

From this we have the desired the desired result.

(a) ⇒ (d): Assume (a) and fix $\beta > \alpha$ and observe that there exists a constant $C_1 > 0$ such that

$$\left|\nabla g(0)\right|^p = \left(\sum_i \left|D_i g(0)\right|^2\right)^{p/2} \leqslant C_1 \int_B \left|g(y)\right|^p dV_\beta(y)$$

for all harmonic g in B; See Corollary 8.2 in [2]. Let $g = f \circ \phi_x$ where $x \in B$. Making a change of variables according to Proposition 3.1, we obtain

$$\begin{aligned}\left|\widetilde{\nabla} f(x)\right|^p &= \left|\nabla(f\circ\phi_x)(0)\right|^p = \left|g(0)\right|^p \leqslant C_1 \int_B \left|g(y)\right|^p \, dV_\beta(y)\\ &= C_1 \int_B \left|f\circ\phi_x(y)\right|^p \, dV_\beta(y) = C_1\left(1-|x|^2\right)^{n+\beta} \int_B \frac{|f(y)|^p \, dV_\beta(y)}{[x,y]^{2(n+\beta)}}.\end{aligned}$$

An application of Fubini's theorem and Lemma 3.2 then gives

$$\begin{aligned}\int_B \left|\widetilde{\nabla} f(x)\right|^p dV_\alpha(x) &\leqslant C_1 \int_B \left(1-|x|^2\right)^{n+\beta} \int_B \frac{|f(y)|^p}{[x,y]^{2(n+\beta)}} \, dV_\beta(y)\, dV_\alpha(x)\\ &= C_1 \int_B \left|f(y)\right|^p dV_\beta(y) \int_B \frac{(1-|x|^2)^{n+\beta+\alpha}}{[x,y]^{2(n+\beta)}} \, dV(x)\\ &\leqslant C_2 \int_B \frac{|f(y)|^p}{(1-|y|^2)^{\beta-\alpha}} \, dV_\beta(y)\\ &= C_2 \int_B \left|f(y)\right|^p dV_\alpha(y)\end{aligned}$$

for some constant $C_2 > 0$ and all f harmonic in B. Actually, replacing f by $f - f(0)$, we have

$$\int_B \left|\widetilde{\nabla} f(x)\right|^p dV_\alpha(x) \leqslant C_2 \int_B \left|f(x) - f(0)\right|^p dV_\alpha(x).$$

This completes the proof. □

For harmonic Bloch space, we have the following result.

Theorem 3.8. *Let $\alpha > -1$. Then the following conditions are equivalent*:

(a) $f \in \mathcal{B}$ ($\mathcal{B}_0$ *resp.*).
(b) $|\widetilde{\nabla} f(x)| \in L^\infty$ (C_0 *resp.*).
(c) $(1-|x|^2)\mathcal{D}f(x) \in L^\infty$ (C_0 *resp.*).
(d) $f = Q_\alpha g$ *for some* $g \in L^\infty$ (C_0 *resp.*).

Proof. Lemma 3.6 and Theorem 1.4 of [4] imply that (a) ⇔ (b) ⇒ (c). Thus we just show that (c) ⇒ (d) and (d) ⇒ (b).

(c) ⇒ (d): Assume (c). This gives $f \in \mathcal{B}$ by Theorem 1.4 of [4]. We let Sf denote the function defined by

$$Sf(x) = \left(1-|x|^2\right)\left[\mathcal{D}f(x) + (n/2+1)f(x)\right]$$

for $x \in B$. Then we have $S : \mathcal{B} \to L^\infty$ is bounded and $S\mathcal{B}_0 \subset C_0$. We also have $Q_\alpha S$ is the identity on $\mathcal{B}$ from Lemma 10 of [5]. Taking $g = Sf$, we have $g \in L^\infty$ and easily check that $f = Q_\alpha(Sf)$. Thus we have the desired result.

(d) ⇒ (b): Assume that $f = Q_\alpha g$ for some $g \in L^\infty$.

Fix $x \in B$. Making the change of variables with (2.1) and (2.3) implies that

$$\begin{aligned}f\circ\phi_x(z) &= \int_B g(y) R^\alpha\left(\phi_x(z), y\right) dV_\alpha(y)\\ &= \int_B g\left(\phi_x(y)\right) R^\alpha\left(\phi_x(z), \phi_x(y)\right)\left(1-\left|\phi_x(y)\right|^2\right)^\alpha J\phi_x(y)\, dy\\ &= \int_B g\left(\phi_x(y)\right) R^\alpha\left(\phi_x(z), \phi_x(y)\right)\left(\frac{(1-|y|^2)(1-|x|^2)}{[y,x]^2}\right)^\alpha \left(\frac{1-|x|^2}{[y,x]^2}\right)^n dy\\ &= \int_B g\left(\phi_x(y)\right) R^\alpha\left(\phi_x(z), \phi_x(y)\right)\frac{(1-|y|^2)^\alpha(1-|x|^2)^{n+\alpha}}{[y,x]^{2(n+\alpha)}}\, dy.\end{aligned}$$

Using Lemma 3.6 with (1.1) and (2.2), we get

$$\begin{aligned}\left|\widetilde{\nabla} f(x)\right| &\leqslant \int_B \left|g\big(\phi_x(y)\big)\right|\left|\nabla R^\alpha\big(\phi_x(0),\phi_x(y)\big)\right|\frac{(1-|y|^2)^\alpha(1-|x|^2)^{n+\alpha+1}}{[y,x]^{2(n+\alpha)}}\,dy\\ &\lesssim \|g\|_\infty \int_B \frac{1}{[x,\phi_x(y)]^{n+\alpha+1}}\frac{(1-|y|^2)^\alpha(1-|x|^2)^{n+\alpha+1}}{[y,x]^{2(n+\alpha)}}\,dy\\ &= \|g\|_\infty \int_B \frac{(1-|y|^2)^\alpha}{[y,x]^{n+\alpha-1}}\,dy.\end{aligned}$$

From Lemma 3.2, we have (b).

Similarly, the left part can be shown, and so its proof is omitted. □

4. Proof of the main theorems

Finally, we prove our first main theorem.

Proof of Theorem 1.1. (b) ⇒ (a): Assume (b) and fix $x \in B$. For a scalar t, write $y = tx$. Then we have

$$\frac{|f(x)-f(y)|}{|x-y|} \leqslant \frac{\rho(x,y)}{|x-y|}\big[g(x)+g(y)\big]$$

for all $x, y \in B$ with $x \neq y$. Let y approach x in the radial direction. Applying Lemma 2.2, we have

$$\big(1-|x|^2\big)\left|\mathcal{D}f(x)\right| \leqslant 2g(x)$$

for all $x \in B$. According to Theorem 3.7, we have desired result.

(a) ⇒ (b): For any $f \in b^p_\alpha$ and for any $x \in B$, we have

$$f(x)-f(0) = \int_0^1 x\cdot\nabla f(tx)\,dt.$$

For a fixed $r \in (0,1)$, let $\rho(x,0) < r$. Then we have

$$\left|f(x)-f(0)\right| \leqslant |x|\sup\big\{\left|\nabla f(y)\right|:\ y \in E_r(0)\big\}.$$

Note that $|\nabla f(y)| \approx |\widetilde{\nabla} f(y)|$ for $y \in E_r(0)$. So, there exists a positive constant $C_1 = C_1(r)$ such that

$$\left|f(x)-f(0)\right| \leqslant C_1\rho(x,0)\sup\big\{\left|\widetilde{\nabla} f(y)\right|:\ y \in E_r(0)\big\}$$

for all $x \in E_r(0)$.

Replace f by $f\circ\phi_y$ then replace x by $\phi_y(x)$ and use the Möbius invariance of the pseudo-hyperbolic metric and Proposition 3.5. Then we obtain

$$\left|f(x)-f(y)\right| \leqslant C_1\rho(x,y)\sup\big\{\left|\widetilde{\nabla} f(z)\right|:\ z \in E_r(x)\big\}$$

for all $x, y \in B$ with $\rho(x,y) < r$. Let

$$g(x) = \frac{|f(x)|}{r} + C_1\sup\big\{\left|\widetilde{\nabla} f(z)\right|:\ z \in E_r(x)\big\}.$$

Thus g is continuous on B and

$$\left|f(x)-f(y)\right| \leqslant \rho(x,y)\big[g(x)+g(y)\big]$$

for all $x, y \in B$.

Now, it remains to show that $g \in L^p_\alpha$. Let

$$g(x) = \frac{|f(x)|}{r} + h(x),$$

where $h(x) = C_1\sup\{|\widetilde{\nabla} f(z)|:\ z \in E_r(x)\}$. Since $f \in L^p_\alpha$, it suffice to show that $h \in L^p_\alpha$. By Lemma 2.3, there exists a positive constant $C_2 = C_2(r)$ such that

$$h(x)^p \leqslant \frac{C_2}{(1-|x|^2)^n}\int_{E_r(x)} h(y)^p\,dy \lesssim \frac{1}{(1-|x|^2)^{n-p}}\int_{E_r(x)} \left|\nabla f(y)\right|^p dy$$

for all $x \in B$. Let χ_x denote the characteristic function of the set $E_r(x)$, then we have

$$h(x)^p \lesssim \frac{1}{(1-|x|^2)^{n-p}} \int_B \left|\nabla f(y)\right|^p \chi_x(y)\,dy$$

for all $x \in B$. Using Fubini's theorem, we obtain

$$\begin{aligned}\int_B h(x)^p\,dV_\alpha(x) &\lesssim \int_B \left(1-|x|^2\right)^{p+\alpha-n} \int_B \left|\nabla f(y)\right|^p \chi_x(y)\,dy\,dx \\ &= \int_B \left|\nabla f(y)\right|^p \int_B \left(1-|x|^2\right)^{p+\alpha-n} \chi_y(x)\,dx\,dy \\ &= \int_B \left|\nabla f(y)\right|^p \int_{E_r(y)} \left(1-|x|^2\right)^{p+\alpha-n}\,dx\,dy.\end{aligned}$$

Combining this with Lemma 2.1, we have

$$\int_B h(x)^p\,dV_\alpha(x) \lesssim \int_B \left|\nabla f(y)\right|^p \left(1-|y|^2\right)^{p+\alpha}\,dy.$$

This implies that $h \in L^p_\alpha$ by Lemma 3.7. Thus, we have (b).

(b) $\Rightarrow$ (c): Since $\rho \leqslant \beta$, if the condition (b) holds then the condition (c) holds for the same function g.

(c) $\Rightarrow$ (a): If the condition (c) holds, then we have by Lemma 2.2

$$\left(1-|x|^2\right)\left|\mathcal{D}f(x)\right| \leqslant 2g(x)$$

for all $x \in B$. Thus Lemma 3.7 implies (a).

(d) $\Rightarrow$ (a): Assume that (d) holds, namely, there exists a continuous function $g \in L^p_{\alpha+p}$ such that

$$\left|f(x)-f(y)\right| \leqslant |x-y|\left[g(x)+g(y)\right] \tag{4.1}$$

for all $x, y \in B$. Letting y approach x in the direction of a real coordinate axis we have

$$\left|\frac{\partial f}{\partial x_k}(x)\right| \leqslant 2g(x), \quad 1 \leqslant k \leqslant n$$

so that

$$\left|\nabla f(x)\right| \leqslant 2\sqrt{n}\,g(x)$$

for all $x \in B$. This together with the assumption $g \in L^p_{\alpha+p}$ implies that $(1-|x|^2)|\nabla f(x)|$ is in L^p_α. Thus we have $f \in b^p_\alpha$ by Theorem 3.7.

(b) $\Rightarrow$ (d): Assume (b). Since $\rho(x,y) = |x-y|/[x,y]$, we have by the assumption there exists a continuous function $h \in L^p_\alpha$ such that

$$\left|f(x)-f(y)\right| \leqslant \rho(x,y)\left[h(x)+h(y)\right] = |x-y|\left\{\frac{h(x)}{[x,y]} + \frac{h(y)}{[y,x]}\right\}.$$

Note that

$$[x,y] = \left|\frac{x}{|x|} - y|x|\right| \geqslant 1-|x||y|$$

for all $x, y \in B$. Thus we have

$$\left|f(x)-f(y)\right| \leqslant |x-y|\left\{\frac{h(x)}{1-|x|} + \frac{h(y)}{1-|y|}\right\} = |x-y|\left[g(x)+g(y)\right],$$

where

$$g(x) = \frac{h(x)}{1-|x|} \leqslant \frac{2h(x)}{1-|x|^2}.$$

Since $h \in L^p_\alpha$, we have $g \in L^p_{\alpha+p}$. The proof is complete. $\square$

In the same manner we can prove Theorem 1.2.

Proof of Theorem 1.2. (a) $\Rightarrow$ (b): For any $f \in \mathcal{B}$ and for any $x \in B$, we have

$$f(x) - f(0) = \int_0^1 x \cdot \nabla f(tx)\, dt.$$

For a fixed $r \in (0,1)$, let $\rho(x,0) < r$. Then we have

$$\left|f(x) - f(0)\right| \leqslant |x| \sup\left\{\left|\nabla f(y)\right|:\ y \in E_r(0)\right\}.$$

Note that $|\nabla f(y)| \approx |\widetilde{\nabla} f(y)|$ for $y \in E_r(0)$. So, there exists a positive constant $C_1 = C_1(r)$ such that

$$\left|f(x) - f(0)\right| \leqslant C_1 \rho(x,0) \sup\left\{\left|\widetilde{\nabla} f(y)\right|:\ y \in E_r(0)\right\}$$

for all $x \in E_r(0)$.

Replace f by $f \circ \phi_y$ then replace x by $\phi_y(x)$ and use the Möbius invariance of the pseudo-hyperbolic metric and Proposition 3.5. Then we obtain

$$\left|f(x) - f(y)\right| \leqslant C_1 \rho(x,y) \sup\left\{\left|\widetilde{\nabla} f(z)\right|:\ z \in E_r(x)\right\}$$

for all $x, y \in B$ with $\rho(x,y) < r$. Let

$$g(x) = \frac{|f(x)|}{r} + C_1 \sup\left\{\left|\widetilde{\nabla} f(z)\right|:\ z \in E_r(x)\right\}.$$

Thus g is continuous on B and

$$\left|f(x) - f(y)\right| \leqslant \rho(x,y)\left[g(x) + g(y)\right]$$

for all $x, y \in B$.

Now, it remains to show that $g \in L^\infty$. Let

$$g(x) = \frac{|f(x)|}{r} + h(x),$$

where $h(x) = C_1 \sup\{|\widetilde{\nabla} f(z)|:\ z \in E_r(x)\}$. By Theorem 3.8, we have $h \in L^\infty$ and $f = Q_\alpha g$ for some $g \in L^\infty$. From the boundedness of Q_α on L^2_α, we get $f \in L^\infty$.

All the other proof is followed by the same method as in the proof of Theorem 1.1 with Theorem 3.8. The proof is complete. $\square$

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