



Asymptotics for polynomials orthogonal over the unit disk with respect to a positive polynomial weight

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ABSTRACT

We derive asymptotics for polynomials orthogonal over the complex unit disk with respect to a weight of the form $|h(z)|^2$, with $h(z)$ a polynomial without zeros in $|z| < 1$. The behavior of the polynomials is established at every point of the complex plane. The proofs are based on adapting to the unit disk a technique of J. Szabados for the asymptotic analysis of polynomials orthogonal over the unit circle with respect to the same type of weight.

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1. Introduction and results

Asymptotic results for polynomials of a complex variable that are orthogonal over the unit disk are rather scarce, and those existing have been mainly obtained in the more general context of orthogonality over Jordan domains, without taking any possible advantage that the unit disk, as a simpler orthogonality domain, could afford. In contrast, for polynomials orthogonal over the unit circle, a rich theory has been developed [11,7,8], and particularly, their asymptotic properties have been established under very general conditions on the orthogonality measure (e.g. Szegő's condition, Rakhmanov's condition).

As we show in this paper, it is possible, however, that some techniques that work well for the unit circle be equally satisfactory for dealing with orthogonality over the unit disk.

In a notable paper [10], J. Szabados obtained asymptotic formulae for polynomials orthogonal over the unit circle $|z| = 1$ with respect to a weight of the form $|h(z)|^2$, $h(z)$ an arbitrary polynomial. Szabados's paper had the novelty of exhibiting a weight on the unit circle of considerable generality, for which the asymptotic behavior of the corresponding orthonormal polynomials is determined at every point of the complex plane. Moreover, his formulas enabled him to establish very fine results on the limiting distribution of the zeros of the polynomials.

In the present paper, we retake the ideas of Szabados and suitably modify them to derive asymptotics for polynomials $\{p_n(z)\}_{n=0}^\infty$ that are orthonormal over the unit disk $\mathbb{D} := \{z: |z| < 1\}$ with respect to a weight of the form $|h_m(z)|^2/\pi$, with $h_m(z)$ a polynomial of degree $m \geq 1$ without zeros in \mathbb{D} . These are uniquely determined by the conditions

$$p_n(z) = \lambda_n z^n + \text{lower degree terms}, \quad \lambda_n > 0, \quad n \geq 0, \quad (1)$$

$$\frac{1}{\pi} \int_{\mathbb{D}} p_n(z) \overline{p_l(z)} |h_m(z)|^2 dx dy = \delta_{n,l}, \quad n, l \geq 0. \quad (2)$$

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Our asymptotic formulas describe the behavior of $p_n(z)$ as $n \rightarrow \infty$ at every point z of the complex plane. They are very similar in structure to those obtained by Szabados in [10], and as such, they also yield fine results on the zero distribution of the p_n 's.

An essential difference between orthogonality over the unit circle and orthogonality over the unit disk, both with respect to a weight of the form $|h(z)|^2$, $h(z)$ an arbitrary polynomial, is that in the former one, it is irrelevant whether some zeros of $h(z)$ lie inside the unit circle or not, because replacing a zero of $h(z)$ by its reflection about the unit circle only changes the value of $|h(z)|^2$ on $|z| = 1$ by a multiplicative constant. This is certainly not true on $|z| < 1$, which makes the case of orthogonality over the unit disk more interesting and challenging.

Generalizations of Szabados's results have been recently carried out in [1,2], thus we expect analogous generalizations for the unit disk to be possible as well. Some earlier results by Suetin, Korovkin and Smirnov, that are of relevance to our investigation, are discussed at the end of this section in Remarks 5 and 7.

We shall preserve the notation of [10], as this will benefit the reader when referring to some facts from that paper and comparing our asymptotic formulas with those obtained therein.

Let $h_m(z)$ be a monic polynomial of degree $m \geq 1$ without zeros in the open unit disk, that is

$$h_m(z) = (z - z_1)^{m_1} (z - z_2)^{m_2} \cdots (z - z_s)^{m_s}, \quad s \geq 1,$$

where the z_k 's are pairwise distinct and $|z_k| \geq 1$, $1 \leq k \leq s$. Here $m_k \geq 1$ is the multiplicity of the zero z_k , so that

$$m_1 + m_2 + \cdots + m_s = m.$$

Define

$$h_m^*(z) := \prod_{k=1}^s (1 - z\bar{z}_k)^{m_k}, \quad z_k^* := 1/\bar{z}_k, \quad \kappa_m := h_m(0) = \prod_{k=1}^s (-z_k)^{m_k},$$

and the important quantities

$$\rho := \max_{1 \leq k \leq s} |z_k^*|, \quad M := \max_{|z_k^*|=\rho} m_k.$$

The following two theorems give asymptotic formulas for the orthonormal polynomials $p_n(z)$ defined by (1) and (2). These formulas hold uniformly as $n \rightarrow \infty$ on closed subsets of the specified sets. First, we consider the case of $h_m(z)$ having zeros on the unit circle.

Theorem 1. If $\rho = 1$, then

$$\begin{aligned} p_n(z) &= \frac{|\kappa_m| \sqrt{n} z^{n+m}}{\kappa_m h_m^*(z)} + O(n^{-1/2} |z|^n), \quad |z| \geq 1, \quad z \neq z_k^*, \quad 1 \leq k \leq s; \\ p_n(z_k^*) &= \frac{|\kappa_m| z_k^{*n+m-m_k} n^{m_k+1/2}}{\kappa_m h_m^*(m_k)(z_k^*)^{(2m_k+1)/m_k}} + O(n^{m_k-1/2}), \quad |z_k^*| = 1, \quad 1 \leq k \leq s; \\ p_n(z) &= \frac{|\kappa_m| n^{-3/2}}{\kappa_m h_m(z)} \sum_{|z_k^*|=1} \frac{m_k(m_k+1) h_m^{(m_k)}(z_k^*) z_k^{*n+m+2}}{(-1)^{m_k} h_m^{*(m_k)}(z_k^*) (z - z_k^*)^2} + O(n^{-5/2}), \quad |z| < 1. \end{aligned} \quad (3)$$

If $h_m(z)$ has all its zeros outside the unit circle, then we have the following

Theorem 2. If $\rho < 1$, then

$$p_n(z) = \frac{|\kappa_m| \sqrt{n} z^{n+m}}{\kappa_m h_m^*(z)} + O(n^{-1/2} |z|^n), \quad |z| > \rho; \quad (4)$$

$$p_n(z) = \frac{|\kappa_m| M n^{M-3/2}}{\kappa_m h_m(z)} \sum_{\substack{|z_k^*|=\rho \\ m_k=M}} \frac{h_m(z_k^*) z_k^{*n+m+2-M}}{h_m^{(M)}(z_k^*) (z - z_k^*)^2} + O(n^{M-5/2} \rho^n), \quad |z| < \rho. \quad (5)$$

Remark 3. When $\rho < 1$, the behavior of $p_n(z)$ on the circle $|z| = \rho$ depends on the value of M and can be determined from the following more specific asymptotic formulas:

$$\frac{\kappa_m h_m(z) p_n(z)}{\sqrt{n} |\kappa_m|} = \frac{z^{n+m} h_m(z)}{h_m^*(z)} + Mn^{M-2} \sum_{\substack{|z_k^*|=\rho \\ m_k=M}} \frac{h_m(z_k^*) z_k^{*n+m+2-M}}{h_m^{*(M)}(z_k^*) (z - z_k^*)^2} \\ + O(n^{-1} |z|^n) + O(n^{M-3} \rho^n), \quad z \neq z_k^*, \quad 1 \leq k \leq s, \quad (6)$$

while for every z_μ^* with $|z_\mu^*| = \rho$,

$$\frac{\kappa_m h_m(z_\mu^*) p_n(z_\mu^*)}{|\kappa_m| \sqrt{n}} = \frac{h_m(z_\mu^*) z_\mu^{*n+m-m_\mu} n^{m_\mu}}{(m_\mu + 1) h_m^{*(m_\mu)}(z_\mu^*)} + Mn^{M-2} \sum_{\substack{|z_k^*|=\rho \\ m_k=M, k \neq \mu}} \frac{h_m(z_k^*) z_k^{*n+m+2-M}}{h_m^{*(M)}(z_k^*) (z_\mu^* - z_k^*)^2} \\ + O(\rho^n n^{m_\mu-1}) + O(\rho^n n^{M-3}). \quad (7)$$

For instance, we find from (6) that if $M = 1$, then (4) holds for $|z| \geq \rho$, $z \neq z_k^*$, $1 \leq k \leq s$, and that if $M \geq 3$, then (5) holds for $|z| \leq \rho$, $z \notin \{z_k^* : |z_k^*| = \rho\}$.

Remark 4. It is well known that asymptotic formulas with the structure of those in Theorems 1 and 2 yield fine results on the limiting distribution of the zeros of the polynomials they represent. For instance, we can deduce that for every closed set $K \subset \{z : |z| > \rho\}$, it is possible to find a number n_K such that every p_n of degree $n > n_K$ has no zeros in K . If \mathcal{Z} denotes the set of points $t \in \mathbb{C}$ such that every neighborhood of t contains zeros of infinitely many p_n 's, then $\{z : |z| = \rho\} \subset \mathcal{Z} \subset \{z : |z| \leq \rho\}$. It is possible to characterize $\mathcal{Z} \cap \{z : |z| < \rho\}$, a set that could be finite, or consisting of traces of algebraic curves, or a two-dimensional domain bounded by algebraic curves, all depending on the arguments of those z_k^* with largest modulus. Also, the sequence of normalized counting measures of the zeros of the p_n 's converges in the weak star topology to the normalized arc-length measure $(2\pi\rho)^{-1} |dz|$ of the circle $|z| = \rho$.

The procedure to derive these results is quite standard and can be found, for instance, in the original paper by Szabados [10], and the more recent ones [2,4].

Remark 5. Formula (4) of Theorem 2 is partially contained in a result by Suetin for polynomials orthogonal over the interior of an analytic Jordan curve with respect to a continuous weight. For the case of the unit disk, the statement of that result is as follows. Let $w(z)$ be a continuous weight function that is strictly positive and satisfies a Lipschitz condition of order $0 < \alpha < 1$ across $\overline{\mathbb{D}}$, and let $q_n(z)$ be the n th orthonormal polynomial with respect to $w(z)/\pi$ over the unit disk \mathbb{D} . Suetin proved [9, Thm. 3.3] that

$$q_n(z) = \sqrt{n} z^n D(z; w) (1 + O([\log n/n]^{\alpha/2})), \quad |z| > 1, \quad (8)$$

where

$$D(z; w) := \exp \left\{ \frac{1}{4\pi} \int_{|\zeta|=1} \log w(\zeta) \frac{\zeta + z}{\zeta - z} |d\zeta| \right\}, \quad |z| > 1$$

is the so-called exterior Szegő function for w . This is the unique function that is analytic and never zero in $|z| > 1$, positive at ∞ , and with boundary values on the unit circle that satisfy

$$|D(z; w)|^{-2} = w(z), \quad |z| = 1$$

(for more about Szegő functions, see [11, Chap. X]).

In the case $w(z) = |h_m(z)|^2$, with $h_m(z)$ a polynomial without zeros in $\overline{\mathbb{D}}$, we have

$$D(z; w) = \frac{|\kappa_m| z^m}{\kappa_m h_m^*(z)},$$

and (4) shows that, in this particular case, (8) indeed holds on the larger domain $|z| > \rho$, with the error term decaying faster.

Remark 6. The estimates given in Theorems 1 and 2 for the rate of decay of the O -error terms are best possible. This can be seen by analyzing the simpler case of $h_m(z)$ having just one zero. In fact, the proof of these two theorems yields that if z_1 is an arbitrary complex number (not necessarily with $|z_1| \geq 1$), then the polynomials $p_n(z)$ orthonormal over \mathbb{D} with respect to the weight $|z - z_1|^2/\pi$ admit the representation

$$p_n(z) = \frac{(1 - n\bar{z}_1^{n+2})^{-1/2}}{\sqrt{n+1}(z - z_1)} \left[(n+2)z^{n+1} - n \left(\frac{1 - (z\bar{z}_1)^{n+3}}{1 - z\bar{z}_1} \right)' \right], \quad n \geq 0, \quad (9)$$

where n is the unique constant that makes the right-hand side of (9) a polynomial. Hence we obtain after some computations that

$$p_n(z) = \frac{\sqrt{n+2} \sum_{k=0}^n (z_1^{n-k} \sum_{j=0}^k (j+1) |z_1|^{2j}) z^k}{\sqrt{\sum_{j=0}^{n+1} (j+1) |z_1|^{2j}} \sqrt{\sum_{j=0}^n (j+1) |z_1|^{2j}}}, \quad n \geq 0, \quad (10)$$

and for the leading coefficient,

$$\lambda_n = \sqrt{\frac{(n+2) \sum_{j=0}^n (j+1) |z_1|^{2j}}{\sum_{j=0}^{n+1} (j+1) |z_1|^{2j}}}, \quad n \geq 0. \quad (11)$$

For $z_1 = 1$, formula (10) was previously obtained by Suetin [9, Sec. V.6] using a different method.

Thus, for instance, by evaluating (3) and (4) at ∞ we obtain that if $|z_1| \geq 1$, then

$$\lambda_n = \sqrt{n+2} |z_1|^{-1} (1 + O(n^{-1})),$$

which is sharp, as we get from (11) that

$$\lambda_n = \frac{\sqrt{n+2}}{|z_1|} \sqrt{1 + \frac{(n+2)^{-1} (|z_1|^2 - 1) (1 + |z_1|^{-2n-2})}{(|z_1|^2 - 1) - (n+2)^{-1} + (n+2)^{-1} |z_1|^{-2n-2}}}.$$

Remark 7. If $h_m(z)$ is a polynomial of degree m with all its zeros lying in a closed disk $|z| \leq v < 1$, and if $p_n(z)$ is the n th orthonormal polynomial with respect to $|h_m(z)|^2/\pi$ over \mathbb{D} , then it is known that

$$p_n(z) = \frac{\sqrt{n+m+1} z^{n+m}}{h_m(z)} + O(n\tau^n), \quad v < |z| \leq 1, \quad (12)$$

with τ any number such that $v < \tau < 1$. In comparing (12) with (4), we note that the position of the zeros of $h_m(z)$ with respect to the unit circle (i.e. whether they are outside or inside) does have a drastic impact on the rate of decay of the error term.

Formula (12) follows from a result in the monograph by Smirnov and Lebedev [6] concerning orthogonality over a domain bounded by an analytic Jordan curve. When the domain is the unit disk, that result reduces to the following. Let $w(z) \geq 0$ be an integrable weight function in \mathbb{D} , such that for some number $0 < v < 1$,

$$w(z) = |d(z)|^2, \quad v < |z| < 1,$$

where $d(z)$ is some function analytic and without zeros in $v < |z| < \infty$, whose Laurent expansion in that annulus is of the form

$$d(z) = \sum_{k=u}^{\infty} \frac{a_k}{z^k}, \quad a_u > 0, \quad u \in \mathbb{Z}.$$

If $q_n(z)$ is the n th orthonormal polynomial with respect to $w(z)/\pi$ over \mathbb{D} , then for every $v < \tau < 1$,

$$q_n(z) = \frac{\sqrt{n-u+1} z^{n-u}}{d(z)} + O(n\tau^n)$$

uniformly as $n \rightarrow \infty$ on compact subsets of $v < |z| \leq 1$.

For $u = 0$, this was proven by Korovkin in [3], while the extension to an arbitrary integer u was carried out in [6, Sec. 3.3.4].

2. Proofs

We shall prove Theorems 1 and 2 in unison, after a number of lemmas have been established.

For integers $1 \leq k \leq s$ and $n \geq 0$, we consider the following m_k rational functions introduced by Szabados [10]:

$$r_{k,\ell,n}(z) := \left(\frac{z}{z - z_k^*} \right)^\ell - \left(\frac{z_k^*}{z} \right)^{n+m} \sum_{j=0}^{\ell-1} \binom{n+m+\ell}{j} \left(\frac{z_k^*}{z - z_k^*} \right)^{\ell-j}, \quad 1 \leq \ell \leq m_k. \quad (13)$$

The following facts are proven in [10]. For easier reference, we group them in a lemma.

Lemma 8.

(a) $z^{n+m}r_{k,\ell,n}(z)$ is a monic polynomial of exact degree $n+m$. Indeed,

$$z^{n+m}r_{k,\ell,n}(z) = \sum_{j=\ell}^{n+m+\ell} \binom{n+m+\ell}{j} (z_k^*)^{n+m+\ell-j} (z - z_k^*)^{j-\ell}. \quad (14)$$

(b) For any choice of numbers $a_{k,\ell,n}$, the rational function

$$\frac{z^{n+m} - \sum_{k=1}^s \sum_{\ell=1}^{m_k} a_{k,\ell,n} z^{n+m} r_{k,\ell,n}(z)}{h_m(z)}$$

is orthogonal to all integer powers z^p , $0 \leq p \leq n-1$, with respect to $|h_m(z)|^2 |dz|$ over the unit circle $\mathbb{T} = \{z: |z| = 1\}$.

(c) For integers $v \geq 0$, $1 \leq \mu \leq s$, we have that as $n \rightarrow \infty$

$$r_{k,\ell,n}^{(v)}(z_\mu) = \left[\left(\frac{z}{z - z_k^*} \right)^\ell \right]_{z=z_\mu}^{(v)} + O(n^{v+\ell-1} |z_k^*/z_\mu|^n), \quad z_\mu \neq z_k^*, \quad (15)$$

$$r_{\mu,\ell,n}^{(v)}(z_\mu) = \frac{(-1)^v z_\mu^{-v} n^{v+\ell}}{(\ell-1)!(v+\ell)} + O(n^{v+\ell-1}), \quad |z_\mu| = 1. \quad (16)$$

We now define the functions

$$R_{k,\ell,n}(z) := r_{k,\ell,n+1}(z) + \frac{zr'_{k,\ell,n+1}(z)}{n+m+1},$$

for which we have the following corresponding lemma.

Lemma 9.

(a) $z^{n+m}R_{k,\ell,n}(z)$ is a monic polynomial of exact degree $n+m$. Indeed,

$$z^{n+m}R_{k,\ell,n}(z) = \frac{[z^{n+m+1}r_{k,\ell,n+1}(z)]'}{n+m+1}. \quad (17)$$

(b) For any choice of numbers $c_{k,\ell,n}$, the rational function

$$\frac{z^{n+m} - \sum_{k=1}^s \sum_{\ell=1}^{m_k} c_{k,\ell,n} z^{n+m} R_{k,\ell,n}(z)}{h_m(z)}$$

is orthogonal to all integer powers z^p , $0 \leq p \leq n-1$, with respect to $|h_m(z)|^2 dx dy$ over the unit disk $\mathbb{D} = \{z: |z| < 1\}$.

(c) For integers $v \geq 0$ and $1 \leq \mu \leq s$, we have that as $n \rightarrow \infty$

$$R_{k,\ell,n}^{(v)}(z_\mu) = \left[\left(\frac{z}{z - z_k^*} \right)^\ell \right]_{z=z_\mu}^{(v)} + O(n^{v+\ell-1} |z_k^*/z_\mu|^n) + O(1/n), \quad z_\mu \neq z_k^*, \quad (18)$$

$$R_{\mu,\ell,n}^{(v)}(z_\mu) = \frac{(-1)^v z_\mu^{-v} n^{v+\ell}}{(\ell-1)!(v+\ell)(v+\ell+1)} + O(n^{v+\ell-1}), \quad |z_\mu| = 1. \quad (19)$$

Remark 10. We remark that both Lemmas 8(b) and 9(b) are indeed valid for a polynomial $h_m(z)$ whose roots z_k are arbitrary non-zero complex numbers (i.e. without assuming that $|z_k| \geq 1$).

Proof of Lemma 9. Part (a) follows from Lemma 8(a). For part (b), we use the complex version of Green's formula (see e.g. [5, pp. 240–241]) together with (17) and Lemma 8(b) to deduce that for every integer $0 \leq p \leq n-1$ (here $\mathbb{T} = \partial\mathbb{D}$),

$$\begin{aligned} & (n+m+1) \int_{\mathbb{D}} \left[\frac{z^{n+m} - \sum_{k=1}^s \sum_{\ell=1}^{m_k} c_{k,\ell,n} z^{n+m} R_{k,\ell,n}(z)}{h_m(z)} \right] z^p |h_m(z)|^2 dx dy \\ &= \int_{\mathbb{D}} \left[\frac{z^{n+m+1} - \sum_{k=1}^s \sum_{\ell=1}^{m_k} c_{k,\ell,n} z^{n+m+1} r_{k,\ell,n+1}(z)}{h_m(z)} \right]' z^p h_m(z) dx dy \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2i} \int_{\mathbb{T}} \overline{z^{n+m+1} - \sum_{k=1}^s \sum_{\ell=1}^{m_k} c_{k,\ell,n} z^{n+m+1} r_{k,\ell,n+1}(z) z^p h_m(z)} dz \\
&= \frac{1}{2} \int_{\mathbb{T}} \left[\frac{z^{n+m+1} - \sum_{k=1}^s \sum_{\ell=1}^{m_k} c_{k,\ell,n} z^{n+m+1} r_{k,\ell,n+1}(z)}{h_m(z)} \right] z^{p+1} |h_m(z)|^2 |dz| = 0.
\end{aligned}$$

As for part (c), if $v \geq 0$ is an integer, then

$$\begin{aligned}
R_{k,\ell,n}^{(v)}(z) &= r_{k,\ell,n+1}^{(v)}(z) + \frac{[zr'_{k,\ell,n+1}(z)]^{(v)}}{n+m+1} \\
&= r_{k,\ell,n+1}^{(v)}(z) + \frac{zr_{k,\ell,n+1}^{(v+1)}(z) + vr_{k,\ell,n+1}^{(v)}(z)}{n+m+1},
\end{aligned}$$

which combined with (15) readily yields (18), while combined with (16) yields

$$R_{\mu,\ell,n}^{(v)}(z_\mu) = \frac{(-1)^v z_\mu^{-v} (n+1)^{v+\ell}}{(\ell-1)!(v+\ell)} - \frac{(-1)^v z_\mu^{-v} (n+1)^{v+\ell}}{(\ell-1)!(v+\ell+1)} \left(1 - \frac{m}{n+m+1}\right) + O(n^{v+\ell-1}), \quad |z_\mu| = 1. \quad \square$$

Lemma 11. For n large enough, the linear system of equations

$$\sum_{k=1}^s \sum_{\ell=1}^{m_k} c_{k,\ell,n} R_{k,\ell,n}^{(v)}(z_\mu) = \delta_{v,0}, \quad v = 0, \dots, m_\mu - 1, \quad \mu = 1, \dots, s, \quad (20)$$

in the unknowns $c_{k,\ell,n}$ has a unique solution which satisfies

$$c_{k,\ell,n} = \begin{cases} b_{k,\ell} + O(1/n), & |z_k^*| < 1, \\ (\ell-1)!n^{-\ell}(b_{k,\ell} + O(1/n)), & |z_k^*| = 1 \end{cases} \quad (21)$$

as $n \rightarrow \infty$, where

$$b_{k,\ell} = \begin{cases} -\frac{(-\bar{z}_k)^{m_k-\ell}}{\kappa_m(m_k-\ell)!} \left[\frac{(1-\bar{z}_k^* z)^{m_k} h_m(1/z)}{h_m^*(1/z)} \right]_{z=\bar{z}_k}^{(m_k-\ell)}, & 1 \leq \ell \leq m_k, \quad |z_k^*| < 1, \\ -\ell \binom{m_k}{\ell} (-m_k-1) \frac{h_m^{(m_k)}(z_k^*)}{\kappa_m h_m^{*(m_k)}(z_k^*)}, & 1 \leq \ell \leq m_k, \quad |z_k^*| = 1. \end{cases} \quad (22)$$

Moreover,

$$\frac{h_m(z)}{\kappa_m h_m^*(z)} = 1 - \sum_{|z_k^*| < 1} \sum_{\ell=1}^{m_k} \frac{b_{k,\ell} z^\ell}{(z - z_k^*)^\ell}, \quad (23)$$

$$\sum_{\ell=1}^{m_k} b_{k,\ell} = \frac{(-1)^{m_k+1} m_k (m_k+1) h_m^{(m_k)}(z_k^*)}{\kappa_m h_m^{*(m_k)}(z_k^*)}, \quad |z_k^*| = 1 \quad (24)$$

and

$$\sum_{\ell=1}^{m_k} \frac{b_{k,\ell}}{(m_k+\ell)(m_k+\ell+1)} = \frac{(-1)^{m_k+1} h_m^{(m_k)}(z_k^*)}{\kappa_m h_m^{*(m_k)}(z_k^*) \binom{2m_k+1}{m_k}}, \quad |z_k^*| = 1. \quad (25)$$

Proof. Setting

$$b_{k,\ell,n} := \begin{cases} c_{k,\ell,n}, & |z_k^*| < 1, \\ c_{k,\ell,n} n^\ell / (\ell-1)!, & |z_k^*| = 1, \end{cases}$$

and using (18) and (19), the system (20) can be written as

$$\sum_{|z_k^*| < 1} \sum_{\ell=1}^{m_k} b_{k,\ell,n} \left\{ \left[\left(\frac{z}{z - z_k^*} \right)^{\ell} \right]_{z=z_\mu}^{(\nu)} + O(1/n) \right\} + \sum_{|z_k^*|=1} \sum_{\ell=1}^{m_k} b_{k,\ell,n} O(n^{-\ell}) = \delta_{v,0}, \quad v = 0, \dots, m_\mu - 1, |z_\mu| > 1, \quad (26)$$

$$\sum_{|z_k^*| < 1} \sum_{\ell=1}^{m_k} \frac{b_{k,\ell,n}}{n^\nu} \left\{ \left[\left(\frac{z}{z - z_k^*} \right)^{\ell} \right]_{z=z_\mu}^{(\nu)} + O(1/n) \right\} + \sum_{\substack{|z_k^*|=1 \\ k \neq \mu}} \sum_{\ell=1}^{m_k} b_{k,\ell,n} O(n^{-1}) + \sum_{\ell=1}^{m_\mu} b_{\mu,\ell,n} \left\{ \frac{(-1)^\nu z_\mu^{-\nu}}{(v+\ell)(v+\ell+1)} + O(n^{-1}) \right\} = \delta_{v,0}, \quad v = 0, \dots, m_\mu - 1, |z_\mu| = 1. \quad (27)$$

Letting $n \rightarrow \infty$ we obtain the following limiting systems of equations in the unknowns $b_{k,\ell}$:

$$\sum_{|z_k^*| < 1} \sum_{\ell=1}^{m_k} b_{k,\ell} \left[\left(\frac{z}{z - z_k^*} \right)^{\ell} \right]_{z=z_\mu}^{(\nu)} = \delta_{v,0}, \quad v = 0, \dots, m_\mu - 1, |z_\mu| > 1, \quad (28)$$

$$\sum_{\ell=1}^{m_\mu} \frac{b_{\mu,\ell}}{(v+\ell)(v+\ell+1)} = \delta_{v,0} \left(1 - \sum_{|z_k^*| < 1} \sum_{\ell=1}^{m_k} b_{k,\ell} \left(\frac{z_\mu}{z_\mu - z_k^*} \right)^\ell \right), \quad v = 0, \dots, m_\mu - 1, |z_\mu| = 1. \quad (29)$$

The system (28) has a solution if and only if numbers $b_{k,\ell}$ and A can be found satisfying

$$\sum_{|z_k^*| < 1} \sum_{\ell=1}^{m_k} b_{k,\ell} \left(\frac{z}{z - z_k^*} \right)^\ell - 1 = \frac{A \prod_{|z_k^*| < 1} (z - z_k^*)^{m_k}}{\prod_{|z_k^*| < 1} (z - z_k^*)^{m_k}} = \frac{A \bar{\kappa}_m h_m(z)}{h_m^*(z)}. \quad (30)$$

It is easy to see that the only such numbers are $A = -|\kappa_m|^{-2}$ and those given by (22) (case $|z_k^*| < 1$). Hence (23) follows from (30).

Then, from (30) we obtain that the system (29), which is actually made up of different independent systems, one for each μ with $|z_\mu| = 1$, takes the form

$$\sum_{\ell=1}^{m_\mu} \frac{b_{\mu,\ell}}{(v+\ell)(v+\ell+1)} = \delta_{v,0} \frac{h_m^{(m_\mu)}(z_\mu)}{\kappa_m h_m^{*(m_\mu)}(z_\mu)}, \quad v = 0, \dots, m_\mu - 1, |z_\mu| = 1. \quad (31)$$

For each fixed μ , the system (31) has also a unique solution and is solved in Appendix A at the end of this paper. Relations (A.1), (A.2) and (A.3) below in Appendix A show that the solution is given by (22) (case $|z_k^*| = 1$), and that (24) and (25) hold true.

Since the limiting system has a unique solution, so does the original system (20), provided n is large enough. Moreover, setting $\epsilon_{k,\ell,n} := b_{k,\ell,n} - b_{k,\ell}$, we obtain from (26), (27), (28) and (29) that

$$\sum_{|z_k^*| < 1} \sum_{\ell=1}^{m_k} \epsilon_{k,\ell,n} \left[\left(\frac{z}{z - z_k^*} \right)^{\ell} \right]_{z=z_\mu}^{(\nu)} = O(1/n), \quad v = 0, \dots, m_\mu - 1, |z_\mu| > 1, \\ \sum_{\ell=1}^{m_\mu} \frac{\epsilon_{\mu,\ell,n}}{(v+\ell)(v+\ell+1)} + \delta_{v,0} \sum_{|z_k^*| < 1} \sum_{\ell=1}^{m_k} \epsilon_{k,\ell,n} \left(\frac{z_\mu}{z_\mu - z_k^*} \right)^\ell = O(1/n), \quad v = 0, \dots, m_\mu - 1, |z_\mu| = 1,$$

whence we get by Cramer's formula that $\epsilon_{k,\ell,n} = O(1/n)$ as $n \rightarrow \infty$, $1 \leq \ell \leq m_k$, $1 \leq k \leq s$, proving (21). \square

Hereafter we use $c_{k,\ell,n}$ to denote the unique solutions of (20).

Lemma 12. For all n large enough,

$$p_n(z) = \frac{\alpha_n [z^{n+m} - \sum_{k=1}^s \sum_{\ell=1}^{m_k} c_{k,\ell,n} z^{n+m} R_{k,\ell,n}(z)]}{h_m(z)} \\ = \frac{\alpha_n (1 - \sum_{k=1}^s \sum_{\ell=1}^{m_k} c_{k,\ell,n} z^{n+m} + \dots)}{h_m(z)} \quad (32)$$

with

$$\alpha_n = \sqrt{\frac{n+m+1}{1 - \sum_{k=1}^s \sum_{\ell=1}^{m_k} c_{k,\ell,n}}} = \sqrt{n+m+1} |\kappa_m| (1 + O(1/n)) \quad (33)$$

as $n \rightarrow \infty$.

Proof. Let us compute the norm of the polynomial

$$q_n(z) := \frac{z^{n+m} - \sum_{k=1}^s \sum_{\ell=1}^{m_k} c_{k,\ell,n} z^{n+m} R_{k,\ell,n}(z)}{h_m(z)},$$

which we already know is orthogonal to all integer powers z^p , $0 \leq p \leq n-1$. For this, we make use of (13), (14) and Green's formula to get (as above, here $\mathbb{T} = \partial \mathbb{D}$)

$$\begin{aligned} & \frac{1}{\pi} \int_{\mathbb{D}} |q_n(z)|^2 |h_m(z)|^2 dx dy \\ &= \frac{1 - \sum_{k=1}^s \sum_{\ell=1}^{m_k} c_{k,\ell,n}}{\pi} \int_{\mathbb{D}} \overline{q_n(z)} z^n |h_m(z)|^2 dx dy \\ &= \frac{1 - \sum_{k=1}^s \sum_{\ell=1}^{m_k} c_{k,\ell,n}}{2\pi i (n+m+1)} \int_{\mathbb{T}} z^{n+m+1} \left[1 - \sum_{k=1}^s \sum_{\ell=1}^{m_k} c_{k,\ell,n} r_{k,\ell,n+1}(z) \right] z^n h_m(z) dz \\ &= \frac{1 - \sum_{k=1}^s \sum_{\ell=1}^{m_k} c_{k,\ell,n}}{n+m+1} \\ & \quad \times \left\{ \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{h_m(z) dz}{z^{m+1}} - \sum_{k=1}^s \sum_{\ell=1}^{m_k} \frac{\overline{c_{k,\ell,n}}}{2\pi i} \int_{\mathbb{T}} \frac{h_m(z) [z_k^\ell - (\frac{z}{z_k})^{n+m+1} \sum_{j=0}^{\ell-1} \binom{n+m+\ell+1}{j} z^{\ell-j} (z_k - z)^j]}{z^{m+1} (z_k - z)^\ell} dz \right\} \\ &= \frac{1 - \sum_{k=1}^s \sum_{\ell=1}^{m_k} c_{k,\ell,n}}{n+m+1}. \end{aligned}$$

Hence Lemma 12 will be proven once we verify that for n large, the real number $1 - \sum_{k=1}^s \sum_{\ell=1}^{m_k} c_{k,\ell,n}$ is different from zero. But this follows from (21) and letting $z \rightarrow \infty$ in (30). In details,

$$1 - \sum_{k=1}^s \sum_{\ell=1}^{m_k} c_{k,\ell,n} = 1 - \sum_{|z_k^*| < 1} \sum_{\ell=1}^{m_k} b_{k,\ell} + O(1/n) = |\kappa_m|^{-2} + O(1/n). \quad \square$$

We are now ready for the

Proof of Theorems 1 and 2. From (13) and (17), we get that

$$\begin{aligned} z^{n+m} R_{k,\ell,n}(z) &= \frac{z^{n+m+\ell}}{(z - z_k^*)^\ell} - \frac{z_k^* z^{n+m+\ell}}{(n+m+1)(z - z_k^*)^{\ell+1}} \\ & \quad + \frac{z_k^{*n+m+1}}{n+m+1} \sum_{j=0}^{\ell-1} \frac{\binom{n+m+\ell+1}{j} (l-j) z_k^{*\ell-j}}{(z - z_k^*)^{\ell+1-j}} \\ &= \frac{z^{n+m+\ell}}{(z - z_k^*)^\ell} + \frac{z_k^{*n+m+2} n^{\ell-2}}{(l-1)!(z - z_k^*)^2} \\ & \quad + O(n^{-1} |z|^n) + O(n^{\ell-3} |z_k^*|^n), \quad z \neq z_k^* \quad (n \rightarrow \infty), \end{aligned} \quad (34)$$

while from (14) we get

$$z^{n+m} R_{k,\ell,n}(z) = \frac{z_k^{*n+m} \binom{n+m+\ell+1}{\ell+1}}{(n+m+1)} = \frac{z_k^{*n+m} n^\ell}{(\ell+1)!} + O(n^{\ell-1} |z_k^*|^n), \quad z = z_k^*. \quad (35)$$

Suppose first that $\rho < 1$, i.e., $|z_k| > 1$ for all $1 \leq k \leq s$. Then we obtain by combining (32), (34), (21), (22) and (23) that

$$\begin{aligned} h_m(z)\alpha_n^{-1}p_n(z) &= z^{n+m}\left(1 - \sum_{k=1}^s \sum_{\ell=1}^{m_k} \frac{b_{k,\ell}z^\ell}{(z-z_k^*)^\ell}\right) + O(n^{-1}|z|^n) - \frac{n^{M-2}}{(M-1)!} \sum_{\substack{|z_k^*|=\rho \\ m_k=M}} \frac{b_{k,m_k}z_k^{*n+m+2}}{(z-z_k^*)^2} + O(n^{M-3}\rho^n) \\ &= \frac{z^{n+m}h_m(z)}{\kappa_m h_m^*(z)} + \frac{Mn^{M-2}}{\kappa_m} \sum_{\substack{|z_k^*|=\rho \\ m_k=M}} \frac{h_m(z_k^*)z_k^{*n+m+2-M}}{h_m^{*(M)}(z_k^*)(z-z_k^*)^2} \\ &\quad + O(n^{-1}|z|^n) + O(n^{M-3}\rho^n), \quad z \neq z_k^*, \quad 1 \leq k \leq s. \end{aligned} \quad (36)$$

Theorem 2 follows from (36), (33) and obvious applications of the maximum modulus principle.

Observe that formula (6) in Remark 8 is nothing but (36) with α_n replaced by its asymptotic estimate (33). In likewise fashion, one derives (7) with the additional use of (35).

Similarly, for the case $\rho = 1$, we obtain from (32), (34), (21), (23) and (24) that

$$\begin{aligned} h_m(z)\alpha_n^{-1}p_n(z) &= z^{n+m}\left(1 - \sum_{\substack{|z_k^*|<1}} \sum_{\ell=1}^{m_k} \frac{b_{k,\ell}z^\ell}{(z-z_k^*)^\ell}\right) + O(n^{-1}|z|^n) - \frac{1}{n^2} \sum_{\substack{|z_k^*|=1}} \sum_{\ell=1}^{m_k} \frac{b_{k,\ell}z_k^{*n+m+2}}{(z-z_k^*)^2} + O(n^{-3}) \\ &= \frac{z^{n+m}h_m(z)}{\kappa_m h_m^*(z)} + \frac{1}{n^2} \sum_{\substack{|z_k^*|=1}} \frac{m_k(m_k+1)h_m^{(m_k)}(z_k)z_k^{*n+m+2}}{(-1)^{m_k}\kappa_m h_m^{*(m_k)}(z_k)(z-z_k^*)^2} \\ &\quad + O(n^{-1}|z|^n) + O(n^{-3}), \quad z \neq z_k^*, \quad 1 \leq k \leq s. \end{aligned}$$

This yields the first and third asymptotic formulas of Theorem 1.

If now $|z_\mu^*| = 1$, then applying L'Hospital's rule in (32) followed by the insertion of (18), (19), (21) and (25) gives

$$\begin{aligned} p_n(z_\mu^*) &= \frac{\alpha_n z_\mu^{*n+m}}{h_m^{(m_\mu)}(z_\mu^*)} \left(- \sum_{k=1}^s \sum_{\ell=1}^{m_k} c_{k,\ell,n} R_{k,\ell,n}^{(m_\mu)}(z_\mu^*) \right) \\ &= - \frac{\alpha_n z_\mu^{*n+m}}{h_m^{(m_\mu)}(z_\mu^*)} \left(\sum_{\ell=1}^{m_\mu} \frac{c_{\mu,\ell,n} (-1)^{m_\mu} z_\mu^{*-m_\mu} n^{m_\mu+\ell}}{(\ell-1)!(m_\mu+\ell)(m_\mu+\ell+1)} + \sum_{\substack{|z_k^*|=1}} \sum_{\ell=1}^{m_k} c_{k,\ell,n} O(n^{m_\mu+\ell-1}) + \sum_{\substack{|z_k^*|<1}} \sum_{\ell=1}^{m_k} c_{k,\ell,n} O(1) \right) \\ &= - \frac{\alpha_n z_\mu^{*n+m}}{h_m^{(m_\mu)}(z_\mu^*)} \left(\frac{(-1)^{m_\mu} n^{m_\mu}}{z_\mu^{*m_\mu}} \sum_{\ell=1}^{m_\mu} \frac{b_{\mu,\ell}}{(m_\mu+\ell)(m_\mu+\ell+1)} + O(n^{m_\mu-1}) \right) \\ &= \frac{\alpha_n z_\mu^{*n+m-m_\mu} n^{m_\mu} (1 + O(n^{-1}))}{\kappa_m h_m^{*(m_\mu)}(z_\mu^*) \binom{2m_\mu+1}{m_\mu}}, \end{aligned}$$

completing the proof of Theorem 1. \square

Appendix A

The following facts are presented without proof, as they are not difficult to verify and were used in the previous section for the sole purpose of solving the linear system of equations (31). We shall denote the determinant of a matrix D by $|D|$.

For an integer $n \geq 1$ and $1 \leq l \leq n$, let

$$D_n(y) := \begin{pmatrix} \frac{1}{y(y+1)} & \frac{1}{(y+1)(y+2)} & \cdots & \frac{1}{(y+n-1)(y+n)} \\ \frac{1}{(y+1)(y+2)} & \frac{1}{(y+2)(y+3)} & \cdots & \frac{1}{(y+n)(y+n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(y+n-1)(y+n)} & \frac{1}{(y+n)(y+n+1)} & \cdots & \frac{1}{(y+2n-2)(y+2n-1)} \end{pmatrix},$$

and let $D_{n,l}(y)$ be the matrix that results from $D_n(y)$ by deleting its first row and l th column. Let $A_n(y)$ denote the matrix obtained from $D_n(y)$ by replacing every element of its first row by 1.

Then, for every integer $n \geq 1$,

$$\begin{aligned} |D_n(y)| &= d_n(y+n)^{-n} \left(\prod_{k=1}^n (y+k-1)^{-k} \right) \left(\prod_{k=1}^{n-1} (y+2n-k)^{-k} \right), \\ |D_{n,l}(y)| &= d_{n,l}(y+n+1)^{-n+1} \left(\prod_{k=1}^n (y+k)^{-k} \right) \left(\prod_{k=1}^{n-2} (y+2n-k)^{-k} \right), \\ |A_n(y)| &= a_n(y+n)^{-n+1} \left(\prod_{k=1}^{n-1} (y+k)^{-k} \right) \left(\prod_{k=1}^{n-1} (y+2n-k)^{-k} \right), \end{aligned}$$

where

$$d_n = \frac{\prod_{k=1}^n (k!)^2}{n!}, \quad d_{n,l} = \frac{\prod_{k=1}^{n-1} (k!)^2}{(l-1)!(n-l)!}, \quad a_n = \frac{(-1)^{n+1} \prod_{k=1}^n (k!)^2}{(n-1)!n!}.$$

Consider now the linear system of equations

$$D_n(y) \cdot \mathbf{x}_n^T = (1, 0, 0, \dots, 0)^T$$

in the unknown $\mathbf{x}_n := (x_1, x_2, \dots, x_n)$. By Cramer's formula, we have for all $1 \leq l \leq n$ that

$$x_l = (-1)^{l+1} \frac{|D_{n,l}(y)|}{|D_n(y)|} = (-1)^{l+1} \frac{(y+l) \cdots (y+n+l-1)y(y+1) \cdots (y+n-1)}{n!(n-l)!(l-1)!},$$

while

$$\sum_{l=1}^n x_l = \frac{|A_n(y)|}{|D_n(y)|} = \frac{(-1)^{n+1} y(y+1) \cdots (y+n)}{(n-1)!},$$

and

$$\begin{aligned} \sum_{l=1}^n \frac{x_l}{(y+n+l-1)(y+n+l)} &= (-1)^{n+1} \frac{|D_n(y+1)|}{|D_n(y)|} \\ &= \frac{(-1)^{n+1} y(y+1) \cdots (y+n-1)}{(y+n+1)(y+n+2) \cdots (y+2n)}. \end{aligned}$$

In particular, when $y = 1$ we get

$$x_l = (-1)^{l+1} \frac{(n+l)!}{l!(l-1)!(n-l)!} = -l \binom{n}{l} \binom{-n-1}{l}, \quad (\text{A.1})$$

$$\sum_{l=1}^n x_l = (-1)^{n+1} n(n+1), \quad (\text{A.2})$$

and

$$\sum_{l=1}^n \frac{x_l}{(n+l)(n+l+1)} = \frac{(-1)^{n+1} n!(n+1)!}{(2n+1)!}. \quad (\text{A.3})$$

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