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Generalizations of weakly peripherally multiplicative maps between uniform algebras

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ABSTRACT

Let $\mathcal{A}$ and $\mathcal{B}$ be uniform algebras on first-countable, compact Hausdorff spaces X and Y, respectively. For $f \in \mathcal{A}$, the *peripheral spectrum* of f, denoted by $\sigma_\pi(f) = \{\lambda \in \sigma(f)\colon |\lambda| = \|f\|\}$, is the set of spectral values of maximum modulus. A map $T : \mathcal{A} \to \mathcal{B}$ is *weakly peripherally multiplicative* if $\sigma_\pi(T(f)T(g)) \cap \sigma_\pi(fg) \neq \emptyset$ for all $f, g \in \mathcal{A}$. We show that if T is a surjective, weakly peripherally multiplicative map, then T is a weighted composition operator, extending earlier results. Furthermore, if $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ are surjective mappings that satisfy $\sigma_\pi(T_1(f)T_2(g)) \cap \sigma_\pi(fg) \neq \emptyset$ for all $f, g \in \mathcal{A}$, then $T_1(f)T_2(1) = T_1(1)T_2(f)$ for all $f \in \mathcal{A}$, and the map $f \mapsto T_1(f)T_2(1)$ is an isometric algebra isomorphism.

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1. Introduction and background

The study of a map, not assumed to be linear, between Banach algebras that preserves some property or subset of the spectrum of elements has become known as a *spectral preserver problem*. If X is a first-countable, compact Hausdorff space, $C(X)$ is the space of complex-valued, continuous functions on X, and $T : C(X) \to C(X)$ is a surjective map that satisfies $\sigma(T(f)T(g)) = \sigma(fg)$ for all $f, g \in C(X)$, Molnár [1] showed that there exists a homeomorphism $\varphi : X \to X$ such that $T(f)(x) = T(1)(x)f(\varphi(x))$ for all $f \in C(X)$ and all $x \in X$, i.e. T is a weighted composition operator. In particular, if $T(1) = 1$, then T is an isometric algebra isomorphism. In [2], Rao and Roy proved that the underlying domain need not be first-countable and that similar results hold when the mapping T is a mapping from a uniform algebra $A \subset C(X)$ – where X is the maximal ideal space of A – onto itself. This was generalized further by Hatori et al. in [3], to the case where the underlying domains of A and B need not be the maximal ideal spaces. Throughout, $A \subset C(X)$ and $B \subset C(Y)$ refer to uniform algebras on an arbitrary compact Hausdorff spaces.

In fact, the full spectrum need not be preserved to achieve results of this type; it can be replaced by the *peripheral spectrum*,

$$\sigma_\pi(f) = \{\lambda \in \sigma(f)\colon |\lambda| = \|f\|\}, \tag{1}$$

the set of spectral values of maximum modulus, where $\|f\|$ denotes the uniform norm of $f \in A$. Mappings that satisfy $\sigma_\pi(T(f)T(g)) = \sigma_\pi(fg)$ are called *peripherally multiplicative*, and it was shown in [4] that if A and B are uniform algebras on compact Hausdorff spaces X and Y, respectively, and $T : A \to B$ is a surjective, peripherally multiplicative mapping, then T is a weighted composition operator. If, in addition, $T(1) = 1$, then T is an isometric algebra isomorphism. Related work on peripherally multiplicative mappings in settings outside of uniform algebras can be found in [5].

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The proofs of each of the results above proceed by constructing a homeomorphism between the Choquet boundaries of the algebras A and B, and this is done by analyzing the effect of T on the peaking functions. The set of *peaking functions* in a uniform algebra A is the collection

$$\mathcal{P}(A) = \{h \in A\colon \sigma_\pi(h) = \{1\}\},$$

i.e. the set of functions h such that $|h(x)| \leqslant 1$ for all $x \in X$ and $|h(x)| = 1$ if and only if $h(x) = 1$.

Again it is possible to generalize these results and ask whether or not the entire peripheral spectrum must be multiplicatively preserved, and it turns out that the answer is no. In [6], it was shown that mappings $T : A \to B$ between uniform algebras that satisfy $T(\mathcal{P}(A)) = \mathcal{P}(B)$, $T(1) = 1$, and

$$\sigma_\pi\big(T(f)T(g)\big) \cap \sigma_\pi(fg) \neq \emptyset \tag{2}$$

must be isometric algebra isomorphisms. A mapping that satisfies (2) is called *weakly peripherally multiplicative*, and the first goal of this work is to show that preserving the peaking functions is an unnecessary assumption to guarantee that such a map is a weighted composition operator, in the case that the underlying domains are first-countable. Whereas A and B refer to uniform algebras on arbitrary compact Hausdorff spaces, we denote by $\mathcal{A}$ and $\mathcal{B}$ uniform algebras on *first-countable* compact Hausdorff spaces.

Theorem 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be uniform algebras on first-countable, compact Hausdorff spaces X and Y, respectively, and let $T : \mathcal{A} \to \mathcal{B}$ be surjective and weakly peripherally multiplicative. Then the map $\Phi : \mathcal{A} \to \mathcal{B}$ defined by $\Phi(f) = T(1)T(f)$ is an isometric algebra isomorphism.*

In fact, in this case it is again true that T is a weighted composition operator. This extends the results in [6], in the case that the underlying spaces X and Y are first-countable; related results have been shown in algebras of Lipschitz functions [7,8] and in function algebras without units [9].

A natural next step is to analyze pairs of mappings that jointly satisfy criteria such as (2). Hatori et al. have shown [10] that if $T_1, T_2 : A \to B$ are surjections between uniform algebras such that $\sigma_\pi(T_1(f)T_2(g)) = \sigma_\pi(fg)$, then T_1 and T_2 must be weighted composition operators. We extend their results – under the assumption of first-countability – and the results above by the following:

Theorem 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be uniform algebras on first-countable, compact Hausdorff spaces X and Y, respectively, and let $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ be surjective maps satisfying*

$$\sigma_\pi\big(T_1(f)T_2(g)\big) \cap \sigma_\pi(fg) \neq \emptyset \tag{3}$$

for all $f, g \in \mathcal{A}$. Then $T_1(f)T_2(1) = T_1(1)T_2(f)$ holds for all $f \in \mathcal{A}$, and the mapping $\Phi : \mathcal{A} \to \mathcal{B}$ defined by $\Phi(f) = T_1(f)T_2(1)$ is an isometric algebra isomorphism.

The maximizing set of $f \in A$, denoted $M(f) = \{x \in X\colon |f(x)| = \|f\|\}$, is the set of points where f attains its maximum modulus; the maximizing set of a peaking function h is often called its *peak set*. The collection of peaking functions that contain a point $x_0 \in X$ in their peak set is denoted by $\mathcal{P}_{x_0}(A) = \{h \in \mathcal{P}(A)\colon x_0 \in M(h)\}$, and a point $x \in X$ is called a *weak peak point* (or strong boundary point or p-point) if $\bigcap_{h \in \mathcal{P}_x(A)} M(h) = \{x\}$. Equivalently, $x \in X$ is a weak peak point if and only if for every open neighborhood U of x, there exists a peaking function $h \in \mathcal{P}_x(A)$ such that $M(h) \subset U$. It is well known that in uniform algebras the *Choquet boundary* of A, denoted δA, is the set of all weak peak points. An important relationship between δA and the peripheral spectrum of functions in a uniform algebra is that, given $\lambda \in \sigma_\pi(f)$, there exists an $x \in \delta A$ such that $f(x) = \lambda$.

A fundamental use of the peaking functions is that they can multiplicatively isolate the values of functions at points of the Choquet boundary. This fact, originally due to Bishop [11, Theorem 2.4.1], has been generalized in many directions (e.g. [6, Lemma 3], [3, Lemma 2.3], among several others). This result is essential to the work here, so we include the version given by Hatori et al. in [12].

Lemma 1. *(See [12, Proposition 2.2].) Let A be a uniform algebra on a compact Hausdorff space X, $x_0 \in \delta A$, and $f \in A$. If $f(x_0) \neq 0$, then there exists $h \in \mathcal{P}_{x_0}(A)$ such that $\sigma_\pi(fh) = \{f(x_0)\}$. If $f(x_0) = 0$, then for any $\varepsilon > 0$ there exists a peaking function $h \in \mathcal{P}_{x_0}(A)$ such that $\|fh\| < \varepsilon$.*

A *strong peak point* is a point $x \in X$ such that $M(f) = \{x\}$ for some $f \in A$. Clearly, if $x \in X$ is a strong peak point, then $x \in \delta A$, but the converse is not true in general, as uniform algebras need not have any strong peak points [13, Ex. 10, p. 54]. If X is first-countable, however, then all weak peak points are strong peak points [14, Lemma 12.1, p. 56], leading to the following corollary:

Corollary 1. *Let $\mathcal{A}$ be a uniform algebra on a first-countable, compact Hausdorff space X; let $x_0 \in \delta\mathcal{A}$; and let $f \in \mathcal{A}$ be such that $f(x_0) \neq 0$. Then there exists $h \in \mathcal{P}(\mathcal{A})$ such that $M(h) = M(fh) = \{x_0\}$. In particular, $\sigma_\pi(fh) = \{f(x_0)\}$.*

Proof. Let $x_0 \in \delta\mathcal{A}$, then, since X is first-countable, $\{x_0\}$ is a G_δ set. Thus there exists a countable collection of open sets $\{U_n\}_{n=1}^\infty$ such that $\bigcap_{n=1}^\infty U_n = \{x_0\}$. Since $x_0 \in \delta\mathcal{A}$, for each n there exists an $h_n \in \mathcal{P}_{x_0}(\mathcal{A})$ such that $M(h_n) \subset U_n$. Thus $\bigcap_{n=1}^\infty M(h_n) = \{x_0\}$. Set $h = \sum_{n=1}^\infty \frac{h_n}{2^n}$, then $h \in \mathcal{P}(\mathcal{A})$ and $M(h) = \{x_0\}$. By Lemma 1, there exists a $k \in \mathcal{P}(\mathcal{A})$ such that $\sigma_\pi(fk) = \{f(x_0)\}$. Therefore $kh \in \mathcal{P}(\mathcal{A})$ is the peaking function we seek, and $M(fkh) = \{x_0\}$. □

Note that the difference between this and Lemma 1 is that this result ensures the peak set of h consists solely of the point x_0.

Following the arguments in [8], for each $x \in X$ we define the set

$$F_x(A) = \{f \in A\colon \|f\| = |f(x)| = 1\}.$$

Notice that $\mathcal{P}_x(A) \subset F_x(A)$ and that $f, g \in F_x(A)$ imply $fg \in F_x(A)$. A useful property of these sets is that they can identify elements of the Choquet boundary, as shown by the following lemma.

Lemma 2. *Let A be a uniform algebra on a compact Hausdorff space X, $x \in \delta A$, and $x' \in X$. Then $x = x'$ if and only if $F_x(A) \subset F_{x'}(A)$.*

Proof. The forward direction is clear, so we suppose $F_x(A) \subset F_{x'}(A)$ and $x \neq x'$. Since X is Hausdorff, there exist open sets U, V such that $x \in U$, $x' \in V$, and $U \cap V = \emptyset$. Thus there exists a peaking function $h \in \mathcal{P}_x(A)$ such that $M(h) \subset U$, hence $|h(x')| < 1$, which contradicts $F_x(A) \subset F_{x'}(A)$. □

The assumption that $x \in \delta A$ is essential to the result of Lemma 2. Consider the *disk algebra*, $A(\overline{\mathbb{D}})$, the set of continuous functions on the closed unit disk that are analytic on the interior of the disk. It is well known that $\delta A(\overline{\mathbb{D}}) = \mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$. If $z \in \mathbb{D}$, then, by the maximum modulus principle, $F_z(A(\overline{\mathbb{D}}))$ consists precisely of the constant functions of modulus one. Hence for any pair $z_1, z_2 \in \mathbb{D}$, we have that $F_{z_1}(A(\overline{\mathbb{D}})) = F_{z_2}(A(\overline{\mathbb{D}}))$.

2. Weak peripheral multiplicativity

As described above, a mapping $T : A \to B$ that satisfies

$$\sigma_\pi\big(T(f)T(g)\big) \cap \sigma_\pi(fg) \neq \emptyset \tag{4}$$

for all $f, g \in A$ is called *weakly peripherally multiplicative*. In general, a weakly peripherally multiplicative map need not be an algebra isomorphism, as is shown in [6, Example 2] where also the following proposition is proven:

Proposition 1. *(See [6, Theorem 3].) Let A and B be uniform algebras on compact Hausdorff spaces X and Y, respectively. If $T : A \to B$ is a weakly peripherally multiplicative map such that $\mathcal{P}(B) = \{T(1)T(h)\colon h \in \mathcal{P}(A)\}$, then the map $\Phi : A \to B$ defined by $\Phi(f) = T(1)T(f)$ is an isometric algebra isomorphism.*

In the case that X and Y are first-countable, Theorem 1 is more general than Proposition 1, as it shows that the requirement that $\mathcal{P}(B) = \{T(1)T(h)\colon h \in \mathcal{P}(A)\}$ is superfluous. The proof of Theorem 1 will follow from Proposition 1 by showing that if T is a weakly peripherally multiplicative map, then the map $f \mapsto T(1)T(f)$ automatically preserves the peaking functions.

2.1. General results on weakly peripherally multiplicative maps

In this section we assume that $\mathcal{A} \subset C(X)$ and $\mathcal{B} \subset C(Y)$ are uniform algebras on first-countable, compact Hausdorff spaces and that $T : \mathcal{A} \to \mathcal{B}$ is a surjective, weakly peripherally multiplicative map. Note that (4) implies

$$\big\|T(f)T(g)\big\| = \|fg\| \tag{5}$$

for all $f, g \in \mathcal{A}$. A mapping that satisfies (5) is called *norm multiplicative*, and the following proposition shows that such a mapping, when restricted to the Choquet boundary, is a composition operator in modulus.

Proposition 2. *(See [15, Theorem 4.1.2].) If $\Psi : \mathcal{A} \to \mathcal{B}$ is surjective and norm multiplicative, then there exists a homeomorphism $\tau : \delta\mathcal{A} \to \delta\mathcal{B}$ such that*

$$\big|\Psi(f)\big(\tau(x)\big)\big| = \big|f(x)\big| \tag{6}$$

holds for all $x \in \delta\mathcal{A}$.

Note that any surjective, weakly peripherally multiplicative map satisfies the hypothesis of Proposition 2, and, as is shown next, must be injective.

Lemma 3. *A surjective, weakly peripherally multiplicative map $T : \mathcal{A} \to \mathcal{B}$ is injective.*

Proof. Let $x_0 \in \delta\mathcal{A}$, $f, g \in \mathcal{A}$, and suppose that $T(f) = T(g)$. If $f(x_0) = 0$, then, by (6), $0 = |f(x_0)| = |T(f)(\tau(x_0))| = |T(g)(\tau(x_0))| = |g(x_0)|$, so $g(x_0) = f(x_0)$.

If $f(x_0) \neq 0$, then (6) implies that $T(f)(\tau(x_0)) \neq 0$. Hence, by Corollary 1, there exists a peaking function $k \in \mathcal{P}_{\tau(x_0)}(\mathcal{B})$ such that $M(k) = M(T(f)k) = \{\tau(x_0)\}$, and thus $\sigma_\pi(T(f)k) = \{T(f)(\tau(x_0))\}$.

Let $h \in \mathcal{A}$ be such that $T(h) = k$, and choose $x' \in M(fh) \cap \delta\mathcal{A}$. By (5) and (6),

$$\begin{aligned}\left|T(f)\left(\tau\left(x'\right)\right)k\left(\tau\left(x'\right)\right)\right| &= \left|T(f)\left(\tau\left(x'\right)\right)T(h)\left(\tau\left(x'\right)\right)\right| = \left|f\left(x'\right)h\left(x'\right)\right| \\ &= \|fh\| = \left\|T(f)T(h)\right\| = \left\|T(f)k\right\|.\end{aligned}$$

Since $M(T(f)k) = \{\tau(x_0)\}$, $\tau(x') = \tau(x_0)$, and the injectivity of τ yields $x' = x_0$. Thus $M(fh) \cap \delta\mathcal{A} = \{x_0\}$, which gives that $\sigma_\pi(fh) = \{f(x_0)h(x_0)\}$. A similar argument shows $\sigma_\pi(gh) = \{g(x_0)h(x_0)\}$.

As $T(f) = T(g)$, (4) implies that

$$\sigma_\pi\left(T(f)T(h)\right) \cap \sigma_\pi(fh) \neq \emptyset,$$

and

$$\sigma_\pi\left(T(f)T(h)\right) = \sigma_\pi\left(T(g)T(h)\right) \cap \sigma_\pi(gh) \neq \emptyset.$$

Since $\sigma_\pi(T(f)T(h)) = \{T(f)(\tau(x_0))\}$, we have that $f(x_0)h(x_0) = T(f)(\tau(x_0))$ and $g(x_0)h(x_0) = T(f)(\tau(x_0))$. Thus $f(x_0)h(x_0) = g(x_0)h(x_0)$, which implies that $f(x_0) = g(x_0)$, since $|h(x_0)| = |T(h)(\tau(x_0))| = |k(\tau(x_0))| = 1$. Therefore $f(x) = g(x)$ for all $x \in \delta\mathcal{A}$, i.e. T is injective. □

If T preserves the peaking functions, then it is straightforward to show that $T(1)^2 = 1$ [6], but the following lemma demonstrates that this is true even when the peaking functions are not assumed to be preserved.

Lemma 4. *A surjective, weakly peripherally multiplicative map T satisfies $T(1)^2 = 1$.*

Proof. Firstly note that $|T(1)(\tau(x))| = |1(x)| = 1$ for all $x \in \delta\mathcal{A}$. Choose $y_0 \in \delta\mathcal{B}$, and let $x_0 \in \delta\mathcal{A}$ be such that $\tau(x_0) = y_0$. Corollary 1 implies that there exists a peaking function $k \in \mathcal{P}(\mathcal{B})$ such that $M(k) = M(T(1)^2k) = \{y_0\}$, so $\sigma_\pi(T(1)^2k) = \{T(1)^2(y_0)\}$. Notice also that $\sigma_\pi(T(1)^2k^2) = \{T(1)^2(y_0)\}$.

Since T is surjective, there exists $h \in \mathcal{A}$ such that $T(h) = T(1)k$, so choose $x' \in M(h) \cap \delta\mathcal{A}$. By (6),

$$\begin{aligned}\left|k\left(\tau\left(x'\right)\right)\right| &= \left|T(1)\left(\tau\left(x'\right)\right)k\left(\tau\left(x'\right)\right)\right| = \left|T(h)\left(\tau\left(x'\right)\right)\right| = \left|h\left(x'\right)\right| = \|h\| = \left\|T(1)T(h)\right\| \\ &= \left\|T(1)^2k\right\| = \left|T(1)^2(y_0)\right| = 1.\end{aligned}$$

Now $|k(y)| = 1$ if and only if $y = y_0$, so $\tau(x') = y_0 = \tau(x_0)$, which yields that $x' = x_0$. Therefore $M(h) \cap \delta\mathcal{A} = \{x_0\}$, and $\sigma_\pi(h) = \{h(x_0)\}$. Note further that $\sigma_\pi(h^2) = \{h^2(x_0)\}$.

By (4)

$$\sigma_\pi\left(T(1)^2k\right) \cap \sigma_\pi(h) \neq \emptyset,$$

and

$$\sigma_\pi\left(T(1)^2k^2\right) \cap \sigma_\pi\left(h^2\right) \neq \emptyset,$$

so $h(x_0) = T(1)^2(y_0)$ and $h^2(x_0) = T(1)^2(y_0)$. Now $h(x_0) \neq 0$, since

$$\left|h(x_0)\right| = \left|T(h)\left(\tau(x_0)\right)\right| = \left|T(1)\left(\tau(x_0)\right)k\left(\tau(x_0)\right)\right| = \left|T(1)\left(\tau(x_0)\right)\right| \cdot \left|k(y_0)\right| = \left|T(1)\left(\tau(x_0)\right)\right| = 1.$$

Thus $T(1)^2(y_0) = (T(1)^2(y_0))^2$, which implies that $T(1)^2(y_0) = 1$. Since $T(1)^2$ is identically 1 on $\delta\mathcal{B}$, it is identically 1 on all of Y. □

2.2. Proof of Theorem 1

Given a surjective, weakly peripherally multiplicative map $T : \mathcal{A} \to \mathcal{B}$, define $\Phi : \mathcal{A} \to \mathcal{B}$ by

$$\Phi(f) = T(1)T(f). \tag{7}$$

Lemma 5. *The mapping Φ is surjective, unital, weakly peripherally multiplicative, and satisfies $\sigma_\pi(\Phi(f)) \cap \sigma_\pi(f) \neq \emptyset$.*

Proof. Lemma 4 implies that $T(1)^2 = 1$, so $\Phi(1) = T(1)^2 = 1$, i.e. Φ is unital.

In addition, $\Phi(f)\Phi(g) = T(1)T(f)T(1)T(g) = T(f)T(g)$ holds for all $f, g \in \mathcal{A}$. Hence, the weak peripheral multiplicativity of T implies that Φ is weakly peripherally multiplicative. The fact that Φ is weakly peripherally multiplicative and unital immediately give

$$\sigma_\pi\big(\Phi(f)\big) \cap \sigma_\pi(f) \neq \emptyset \tag{8}$$

for all $f \in \mathcal{A}$.

Let $g \in \mathcal{B}$, then, by the surjectivity of T, there exists an $f \in \mathcal{A}$ such that $T(f) = T(1)g$. Thus $\Phi(f) = T(1)T(f) = T(1)^2 g = g$, which implies that Φ is surjective. □

Since Φ is surjective and weakly peripherally multiplicative, all of the results of Section 2.1 hold for Φ. Thus, by Lemma 3, Φ is injective and has a formal inverse $\Phi^{-1} : \mathcal{B} \to \mathcal{A}$, which satisfies

$$\sigma_\pi\big(\Phi^{-1}(f)\Phi^{-1}(g)\big) \cap \sigma_\pi(fg) \neq \emptyset \tag{9}$$

for all $f, g \in \mathcal{B}$. Thus Φ^{-1} is a bijective, unital, weakly peripherally multiplicative map.

As noted before, Theorem 1 follows if the map Φ preserves the peaking functions. We first show that Φ preserves functions whose maximizing sets are singletons.

Lemma 6. *Let $h \in \mathcal{A}$ and $x_0 \in \delta\mathcal{A}$. Then $M(h) \cap \delta\mathcal{A} = \{x_0\}$ if and only if $M(\Phi(h)) \cap \delta\mathcal{B} = \{\tau(x_0)\}$, in which case $h(x_0) = \Phi(h)(\tau(x_0))$.*

Proof. Note that $M(h) \cap \delta\mathcal{A} = \{x_0\}$ if and only if $|h(x)| < \|h\|$ for all $x \in \delta\mathcal{A} \setminus \{x_0\}$, which holds if and only if $|\Phi(h)(\tau(x))| < \|\Phi(h)\|$ for all $\tau(x) \in \delta\mathcal{B} \setminus \{\tau(x_0)\}$, i.e. if and only if $M(\Phi(h)) \cap \delta\mathcal{B} = \{\tau(x_0)\}$.

In this case, $\sigma_\pi(h) = \{h(x_0)\}$ and $\sigma_\pi(\Phi(h)) = \{\Phi(h)(\tau(x_0))\}$, so (8) implies $h(x_0) = \Phi(h)(\tau(x_0))$. □

Note that this implies a certain class of peaking functions are preserved – the peaking functions that peak at a single point – but not necessarily that all peaking functions are preserved.

We now proceed with the proof of Theorem 1.

Theorem 1. *Let $\mathcal{A}$ and $\mathcal{B}$ be uniform algebras on first-countable compact Hausdorff spaces X and Y, respectively, and let $T : \mathcal{A} \to \mathcal{B}$ be surjective and weakly peripherally multiplicative. Then the map $\Phi : \mathcal{A} \to \mathcal{B}$ defined by $\Phi(f) = T(1)T(f)$ is an isometric algebra isomorphism.*

Proof. By Lemma 5, Φ is a surjective, unital, weakly peripherally multiplicative operator, so it is only to show that Φ satisfies $\Phi[\mathcal{P}(\mathcal{A})] = \mathcal{P}(\mathcal{B})$.

Choose $h \in \mathcal{P}(\mathcal{A})$ and $y_0 \in M(\Phi(h)) \cap \delta\mathcal{B}$. Since Φ preserves norm, $1 = \|h\| = \|\Phi(h)\|$, so $|\Phi(h)(y_0)| = 1$. By Corollary 1, there exists $k \in \mathcal{P}(\mathcal{B})$ such that $M(k) = M(\Phi(h)k) = \{y_0\}$.

Set $x_0 = \tau^{-1}(y_0)$ and $g = \Phi^{-1}(k)$, and note that $M(g) = \{x_0\}$ and $g(x_0) = 1$, by Lemma 6. If $x' \in M(hg) \cap \delta\mathcal{A}$, then

$$\big|\Phi(h)\big(\tau(x')\big)\Phi(g)\big(\tau(x')\big)\big| = \big|h(x')g(x')\big| = \|hg\| = \big\|\Phi(h)\Phi(g)\big\| = \big\|\Phi(h)k\big\|,$$

which implies that $\tau(x') = y_0 = \tau(x_0)$, i.e. $x' = x_0$. Therefore $M(hg) \cap \delta\mathcal{A} = \{x_0\}$, and

$$\big|h(x_0)\big| = \big|\Phi(h)\big(\tau(x_0)\big)\big| = \big|\Phi(h)(y_0)\big| = 1.$$

Since h is a peaking function, $h(x_0) = 1$, so $\sigma_\pi(hg) = \{1\}$. Thus $1 \in \sigma_\pi(\Phi(h)k) = \{\Phi(h)(y_0)\}$, i.e. $\Phi(h)(y_0) = 1$. Since this holds for any $y_0 \in M(\Phi(h)) \cap \delta\mathcal{B}$, $\sigma_\pi(\Phi(h)) = \{1\}$, which is to say $\Phi(h) \in \mathcal{P}(\mathcal{B})$. This shows that $\Phi[\mathcal{P}(\mathcal{A})] \subset \mathcal{P}(\mathcal{B})$, and a similar argument with Φ^{-1} proves the reverse inclusion.

Since Φ is a surjective, unital, weakly peripherally multiplicative operator that preserves the peaking functions, Proposition 1 gives that Φ is an isometric algebra isomorphism. □

3. Jointly weakly peripherally multiplicative maps

Given the above results on single weakly peripherally multiplicative operators, it is natural to analyze pairs of maps that jointly satisfy related conditions. Throughout this section we assume that $\mathcal{A} \subset C(X)$ and $\mathcal{B} \subset C(Y)$ are uniform algebras on first-countable, compact Hausdorff spaces X and Y, and $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ are surjective mappings that satisfy

$$\sigma_\pi\big(T_1(f)T_2(g)\big) \cap \sigma_\pi(fg) \neq \emptyset \tag{10}$$

for all $f, g \in \mathcal{A}$. Notice that (10) implies that

$$\big\|T_1(f)T_2(g)\big\| = \|fg\| \tag{11}$$

holds for all $f, g \in \mathcal{A}$.

3.1. General results on T_1 and T_2

We begin with some properties of T_1 and T_2.

Lemma 7. *Let $f, g \in \mathcal{A}$, then the following are equivalent*:

(a) $|f(x)| \leqslant |g(x)|$ *for all* $x \in \delta\mathcal{A}$,
(b) $|T_1(f)(y)| \leqslant |T_1(g)(y)|$ *for all* $y \in \delta\mathcal{B}$, *and*
(c) $|T_2(f)(y)| \leqslant |T_2(g)(y)|$ *for all* $y \in \delta\mathcal{B}$.

Proof. (a) $\Rightarrow$ (b). Let $f, g \in \mathcal{A}$ be such that $|f(x)| \leqslant |g(x)|$ on $\delta\mathcal{A}$, then $\|fh\| \leqslant \|gh\|$ for $h \in \mathcal{A}$. If $k \in \mathcal{P}(\mathcal{B})$ and $h \in \mathcal{A}$ is such that $T_2(h) = k$, then, by (11),

$$\big\|T_1(f)k\big\| = \big\|T_1(f)T_2(h)\big\| = \|fh\| \leqslant \|gh\| = \big\|T_1(g)T_2(h)\big\| = \big\|T_1(g)k\big\|.$$

Since $k \in \mathcal{P}(\mathcal{B})$ was chosen arbitrarily, $|T_1(f)(y)| \leqslant |T_1(g)(y)|$ for all $y \in \delta\mathcal{B}$ (see e.g. [4, Lemma 2]).

(b) $\Rightarrow$ (c). Let $f, g \in \mathcal{A}$ be such that $|T_1(f)(y)| \leqslant |T_1(g)(y)|$ for all $y \in \delta\mathcal{B}$, then $\|T_1(f)k\| \leqslant \|T_1(g)k\|$ for all $k \in \mathcal{B}$. If $k \in \mathcal{P}(\mathcal{B})$ and $h \in \mathcal{A}$ is such that $T_1(h) = k$, then

$$\big\|T_2(f)k\big\| = \|fh\| = \big\|T_1(f)T_2(h)\big\| \leqslant \big\|T_1(g)T_2(h)\big\| = \|gh\| = \big\|T_2(g)k\big\|.$$

As $k \in \mathcal{P}(\mathcal{B})$ was arbitrarily chosen, $|T_2(f)(y)| \leqslant |T_2(g)(y)|$ for all $y \in \delta\mathcal{B}$.

(c) $\Rightarrow$ (a). Suppose that $f, g \in \mathcal{A}$ are such that $|T_2(f)(y)| \leqslant |T_2(g)(y)|$ for all $y \in \delta\mathcal{B}$, then $\|T_2(f)k\| \leqslant \|T_2(g)k\|$ for any $k \in \mathcal{B}$. If $h \in \mathcal{P}(\mathcal{A})$, then (11) implies that

$$\|fh\| = \big\|T_1(f)T_2(h)\big\| \leqslant \big\|T_2(g)T_2(h)\big\| = \|gh\|.$$

By the liberty of the choice of h, $|f(x)| \leqslant |g(x)|$ for all $x \in \delta\mathcal{A}$. □

Given $h, k \in F_x(\mathcal{A})$, (11) implies that $\|T_1(h)T_2(k)\| = \|hk\| = 1$, hence $M(T_1(h)T_2(k)) = \{y \in Y\colon\ |T_1(h)T_2(k)(y)| = 1\}$. Following the argument pioneered by Molnár [1], for each $x \in \delta\mathcal{A}$, we define the set

$$A_x = \bigcap_{h,k \in F_x(X)} M\big(T_1(h)T_2(k)\big). \tag{12}$$

Lemma 8. *For each $x \in \delta\mathcal{A}$, the set A_x is non-empty.*

Proof. We will show that the family $\{M(T_1(h)T_2(k))\colon\ h, k \in F_x(\mathcal{A})\}$ has the finite intersection property. Let $h_1, \ldots, h_n, k_1, \ldots, k_n \in F_x(\mathcal{A})$ and set $h = h_1 \cdot \cdots \cdot h_n \in F_x(\mathcal{A})$ and $k = k_1 \cdot \cdots \cdot k_n \in F_x(\mathcal{A})$. Since $|h_i(\zeta)| \leqslant 1$ and $|k_i(\zeta)| \leqslant 1$ for all $1 \leqslant i \leqslant n$ and all $\zeta \in \delta\mathcal{A}$, $|h(\zeta)| \leqslant |h_i(\zeta)|$ and $|k(\zeta)| \leqslant |k_i(\zeta)|$ for any $1 \leqslant i \leqslant n$ and all $\zeta \in \delta\mathcal{A}$. Lemma 7 implies that $|T_1(h)(\eta)| \leqslant |T_1(h_i)(\eta)|$ and $|T_2(k)(\eta)| \leqslant |T_2(k_i)(\eta)|$ for any $1 \leqslant i \leqslant n$ and all $\eta \in \delta\mathcal{B}$. Since maximizing sets meet the Choquet boundary [15, Lemma 3.2.3], there exists a $y \in M(T_1(h)T_2(k)) \cap \delta\mathcal{B}$. Hence $1 = |T_1(h)(y)T_2(k)(y)| \leqslant |T_1(h_i)(y)T_2(k_i)(y)| \leqslant 1$ for each $1 \leqslant i \leqslant n$, thus $|T_1(h_i)(y)T_2(k_i)| = 1$ for each $1 \leqslant i \leqslant n$. Thus $y \in \bigcap_{i=1}^n M(T_1(h_i)T_2(k_i))$, and $\{M(T_1(h)T_2(k))\colon\ h, k \in F_x(\mathcal{A})\}$ has the finite intersection property as claimed. Since maximizing sets are compact subsets of the compact set Y, A_x is non-empty. □

Since A_x is a non-empty intersection of maximizing sets, it meets the Choquet boundary [15, Lemma 3.2.3].

Lemma 9. *Let $f, g \in \mathcal{A}$. Then for each $x \in \delta\mathcal{A}$ and each $y \in A_x \cap \delta\mathcal{B}$, $fg \in F_x(\mathcal{A})$ if and only if $T_1(f)T_2(g) \in F_y(\mathcal{B})$.*

Proof. Fix $x \in \delta\mathcal{A}$, and choose $y \in A_x \cap \delta\mathcal{B}$. If $T_1(f)T_2(g) \in F_y(\mathcal{B})$, then, by (11), $1 = \|T_1(f)T_2(g)\| = \|fg\|$. Thus we need only show that $|f(x)g(x)| = 1$. If $f(x)g(x) = 0$, then, without loss of generality, assume $f(x) = 0$. Hence Lemma 1 implies that there exists a peaking function $h \in \mathcal{P}_x(\mathcal{A})$ such that $\|fh\| < \frac{1}{\|g\|}$. As $h \in F_x(\mathcal{A})$, Lemma 8 yields $T_1(h)T_2(h) \in F_y(\mathcal{B})$, so

$$1 = \left\|T_1(f)T_2(g)T_1(h)T_2(h)\right\| \leqslant \left\|T_1(f)T_2(h)\right\| \cdot \left\|T_1(h)T_2(g)\right\| = \|fh\| \cdot \|gh\| < \frac{1}{\|g\|} \cdot \|g\| = 1,$$

which is a contradiction. Hence $f(x)g(x) \neq 0$, i.e. $f(x), g(x) \neq 0$, thus, by Corollary 1, there exist peaking functions $h_1, h_2 \in \mathcal{P}_x(\mathcal{A})$ such that $M(h_2) = M(fh_2) = \{x\}$ and $M(h_1) = M(gh_1) = \{x\}$. As $h_1, h_2 \in F_x(\mathcal{A})$, again Lemma 8 implies that $T_1(h_1)T_2(h_2) \in F_y(\mathcal{B})$, thus

$$\left|f(x)g(x)\right| = \|fh_2\| \cdot \|gh_1\| = \left\|T_1(f)T_2(h_2)\right\| \cdot \left\|T_1(h_1)T_2(g)\right\| \geqslant \left\|T_1(f)T_2(g)T_1(h_1)T_2(h_2)\right\| = 1.$$

Therefore $|f(x)g(x)| = 1$, yielding $fg \in F_x(\mathcal{A})$.

Conversely, suppose that $fg \in F_x(\mathcal{A})$, then $1 = \|fg\| = \|T_1(f)T_2(g)\|$, and it is only to show that $|T_1(f)(y)T_2(g)(y)| = 1$. If $T_1(f)(y)T_2(g)(y) = 0$, then, without loss of generality, assume $T_1(f)(y) = 0$. Lemma 1 implies that there exists an $k \in \mathcal{P}_y(\mathcal{B})$ such that $\|T_1(f)k\| < \frac{1}{\|T_2(g)\|}$. Let $h_1, h_2 \in \mathcal{A}$ be such that $T_1(h_1) = T_2(h_2) = k$, then, as $T_1(h_1)T_2(h_2) \in F_y(\mathcal{B})$, we have that $h_1h_2 \in F_x(\mathcal{A})$, thus

$$1 = \|fgh_1h_2\| \leqslant \|fh_1\| \cdot \|gh_2\| = \left\|T_1(f)k\right\| \cdot \left\|T_2(g)k\right\| < \frac{1}{\|T_2(g)\|} \cdot \left\|T_2(g)\right\| = 1,$$

which is a contradiction. Hence $T_1(f)(y)T_2(g)(y) \neq 0$, thus $T_1(f)(y), T_2(g)(y) \neq 0$, so by Corollary 1, there exist peaking functions $k_1, k_2 \in \mathcal{P}_y(\mathcal{B})$ such that $M(k_1) = M(T_1(f)k_1) = \{y\}$ and $M(k_2) = M(T_2(g)k_2) = \{y\}$. Let $h_1, h_2 \in \mathcal{A}$ be such that $T_1(h_1) = k_2$ and $T_2(h_2) = k_1$, then $h_1h_2 \in F_x(\mathcal{A})$, hence

$$\left|T_1(f)(y)T_2(g)(y)\right| = \left\|T_1(f)k_1\right\| \cdot \left\|T_2(g)k_2\right\| = \|fh_2\| \cdot \|gh_1\| \geqslant \|fgh_1h_2\| = 1,$$

which implies that $|T_1(f)(y)T_2(g)(y)| = 1$. Therefore $T_1(f)T_2(g) \in F_y(\mathcal{B})$. □

Not only is $A_x \cap \delta\mathcal{B}$ non-empty, but the following lemma shows that it is, in fact, a singleton.

Lemma 10. *For each $x \in \delta\mathcal{A}$, the set $A_x \cap \delta\mathcal{B}$ is a singleton.*

Proof. Fix $x \in \delta\mathcal{A}$, and let $y, y' \in A_x \cap \delta\mathcal{B}$. If $y \neq y'$, then there exist open sets U and V such that $y \in U$, $y' \in V$, and $U \cap V = \emptyset$. Since $y \in \delta\mathcal{B}$, there exists a peaking function $k \in \mathcal{P}_y(\mathcal{B})$ such that $M(k) \subset U$. If $h_1, h_2 \in \mathcal{A}$ are such that $T_1(h_1) = T_2(h_2) = k$, then Lemma 9 implies that $h_1h_2 \in F_x(\mathcal{A})$. Hence, by Lemma 9, $k^2 = T_1(h_1)T_2(h_2) \in F_{y'}(\mathcal{B})$, which is a contradiction. Thus no two distinct points y and y' are elements of $A_x \cap \delta\mathcal{B}$, i.e. $A_x \cap \delta\mathcal{B}$ is a singleton. □

Given that $A_x \cap \delta\mathcal{B}$ is a singleton for each $x \in \delta\mathcal{A}$, we define the map $\tau : \delta\mathcal{A} \to \delta\mathcal{B}$ by

$$\left\{\tau(x)\right\} = A_x \cap \delta\mathcal{B}. \tag{13}$$

Lemma 11. *The map $\tau : \delta\mathcal{A} \to \delta\mathcal{B}$ defined by* (13) *is injective.*

Proof. Let $x, x' \in \delta\mathcal{A}$, and choose $h \in F_x(\mathcal{A})$. By Lemma 9, $T_1(h)T_2(h) \in F_{\tau(x)}(\mathcal{B})$. If $\tau(x) = \tau(x')$, then $T_1(h)T_2(h) \in F_{\tau(x')}(\mathcal{B})$, which, again by Lemma 9, shows $h^2 \in F_{x'}(\mathcal{A})$ and thus $h \in F_{x'}(\mathcal{A})$. Lemma 2 then gives $x = x'$. □

Lemma 12. *Let $f, g \in \mathcal{A}$, then $|T_1(f)(\tau(x))T_2(g)(\tau(x))| = |f(x)g(x)|$ holds for all $x \in \delta\mathcal{A}$.*

Proof. If any of $f, g, T_1(f), T_2(g)$ is identically 0, then the result follows by (11), thus we can assume that $f, g, T_1(f), T_2(g) \neq 0$.

Let $x \in \delta\mathcal{A}$. If $f(x)g(x) = 0$, then, without loss of generality, we can assume that $f(x) = 0$. Given $\varepsilon > 0$, Lemma 1 implies that there exists a peaking function $h \in \mathcal{P}_x(\mathcal{A})$ such that $\|fh\| < \frac{\varepsilon}{\|g\|}$. As $h^2 \in F_x(\mathcal{A})$, Lemma 9 gives $T_1(h)T_2(h) \in F_{\tau(x)}(\mathcal{B})$, so

$$\begin{aligned}\left|T_1(f)\big(\tau(x)\big)T_2(g)\big(\tau(x)\big)\right| &\leqslant \left\|T_1(f)T_2(g)T_1(h)T_2(h)\right\| \leqslant \left\|T_1(f)T_2(h)\right\| \cdot \left\|T_1(h)T_2(g)\right\| \\ &= \|fh\| \cdot \|gh\| < \frac{\varepsilon}{\|g\|} \cdot \|g\| = \varepsilon.\end{aligned}$$

Therefore $T_1(f)(\tau(x))T_2(g)(\tau(x)) = 0$, by the liberty of the choice of ε. A symmetric argument shows that $T_1(f)(\tau(x))T_2(g)(\tau(x)) = 0$ implies $f(x)g(x) = 0$.

If $f(x)g(x) \neq 0$, then $f(x), g(x) \neq 0$. Hence, by Corollary 1, there exist peaking functions $h_1, h_2 \in \mathcal{P}_x(\mathcal{A})$ such that $M(h_2) = M(fh_2) = \{x\}$ and $M(h_1) = M(gh_1) = \{x\}$. Since $h_1h_2 \in F_x(\mathcal{A})$, Lemma 9 implies that $T_1(h_1)T_2(h_2) \in F_{\tau(x)}(\mathcal{B})$, hence

$$\begin{aligned}\left|T_1(f)\big(\tau(x)\big)T_2(g)\big(\tau(x)\big)\right| &\leqslant \left\|T_1(f)T_2(g)T_1(h_1)T_2(h_2)\right\| \leqslant \left\|T_1(f)T_2(h_2)\right\| \cdot \left\|T_1(h_1)T_2(g)\right\| \\ &= \|fh_2\| \cdot \|gh_1\| = \left|f(x)g(x)\right|.\end{aligned}$$

Since $T_1(f)(\tau(x))T_2(g)(\tau(x)) = 0$ if and only if $f(x)g(x) = 0$, the assumption that $f(x)g(x) \neq 0$ implies that $T_1(f)(\tau(x))T_2(g)(\tau(x)) \neq 0$, then $T_1(f)(\tau(x)), T_2(g)(\tau(x)) \neq 0$. Corollary 1 implies that there exist $k_1, k_2 \in \mathcal{P}_{\tau(x)}(\mathcal{B})$ such that $M(k_2) = M(T_1(f)k_2) = \{\tau(x)\}$ and $M(k_1) = M(T_2(g)k_1) = \{\tau(x)\}$. Let $h_1, h_2 \in \mathcal{A}$ be such that $T_1(h_1) = k_1$ and $T_2(h_2) = k_2$. As $k_1k_2 \in F_{\tau(x)}(\mathcal{B})$, Lemma 9 yields that $h_1h_2 \in F_x(\mathcal{A})$, thus

$$\left|f(x)g(x)\right| \leqslant \|fgh_1h_2\| \leqslant \|fh_2\| \cdot \|gh_1\| = \left\|T_1(f)k_2\right\| \cdot \left\|T_2(g)k_1\right\| = \left|T_1(f)\big(\tau(x)\big)T_2(g)\big(\tau(x)\big)\right|.$$

Therefore $|T_1(f)(\tau(x))T_2(g)(\tau(x))| = |f(x)g(x)|$ holds for all $x \in \delta\mathcal{A}$. □

Lemma 13. *The mappings T_1 and T_2 are injective.*

Proof. Let $f, g \in \mathcal{A}$ be such that $T_1(f) = T_1(g)$ and let $x_0 \in \delta\mathcal{A}$. If $f(x_0) = 0$, then, by Lemma 12, $|f(x_0)| = |T_1(f)(\tau(x_0))T_2(1)(\tau(x_0))| = |T_1(g)(\tau(x_0))T_2(1)(\tau(x_0))| = |g(x_0)|$, thus $f(x_0) = g(x_0)$.

If $f(x_0) \neq 0$, then Lemma 12 implies that $T_1(f)(\tau(x_0)) \neq 0$. Hence, by Corollary 1, there exists a peaking function $k \in \mathcal{P}_{\tau(x_0)}(\mathcal{B})$ such that $M(T_1(f)k) = \{\tau(x_0)\}$ and $\sigma_\pi(T_1(f)k) = \{T_1(f)(\tau(x_0))\}$.

Let $h \in \mathcal{A}$ be such that $T_2(h) = k$. We claim that $\sigma_\pi(fh) = \{f(x_0)h(x_0)\}$. Indeed, let $x' \in M(fh) \cap \delta\mathcal{A}$, then, by (11) and Lemma 12,

$$\begin{aligned}\left|T_1(f)\big(\tau(x')\big)k\big(\tau(x')\big)\right| &= \left|T_1(f)\big(\tau(x')\big)T_2(h)\big(\tau(x')\big)\right| = \left|f(x')h(x')\right| \\ &= \|fh\| = \left\|T_1(f)T_2(h)\right\| = \left\|T_1(f)k\right\|.\end{aligned}$$

Since $M(T_1(f)k) = \{\tau(x_0)\}$, $\tau(x') = \tau(x_0)$, so $x' = x_0$. Thus $M(fh) \cap \delta\mathcal{A} = \{x_0\}$, i.e. $\sigma_\pi(fh) = \{f(x_0)h(x_0)\}$. A similar argument shows that $\sigma_\pi(gh) = \{g(x_0)h(x_0)\}$.

As $T_1(f) = T_1(g)$, (10) implies that

$$\sigma_\pi\big(T_1(f)T_2(h)\big) \cap \sigma_\pi(fh) \neq \emptyset,$$

and

$$\sigma_\pi\big(T_1(f)T_2(h)\big) \cap \sigma_\pi(gh) \neq \emptyset.$$

Thus $f(x_0)h(x_0) = T_1(f)(\tau(x_0)) = g(x_0)h(x_0)$. Let $l \in \mathcal{A}$ be such that $T_1(l) = k$, then $|l(x_0)h(x_0)| = |T_1(l)(\tau(x_0))T_2(h)(\tau(x_0))| = |k(\tau(x_0))|^2 = 1$, thus $h(x_0) \neq 0$, which yields that $f(x_0) = g(x_0)$. Therefore $f(x) = g(x)$ for all $x \in \delta\mathcal{A}$, i.e. T_1 is injective. A similar argument applies to T_2. □

Since T_1, T_2 are injective, there exist formal inverses $T_1^{-1}, T_2^{-1} : \mathcal{B} \to \mathcal{A}$. In addition, T_1^{-1} and T_2^{-1} are bijective mappings that satisfy

$$\sigma_\pi\big(T_1^{-1}(f)T_2^{-1}(g)\big) \cap \sigma_\pi(fg) \neq \emptyset \tag{14}$$

for all $f, g \in \mathcal{B}$, thus all of the previous results for T_1 and T_2 hold for T_1^{-1} and T_2^{-1}. Hence there exists an injective mapping $\psi : \delta\mathcal{B} \to \delta\mathcal{A}$ such that

$$\left|T_1^{-1}(f)(y)T_2^{-1}(g)(y)\right| = \left|f\big(\psi(y)\big)g\big(\psi(y)\big)\right| \tag{15}$$

holds for all $y \in \delta\mathcal{B}$ and $f, g \in \mathcal{B}$.

Lemma 14. *The map $\tau : \delta\mathcal{A} \to \delta\mathcal{B}$ defined by* (13) *is surjective.*

Proof. Let $y \in \delta\mathcal{B}$ and let $k \in F_{\tau(\psi(y))}(\mathcal{B})$. If $h_1, h_2 \in \mathcal{A}$ are such that $T_1(h_1) = T_2(h_2) = k$, then Lemma 9 yields that $h_1h_2 \in F_{\psi(y)}(\mathcal{A})$. Hence, by (15), $1 = |h_1(\psi(y))h_2(\psi(y))| = |k(y)|^2$, thus $k \in F_y(\mathcal{B})$. Therefore, by Lemma 2, $y = \tau(\psi(y))$. □

Of course a similar argument applies to ψ, showing that τ and ψ are mutual inverses of each other. As τ is a bijective mapping between $\delta\mathcal{A}$ and $\delta\mathcal{B}$, Lemma 12 implies that $|T_1(1)T_2(1)| = 1$ on $\delta\mathcal{B}$. In fact, as the following lemma shows, $T_1(1)T_2(1) = 1$.

Lemma 15. *The mappings T_1 and T_2 satisfy $T_1(1)T_2(1) = 1$.*

Proof. Let $y_0 \in \delta\mathcal{B}$ and $x_0 = \psi(y_0)$. Lemma 12 implies that $|T_1(1)(y_0)T_2(1)(y_0)| = 1$, thus, by Corollary 1, there exists a peaking function $k \in \mathcal{P}_{y_0}(\mathcal{B})$ such that $M(T_1(1)T_2(1)k) = \{y_0\}$. Notice that $\sigma_\pi(T_1(1)T_2(1)k) = \{T_1(1)(y_0)T_2(1)(y_0)\}$ and $\sigma_\pi(T_1(1)T_2(1)k^2) = \{T_1(1)(y_0)T_2(1)(y_0)\}$.

Let $h_1, h_2 \in \mathcal{A}$ be such that $T_1(h_1) = T_1(1)k$ and $T_2(h_2) = T_2(1)k$. If $x \in M(h_1) \cap \delta\mathcal{A}$, then, by Lemma 12,

$$\begin{aligned}\|T_1(1)T_2(1)k\| &= \|T_1(h_1)T_2(1)\| = \|h_1\| = |h_1(x)| \\ &= \big|T_1(h_1)\big(\tau(x)\big)T_2(1)\big(\tau(x)\big)\big| = \big|T_1(1)\big(\tau(x)\big)T_2(1)\big(\tau(x)\big)k\big(\tau(x)\big)\big|.\end{aligned}$$

Since $M(T_1(1)T_2(1)k) = \{y_0\}$, we have that $\tau(x) = y_0 = \tau(x_0)$, so the injectivity of τ implies that $M(h_1) \cap \delta\mathcal{A} = \{x_0\}$. A similar argument yields that $M(h_2) \cap \delta\mathcal{A} = \{x_0\}$, hence $\sigma_\pi(h_1) = \{h_1(x_0)\}$, $\sigma_\pi(h_2) = \{h_2(x_0)\}$, and $\sigma_\pi(h_1h_2) = \{h_1(x_0)h_2(x_0)\}$.

By (10),

$$\sigma_\pi(h_1) \cap \sigma_\pi\big(T_1(1)T_2(1)k\big) \neq \emptyset,$$
$$\sigma_\pi(h_2) \cap \sigma_\pi\big(T_1(1)T_2(1)k\big) \neq \emptyset,$$

and

$$\sigma_\pi(h_1h_2) \cap \sigma_\pi\big(T_1(1)T_2(1)k^2\big) \neq \emptyset,$$

so $h_1(x_0) = h_2(x_0) = h_1(x_0)h_2(x_0) = T_1(1)(y_0)T_2(1)(y_0)$, which yields that $T_1(1)(y_0)T_2(1)(y_0) = (T_1(1)(y_0)T_2(1)(y_0))^2$. Therefore, $T_1(1)(y_0)T_2(1)(y_0) = 1$, i.e. $T_1(1)(y)T_2(1)(y) = 1$ for all $y \in \delta\mathcal{B}$. □

3.2. Proof of Theorem 2

Define the mappings $\Phi_1, \Phi_2 : \mathcal{A} \to \mathcal{B}$ by $\Phi_1(f) = T_1(f)T_2(1)$ and $\Phi_2(f) = T_1(1)T_2(f)$.

Lemma 16. *The mappings $\Phi_1, \Phi_2 : \mathcal{A} \to \mathcal{B}$ are surjective, unital, and satisfy $\sigma_\pi(\Phi_1(f)\Phi_2(g)) \cap \sigma_\pi(fg) \neq \emptyset$ for all $f, g \in \mathcal{A}$.*

Proof. By Lemma 15, $\Phi_1(1) = \Phi_2(1) = T_1(1)T_2(1) = 1$, hence Φ_1 and Φ_2 are unital. Additionally, $\Phi_1(f)\Phi_2(g) = T_1(f)T_2(1)T_1(1)T_2(g) = T_1(f)T_2(g)$ holds for all $f, g \in \mathcal{A}$. Thus, by (10), $\sigma_\pi(\Phi_1(f)\Phi_2(g)) \cap \sigma_\pi(fg) \neq \emptyset$. Let $g \in \mathcal{B}$, then the surjectivity of T_1 implies that there exists an $f \in \mathcal{A}$ such that $T_1(f) = T_1(1)g$. Thus $\Phi_1(f) = T_1(f)T_2(1) = T_1(1)T_2(1)g = g$, which implies that $\Phi_1(f)$ is surjective. The surjectivity of Φ_2 is proved similarly. □

As Φ_1 and Φ_2 are unital and satisfy $\sigma_\pi(\Phi_1(f)\Phi_2(g)) \cap \sigma_\pi(fg) \neq \emptyset$, we have also that

$$\sigma_\pi\big(\Phi_1(f)\big) \cap \sigma_\pi(f) \neq \emptyset, \tag{16}$$
$$\sigma_\pi\big(\Phi_2(f)\big) \cap \sigma_\pi(f) \neq \emptyset, \tag{17}$$

and

$$\|\Phi_1(f)\| = \|\Phi_2(f)\| = \|f\| \tag{18}$$

holds for all $f \in \mathcal{A}$. In addition, Φ_1 and Φ_2 are surjective, thus the results of Section 3.1 hold for Φ_1 and Φ_2. In particular, Lemma 12 yields that

$$\big|\Phi_1(f)\big(\tau(x)\big)\big| = \big|f(x)\big| = \big|\Phi_2(f)\big(\tau(x)\big)\big| \tag{19}$$

holds for all $f \in \mathcal{A}$ and $x \in \delta\mathcal{A}$.

Lemma 17. *Let $h \in \mathcal{A}$ and $x_0 \in \delta\mathcal{A}$, then the following are equivalent*:

(a) $M(h) \cap \delta\mathcal{A} = \{x_0\}$,
(b) $M(\Phi_1(h)) \cap \delta\mathcal{B} = \{\tau(x_0)\}$, *and*
(c) $M(\Phi_2(h)) \cap \delta\mathcal{B} = \{\tau(x_0)\}$.

In any case $\Phi_1(h)(\tau(x_0)) = h(x_0) = \Phi_2(h)(\tau(x_0))$.

The proof of Lemma 17 is similar to the proof of Lemma 6. Notice that if $\Phi_1 = \Phi_2$, then Φ_1 is a surjective, unital, weakly peripherally multiplicative map, hence Theorem 1 implies that Φ_1 is an isometric algebra isomorphism. Therefore the proof of Theorem 2 follows from showing that $\Phi_1(f) = \Phi_2(f)$ holds for all $f \in \mathcal{A}$.

Theorem 2. *Let $\mathcal{A}$ and $\mathcal{B}$ be uniform algebras on first-countable compact Hausdorff spaces X and Y, respectively, and let $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ be surjective mappings that satisfy $\sigma_\pi(T_1(f)T_2(g)) \cap \sigma_\pi(fg) \neq \emptyset$ for all $f, g \in \mathcal{A}$. Then $T_1(f)T_2(1) = T_1(1)T_2(f)$ holds for all $f \in \mathcal{A}$, and the mapping $\Phi : \mathcal{A} \to \mathcal{B}$ defined by $\Phi(f) = T_1(f)T_2(1)$ is an isometric algebra isomorphism.*

Proof. Let Φ_1 and Φ_2 be defined as above, and let $f \in \mathcal{A}$ and $y_0 \in \delta\mathcal{B}$. If $x_0 \in \delta\mathcal{A}$ satisfies $\tau(x_0) = y_0$ and $\Phi_1(f)(y_0) = 0$, then by (19), $|\Phi_1(f)(y_0)| = |f(x_0)| = |\Phi_2(f)(y_0)|$, so $\Phi_1(f)(y_0) = \Phi_2(f)(y_0)$.

Suppose that $\Phi_1(f)(y_0) \neq 0$, then (19) implies that $f(x_0) \neq 0$, hence Corollary 1 implies that there exists a peaking function $h \in \mathcal{P}_{x_0}(\mathcal{A})$ such that $M(h) = M(fh) = \{x_0\}$ and $\sigma_\pi(fh) = \{f(x_0)\}$. If $y' \in M(\Phi_1(f)\Phi_2(h)) \cap \delta\mathcal{B}$, then, by (19),

$$\left|f\left(\tau^{-1}\left(y'\right)\right)h\left(\tau^{-1}\left(y'\right)\right)\right| = \left|\Phi_1(f)\left(y'\right)\Phi_2(h)\left(y'\right)\right| = \left\|\Phi_1(f)\Phi_2(h)\right\| = \|fh\|.$$

Since $M(fh) = \{x_0\}$, $\tau^{-1}(y') = x_0$, thus $y' = \tau(x_0) = y_0$. Therefore $M(\Phi_1(f)\Phi_2(h)) \cap \delta\mathcal{B} = \{y_0\}$, hence $\sigma_\pi(\Phi_1(f)\Phi_2(h)) = \{\Phi_1(f)(y_0)\Phi_2(h)(y_0)\}$. As $M(h) \cap \delta\mathcal{A} = \{x_0\}$, Lemma 17 gives $M(\Phi_2(h)) \cap \delta\mathcal{B} = \{y_0\}$ and $\Phi_2(h)(y_0) = h(x_0) = 1$, which implies that $\sigma_\pi(\Phi_1(f)\Phi_2(h)) = \{\Phi_1(f)(y_0)\}$. A similar argument shows that $\sigma_\pi(\Phi_1(h)\Phi_2(f)) = \{\Phi_2(f)(y_0)\}$.

By Lemma 16,

$$\sigma_\pi\left(\Phi_1(f)\Phi_2(h)\right) \cap \sigma_\pi(fh) \neq \emptyset$$

and

$$\sigma_\pi\left(\Phi_1(h)\Phi_2(f)\right) \cap \sigma_\pi(fh) \neq \emptyset.$$

Thus $\Phi_1(f)(y_0) = f(x_0) = \Phi_2(f)(y_0)$, therefore $\Phi_1(f)(y) = \Phi_2(f)(y)$ for all $y \in \delta\mathcal{B}$, which yields that $\Phi_1(f) = \Phi_2(f)$.

Since $\Phi_1 = \Phi_2$, we have that $\Phi = \Phi_1$ is a unital, surjective, weakly peripheral multiplicative map. By Theorem 1, Φ is an isometric algebra isomorphism. □

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