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Multivariate generalized sampling in shift-invariant spaces and its approximation properties

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ABSTRACT

Nowadays the topic of sampling in a shift-invariant space is having a significant impact: it avoids most of the problems associated with classical Shannon's theory. Under appropriate hypotheses, any multivariate function in a shift-invariant space can be recovered from its samples at $\mathbb{Z}^d$. However, in many common situations the available data are samples of some convolution operators acting on the function itself: this leads to the problem of multivariate generalized sampling in shift-invariant spaces. This extra information on the functions in the shift-invariant space will allow to sample in an appropriate sub-lattice of $\mathbb{Z}^d$. In this paper an $L^2(\mathbb{R}^d)$ theory involving the frame theory is exhibited. Sampling formulas which are frame expansions for the shift-invariant space are obtained. In the case of overcomplete frame formulas, the search of reconstruction functions with prescribed good properties is allowed. Finally, approximation schemes using these generalized sampling formulas are included.

1. Statement of the problem

The classical Whittaker–Shannon–Kotel'nikov sampling theorem states that any function f band-limited to $[-1/2, 1/2]$, i.e., $f(t) = \int_{-1/2}^{1/2} \widehat{f}(w) e^{2\pi i t w}\, dw$, $t \in \mathbb{R}$, may be reconstructed from its sequence of samples $\{f(n)\}_{n\in\mathbb{Z}}$ as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\, \mathrm{sinc}(t-n), \quad t \in \mathbb{R},$$

where sinc denotes the cardinal sine function, $\mathrm{sinc}(t) = \sin \pi t/\pi t$. Thus, the Paley–Wiener space of functions band-limited to $[-1/2, 1/2]$ is generated by the integer shifts of the sinc function. The WSK sampling formula has its counterpart in d dimensions. It reads:

$$f(t) = \sum_{\alpha\in\mathbb{Z}^d} f(\alpha)\, \mathrm{sinc}(t_1 - \alpha_1) \ldots \mathrm{sinc}(t_d - \alpha_d), \quad t = (t_1, \ldots, t_d) \in \mathbb{R}^d,$$

where now the function f is band-limited to the d-dimensional cube $[-1/2, 1/2]^d$, i.e., $f(t) = \int_{[-1/2,1/2]^d} \widehat{f}(x) e^{2\pi i x^\top t}\, dx$, $t \in \mathbb{R}^d$.

Although Shannon's sampling theory has had an enormous impact, it has a number of problems, as pointed out by Unser in [28,29]: It relies on the use of ideal filters; the band-limited hypothesis is in contradiction with the idea of a finite

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duration signal; the band-limiting operation generates Gibbs oscillations; and finally, the sinc function has a very slow decay, which makes computation in the signal domain very inefficient. Besides, in several dimensions it is also inefficient to assume that a multidimensional signal is band-limited to a d-dimensional interval.

Moreover, many applied problems impose different a priori constraints on the type of functions. For this reason, sampling and reconstruction problems have been investigated in spline spaces, wavelet spaces, and general shift-invariant spaces. See, for instance, [1,29,36] and the references therein.

In many practical applications, signals are assumed to belong to some shift-invariant space of the form: $V_\varphi^2 := \overline{\operatorname{span}}_{L^2}\{\varphi(t-\alpha)\colon\ \alpha \in \mathbb{Z}^d\}$ where the function φ in $L^2(\mathbb{R}^d)$ is called the generator of V_φ^2. Assuming that $\varphi \in L^2(\mathbb{R}^d)$ is a stable generator, i.e., the sequence $\{\varphi(t-\alpha)\}_{\alpha\in\mathbb{Z}^d}$ is a Riesz basis for V_φ^2, the shift-invariant space V_φ^2 can be described as

$$V_\varphi^2 = \left\{ \sum_{\alpha\in\mathbb{Z}^d} a_\alpha \varphi(t-\alpha)\colon\ \{a_\alpha\} \in \ell^2\big(\mathbb{Z}^d\big) \right\} \subset L^2\big(\mathbb{R}^d\big). \tag{1}$$

On the other hand, in many common situations the available data are samples of some filtered versions of the signal itself. This leads to generalized sampling (or average sampling following some recent authors [3]) in V_φ^2: Suppose that s linear time-invariant systems (filters) $\mathcal{L}_j$, $j = 1, 2, \ldots, s$, are defined on the shift-invariant subspace V_φ^2 of $L^2(\mathbb{R}^d)$. In mathematical terms we are dealing with (continuous) operators which commute with shifts. The goal is to recover any function f in V_φ^2 from an appropriate subsequence of the set of samples $\{(\mathcal{L}_j f)(\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\ldots,s}$, by means of a sampling formula which is a frame expansion in V_φ^2. Recall that a sequence $\{f_n\}$ is a frame for a separable Hilbert space $\mathcal{H}$ if there exist two constants $A, B > 0$ (frame bounds) such that

$$A\|f\|^2 \leqslant \sum_n \big|\langle f, f_n\rangle\big|^2 \leqslant B\|f\|^2 \quad \text{for all } f \in \mathcal{H}.$$

Given a frame $\{f_n\}$ for $\mathcal{H}$ the representation property of any vector $f \in \mathcal{H}$ as a series $f = \sum_n c_n f_n$ is retained, but, unlike the case of Riesz bases, the uniqueness of this representation (for overcomplete frames) is sacrificed. Suitable frame coefficients c_n which depend continuously and linearly on f are obtained by using the dual frames $\{g_n\}$ of $\{f_n\}$, i.e., $\{g_n\}$ is another frame for $\mathcal{H}$ such that $f = \sum_n \langle f, g_n\rangle f_n = \sum_n \langle f, f_n\rangle g_n$ for each $f \in \mathcal{H}$. For more details on the frame theory see the superb monograph [5] and the references therein.

Under appropriate hypotheses, any function in a shift-invariant space in $L^2(\mathbb{R}^d)$ can be recovered from its samples in the lattice $\mathbb{Z}^d$ of $\mathbb{R}^d$ (see [26]). If we sample the function on the sub-lattice $M\mathbb{Z}^d$, where M denotes a matrix of integer entries with positive determinant, we are using the sampling rate $1/(\det M)$ and, roughly speaking, we will need the generalized samples $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\ldots,s}$ from $s \geqslant \det M$ linear systems $\mathcal{L}_j$ for the recovery of f. The one-dimensional case has been treated in [8,12,30]: Under suitable hypotheses, we can recover any function in V_φ^2 from the sequence of generalized samples $\{(\mathcal{L}_j f(rn)\}_{n\in\mathbb{Z},\, j=1,2,\ldots,s}$, where the number of channels is $s \geqslant r \in \mathbb{N}$.

In this work we obtain, in the light of the $L^2(\mathbb{R}^d)$-theory, sampling formulas for V_φ^2 of the type

$$f(t) = (\det M) \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_j(t - M\alpha), \quad t \in \mathbb{R}^d, \tag{2}$$

where the sequence of reconstruction functions $\{S_j(\cdot - M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\ldots,s}$ forms a frame for the shift-invariant space V_φ^2. To this end, first observe that the shift-invariant space V_φ^2 is the image of $L^2[0,1)^d$ under the isomorphism $\mathcal{T}_\varphi : L^2[0,1)^d \to V_\varphi^2$, which maps the orthonormal basis $\{e^{-2\pi i\alpha^\top x}\}_{\alpha\in\mathbb{Z}^d}$ for $L^2[0,1)^d$ onto the Riesz basis $\{\varphi(t-\alpha)\}_{\alpha\in\mathbb{Z}^d}$ for V_φ^2.

Next we express the generalized samples $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\ldots,s}$ as the inner products of the function $F = \mathcal{T}_\varphi^{-1} f \in L^2[0,1)^d$ with respect to a particular frame in $L^2[0,1)^d$. Searching for its dual frames we obtain those expansions for F in $L^2[0,1)^d$ having the samples $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\ldots,s}$ as the frame coefficients. These frame expansions have precisely the form

$$F = (\det M) \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) d_j(x) e^{-2\pi i\alpha^\top M^\top x} \quad \text{in } L^2[0,1)^d, \tag{3}$$

where the functions $d_j \in L^2[0,1)^d$, $j = 1, 2, \ldots, s$, are obtained by solving a matrix equation

$$\big[d_1(x), \ldots, d_s(x)\big]\, \mathbf{G}(x) = [1, 0, \ldots, 0] \quad \text{a.e. in } [0,1)^d, \tag{4}$$

where $\mathbf{G}(x)$ is an $s \times (\det M)$ matrix of functions defined in $[0,1)^d$ (the so-called modulation matrix in the filter-bank jargon) which only depends on the generator φ and on the systems $\mathcal{L}_j$, $j = 1, 2, \ldots, s$ (see (12) infra).

Finally, applying the isomorphism $\mathcal{T}_\varphi$ to the frame expansion (3) for F we will obtain the aforesaid sampling expansions for $f = \mathcal{T}_\varphi F$ in V_φ^2, where $S_j = \mathcal{T}_\varphi d_j$, $j = 1, 2, \ldots, s$.

Besides, the perturbation theory for frames gives generalized irregular sampling for appropriate sequences of perturbed generalized samples $\{(\mathcal{L}_j f)(M\alpha + \varepsilon_{j,\alpha})\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\ldots,s}$.

Moreover, in the oversampling case, i.e., whenever $s > \det M$, we are dealing with overcomplete frames and several different dual frames allow us to obtain a variety of reconstruction functions. Thus we can try to find some reconstruction functions S_j, $j = 1, 2, \ldots, s$, with "good properties", such as compact support, exponential decay, etc. As one can see in the present paper, this relies on the search of solutions of Eq. (4) with prescribed properties. From a mathematical point of view, this is equivalent to solving Eq. (4) whenever the entries of the matrix function $\mathbf{G}(x)$ belong to a prescribed algebra of functions.

On the other hand, as pointed out in [21] by Lei et al., there are many ways to construct approximation schemes with shift-invariant spaces. Among them, they cite cardinal interpolation, quasi-interpolation, projection and convolution (see also [4,17,18]). This paper shows that a generalized sampling formula like (2) allows to construct an $L^2(\mathbb{R}^d)$-approximation scheme as follows: For a suitable smooth function f (in a Sobolev space), consider the operator Γ, formally defined as

$$(\Gamma f)(t) := (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)S_j(t - M\alpha), \quad t\in\mathbb{R}^d.$$

The aim is to obtain a good approximation of f by means of the scaled operator Γ^h given by $\Gamma^h := \sigma_h \Gamma \sigma_{1/h}$, where $\sigma_h f := f(\cdot/h)$, $h > 0$. For f in an appropriate Sobolev space we obtain an estimation for the L^2-approximation error of the type $\|\Gamma^h f - f\|_2 = \mathcal{O}(h^r)$ as $h \to 0^+$, where $r \in \mathbb{N}$ denotes the approximation order which coincides with the order of the Strang–Fix conditions satisfied by the generator φ.

Looking for an estimation like the one above with respect to the L^∞-norm leads to extend the sampling formula (2) to the larger space

$$V_\varphi^\infty := \overline{\operatorname{span}}_{L^\infty}\bigl\{\varphi(t-\alpha):\ \alpha\in\mathbb{Z}^d\bigr\}.$$

Thus, for any function f in an appropriate Sobolev space, we obtain an analogous estimation for the L^∞-approximation error: Namely, $\|\Gamma^h f - f\|_\infty = \mathcal{O}(h^r)$ as $h \to 0^+$ where now the approximation order r depends both on the order of the Strang–Fix conditions satisfied by the generator φ, and on the greatest order of the partial derivatives appearing in the systems $\mathcal{L}_j$, if any.

All these steps will be carried out throughout the remaining sections.

2. Introducing multivariate generalized sampling in V_φ^2

In this section we introduce the needed preliminaries on the shift-invariant space V_φ^2, on the linear time-invariant systems $\mathcal{L}_j$, and on the lattices in $\mathbb{Z}^d$ in order to derive a generalized sampling theory in V_φ^2. Moreover, we study some sequences in $L^2[0,1)^d$ which play a crucial role in what follows.

2.1. Preliminaries on the shift-invariant space V_φ^2

Let $\varphi \in L^2(\mathbb{R}^d)$ be a stable generator for the shift-invariant space

$$V_\varphi^2 := \left\{\sum_{\alpha\in\mathbb{Z}^d} a_\alpha \varphi(\cdot - \alpha):\ \{a_\alpha\}_{\alpha\in\mathbb{Z}^d} \in \ell^2\bigl(\mathbb{Z}^d\bigr)\right\} \subset L^2\bigl(\mathbb{R}^d\bigr),$$

i.e., the sequence $\{\varphi(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d}$ is a Riesz basis for V_φ^2. A Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Recall that the sequence $\{\varphi(\cdot - \alpha)\}_{\alpha\in\mathbb{Z}^d}$ is a Riesz sequence in $L^2(\mathbb{R}^d)$, i.e., a Riesz basis for V_φ^2 if and only if

$$0 < \|\Phi\|_0 \leqslant \|\Phi\|_\infty < \infty,$$

where $\|\Phi\|_0$ denotes the essential infimum of the function $\Phi(w) := \sum_{\beta\in\mathbb{Z}^d} |\widehat{\varphi}(w+\beta)|^2$ in $[0,1)^d$, and $\|\Phi\|_\infty$ its essential supremum. Furthermore, $\|\Phi\|_0$ and $\|\Phi\|_\infty$ are the optimal Riesz bounds [5, p. 143].

We assume throughout the paper that the functions in the shift-invariant space V_φ^2 are continuous on $\mathbb{R}^d$. Equivalently, that the generator φ is continuous on $\mathbb{R}^d$ and the function $\sum_{\alpha\in\mathbb{Z}^d} |\varphi(t-\alpha)|^2$ is uniformly bounded on $\mathbb{R}^d$ (see [26,36]). Thus, any $f \in V_\varphi^2$ is defined on $\mathbb{R}^d$ as the pointwise sum $f(t) = \sum_{\alpha\in\mathbb{Z}^d} a_\alpha\varphi(t-\alpha)$ for each $t\in\mathbb{R}^d$.

Besides, V_φ^2 is a reproducing kernel Hilbert space (RKHS) since the evaluation functionals are bounded in V_φ^2. Indeed, for each fixed $t \in \mathbb{R}^d$ we have

$$\bigl|f(t)\bigr|^2 \leqslant \frac{\|f\|^2}{\|\Phi\|_0}\sum_{\alpha\in\mathbb{Z}^d}\bigl|\varphi(t-\alpha)\bigr|^2, \quad f\in V_\varphi^2, \tag{5}$$

where we have used Cauchy–Schwartz's inequality on $f(t)=\sum_{\alpha\in\mathbb{Z}^d} a_\alpha\varphi(t-\alpha)$, and the Riesz basis condition

$$\|\Phi\|_0 \sum_{\alpha\in\mathbb{Z}^d}|a_\alpha|^2 \leqslant \|f\|^2 \leqslant \|\Phi\|_\infty \sum_{\alpha\in\mathbb{Z}^d}|a_\alpha|^2, \quad f\in V_\varphi^2.$$

Inequality (5) shows that convergence in the $L^2(\mathbb{R}^d)$-norm implies pointwise convergence which is uniform on $\mathbb{R}^d$.

The reproducing kernel of V_φ^2 is given by $k(t,s)=\sum_{\alpha\in\mathbb{Z}^d}\varphi(t-\alpha)\overline{\varphi^*(s-\alpha)}$ where the sequence $\{\varphi^*(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d}$ denotes the dual Riesz basis of $\{\varphi(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d}$. Recall that the Fourier transform of the function φ^* is $\widehat{\varphi^*}=\widehat{\varphi}/\Phi$.

On the other hand, the space V_φ^2 is the image of $L^2[0,1)^d$ by means of the isomorphism $\mathcal{T}_\varphi: L^2[0,1)^d\to V_\varphi^2$ which maps the orthonormal basis $\{e^{-2\pi i\alpha^\top x}\}_{\alpha\in\mathbb{Z}^d}$ for $L^2[0,1)^d$ onto the Riesz basis $\{\varphi(t-\alpha)\}_{\alpha\in\mathbb{Z}^d}$ for V_φ^2: For $F\in L^2[0,1)^d$ we have

$$(\mathcal{T}_\varphi F)(t):=\sum_{\alpha\in\mathbb{Z}^d}\widehat{F}(\alpha)\varphi(t-\alpha), \quad t\in\mathbb{R}^d,$$

where $\widehat{F}(\alpha)$, $\alpha\in\mathbb{Z}^d$, are the Fourier coefficients of F, i.e., for each $\alpha\in\mathbb{Z}^d$, $\widehat{F}(\alpha):=\int_{[0,1)^d}F(x)e^{2\pi i\alpha^\top x}\,dx$.

Notice that any function $f=\mathcal{T}_\varphi F$ in V_φ^2, where $F\in L^2[0,1)^d$, can be expressed as

$$f(t)=\big\langle F,\overline{Z\varphi}(t,\cdot)\big\rangle_{L^2[0,1)^d}, \quad t\in\mathbb{R}^d,$$

where $Z\varphi$ denotes the Zak transform of φ. Recall that the Zak transform of $f\in L^2(\mathbb{R}^d)$ is formally defined in $\mathbb{R}^{2d}$ as $(Zf)(t,x):=\sum_{\beta\in\mathbb{Z}^d}f(t+\beta)e^{-2\pi i\beta^\top x}$. See [15] for properties and uses of the Zak transform.

The following shifting property of $\mathcal{T}_\varphi$ will be used later: For $F\in L^2[0,1)^d$ and $\alpha\in\mathbb{Z}^d$ we have

$$\mathcal{T}_\varphi\big[F(\cdot)e^{-2\pi i\alpha^\top\cdot}\big](t)=\mathcal{T}_\varphi[F](t-\alpha), \quad t\in\mathbb{R}^d. \tag{6}$$

2.2. The linear time-invariant systems $\mathcal{L}_j$

We consider s linear time-invariant systems $\mathcal{L}_j$ in $L^2(\mathbb{R}^d)$ such that $\mathcal{L}_j f=\mathsf{h}_j * f$, $j=1,2,\dots,s$, of the following types:

(a) The impulse response h_j of $\mathcal{L}_j$ belongs to $L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. Thus, for any $f\in V_\varphi^2$ we have

$$(\mathcal{L}_j f)(t):=[f*\mathsf{h}_j](t)=\int_{\mathbb{R}^d} f(x)\mathsf{h}_j(t-x)\,dx, \quad t\in\mathbb{R}^d.$$

(b) The impulse response h_j is a linear combination of partial derivatives of shifted delta functionals, i.e.,

$$(\mathcal{L}_j f)(t):=\sum_{|\beta|\leqslant N_j} c_{j,\beta}D^\beta f(t+d_{j,\beta}), \quad t\in\mathbb{R}^d.$$

If there is a system of this type, we also assume that $\sum_{\alpha\in\mathbb{Z}^d}|D^\beta\varphi(t-\alpha)|^2$ is uniformly bounded on $\mathbb{R}^d$ for $|\beta|\leqslant N_j$.

Whenever the linear system $\mathcal{L}_j$ is of type (a), the Minkowski inequality for integrals shows that the sequence $\{(\mathcal{L}_j\varphi)(t+\beta)\}_{\beta\in\mathbb{Z}^d}\in\ell^2(\mathbb{Z}^d)$ for any fixed $t\in\mathbb{R}^d$ (see [12, Lemma 1]). Trivially, the same applies for $\mathcal{L}_j$ of type (b). Therefore, for any fixed $t\in\mathbb{R}^d$, the function $(Z\mathcal{L}_j\varphi)(t,x):=\sum_{\beta\in\mathbb{Z}^d}(\mathcal{L}_j\varphi)(t+\beta)e^{-2\pi i\beta^\top x}$ belongs to $L^2[0,1)^d$ and the following expression for $\mathcal{L}_j$ holds: For any $f=\mathcal{T}_\varphi F\in V_\varphi^2$ we have

$$(\mathcal{L}_j f)(t)=\big\langle F,\overline{Z\mathcal{L}_j\varphi}(t,\cdot)\big\rangle_{L^2[0,1)^d}, \quad t\in\mathbb{R}^d. \tag{7}$$

The proof is analogous to the one of [12, Lemma 2]. In particular, for any $\alpha\in\mathbb{Z}^d$ we have

$$(\mathcal{L}_j f)(\alpha)=\big\langle F,\overline{Z\mathcal{L}_j\varphi}(0,\cdot)e^{-2\pi i\alpha^\top\cdot}\big\rangle_{L^2[0,1)^d}=\int_{[0,1)^d}F(x)g_j(x)e^{2\pi i\alpha^\top x}\,dx, \tag{8}$$

where the functions g_j, $j=1,2,\dots,s$, given by

$$g_j(x):=Z\mathcal{L}_j\varphi(0,x)=\sum_{\beta\in\mathbb{Z}^d}(\mathcal{L}_j\varphi)(\beta)e^{-2\pi i\beta^\top x}\in L^2[0,1)^d, \tag{9}$$

will play a central role throughout this paper.

2.3. Lattices in $\mathbb{Z}^d$

Given a nonsingular matrix M with integer entries, we consider the lattice in $\mathbb{Z}^d$ generated by M, i.e.,

$$\mathrm{Lat}(M) := \{M\alpha:\ \alpha \in \mathbb{Z}^d\} \subset \mathbb{Z}^d.$$

Without loss of generality we can assume that $\det M > 0$; otherwise we can consider $M' = ME$ where E is some $d \times d$ integer matrix satisfying $\det E = -1$. Trivially, $\mathrm{Lat}\, M = \mathrm{Lat}\, M'$. We denote by $M^\top$ and $M^{-\top}$ the transpose matrices of M and M^{-1} respectively. The following useful generalized orthogonal relationship holds (see [31]):

$$\sum_{k \in \mathcal{N}(M^\top)} e^{-2\pi i \alpha^\top M^{-T} k} = \begin{cases} \det M, & \alpha \in \mathrm{Lat}(M), \\ 0, & \alpha \in \mathbb{Z}^d \setminus \mathrm{Lat}(M), \end{cases} \tag{10}$$

where

$$\mathcal{N}(M^\top) := \mathbb{Z}^d \cap \{M^\top x:\ x \in [0,1)^d\}.$$

The set $\mathcal{N}(M^\top)$ has $\det M$ elements (see [31] or [32]). One of these elements is zero, say $i_1 = 0$; we denote the rest of elements by $i_2, \ldots, i_{\det M}$ ordered in any form.

Note that the sets, defined as $Q_k := M^{-\top} i_k + M^{-\top}[0,1)^d$, $k = 1, 2, \ldots, \det M$, satisfy (see [32, p. 110]):

$$Q_k \cap Q_{k'} = \emptyset \quad \text{if } k \neq k' \quad \text{and} \quad \mathrm{Vol}\left(\bigcup_{k=1}^{\det M} Q_k\right) = 1.$$

Thus, for any function F integrable in $[0,1)^d$ and $\mathbb{Z}^d$-periodic we have $\int_{[0,1)^d} F(x)\,dx = \sum_{k=1}^{\det M} \int_{Q_k} F(x)\,dx$.

Given s linear time-invariant systems $\mathcal{L}_j$, $j = 1, 2, \ldots, s$, the aim is to recover any function $f \in V_\varphi^2$ from its generalized samples at a lattice $\mathrm{Lat}(M)$ of $\mathbb{Z}^d$, i.e., from the sequence of samples $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha \in \mathbb{Z}^d,\, j=1,2,\ldots,s}$. Eq. (8) gives

$$(\mathcal{L}_j f)(M\alpha) = \left\langle F, \overline{g_j(x)} e^{-2\pi i \alpha^\top M^\top x} \right\rangle_{L^2[0,1)^d}, \quad \alpha \in \mathbb{Z}^d \text{ and } j = 1, 2, \ldots, s. \tag{11}$$

As a consequence, the recovery of the function $F = \mathcal{T}_\varphi^{-1} f \in L^2[0,1)^d$, and hence of $f \in V_\varphi^2$, from the sequence of generalized samples leads us to study the properties (completeness, Bessel, frame, or Riesz basis properties) of the sequence $\{\overline{g_j(x)} e^{-2\pi i \alpha^\top M^\top x}\}_{\alpha \in \mathbb{Z}^d,\, j=1,2,\ldots,s}$ in $L^2[0,1)^d$.

2.4. The sequence $\{\overline{g_j(x)} e^{-2\pi i \alpha^\top M^\top x}\}_{\alpha \in \mathbb{Z}^d,\, j=1,2,\ldots,s}$ in $L^2[0,1)^d$

Next, we carry out the study of the completeness, Bessel, frame, or Riesz basis properties of the sequence $\{\overline{g_j(x)} e^{-2\pi i \alpha^\top M^\top x}\}_{\alpha \in \mathbb{Z}^d,\, j=1,2,\ldots,s}$ in $L^2[0,1)^d$. To this end, we introduce the $s \times (\det M)$ matrix of functions

$$\mathbf{G}(x) := \begin{bmatrix} g_1(x) & g_1(x + M^{-\top} i_2) & \cdots & g_1(x + M^{-\top} i_{\det M}) \\ g_2(x) & g_2(x + M^{-\top} i_2) & \cdots & g_2(x + M^{-\top} i_{\det M}) \\ \vdots & \vdots & \vdots & \vdots \\ g_s(x) & g_s(x + M^{-\top} i_2) & \cdots & g_s(x + M^{-\top} i_{\det M}) \end{bmatrix} = \left[g_j(x + M^{-\top} i_k) \right]_{\substack{j=1,2,\ldots,s \\ k=1,2,\ldots,\det M}}, \tag{12}$$

and its related constants

$$A_{\mathbf{G}} := \operatorname*{ess\,inf}_{x \in [0,1)^d} \lambda_{\min}\left[\mathbf{G}^*(x)\mathbf{G}(x)\right], \qquad B_{\mathbf{G}} := \operatorname*{ess\,sup}_{x \in [0,1)^d} \lambda_{\max}\left[\mathbf{G}^*(x)\mathbf{G}(x)\right],$$

where $\mathbf{G}^*(x)$ denotes the transpose conjugate of the matrix $\mathbf{G}(x)$, and $\lambda_{\min}$ (respectively $\lambda_{\max}$) the smallest (respectively the largest) eigenvalue of the positive semidefinite matrix $\mathbf{G}^*(x)\mathbf{G}(x)$. Observe that $0 \leqslant A_{\mathbf{G}} \leqslant B_{\mathbf{G}} \leqslant \infty$. Note that in the definition of the matrix $\mathbf{G}(x)$ we are considering the $\mathbb{Z}^d$-periodic extension of the involved functions g_j, $j = 1, 2, \ldots, s$.

The following result remains true for arbitrary functions g_j in $L^2[0,1)^d$, $j = 1, 2, \ldots, s$, not necessarily given by (9).

Lemma 1. *Let g_j be in $L^2[0,1)^d$ for $j = 1, 2, \ldots, s$ and let $\mathbf{G}(x)$ be its associated matrix as in* (12)*. Then*:

(a) *The sequence $\{\overline{g_j(x)} e^{-2\pi i \alpha^\top M^\top x}\}_{\alpha \in \mathbb{Z}^d,\, j=1,2,\ldots,s}$ is a complete system for $L^2[0,1)^d$ if and only if the rank of the matrix $\mathbf{G}(x)$ is $\det M$ a.e. in $[0,1)^d$.*

(b) *The sequence $\{\overline{g_j(x)} e^{-2\pi i \alpha^\top M^\top x}\}_{\alpha \in \mathbb{Z}^d,\, j=1,2,\ldots,s}$ is a Bessel sequence for $L^2[0,1)^d$ if and only if $g_j \in L^\infty[0,1)^d$ (or equivalently $B_{\mathbf{G}} < \infty$). In this case, the optimal Bessel bound is $B_{\mathbf{G}}/(\det M)$.*

(c) *The sequence* $\{\overline{g_j(x)}e^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\ldots,s}$ *is a frame for* $L^2[0,1)^d$ *if and only if* $0 < A_{\mathbf{G}} \leqslant B_{\mathbf{G}} < \infty$. *In this case, the optimal frame bounds are* $A_{\mathbf{G}}/(\det M)$ *and* $B_{\mathbf{G}}/(\det M)$.

(d) *The sequence* $\{\overline{g_j(x)}e^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\ldots,s}$ *is a Riesz basis for* $L^2[0,1)^d$ *if and only if it is a frame and* $s = \det M$.

Proof. Properties (a), (b) and (c) depend on the behaviour of the ℓ^2-norm of the sequence of inner products $\{\langle F, \overline{g_j(\cdot)}e^{-2\pi i\alpha^\top M^\top \cdot}\rangle_{L^2[0,1)^d}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\ldots,s}$ for any function $F \in L^2[0,1)^d$. First, we obtain a representation for this ℓ^2-norm by using that the sequence $\{e^{2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d}$ is an orthogonal basis for $L^2(M^{-\top}[0,1)^d)$. For any $F \in L^2[0,1)^d$ we have

$$\begin{aligned}\big\langle F(x), \overline{g_j(x)}e^{-2\pi i\alpha^\top M^\top x}\big\rangle_{L^2[0,1)^d} &= \int_{[0,1)^d} F(x)g_j(x)e^{2\pi i\alpha^\top M^\top x}\,dx = \sum_{k=1}^{\det M}\int_{Q_k} F(x)g_j(x)e^{2\pi i\alpha^\top M^\top x}\,dx \\ &= \int_{M^{-\top}[0,1)^d}\sum_{k=1}^{\det M} F\big(x+M^{-\top}i_k\big)g_j\big(x+M^{-\top}i_k\big)e^{2\pi i\alpha^\top M^\top x}\,dx, \end{aligned}\tag{13}$$

where we have considered the $\mathbb{Z}^d$-periodic extension of F. Then,

$$\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big|\big\langle F(x), \overline{g_j(x)}e^{-2\pi i\alpha^\top M^\top x}\big\rangle_{L^2[0,1)^d}\big|^2 = \frac{1}{\det M}\sum_{j=1}^{s}\bigg\|\sum_{k=1}^{\det M} F\big(x+M^{-\top}i_k\big)g_j\big(x+M^{-\top}i_k\big)\bigg\|^2_{L^2(M^{-\top}[0,1)^d)}.$$

Denoting $\mathbb{F}(x) := [F(x), F(x+M^{-\top}i_2), \ldots, F(x+M^{-\top}i_{\det M})]^\top$ the equality above reads

$$\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big|\big\langle F(x), \overline{g_j(x)}e^{-2\pi i\alpha^\top M^\top x}\big\rangle_{L^2[0,1)^d}\big|^2 = \frac{1}{\det M}\big\|\mathbf{G}(x)\mathbb{F}(x)\big\|^2_{L^2_s(M^{-\top}[0,1)^d)},\tag{14}$$

where we have denoted $L^2_s(M^{-\top}[0,1)^d) := L^2(M^{-\top}[0,1)^d)\times\cdots\times L^2(M^{-\top}[0,1)^d)$ (s times) with the usual norm.

On the other hand, using that the function g_j is $\mathbb{Z}^d$-periodic, we obtain that the set $\{g_j(x+M^{-\top}i_k+M^{-\top}i_1), g_j(x+M^{-\top}i_k+M^{-\top}i_2), \ldots, g_j(x+M^{-\top}i_k+M^{-\top}i_{\det M})\}$ has the same elements as $\{g_j(x+M^{-\top}i_1), g_j(x+M^{-\top}i_2), \ldots, g_j(x+M^{-\top}i_{\det M})\}$. Thus the matrix $\mathbf{G}(x+M^{-\top}i_k)$ has the same columns of $\mathbf{G}(x)$, possibly in a different order. Hence, $\operatorname{rank}\mathbf{G}(x) = \det M$ a.e. in $[0,1)^d$ if and only if $\operatorname{rank}\mathbf{G}(x) = \det M$ a.e. in $M^{-\top}[0,1)^d$. Moreover,

$$A_{\mathbf{G}} = \operatorname*{ess\,inf}_{x\in M^{-\top}[0,1)^d}\lambda_{\min}\big[\mathbf{G}^*(x)\mathbf{G}(x)\big], \qquad B_{\mathbf{G}} = \operatorname*{ess\,sup}_{x\in M^{-\top}[0,1)^d}\lambda_{\max}\big[\mathbf{G}^*(x)\mathbf{G}(x)\big].\tag{15}$$

To prove (a), assume that there exists a set $\Omega \subseteq M^{-\top}[0,1)^d$ with positive measure such that $\operatorname{rank}\mathbf{G}(x) < \det M$, $x\in\Omega$. Then, there exists a measurable function $v(x)$, $x\in\Omega$, such that $\mathbf{G}(x)v(x) = 0$ and $|v(x)| = 1$ in Ω. This function can be constructed as in [20, Lemma 2.4]. Define $F\in L^2[0,1)^d$ such that $\mathbb{F}(x) = v(x)$ if $x\in\Omega$, and $\mathbb{F}(x) = 0$ if $x\in M^{-\top}[0,1)^d\setminus\Omega$. Hence, from (14) we obtain that the system is not complete. Conversely, if the system is not complete, by using (14) we obtain a $\mathbb{F}(x)$ different from 0 in a set with positive measure such that $\mathbf{G}(x)\mathbb{F}(x) = 0$. Thus $\operatorname{rank}\mathbf{G}(x) < \det M$ on a set with positive measure.

Parts (b) and (c) in Lemma 1 have been proved in [12, Lemma 3] for the univariate case. By using (14) and (15), the proofs for the general case are completely analogous.

To prove (d) we assume that $\det M = s$ and that the sequence is a frame. We see that it is also a Riesz basis by proving that the analysis operator

$$\Lambda : L^2[0,1)^d \to \ell^2_s, \quad \Lambda(F) := \big\{\big\langle F(x), \overline{g_j(x)}e^{-2\pi i\alpha^\top M^\top x}\big\rangle_{L^2[0,1)^d}\big\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\ldots,s}$$

is surjective (see [5, Theorem 6.5.1]). To this end, notice that when $\det M = s$ the matrix $\mathbf{G}(x)$ is a square matrix and hence, the condition $A_{\mathbf{G}} > 0$ implies that the inverse matrix $\mathbf{G}^{-1}(x)$ exists and its entries are essentially bounded. Let $\{c_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\ldots,s}$ be an element of ℓ^2_s. For $j = 1,2,\ldots,s$ we define the function

$$\xi_j(x) := (\det M)\sum_{\alpha\in\mathbb{Z}^d} c_{j,\alpha}e^{-2\pi i\alpha^\top M^\top x},$$

and let F be the function such that $\mathbb{F}(x) = \mathbf{G}^{-1}(x)[\xi_1(x),\ldots,\xi_s(x)]^\top$, $x\in M^{-\top}[0,1)^d$. This function belongs to $L^2[0,1)^d$ because the entries of $\mathbf{G}^{-1}(x)$ are essentially bounded. We have that $\mathbf{G}(x)\mathbb{F}(x) = [\xi_1(x),\ldots,\xi_s(x)]^\top$, and using (13) we obtain that

$$\begin{aligned}\langle F(x), g_j(x)e^{-2\pi i\alpha^\top M^\top x}\rangle &= \int_{M^{-\top}[0,1)^d} \sum_{k=1}^{\det M} F\big(x+M^{-\top}i_k\big) g_j\big(x+M^{-\top}i_k\big) e^{2\pi i\alpha^\top M^\top x}\,dx \\ &= \int_{M^{-\top}[0,1)^d} \xi_j(x)e^{2\pi i\alpha^\top M^\top x}\,dx = c_{j,\alpha},\end{aligned}$$

and consequently, $\Lambda(F) = \{c_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$.

Conversely, assume that the sequence $\{\overline{g_j(x)}e^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a Riesz basis. Let $\{f_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ be its dual Riesz basis. Then, by using (13) we obtain

$$\int_{M^{-\top}[0,1)^d} \sum_{k=1}^{\det M} f_{j',0}\big(x+M^{-\top}i_k\big) g_j\big(x+M^{-\top}i_k\big) e^{2\pi i\alpha^\top M^\top x}\,dx = \delta_{\alpha,0}\delta_{j,j'}.$$

Therefore, for $j, j' = 1,2,\dots,s$, we have

$$\sum_{k=1}^{\det M} f_{j',0}\big(x+M^{-\top}i_k\big) g_j\big(x+M^{-\top}i_k\big) = (\det M)\delta_{j,j'} \quad \text{a.e. in } [0,1)^d.$$

Thus the matrix $\mathbf{G}(x)$ has a right inverse; in particular, $s \leqslant \det M$. As a consequence of (a) we have $s \geqslant \det M$ and, finally, $s = \det M$. □

Next we discuss the meaning of Lemma 1, whenever the functions g_j, $j = 1,2,\dots,s$, are given by (9), in terms of the average sampling terminology introduced by Aldroubi et al. in [3]. Thus, following [3], we say that:

1. The set of systems $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an *M-determining filtering sampler* for V_φ^2 if the only function $f \in V_\varphi^2$ satisfying $\mathcal{L}_j f(M\alpha) = 0$ for all $j = 1,2,\dots,s$ and $\alpha \in \mathbb{Z}^d$ is the zero function.
2. The set of systems $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an *M-stable filtering sampler* for V_φ^2 if there exist two positive constants C_1 and C_2 such that

$$C_1\|f\|^2 \leqslant \sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d} \big|\mathcal{L}_j f(M\alpha)\big|^2 \leqslant C_2\|f\|^2 \quad \text{for all } f \in V_\varphi^2.$$

If $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an M-stable filtering sampler for V_φ^2, then any function $f \in V_\varphi^2$ can be recovered, in a stable way, from the sequence of generalized samples. Roughly speaking, the operator which maps $f \in V_\varphi^2$ into the sequence of samples $\{\mathcal{L}_j f(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ has a bounded inverse. An M-determining filtering sampler for V_φ^2 can distinguish between two distinct functions in V_φ^2, but the recovery is not necessarily stable. Notice that from (11), parts (a) and (c) of Lemma 1 read as follows:

1. The set of systems $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an M-determining filtering sampler for V_φ^2 if and only if $\operatorname{rank}\mathbf{G}(x) = \det M$ a.e. in $[0,1)^d$ (and hence, necessarily, $s \geqslant \det M$).
2. The set of systems $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an M-stable filtering sampler for V_φ^2 if and only if $0 < A_{\mathbf{G}} \leqslant B_{\mathbf{G}} < \infty$.

These properties can be expressed in terms of the function $\det[\mathbf{G}^*(x)\mathbf{G}(x)]$. Indeed, as $\operatorname{rank}\mathbf{G}(x) = \det M$ if and only if $\det[\mathbf{G}^*(x)\mathbf{G}(x)] \neq 0$, we have that the set of systems $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an M-determining filtering sampler for V_φ^2 if and only if

$$\det\big[\mathbf{G}^*(x)\mathbf{G}(x)\big] \neq 0 \quad \text{a.e. in } [0,1)^d.$$

Provided that the function $g_j \in L^\infty[0,1)^d$ for each $j = 1,2,\dots,s$ (or equivalently $B_{\mathbf{G}} < \infty$), since $\det[\mathbf{G}^*(x)\mathbf{G}(x)]$ is the product of the eigenvalues, we have that

$$\begin{aligned}\big(\lambda_{\min}\big[\mathbf{G}^*(x)\mathbf{G}(x)\big]\big)^{\det M} &\leqslant \det \mathbf{G}^*(x)\mathbf{G}(x) \leqslant \big(\lambda_{\max}\big[\mathbf{G}^*(x)\mathbf{G}(x)\big]\big)^{(\det M)-1}\lambda_{\min}\big[\mathbf{G}^*(x)\mathbf{G}(x)\big] \\ &\leqslant B_{\mathbf{G}}^{(\det M)-1}\lambda_{\min}\big[\mathbf{G}^*(x)\mathbf{G}(x)\big],\end{aligned}$$

and therefore, the set of systems $\{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_s\}$ is an M-stable filtering sampler for V_φ^2 if and only if

$$\operatorname*{ess\,inf}_{x\in[0,1)^d} \det\big[\mathbf{G}^*(x)\mathbf{G}(x)\big] > 0.$$

If the functions g_j, $j=1,2,\dots,s$, are continuous on $\mathbb{R}^d$, the above condition reads: $\det[\mathbf{G}^*(x)\mathbf{G}(x)]\neq 0$ for all $x\in[0,1)^d$. Hence, the set $\{\mathcal{L}_1,\mathcal{L}_2,\dots,\mathcal{L}_s\}$ is an M-stable filtering sampler for V_φ^2 if and only if

$$\operatorname{rank}\mathbf{G}(x)=\det M \quad \text{for all } x\in[0,1)^d. \tag{16}$$

3. Generalized sampling in V_φ^2

In the above section we have proved that, provided that the functions $g_j\in L^\infty[0,1)^d$ for each $j=1,2,\dots,s$, the set $\{\mathcal{L}_1,\mathcal{L}_2,\dots,\mathcal{L}_s\}$ is an M-stable filtering sampler for V_φ^2 if and only if $A_\mathbf{G}>0$. In this section we obtain the corresponding stable sampling formulas leading to the recovery of any function $f\in V_\varphi^2$ from the sequence of its generalized samples $\{\mathcal{L}_j f(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$. The sampling formula will be unique in the case $s=\det M$. These, explicitly given, sampling formulas consist of the major difference with the analogous results included in [3].

3.1. Generalized regular sampling

Now we prove that the expression (8) allows us to obtain F from the generalized samples $\{\mathcal{L}_j f(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$. Applying the isomorphism $\mathcal{T}_\varphi$ we get generalized regular sampling formulas in V_φ^2.

Assume that $g_j\in L^\infty[0,1)^d$ for $j=1,2,\dots,s$; then, $F(x)g_j(x)\in L^2[0,1)^d$. Hence using (10) and (8), for $j=1,2,\dots,s$ we obtain that

$$\begin{aligned}(\det M)\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)e^{-2\pi i\alpha^\top M^\top x} &= \sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(\alpha)e^{-2\pi i\alpha^\top x}\sum_{k\in\mathcal{N}(M^\top)}e^{-2\pi i\alpha^\top M^{-\top}k}\\ &= \sum_{k\in\mathcal{N}(M^\top)}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(\alpha)e^{-2\pi i\alpha^\top(x+M^{-\top}k)}\\ &= \sum_{k\in\mathcal{N}(M^\top)}F\big(x+M^{-\top}k\big)g_j\big(x+M^{-\top}k\big).\end{aligned}$$

This can be written in matrix form as

$$\mathbf{G}(x)\mathbb{F}(x)=(\det M)\Bigg[\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_1 f)(M\alpha)e^{-2\pi i\alpha^\top M^\top x},\dots,\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_s f)(M\alpha)e^{-2\pi i\alpha^\top M^\top x}\Bigg]^\top$$

in $L^2[0,1)^d$, where the matrix function $\mathbf{G}(x)$ is given in (12) and $\mathbb{F}(x)$ denotes the vector $\mathbb{F}(x):=[F(x),F(x+M^{-\top}i_2),\dots,F(x+M^{-\top}i_{\det M})]^\top$.

In order to recover the function F, let $[d_1(x),\dots,d_s(x)]$ be a vector with entries in $L^\infty[0,1)^d$ such that

$$\big[d_1(x),\dots,d_s(x)\big]\mathbf{G}(x)=[1,0,\dots,0] \quad \text{a.e. in } [0,1)^d.$$

Later, we will show that a necessary and sufficient condition for the existence of such a vector is that $A_\mathbf{G}>0$. As a consequence, we get

$$F(x)=(\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)d_j(x)e^{-2\pi i\alpha^\top M^\top x} \quad \text{in } L^2[0,1)^d. \tag{17}$$

Finally, the isomorphism $\mathcal{T}_\varphi$ gives

$$f(t)=(\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)(\mathcal{T}_\varphi d_j)(t-M\alpha), \quad t\in\mathbb{R}^d,$$

where we have used the shifting property (6) and that the shift-invariant space V_φ^2 is a RKHS. In addition, much more can be said about the above sampling expansion. In fact, the following result holds:

Theorem 1. *Assume that the functions g_j given in* (9) *belong to $L^\infty[0,1)^d$ for each $j=1,2,\dots,s$. Let $\mathbf{G}(x)$ be the associated matrix defined in $[0,1)^d$ as in* (12). *The following statements are equivalents*:

(a) $A_\mathbf{G}>0$;
(b) *The set of systems $\{\mathcal{L}_1,\mathcal{L}_2,\dots,\mathcal{L}_s\}$ is an M-stable filtering sampler for V_φ^2*;
(c) *There exists a vector $[d_1(x),\dots,d_s(x)]$ with entries $d_j\in L^\infty[0,1)^d$ satisfying*

$$\big[d_1(x),\dots,d_s(x)\big]\mathbf{G}(x)=[1,0,\dots,0] \quad \textit{a.e. in } [0,1)^d; \tag{18}$$

(d) *There exists a frame for V_φ^2 having the form $\{S_j(\cdot - M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ such that for any $f \in V_\varphi^2$*

$$f = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)S_j(\cdot - M\alpha) \quad \text{in } L^2\big(\mathbb{R}^d\big). \tag{19}$$

In case the equivalent conditions are satisfied we have that the reconstruction functions S_j, $j = 1,2,\dots,s$, in the sampling formula (19) *are necessarily given through a vector $[d_1(x),\dots,d_s(x)]$ satisfying* (18)*, by*

$$S_j(t) = \sum_{\alpha\in\mathbb{Z}^d}\widehat{d}_j(\alpha)\varphi(t-\alpha), \quad t\in\mathbb{R}^d, \tag{20}$$

where $\widehat{d}_j(\alpha)$, $\alpha\in\mathbb{Z}^d$, are the Fourier coefficients of d_j, i.e., $d_j(x) = \sum_{\alpha\in\mathbb{Z}^d}\widehat{d}_j(\alpha)e^{-2\pi i\alpha^\top x}$. The sampling series in (19) *also converges absolutely and uniformly on $\mathbb{R}^d$.*

If $s = \det M$ then the sequence $\{S_j(\cdot - M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ is a Riesz basis for V_φ^2 and the sampling functions S_j, $j = 1,2,\dots,s$, satisfy the interpolation property $(\mathcal{L}_{j'}S_j)(M\alpha) = \delta_{j,j'}\delta_{\alpha,0}$, where $j, j' = 1,2,\dots,s$ and $\alpha\in\mathbb{Z}^d$.

Proof. Part (c) in Lemma 1 proves that conditions (a) and (b) are equivalent.

If $A_{\mathbf{G}} > 0$ then $\operatorname{ess\,inf}_{x\in[0,1)^d}\det[\mathbf{G}^*(x)\mathbf{G}(x)] > 0$ and, consequently, there exists the pseudo-inverse matrix $\mathbf{G}^\dagger(x) = [\mathbf{G}^*(x)\mathbf{G}(x)]^{-1}\mathbf{G}^*(x)$; its entries are essentially bounded and its first row satisfies (18); therefore (a) implies (c).

If $[d_1(x),\dots,d_s(x)]$ satisfies (18) with $d_j\in L^\infty[0,1)^d$, we have proved earlier that formula (19) holds in $L^2(\mathbb{R}^d)$ where S_j is equal to $\mathcal{T}_\varphi d_j$ or, equivalently, is given by (20). Since we have assumed that $g_j\in L^\infty[0,1)^d$ for each $j = 1,2,\dots,s$, Lemma 1(b) proves that $\{\overline{g_j(x)}e^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ is a Bessel sequence in $L^2[0,1)^d$. The same argument proves that $\{(\det M)d_j(x)e^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ is also a Bessel sequence in $L^2[0,1)^d$. These two Bessel sequences satisfy (see (11) and (17)):

$$F(x) = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big\langle F, \overline{g_j(x)}e^{-2\pi i\alpha^\top M^\top x}\big\rangle_{L^2[0,1)^d}\, d_j(x)e^{-2\pi i\alpha^\top M^\top x}, \quad F\in L^2[0,1)^d.$$

Hence, they form a pair of dual frames for $L^2[0,1)^d$ (see [5, Lemma 5.6.2]). Since $S_j(t-M\alpha) = \mathcal{T}_\varphi[d_j(\cdot)e^{-2\pi i\alpha^\top M^\top\cdot}](t)$ and $\mathcal{T}_\varphi$ is an isomorphism, the sequence $\{S_j(t-M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ is a frame for V_φ^2; hence (c) implies (d).

Notice that since we have assumed that $\{\overline{g_j(x)}e^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ is a Bessel sequence with bound $B_{\mathbf{G}}/(\det M)$ and $(\mathcal{L}_j f)(M\alpha) = \langle F, \overline{g_j(x)}e^{-2\pi i\alpha^\top M^\top x}\rangle_{L^2[0,1)^d}$, we have

$$\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big|\mathcal{L}_j f(M\alpha)\big|^2 \leqslant \frac{B_{\mathbf{G}}}{\det M}\|F\|^2 \leqslant \frac{B_{\mathbf{G}}\|\mathcal{T}_\varphi^{-1}\|^2}{\det M}\|f\|^2, \quad f\in V_\varphi^2.$$

If $\{S_j(t-M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ is a frame for V_φ^2, then formula (19) gives a stable way to recover $f\in V_\varphi^2$ from its generalized samples. Indeed,

$$\|f\|^2 = (\det M)^2\left\|\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)S_j(\cdot - M\alpha)\right\|^2 \leqslant (\det M)^2 C\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}\big|\mathcal{L}_j f(M\alpha)\big|^2,$$

where C is a Bessel bound for $\{S_j(t-M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$. Hence the set $\{\mathcal{L}_1,\mathcal{L}_2,\dots,\mathcal{L}_s\}$ is an M-stable filtering sampler for V_φ^2. Therefore (d) implies (b).

The pointwise convergence in the sampling series is absolute due to the unconditional convergence of a frame expansion; it is uniform on $\mathbb{R}^d$ as a consequence of (5).

If $s = \det M$ then, according to Lemma 1(d), the frame $\{S_j(\cdot - M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s} = \{\mathcal{T}_\varphi d_j e^{-2\pi i\alpha^\top M^\top\cdot}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ is a Riesz basis for V_φ^2. Applying formula (19) for $f = S_{j'}$ and having in mind the uniqueness of the coefficients in a Riesz basis, we get the interpolatory property $(\mathcal{L}_{j'}S_j)(M\alpha) = \delta_{j,j'}\delta_{\alpha,0}$. □

The equivalence between conditions (a), (b) and (d) in Theorem 1 was established in [3] for average sampling, at the lattice $\mathbb{Z}^d$, in finitely-generated shift-invariant spaces by using another techniques. Notice that our generalized sampling on the more general sampling lattice $M\mathbb{Z}^d$ can be seen as a problem of generalized sampling in a finitely-generated shift-invariant space on the sampling lattice $\mathbb{Z}^d$. Indeed, the generalized sampling of the functions $f = \sum_{\alpha\in\mathbb{Z}^d}a_\alpha\varphi(\cdot-\alpha)$ at $M\mathbb{Z}^d$ can be thought as a bounded map from $\ell(\mathbb{Z}^d)$ to $(\ell^2(M\mathbb{Z}^d))^s$:

$$\{a_\alpha\}_{\alpha\in\mathbb{Z}^d} \mapsto \left\{\sum_{\alpha\in\mathbb{Z}^d}a_\alpha\mathcal{L}_j\varphi(M\beta-\alpha)\right\}_{1\leqslant j\leqslant s,\, \beta\in\mathbb{Z}^d},$$

or also as

$$\{a_\alpha^t\}_{1\leqslant t\leqslant \det M,\,\alpha\in\mathbb{Z}^d}\mapsto\left\{\sum_{t=1}^{\det M}\sum_{\alpha\in\mathbb{Z}^d}a_\alpha^t\mathcal{J}_j\varphi_t(\beta-\alpha)\right\}_{1\leqslant j\leqslant s,\,\beta\in\mathbb{Z}^d},$$

where $\mathcal{J}_j f(u):=[\mathcal{L}_j\{f(u+M^{-1}\cdot)\}](0)$, $\varphi_t(\cdot):=\varphi(M\cdot-j_t)$, and $\{j_1,j_2,\dots,j_{\det M}\}=\mathbb{Z}^d\cap(M[0,1)^d)$, which can be seen as generalized sampling at $\mathbb{Z}^d$ of the functions f having the form: $f=\sum_{t=1}^{\det M}\sum_{\alpha\in\mathbb{Z}^d}a_\alpha^t\varphi_t(\cdot-\alpha)$. The sampling formulas (19), explicitly given by using (c), are the novelty of the result proved here.

The solutions of (18) with entries in $L^\infty[0,1)^d$ are exactly the first row of the $(\det M)\times s$ matrices of the form

$$\mathbf{D}(x)=\mathbf{G}^\dagger(x)+\mathbf{U}(x)\big[\mathbf{I}_s-\mathbf{G}(x)\mathbf{G}^\dagger(x)\big], \tag{21}$$

where $\mathbf{G}^\dagger(x)$ is the pseudo-inverse matrix of $\mathbf{G}(x)$, $\mathbf{G}^\dagger(x)=[\mathbf{G}^*(x)\mathbf{G}(x)]^{-1}\mathbf{G}^*(x)$, and $\mathbf{U}(x)$ is an arbitrary $(\det M)\times s$ matrix with entries in $L^\infty[0,1)^d$. Indeed, if the vector $[d_1(x),\dots,d_s(x)]$ satisfies (18), it can be easily checked that the $(\det M)\times s$ matrix $\mathbf{D}(x):=[d_j(x+M^{-\top}i_k)]_{\substack{k=1,2,\dots,\det M\\ j=1,2,\dots,s}}$ is a left inverse of the matrix $\mathbf{G}(x)$, and it can be expressed in the form (21) by taking $\mathbf{U}(x)=\mathbf{D}(x)$. Conversely, any matrix of the form (21) is a left inverse of $\mathbf{G}(x)$ and its first row satisfies (18).

Finally, notice that if the functions g_j, $j=1,2,\dots,s$, are continuous on $\mathbb{R}^d$, the condition (a) in Theorem 1 reads: $\operatorname{rank}\mathbf{G}(x)=\det M$ for all $x\in\mathbb{R}^d$ (see (16)).

3.2. Irregular sampling: jitter error

Given an error sequence $\varepsilon:=\{\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\,j=1,2,\dots,s}$ in $\mathbb{R}^d$, the aim in this section is to study when it is possible to recover any function $f\in V_\varphi^2$ from the sequence of perturbed samples $\{(\mathcal{L}_jf)(M\alpha+\varepsilon_{j,\alpha})\}_{\alpha\in\mathbb{Z}^d,\,j=1,2,\dots,s}$. Having in mind expression (7) for the systems $\mathcal{L}_j$, $j=1,2,\dots,s$, for $f=\mathcal{T}_\varphi F\in V_\varphi^2$ we have

$$(\mathcal{L}_jf)(M\alpha+\varepsilon_{j,\alpha})=\big\langle F,(\overline{Z\mathcal{L}_j\varphi})(\varepsilon_{j,\alpha},\cdot)e^{-2\pi i\alpha^\top M^\top\cdot}\big\rangle_{L^2[0,1)^d},\quad \alpha\in\mathbb{Z}^d, \tag{22}$$

where we have used that $(Z\mathcal{L}_j\varphi)(M\beta+\varepsilon_{j,\beta},x)=(Z\mathcal{L}_j\varphi)(\varepsilon_{j,\beta},x)e^{2\pi i\beta^\top M^\top x}$ for any $\beta\in\mathbb{Z}^d$. Eq. (22) leads us to study the frame property of the perturbed sequence $\{(\overline{Z\mathcal{L}_j\varphi})(\varepsilon_{j,\alpha},\cdot)e^{-2\pi i\alpha^\top M^\top\cdot}\}_{\alpha\in\mathbb{Z}^d,\,j=1,2,\dots,s}$ in $L^2[0,1)^d$.

On the other hand, we know that, whenever $0<A_{\mathbf{G}}\leqslant B_{\mathbf{G}}<\infty$, the sequence $\{(\overline{Z\mathcal{L}_j\varphi})(0,\cdot)e^{-2\pi i\alpha^\top M^\top\cdot}\}_{\alpha\in\mathbb{Z}^d,\,j=1,2,\dots,s}$ is a frame for $L^2[0,1)^d$ with optimal frame bounds $A_{\mathbf{G}}/(\det M)$ and $B_{\mathbf{G}}/(\det M)$. In the case of $s=\det M$, the above sequence is a Riesz basis for $L^2[0,1)^d$.

One possibility is to use frame perturbation theory in order to find the suitable error sequences for which the sequence $\{(\overline{Z\mathcal{L}_j\varphi})(\varepsilon_{j,\alpha},\cdot)e^{-2\pi i\alpha^\top M^\top\cdot}\}_{\alpha\in\mathbb{Z}^d,\,j=1,2,\dots,s}$ is a frame for $L^2[0,1)^d$. The following result on frame perturbation, which proof can be found in [5, p. 354] will be used later:

Lemma 2. *Let $\{f_n\}_{n=1}^\infty$ be a frame for the Hilbert space $\mathcal{H}$ with frame bounds A, B, and let $\{g_n\}_{n=1}^\infty$ be a sequence in $\mathcal{H}$. If there exists a constant $R<A$ such that*

$$\sum_{n=1}^\infty\big|\langle f_n-g_n,f\rangle\big|^2\leqslant R\|f\|^2\quad\text{for each } f\in\mathcal{H},$$

then $\{g_n\}_{n=1}^\infty$ is a frame for $\mathcal{H}$ with bounds $A(1-\sqrt{R/A})^2$ and $B(1+\sqrt{R/B})^2$.

If $\{f_n\}_{n=1}^\infty$ is a Riesz basis, then $\{g_n\}_{n=1}^\infty$ is a Riesz basis.

Given an error sequence $\varepsilon:=\{\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\,j=1,2,\dots,s}\subset\mathbb{R}^d$ we define on $\ell^2(\mathbb{Z}^d)$ the operator $D_\varepsilon=[D_{\varepsilon,1},\dots,D_{\varepsilon,s}]$, where

$$D_{\varepsilon,j}c:=\left\{\sum_{\beta\in\mathbb{Z}^d}\big[\mathcal{L}_j\varphi(M\alpha-\beta+\varepsilon_{j,\alpha})-\mathcal{L}_j\varphi(M\alpha-\beta)\big]c_\beta\right\}_{\alpha\in\mathbb{Z}^d}$$

for each $c=\{c_\beta\}_{\beta\in\mathbb{Z}^d}\in\ell^2(\mathbb{Z}^d)$. The operator norm is defined as usual

$$\|D_\varepsilon\|:=\sup_{c\in\ell^2(\mathbb{Z}^d)\setminus\{0\}}\frac{\|D_\varepsilon c\|_{\ell_s^2(\mathbb{Z}^d)}}{\|c\|_{\ell^2(\mathbb{Z}^d)}},$$

where $\|D_\varepsilon c\|_{\ell_s^2(\mathbb{Z}^d)}^2:=\sum_{j=1}^s\|D_{\varepsilon,j}c\|_{\ell^2(\mathbb{Z}^d)}^2$ for each $c\in\ell^2(\mathbb{Z}^d)$.

Theorem 2. *Assume that $g_j \in L^\infty[0,1)^d$ for $j=1,2,\dots,s$ with $A_{\mathbf{G}} > 0$. If the error sequence $\varepsilon := \{\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ satisfies the inequality $\|D_\varepsilon\|^2 < A_{\mathbf{G}}/(\det M)$, then there exists a frame $\{S^\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ for V^2_φ such that, for any $f \in V^2_\varphi$*

$$f(t) = \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha + \varepsilon_{j,\alpha}) S^\varepsilon_{j,\alpha}(t), \quad t \in \mathbb{R}^d, \tag{23}$$

where the convergence of the series is in the $L^2(\mathbb{R}^d)$-sense, absolute and uniform on $\mathbb{R}^d$. Moreover, when $s = \det M$ the sequence $\{S^\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ is a Riesz basis for V^2_φ, and the interpolation property $(\mathcal{L}_l S^\varepsilon_{j,\alpha})(M\beta + \varepsilon_{j,\beta}) = \delta_{j,l}\delta_{\alpha,\beta}$ holds.

Proof. The sequence $\{(\overline{Z\mathcal{L}_j\varphi})(0,\cdot)e^{-2\pi i\alpha^\top M^\top\cdot}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ is a frame (a Riesz basis if $s=\det M$) for $L^2[0,1)^d$ with frame (Riesz) bounds $A_{\mathbf{G}}/(\det M)$ and $B_{\mathbf{G}}/(\det M)$. For any $F(x) = \sum_{\gamma\in\mathbb{Z}^d} c_\gamma e^{-2\pi i\gamma^\top x}$ in $L^2[0,1)^d$ we have

$$\begin{aligned}
&\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d} \left|\left\langle (\overline{Z\mathcal{L}_j\varphi})(\varepsilon_{j,\alpha},\cdot)e^{-2\pi i\alpha^\top M^\top\cdot} - (\overline{Z\mathcal{L}_j\varphi})(0,\cdot)e^{-2\pi i\alpha^\top M^\top\cdot}, F(\cdot)\right\rangle_{L^2[0,1)^d}\right|^2 \\
&\quad= \sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d} \left|\left\langle \sum_{\beta\in\mathbb{Z}^d} \left[\overline{\mathcal{L}_j\varphi}(\beta+\varepsilon_{j,\alpha}) - \overline{\mathcal{L}_j\varphi}(\beta)\right] e^{-2\pi i(M\alpha-\beta)^\top\cdot}, \sum_{\gamma\in\mathbb{Z}^d} c_\gamma e^{-2\pi i\gamma^\top\cdot}\right\rangle_{L^2[0,1)^d}\right|^2 \\
&\quad= \sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d} \left| \sum_{\beta\in\mathbb{Z}^d} \left[\overline{\mathcal{L}_j\varphi}(\beta+\varepsilon_{j,\alpha}) - \overline{\mathcal{L}_j\varphi}(\beta)\right]\bar{c}_{M\alpha-\beta}\right|^2 \\
&\quad= \sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d} \left| \sum_{\beta\in\mathbb{Z}^d} \left[\mathcal{L}_j\varphi(M\alpha-\beta+\varepsilon_{j,\alpha}) - \mathcal{L}_j\varphi(M\alpha-\beta)\right]c_\beta\right|^2 \\
&\quad= \sum_{j=1}^{s} \left\|D_{\varepsilon,j}\{c_\gamma\}_{\gamma\in\mathbb{Z}^d}\right\|^2_{\ell^2(\mathbb{Z}^d)} \leqslant \|D_\varepsilon\|^2 \left\|\{c_\gamma\}_{\gamma\in\mathbb{Z}^d}\right\|^2_{\ell^2(\mathbb{Z}^d)} = \|D_\varepsilon\|^2\|F\|^2_{L^2[0,1)^d}.
\end{aligned}$$

By using Lemma 2, the sequence $\{(\overline{Z\mathcal{L}_j\varphi})(\varepsilon_{j,\alpha},\cdot)e^{-2\pi i\alpha^\top M^\top\cdot}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ is a frame for $L^2[0,1)^d$ (a Riesz basis if $s = \det M$). Let $\{h^\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ be its canonical dual frame. Hence, for any $F \in L^2[0,1)^d$ we have

$$F = \sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d} \left\langle F(\cdot), (\overline{Z\mathcal{L}_j\varphi})(\varepsilon_{j,\alpha},\cdot)e^{-2\pi i\alpha^\top M^\top\cdot}\right\rangle_{L^2[0,1)^d} h^\varepsilon_{j,\alpha} = \sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha+\varepsilon_{j,\alpha})h^\varepsilon_{j,\alpha} \quad \text{in } L^2[0,1)^d.$$

Applying the isomorphism $\mathcal{T}_\varphi$, one gets (23) in $L^2(\mathbb{R}^d)$ where $S^\varepsilon_{j,\alpha} = \mathcal{T}_\varphi h^\varepsilon_{j,\alpha}$. Since $\mathcal{T}_\varphi$ is an isomorphism between $L^2[0,1)^d$ and V^2_φ, the sequence $\{S^\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ is a frame for V^2_φ (a Riesz basis if $s = \det M$).

Pointwise convergence in the sampling series is absolute due to the unconditional character of a frame. The uniform convergence on $\mathbb{R}^d$ is a consequence of the reproducing property (5) in V^2_φ. The interpolatory property in the case $s = \det M$ follows from the uniqueness of the coefficients with respect to a Riesz basis. □

Formula (23) in Theorem 2 is useless from a practical point of view, since the frame $\{S^\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$, which depends on the error sequence $\{\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$, is impossible to determine. As a consequence, in order to recover any function $f \in V^2_\varphi$ from the samples $\{\mathcal{L}_j f(M\alpha+\varepsilon_{j,\alpha})\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ we should use the frame algorithm (see [10]). In order to approximate the sequence $\{a_\alpha\}_{\alpha\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$ defining $f \in V^2_\varphi$, the frame algorithm can be implemented in the $\ell^2(\mathbb{Z}^d)$ setting as in Ref. [13].

Following the techniques in [13] (see also Refs. [9,26]), whenever the generator φ and the impulse responses of the systems $\mathcal{L}_j$, $j=1,2,\dots,s$, are compactly supported one could obtain a bound for $\|D_\varepsilon\|$ in terms of $\delta := \sup_{j,\alpha}\|\varepsilon_{j,\alpha}\|_\infty$. Finally, it is worth to mention the recent related Refs. [6,22,33].

3.3. L^2-approximation properties

We denote by $W^r_2(\mathbb{R}^d) := \{f\colon\ \|D^\gamma f\|_2 < \infty,\ |\gamma| \leqslant r\}$ the usual Sobolev space, and by $|f|_{j,2} := \sum_{|\beta|=j}\|D^\beta f\|_2$, $0 \leqslant j \leqslant r$, the corresponding seminorm of a function $f \in W^r_2(\mathbb{R}^d)$. We assume here that all the systems $\mathcal{L}_j$, $j=1,2,\dots,s$, are of type (a), i.e., $\mathcal{L}_j f = \mathsf{h}_j * f$, belonging the impulse response h_j to the Hilbert space $\mathcal{L}^2(\mathbb{R}^d)$. Recall that a Lebesgue measurable function $\mathsf{h}: \mathbb{R}^d \to \mathbb{C}$ belongs to the Hilbert space $\mathcal{L}^2(\mathbb{R}^d)$ if

$$|\mathsf{h}|_2 := \left(\int_{[0,1)^d} \left(\sum_{\alpha\in\mathbb{Z}^d} \left|\mathsf{h}(t-\alpha)\right| \right)^2 dt \right)^{1/2} < \infty.$$

Notice that $\mathcal{L}^2(\mathbb{R}^d) \subset L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Moreover, $\|\{\mathsf{h} * f(\alpha)\}_{\alpha\in\mathbb{Z}^d}\|_2 \leqslant |\mathsf{h}|_2 \|f\|_2$ (see [19, Theorem 3.1]); thus the sequence of generalized samples $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ belongs to $\ell^2(\mathbb{Z}^d)$ for any $f \in L^2(\mathbb{R}^d)$. Besides, we assume that the generator φ satisfies the Strang–Fix conditions of order r, i.e.,

$$\widehat{\varphi}(0) \neq 0, \qquad D^\beta \widehat{\varphi}(\alpha) = 0, \quad |\beta| < r, \ \alpha \in \mathbb{Z}^d \setminus \{0\}.$$

Given a vector $\mathbf{d} := [d_1, \dots, d_s]$ with entries in $L^\infty[0,1)^d$ and satisfying (18), first we note that the operator $\Gamma_{\mathbf{d}} : (L^2(\mathbb{R}^d), \|\cdot\|_2) \to (V_\varphi^2, \|\cdot\|_2)$ given by

$$(\Gamma_{\mathbf{d}} f)(t) := (\det M) \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j,\mathbf{d}}(t - M\alpha), \quad t \in \mathbb{R}^d,$$

is a well-defined bounded operator onto V_φ^2. Besides, $\Gamma_{\mathbf{d}} f = f$ for all $f \in V_\varphi^2$.

Under appropriate hypotheses we prove that the scaled operator $\Gamma_{\mathbf{d}}^h := \sigma_h \Gamma_{\mathbf{d}} \sigma_{1/h}$, where $\sigma_h f := f(\cdot/h)$ for $h > 0$, approximates, in the L^2-norm sense, any function f in the Sobolev space $W_2^r(\mathbb{R}^d)$ as $h \to 0^+$. Concretely we have:

Theorem 3. *Assume that* $\operatorname{ess\,sup}_{t\in\mathbb{R}^d} \sum_{\alpha\in\mathbb{Z}^d} |\varphi(t+\alpha)|(1+|t+\alpha|)^r < \infty$ *for some* $r \in \mathbb{N}$. *Let* $\mathbf{d}$ *be a vector with entries in* $L^\infty[0,1)^d$ *and satisfying* (18). *If the generator* φ *satisfies the Strang–Fix conditions of order* r, *then, for each* $f \in W_2^r(\mathbb{R}^d)$ *and* $h > 0$, *the* L^2-*approximation error satisfies*

$$\left\| f - \Gamma_{\mathbf{d}}^h f \right\|_2 \leqslant K |f|_{r,2} h^r,$$

where the constant K *is independent of* f *and* h.

Proof. Using that $\Gamma_{\mathbf{d}}^h \xi = \xi$ for each $\xi \in \sigma_h V_\varphi^2$ then, for each $f \in L^2(\mathbb{R}^d)$ and $\xi \in \sigma_h V_\varphi^2$, Lebesgue's Lemma [7, p. 30] gives

$$\left\| f - \Gamma_{\mathbf{d}}^h f \right\|_2 \leqslant \left(1 + \|\Gamma_{\mathbf{d}}\|\right) \min_{\xi\in\sigma_h V_\varphi^2} \|f - \xi\|_2,$$

where we have used that $\|\Gamma_{\mathbf{d}}^h\| = \|\Gamma_{\mathbf{d}}\|$. Now, for each $f \in W_2^r(\mathbb{R}^d)$ and $h > 0$ there exists a function $\xi_h \in \sigma_h V_\varphi^2$ such that

$$\|\xi_h - f\|_2 \leqslant \widetilde{K} |f|_{r,2} h^r,$$

where the constant $\widetilde{K}$ is independent of f and h (see [21, Theorem 5.2]), from which we obtain the desired result. □

Notice that the approximation property given in Theorem 3 is similar to those given by integral operators in [21].

4. Decay properties of the reconstruction functions

For the efficiency and stability of the reconstruction process given in Theorem 1, it is very desirable for the reconstruction functions S_j, $j = 1, 2, \dots, s$ to be well localized; see the papers [11,16,24] and the references therein. In this section we study two particular cases, reconstruction functions with exponential decay and reconstruction functions with compact support, by using directly formulas (20).

Thus we prove that whenever the generator φ and the functions $\mathcal{L}_j \varphi$, $j = 1, 2, \dots, s$, decay exponentially fast, there are many sampling formulas like (19) involving reconstruction functions S_j with exponential decay, i.e., there exist constants $C > 0$ and $q \in (0,1)$ such that

$$\left|S_j(t)\right| \leqslant C q^{|t|}, \quad t \in \mathbb{R}^d.$$

First we introduce some complex notation, more convenient for this study. We denote $\mathbf{z}^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \dots z_d^{\alpha_d}$ for $\mathbf{z} = [z_1, \dots, z_d] \in \mathbb{C}^d$, $\alpha = [\alpha_1, \dots, \alpha_d] \in \mathbb{Z}^d$, and the d-torus by $T^d := \{\mathbf{z} \in \mathbb{C}^d\colon\ |z_1| = |z_2| = \dots = |z_d| = 1\}$. We define

$$\mathsf{g}_j(\mathbf{z}) := \sum_{\mu\in\mathbb{Z}^d} \mathcal{L}_j \varphi(\mu) \mathbf{z}^{-\mu}, \qquad \mathsf{G}(\mathbf{z}) := \left[\mathsf{g}_j\left(\left[z_1 e^{2\pi i m_1^\top i_k}, \dots, z_d e^{2\pi i m_d^\top i_k}\right]\right)\right]_{\substack{j=1,2,\dots,s \\ k=1,2,\dots,\det M}},$$

where $m_1, \dots, m_d$ denote the columns of the matrix M^{-1}. Note that for the vector $\mathbf{z} = [e^{2\pi i x_1}, \dots, e^{2\pi i x_d}]$ we have $\mathbf{G}(x) = \mathsf{G}(\mathbf{z})$. Provided that the functions g_j are continuous on $\mathbb{R}^d$, we have the following result: There exists a vector $\mathsf{d} = [\mathsf{d}_1, \dots, \mathsf{d}_s]$ with entries essentially bounded in T^d and satisfying

$$\mathsf{d}(\mathbf{z})\mathsf{G}(\mathbf{z}) = [1, 0, \dots, 0] \quad \text{for all } \mathbf{z} \in T^d \tag{24}$$

if and only if

$$\operatorname{rank} \mathsf{G}(\mathbf{z}) = \det M \quad \text{for all } \mathbf{z} \in T^d. \tag{25}$$

For $j = 1, 2, \dots, s$, the corresponding reconstruction function $S_{j,\mathsf{d}}$ in the sampling formula (19) is

$$S_{j,\mathsf{d}}(t) = \sum_{\alpha \in \mathbb{Z}^d} \widehat{\mathsf{d}_j}(\alpha)\varphi(t - \alpha), \tag{26}$$

where $\widehat{\mathsf{d}_j}(\alpha)$, $\alpha \in \mathbb{Z}^d$, are the Laurent coefficients of the functions d_j, i.e., $\mathsf{d}_j(\mathbf{z}) = \sum_{\alpha \in \mathbb{Z}^d} \widehat{\mathsf{d}_j}(\alpha)\mathbf{z}^{-\alpha}$.

Let $\mathcal{H}$ denote the algebra of all holomorphic functions in a neighborhood of the d-torus T^d. Note that the elements in $\mathcal{H}$ are characterized as admitting a Laurent series where the sequence of coefficients decays exponentially fast [19].

The following theorem shows that, whenever the generator φ and the functions $\mathcal{L}_j\varphi$, $j = 1, 2, \dots, s$ have exponential decay, if the vector d has entries in $\mathcal{H}$ then the reconstruction function $S_{j,\mathsf{d}}$ has also exponential decay. It also proves that condition (25) is also sufficient for the existence of a vector d with entries in $\mathcal{H}$ and satisfying (24). Its proof uses the standard technique for proving extensions of Wiener $1/f$ Lemma in group algebras.

Theorem 4. *Assume that the generator φ and the functions $\mathcal{L}_j\varphi$, $j = 1, 2, \dots, s$, have exponential decay. Then, there exists a vector* $\mathsf{d} = [\mathsf{d}_1, \dots, \mathsf{d}_s]$ *with entries in $\mathcal{H}$ and satisfying* $\mathsf{d}(\mathbf{z})\mathsf{G}(\mathbf{z}) = [1, 0, \dots, 0]$ *for all $\mathbf{z} \in T^d$ if and only if* $\operatorname{rank} \mathsf{G}(\mathbf{z}) = \det M$ *for all $\mathbf{z} \in T^d$.*

In this case, all of such vectors d *are given as the first row of a* $(\det M) \times s$ *matrix* $\mathsf{D}(\mathbf{z})$ *of the form*

$$\mathsf{D}(\mathbf{z}) = \mathsf{G}^\dagger(\mathbf{z}) + \mathsf{U}(\mathbf{z})\big[\mathsf{I}_s - \mathsf{G}(\mathbf{z})\mathsf{G}^\dagger(\mathbf{z})\big], \tag{27}$$

where $\mathsf{U}(\mathbf{z})$ *is any* $(\det M) \times s$ *matrix with entries in $\mathcal{H}$ and* $\mathsf{G}^\dagger(\mathbf{z}) := [\mathsf{G}^*(\mathbf{z})\mathsf{G}(\mathbf{z})]^{-1}\mathsf{G}^*(\mathbf{z})$. *The corresponding reconstruction functions $S_{j,\mathsf{d}}$, $j = 1, 2, \dots, s$, given by* (26) *have exponential decay.*

Proof. The hypotheses say that $\mathsf{g}_j \in \mathcal{H}$, $j = 1, 2, \dots, s$; thus $\det[\mathsf{G}^*(\mathbf{z})\mathsf{G}(\mathbf{z})] \in \mathcal{H}$. Assuming that $\operatorname{rank} \mathsf{G}(\mathbf{z}) = \det M$ for all $\mathbf{z} \in T^d$ we have that $\det[\mathsf{G}^*(\mathbf{z})\mathsf{G}(\mathbf{z})] \neq 0$ for all $\mathbf{z} \in T^d$ and then, the matrix $[\mathsf{G}^*(\mathbf{z})\mathsf{G}(\mathbf{z})]^{-1}$ has entries in $\mathcal{H}$. As a consequence, the entries of $\mathsf{G}^\dagger(\mathbf{z}) = [\mathsf{G}^*(\mathbf{z})\mathsf{G}(\mathbf{z})]^{-1}\mathsf{G}^*(\mathbf{z})$ belong to $\mathcal{H}$. Now it is easy to check, as we did in Section 3, that all the vectors d with entries in $\mathcal{H}$ and satisfying (24) are given as the first row of matrices $\mathsf{D}(\mathbf{z})$ satisfying (27), where the entries of $\mathsf{U}(\mathbf{z})$ belong to $\mathcal{H}$.

Since $\mathsf{d}_j \in \mathcal{H}$, $j = 1, 2, \dots, s$, their Laurent coefficients $\widehat{\mathsf{d}_j}(\alpha)$ have exponential decay, i.e., there exist $C > 0$ and $q \in (0, 1)$ such that $|\widehat{\mathsf{d}_j}(\alpha)| \leqslant Cq^{|\alpha|}$, $\alpha \in \mathbb{Z}^d$, $j = 1, 2, \dots, s$. Without loss of generality, we can also assume that $|\varphi(t)| \leqslant Cq^{|t|}$, for all $t \in \mathbb{R}^d$; then the reconstruction functions $S_{j,\mathsf{d}}(t) = \sum_{\alpha \in \mathbb{Z}^d} \widehat{\mathsf{d}_j}(\alpha)\varphi(t - \alpha)$, $j = 1, 2, \dots, s$, satisfy

$$\big|S_{j,\mathsf{d}}(t)\big| \leqslant C \sum_{\alpha \in \mathbb{Z}^d} q^{|\alpha|}\big|\varphi(t - \alpha)\big| \leqslant C^2\bigg(\sum_{\alpha \in \mathbb{Z}^d} q^{2|\alpha|}\bigg)q^{|t|}, \quad t \in \mathbb{R}^d. \qquad \square$$

Notice that, in particular, the solution obtained from the pseudo-inverse matrix $\mathsf{G}^\dagger(\mathbf{z})$, which is unique in the case $s = \det M$, gives reconstruction functions $S_{j,\mathsf{d}}$ with exponential decay.

4.1. Reconstruction functions with compact support

If the generator φ has compact support, then the reconstruction function $S_{j,\mathsf{d}}$ has compact support whenever the sum in (26) is finite. As a consequence, in order to get generalized sampling formulas with reconstruction functions of compact support, we have to solve the following problem: Given a matrix $\mathbf{G}(z)$ whose entries are Laurent polynomials, we have to find a left inverse matrix $\mathbf{D}(z)$ of $\mathbf{G}(z)$ whose entries are also Laurent polynomials. This problem is related to the perfect reconstruction FIR (finite impulse response) filter-banks theory, and it has been studied by several authors. See, for instance, [23,34,35] and the references therein. From [34, Theorems 5.1 and 5.6] the following result can be easily deduced:

Theorem 5. *Let* $\mathsf{G}(\mathbf{z})$ *be an $s \times m$ matrix whose entries are Laurent polynomials. Then, there exists an $m \times s$ matrix* $\mathsf{D}(\mathbf{z})$ *whose entries are also Laurent polynomials satisfying* $\mathsf{D}(\mathbf{z})\mathsf{G}(\mathbf{z}) = \mathsf{I}_m$ *if and only if* $\operatorname{rank} \mathsf{G}(\mathbf{z}) = m$ *for all* $\mathbf{z} \in (\mathbb{C} \setminus \{0\})^d$.

From this theorem, we derive the following corollary:

Corollary 1. *Assume that the generator φ and the functions $\mathcal{L}_j\varphi$, $j = 1, 2, \dots, s$, have compact support. Then, there exists a vector* $\mathsf{d} = [\mathsf{d}_1, \dots, \mathsf{d}_s]$ *whose entries are Laurent polynomials and satisfying* $\mathsf{d}(\mathbf{z})\mathsf{G}(\mathbf{z}) = [1, 0, \dots, 0]$ *if and only if*

$$\operatorname{rank} \mathsf{G}(\mathbf{z}) = \det M \quad \textit{for all } \mathbf{z} \in \big(\mathbb{C} \setminus \{0\}\big)^d.$$

The reconstruction functions $S_{j,\mathsf{d}}$, $j = 1, 2, \dots, s$, obtained from such vectors d *by* (26) *have compact support.*

A vector $\mathsf{d}(\mathbf{z})$ satisfying $\mathsf{d}(\mathbf{z})\mathsf{G}(\mathbf{z}) = [1, 0, \ldots, 0]$ whose entries are Laurent polynomials can be obtained by solving a linear system whose unknowns are precisely the coefficients of $\mathsf{d}_j(\mathbf{z})$, $j = 1, 2, \ldots, s$. From one of these vectors, say $\widetilde{\mathsf{d}} = [\widetilde{\mathsf{d}}_1, \ldots, \widetilde{\mathsf{d}}_s]$, we can get all of them. Indeed, it is easy to check that they are given by the first row of the $(\det M) \times s$ matrices of the form

$$\mathsf{D}(\mathbf{z}) = \widetilde{\mathsf{D}}(\mathbf{z}) + \mathsf{U}(\mathbf{z})\big[\mathsf{I}_s - \mathsf{G}(\mathbf{z})\widetilde{\mathsf{D}}(\mathbf{z})\big], \tag{28}$$

where $\widetilde{\mathsf{D}}(\mathbf{z}) := [\widetilde{\mathsf{d}}_j([z_1 e^{2\pi i m_1^\top i_k}, \ldots, z_d e^{2\pi i m_d^\top i_k}])]_{\substack{k=1,2,\ldots,\det M\\ j=1,2,\ldots,s}}$ and $\mathsf{U}(\mathbf{z})$ is any $(\det M) \times s$ matrix with Laurent polynomial entries.

The interested reader can find in [23,34,35] methods to check if the condition in the theorem holds, and also another method to find a particular solution $\widetilde{\mathsf{D}}(\mathbf{z})$ of (28). Both involve the use of Gröbner bases.

Finally, notice that having reconstruction functions with compact support implies low computational complexity and truncation errors are avoided. A related topic is the local reconstruction in shift-invariant spaces which invokes only finite samples to reconstruct a function on a bounded interval: See the two recent papers [25,27].

5. Sampling formulas in V_φ^∞

The aim in this section is to validate the sampling formulas

$$f(t) = (\det M) \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j,\mathbf{d}}(t - M\alpha), \quad t \in \mathbb{R}^d,$$

obtained in Section 3 for the shift-invariant space V_φ^2, in a larger space. To this end, assume that the generator $\varphi \in \mathcal{L}^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$. Recall that a Lebesgue measurable function $\phi : \mathbb{R}^d \to \mathbb{C}$ belongs to the Banach space $\mathcal{L}^\infty(\mathbb{R}^d)$ if

$$|\phi|_\infty := \operatorname*{ess\,sup}_{t \in [0,1)^d} \sum_{\alpha \in \mathbb{Z}^d} \big|\phi(t - \alpha)\big| < \infty.$$

For $1 \leqslant p \leqslant \infty$ we have that $\mathcal{L}^\infty(\mathbb{R}^d) \subset L^p(\mathbb{R}^d)$; in particular, $\mathcal{L}^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$. Observe that if there are constants $C > 0$ and $\delta > 0$ such that

$$\big|\phi(t)\big| \leqslant \frac{C}{1 + |t|^{d+\delta}}, \quad t \in \mathbb{R}^d,$$

then $\phi \in \mathcal{L}^\infty(\mathbb{R}^d)$.

Let V_φ^∞ be the L^∞-closure of the linear span of the integer translates of φ, i.e.,

$$V_\varphi^\infty := \overline{\operatorname{span}}_{L^\infty}\big\{\varphi(t - \alpha):\ \alpha \in \mathbb{Z}^d\big\}.$$

As the integer translates of φ are ℓ^2-stable (they form a Riesz sequence in $L^2(\mathbb{R}^d)$), then this space can be expressed as $V_\varphi^\infty = \{\varphi *' a:\ a \in \mathsf{c}_0(\mathbb{Z}^d)\}$, where $\varphi *' a$ denotes the semi-discrete convolution $\sum_{\alpha \in \mathbb{Z}^d} a(\alpha)\phi(\cdot - \alpha)$ and $\mathsf{c}_0(\mathbb{Z}^d)$ denotes the space of sequences on $\mathbb{Z}^d$ vanishing at ∞ (see the proof of Lemma 5.1 in [21]). As a consequence, V_φ^∞ is a set of continuous functions on $\mathbb{R}^d$ and the set inclusion $V_\varphi^2 \subset V_\varphi^\infty$ holds.

5.1. Generalized regular sampling in V_φ^∞

Let $\mathcal{A}$ be the set of functions of the form $f(x) = \sum_{\alpha \in \mathbb{Z}^d} a(\alpha) e^{-2\pi i \alpha^\top x}$ with $a \in \ell^1(\mathbb{Z}^d)$. The space $\mathcal{A}$, normed by $\|f\|_{\mathcal{A}} := \|a\|_1$ and with pointwise multiplication is a commutative Banach algebra. If $f \in \mathcal{A}$ and $f(x) \neq 0$ for every $x \in \mathbb{R}^d$, the function $1/f$ is also in $\mathcal{A}$ by Wiener's Lemma.

Consider s linear time-invariant systems $\mathcal{L}_j$, $j = 1, 2, \ldots, s$, as in Section 2. In addition, assume that $D^\beta \varphi \in \mathcal{L}^\infty(\mathbb{R}^d) \cap C(\mathbb{R}^d)$, $|\beta| \leqslant m$, where m is the largest order among the partial derivatives appearing in the systems of type (b) ($m = 0$ if no partial derivatives appear).

Thus we have that $\{\mathcal{L}_j \varphi(\alpha)\}_{\alpha \in \mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$ for the systems of the type (b). This is also true for the systems of type (a) since $\|\{\varphi * \mathsf{h}(\alpha)\}\|_1 \leqslant \|\mathsf{h}\|_1 |\varphi|_\infty$ (see [19, Theorem 3.1]). As a consequence, the Fourier transforms of these sequences, which are precisely the functions g_j, $j = 1, 2, \ldots, s$, defined in (9), belong to the algebra $\mathcal{A}$. The next result describes when Eq. (18) has a solution $\mathbf{d}$ with entries in the algebra $\mathcal{A}$:

Lemma 3. *There exists a vector $\mathbf{d} = [d_1, \ldots, d_s]$ with entries d_j in the algebra $\mathcal{A}$, $j = 1, 2, \ldots, s$, and satisfying*

$$\mathbf{d}(x)\mathbf{G}(x) = [1, 0, \ldots, 0], \quad x \in [0, 1)^d \tag{29}$$

if and only if $\operatorname{rank} \mathbf{G}(x) = \det M$ *for all* $x \in \mathbb{R}^d$.

Proof. The proof is the same as the one in [14, Lemma 1] although for a slightly different matrix **G**. □

For any vector **d** satisfying the above lemma, Theorem 1 gives the corresponding sampling formula in V_φ^2:

$$f(t) = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)S_{j,\mathbf{d}}(t-M\alpha), \quad t\in\mathbb{R}^d, \tag{30}$$

where $S_{j,\mathbf{d}} = \mathcal{T}_\varphi d_j$, $j=1,2,\dots,s$. In particular, formula (30) holds for the space $\operatorname{span}\{\varphi(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d}$.

The reconstruction functions $S_{j,\mathbf{d}}$, $j=1,2,\dots,s$, are determined from the Fourier coefficients of d_j, i.e., $\widehat{d}_j(\alpha) := \int_{[0,1)^d} d_j(x)e^{2\pi i\alpha^\top x}\,dx$, $\alpha\in\mathbb{Z}^d$. More specifically,

$$S_{j,\mathbf{d}}(t) = \sum_{\alpha\in\mathbb{Z}^d}\widehat{d}_j(\alpha)\varphi(t-\alpha), \quad t\in\mathbb{R}^d. \tag{31}$$

Since $\widehat{d}_j\in\ell^1(\mathbb{Z}^d)$, $j=1,2,\dots,s$, and $\varphi\in\mathcal{L}^\infty(\mathbb{R}^d)$, we obtain that the reconstruction functions $S_{j,\mathbf{d}}\in\mathcal{L}^\infty(\mathbb{R}^d)$, $j=1,2,\dots,s$ (notice that $|\phi *' a|_\infty \leqslant |\phi|_\infty\|a\|_1$, see [19, Theorem 2.1]).

By using a density argument, in the next theorem we extend the sampling formula (30) to the whole space V_φ^∞ in a pointwise sense.

Theorem 6. *Let $\mathbf{d}=[d_1,\dots,d_s]$ be a vector with entries d_j in the algebra $\mathcal{A}$, $j=1,2,\dots,s$, and satisfying* (29). *Then, for any $f\in V_\varphi^\infty$ the following sampling formula holds pointwise*:

$$f(t) = (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)S_{j,\mathbf{d}}(t-M\alpha), \quad t\in\mathbb{R}^d, \tag{32}$$

where the reconstruction function $S_{j,\mathbf{d}}$, given by (31), *belongs to $\mathcal{L}^\infty(\mathbb{R}^d)$ for $j=1,2,\dots,s$.*

Moreover, assuming that φ, $D^\beta\varphi\in C_0(\mathbb{R}^d)$ for $|\beta|\leqslant m$, (i.e., φ and its derivatives are continuous on $\mathbb{R}^d$ vanishing at infinity), then the convergence of the sampling series in (32) *is also absolute and uniform on $\mathbb{R}^d$.*

Proof. Consider the Banach space $C_b^m(\mathbb{R}^d)$ of all functions f which, together with all their partial derivatives $D^\beta f$ of order $|\beta|\leqslant m$, are continuous and bounded on $\mathbb{R}^d$ with the norm $\|f\|_{C_b^m} := \max_{|\beta|\leqslant m}\sup_{t\in\mathbb{R}^d}|D^\beta f(t)|$.

For any vector **d** with entries in $\mathcal{A}$ and satisfying (29) there exists a constant $K>0$ such that, for each $f\in C_b^m(\mathbb{R}^d)$,

$$\bigl|(\Gamma_{\mathbf{d}} f)(t)\bigr| \leqslant K\|f\|_{C_b^m} \quad \text{for all } t\in\mathbb{R}^d, \tag{33}$$

where $(\Gamma_{\mathbf{d}} f)(t) := (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)S_{j,\mathbf{d}}(t-M\alpha)$. For the proof, see [14, Lemma 2].

Let $f\in V_\varphi^\infty$ and $a\in c_0(\mathbb{Z}^d)$ such that $f(t)=\sum_{\alpha\in\mathbb{Z}^d}a(\alpha)\varphi(t-\alpha)$. For $n\in\mathbb{N}$ we define

$$f_n(t) := \sum_{|\alpha|\leqslant n}a(\alpha)\varphi(t-\alpha).$$

From the assumptions on φ we have that $f_n\in C_b^m(\mathbb{R}^d)$. Moreover, for $|\beta|\leqslant m$ and $n>l>0$, we have

$$\bigl|D^\beta(f_n-f_l)(t)\bigr| \leqslant \sum_{l<|\alpha|\leqslant n}\bigl|a(\alpha)\bigr|\bigl|D^\beta\varphi(t-\alpha)\bigr| \leqslant \sup_{l<|\alpha|\leqslant n}\bigl|a(\alpha)\bigr|\bigl|D^\beta\varphi\bigr|_\infty, \quad t\in\mathbb{R}^d.$$

Since the sequence $a\in c_0(\mathbb{Z}^d)$, $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in the Banach space $C_b^m(\mathbb{R}^d)$, we deduce that f_n converges in the C_b^m-norm to f as $n\to\infty$. In particular $f\in C_b^m(\mathbb{R}^d)$. Using that the sampling formula holds for $f_n\in\operatorname{span}\{\varphi(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d}$ and inequality (33) we obtain that, for all $t\in\mathbb{R}^d$,

$$0\leqslant\bigl|f_n(t)-\Gamma_{\mathbf{d}}f(t)\bigr| = \bigl|\bigl[\Gamma_{\mathbf{d}}(f_n-f)\bigr](t)\bigr| \leqslant K\|f_n-f\|_{C_b^m}\to 0 \quad \text{as } n\to\infty,$$

and then $\Gamma_{\mathbf{d}}f(t)=f(t)$ for all $t\in\mathbb{R}^d$. This proves that the sampling formula (32) holds pointwise.

It remains to prove the absolute and uniform convergence of the series in (32). Let $|\beta|\leqslant m$. Assuming that $D^\beta\varphi\in C_0(\mathbb{R}^d)$ we have that $D^\beta f_k\in C_0(\mathbb{R}^d)$. Since $D^\beta f_M$ converges uniformly to $D^\beta f$ on $\mathbb{R}^d$, and $C_0(\mathbb{R}^d)$ is a closed subspace in $L^\infty(\mathbb{R}^d)$, we obtain that $D^\beta f\in C_0(\mathbb{R}^d)$. From this fact and using the Lebesgue dominated convergence theorem (whenever $\mathcal{L}_j$ is a system of type (a)), we obtain that $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d}\in c_0(\mathbb{R}^d)$ for each $j=1,2,\dots,s$. Hence, by using that $S_{j,\mathbf{d}}\in\mathcal{L}^\infty(\mathbb{R}^d)$ and the inequality

$$\sum_{|\alpha|>n} \left|(\mathcal{L}_j f)(M\alpha) S_{j,\mathbf{d}}(t - M\alpha)\right| \leqslant \sup_{|\alpha|>n} \left|(\mathcal{L}_j f)(M\alpha)\right| |S_{j,\mathbf{d}}|_\infty, \quad t \in \mathbb{R}^d, \ n \in \mathbb{N},$$

we obtain that the series in (32) converges absolutely and uniformly on $\mathbb{R}^d$. $\square$

Observe that, under the assumed hypotheses, in the proof of the theorem we have obtained that

$$V_\varphi^\infty \subset C_b^m(\mathbb{R}^d). \tag{34}$$

In the case that the continuous functions φ and $D^\beta\varphi$, $|\beta| \leqslant m$, belong to the Wiener space $W(L^\infty, \ell^1) := \{f\colon \sum_{n\in\mathbb{Z}} \|f\chi_{[n,n+1)}\|_\infty < \infty\}$, then the generator φ and its derivatives $D^\beta\varphi$, $|\beta| \leqslant m$, belong to $\mathcal{L}^\infty(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$.

Finally, notice that our space V_φ^∞ differs from the shift-invariant space $V_\infty(\varphi)$ introduced in [2]. Following this reference, under appropriate hypotheses, a similar sampling result can be proved for functions in $V_\infty(\varphi)$ having locally uniform convergence.

5.2. L^∞-approximation properties

We denote by $W_\infty^r(\mathbb{R}^d) := \{f\colon \ \|D^\gamma f\|_\infty < \infty, \ |\gamma| \leqslant r\}$ the usual Sobolev space, and by $|f|_{j,\infty} := \sum_{|\beta|=j} \|D^\beta f\|_\infty$, $0 \leqslant j \leqslant r$, the corresponding seminorm of a function $f \in W_\infty^r(\mathbb{R}^d)$. In [17], Jia proved the following remarkable result about approximation by means of a quasi-projection operator (see [17] for the most general L^p version):

Theorem 7. *Let $\phi \in W_\infty^j(\mathbb{R}^d)$, $0 \leqslant j < r$ and $\widetilde{\phi} \in L^1(\mathbb{R}^d)$ be compactly supported functions and let Q be the quasi-projection operator given by*

$$(Qf)(t) := \sum_{\alpha\in\mathbb{Z}^d} \langle f, \widetilde{\phi}(\cdot - \alpha)\rangle_{L^2(\mathbb{R}^d)} \phi(t - \alpha). \tag{35}$$

If $Q\pi = \pi$ for every polynomial π of degree at most $r - 1$, then

$$|f - Q_h f|_{j,\infty} \leqslant K h^{r-j} |f|_{r,\infty}, \quad f \in W_\infty^r(\mathbb{R}^d),$$

where $Q_h := \sigma_h Q \sigma_{1/h}$ and K is a constant independent of $h > 0$ and f.

From this result, and assuming that the generator φ satisfies the Strang–Fix conditions of order r, we deduce by using Theorem 6 that, for any function $f \in C_0(\mathbb{R}^d) \cap W_\infty^r(\mathbb{R}^d)$

$$\left\|\Gamma_{\mathbf{d}}^h f - f\right\|_\infty = \mathcal{O}\left(h^{r-m}\right) \quad \text{as } h \to 0^+.$$

Theorem 8. *Assume that φ is a compactly supported generator in $W_\infty^r(\mathbb{R}^d)$ where $r > m$, with m being the largest order of the partial derivatives appearing in the systems $\mathcal{L}_j$. Let $\mathbf{d} = [d_1, \ldots, d_s]$ be a vector with entries in the algebra $\mathcal{A}$ and satisfying* (29)*. If the generator φ satisfies the Strang–Fix conditions of order r then, for each $f \in C_0(\mathbb{R}^d) \cap W_\infty^r(\mathbb{R}^d)$ and $0 < h \leqslant 1$, the following inequality holds*:

$$\left\|f - \Gamma_{\mathbf{d}}^h f\right\|_\infty \leqslant K |f|_{r,\infty} h^{r-m},$$

where $\Gamma_{\mathbf{d}}^h := \sigma_h \Gamma_{\mathbf{d}} \sigma_{1/h}$ and the constant K is independent of f and h.

Proof. From Theorem 6 we have that $\Gamma_{\mathbf{d}}^h \xi = \xi$, for any $\xi \in \sigma_h V_\varphi^\infty$ and $h > 0$. From (34), $\xi \in C_b^m(\mathbb{R}^d)$, and from (33), there exists a constant $M > 0$ such that $\|\Gamma_{\mathbf{d}} l\|_\infty \leqslant M\|l\|_{C_b^m}$ for all $l \in C_b^m(\mathbb{R}^d)$. Hence, for any $f \in C_b^m(\mathbb{R}^d)$ and $0 < h \leqslant 1$, we obtain that

$$\begin{aligned}\left\|f - \Gamma_{\mathbf{d}}^h f\right\|_\infty &\leqslant \|f - \xi\|_\infty + \left\|\xi - \Gamma_{\mathbf{d}}^h f\right\|_\infty = \|f - \xi\|_\infty + \left\|\Gamma_{\mathbf{d}}^h(\xi - f)\right\|_\infty = \|f - \xi\|_\infty + \left\|\Gamma_{\mathbf{d}} \sigma_{1/h}(\xi - f)\right\|_\infty \\ &\leqslant \|f - \xi\|_\infty + M\left\|\sigma_{1/h}(\xi - f)\right\|_{C_b^m} \leqslant \|f - \xi\|_\infty + M\|\xi - f\|_{C_b^m} \leqslant (1 + M)\|\xi - f\|_{C_b^m}, \quad \xi \in \sigma_h V_\varphi^\infty,\end{aligned} \tag{36}$$

where we have used that $\sigma_{1/h}(\xi - f) \in C_b^m(\mathbb{R}^d)$.

Given $\phi = \varphi$ satisfying the Strang–Fix conditions of order r, there exists a compactly supported function $\widetilde{\phi} \in L^1(\mathbb{R}^d)$ satisfying the conditions of Theorem 7. An example of such a function $\widetilde{\phi}$ can be found in [21, Theorem 6.1]. Let Q be the quasi-projection operator defined in (35). Note that for $f \in C_0(\mathbb{R}^d)$ we have that $\{\langle f, \widetilde{\phi}(\cdot - \alpha)\rangle_{L^2(\mathbb{R}^d)}\}_{\alpha\in\mathbb{Z}^d} \in c_0(\mathbb{Z}^d)$ and hence $Qf \in V_\varphi^\infty$. Moreover, from Theorem 7, for $j = 0, 1, \ldots, m$, we have that

$$|f - Q_h f|_{j,\infty} \leqslant K_j h^{r-j} |f|_{r,\infty}, \quad f \in W_\infty^r(\mathbb{R}^d),$$

where the constants K_j, $0 \leqslant j \leqslant m$, are independent of f and h. By using (36), for any $f \in C_0(\mathbb{R}^d) \cap W_\infty^r(\mathbb{R}^d)$, we obtain

$$\left\| f - \Gamma_{\mathbf{d}}^h f \right\|_\infty \leqslant C \inf_{\xi \in \sigma_h V_\varphi^\infty} \|\xi - f\|_{C_b^m} \leqslant C \| Q_h f - f\|_{C_b^m} = C \max_{|\beta| \leqslant m} \left\| D^\beta Q_h f - D^\beta f \right\|_\infty \leqslant C \sum_{j=0}^{m} |Q_h f - f|_{j,\infty}$$
$$\leqslant C |f|_{r,\infty} \sum_{j=0}^{m} K_j h^{r-j} \leqslant C \left(\sum_{j=0}^{m} K_j \right) |f|_{r,\infty} h^{r-m},$$

where the constant C is independent of f and h. $\square$

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