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Determining nodes for semilinear parabolic equations

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ABSTRACT

We are concerned with the determination of the asymptotic behavior of strong solutions to the initial-boundary value problem for general semilinear parabolic equations by the asymptotic behavior of these strong solutions on a finite set. More precisely, if the asymptotic behavior of the strong solution is known on a suitable finite set which is called determining nodes, then the asymptotic behavior of the strong solution itself is entirely determined. We prove the above property by the energy method.

1. Introduction

Let $n \in \mathbb{Z}$, $n \geqslant 2$, Ω be a bounded domain in $\mathbb{R}^n$ with its $C^{0,1}$-boundary $\partial\Omega$, H be a closed subspace of $L^2(\Omega)$, $V = H_0^1(\Omega) \cap H$. Our problem is the following strong formulation of the initial-boundary value problem for the semilinear parabolic equation:

$$\begin{aligned} &d_t u + Au + Bu = f \quad \text{in } L^2\big((0,\infty); H\big), \\ &u(0) = u_0 \quad \text{in } V, \end{aligned} \tag{1.1}$$

where u is a strong solution to (1.1), A is a densely defined closed linear operator from $D(A)$ to H, B is a nonlinear operator from $D(B)$ to H, f is an external force, u_0 is an initial data. Moreover, $D(A)$ and $D(B)$ are domains of A and B respectively. As is explained in Section 4, a typical example of $(1.1)_1$ is the following semilinear heat equation:

$$\partial_t u - \kappa\Delta u - |u|^{p-1}u = 0,$$

where u is the absolute temperature, $\kappa > 0$ is the coefficient of heat conductivity, $p > 1$. The existence, uniqueness and regularity of strong solutions to the initial-boundary value problem for the semilinear heat equation has been much studied for fifty years. See, for example, [4] and the references given there on the existence, uniqueness and regularity of strong solutions to the initial-boundary value problem for the semilinear heat equation in $\mathbb{R}^n$ with the Dirichlet boundary condition.

The stationary problem associated with (1.1) is the following boundary value problem for the semilinear elliptic equation:

$$A\bar{u} + B\bar{u} = \bar{f} \quad \text{in } H, \tag{1.2}$$

where $\bar{u}$ is a strong solution to (1.2), $\bar{f}$ is an external force. As is well known in [12], the stationary problem for the semilinear heat equation in $\mathbb{R}^n$ with the Dirichlet boundary condition has a trivial solution and nontrivial solutions for any $1 < p < (n+2)/(n-2)$. It is one of interesting questions whether a strong solution to (1.1) converges to a trivial or nontrivial

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solution to (1.2). According to the previous result by Foias and Temam [3], the conclusion of the asymptotic properties of strong solutions to (1.1) can be given by the theory of determining nodes. An approach of determining nodes is quite natural from the computational point of view. In general, the asymptotic behavior of strong solutions to the initial-boundary value problem for semilinear parabolic equations is uniquely determined by determining nodes which can be obtained from finite many measurements. Some problems related to determining nodes for semilinear parabolic equations have been studied in recent years. Foias and Temam [3] first discussed the existence of determining nodes for the Navier–Stokes equations in $\mathbb{R}^2$ and in $\mathbb{R}^3$ with the Dirichlet and periodic boundary conditions. As for partly dissipative reaction diffusion systems in $\mathbb{R}^2$ and in $\mathbb{R}^3$ with the Dirichlet, Neumann and periodic boundary conditions, Lu and Shao [9] obtained the same results as in [3]. Not only the existence of determining nodes but also the number of determining nodes can be deeply studied in the one-dimensional case. See, for example, [2,7,10] on the theory of determining nodes for the Kuramoto–Sivashinsky equation, the complex Ginzburg–Landau equation and the semilinear Schrödinger equation respectively in $\mathbb{R}$ with various periodic boundary conditions. As is mentioned above, the semilinear heat equation is a typical example of semilinear parabolic equations, but the theory of determining nodes for it has not been constructed yet. It is necessary to discuss the existence of determining nodes for semilinear parabolic equations such as $(1.1)_1$.

In this paper, we are concerned with the determination of the asymptotic behavior of strong solutions to (1.1) by determining nodes. It is an important consequence of our main results that the theory of determining nodes for the Navier–Stokes equations and the semilinear heat equation can be unified. One of our main results is stated as follows: There exists a finite set E in Ω such that if two strong solutions u and v to (1.1) satisfy $u(x,t) - v(x,t) \to 0$ as $t \to \infty$ for any $x \in E$, then $u(\cdot,t) - v(\cdot,t) \to 0$ in $V \cap C^{0,\gamma}(\overline{\Omega})$ as $t \to \infty$ for any $0 < \gamma < 1/2$. We prove the above property by the argument based on [3,9].

This paper is organized as follows: In Section 2, we state our main results concerning the existence of determining nodes for (1.1) after setting up notation and terminology used in this paper. The proofs of our main results are given in Section 3. Finally, we indicate applications of our main results to the semilinear heat equation and the Navier–Stokes equations in Section 4.

2. Preliminaries and main results

2.1. Function spaces

All functions which appear in this paper are either H or H^n-valued. For the sake of notational simplicity, we will not distinguish them from their values, i.e., H^n will also be simply denoted by H.

Function spaces and basic notation which we use throughout this paper are introduced as follows: The norm in $L^p(\Omega)$ $(1 \leqslant p \leqslant \infty)$ and in the Sobolev space $H^k(\Omega)$ $(k \in \mathbb{Z},\ k \geqslant 0)$ are denoted by $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{H^k(\Omega)}$ respectively, $H^0(\Omega) = L^2(\Omega)$. Moreover, the scalar product in $L^2(\Omega)$ and in $H^k(\Omega)$ are denoted by $(\cdot,\cdot)_{L^2(\Omega)}$ and $(\cdot,\cdot)_{H^k(\Omega)}$ respectively. $C_0^\infty(\Omega)$ is the set of all functions which are infinitely differentiable and have compact support in Ω. $H_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. Note that $H_0^1(\Omega)$ is characterized as $H_0^1(\Omega) = \{u \in H^1(\Omega);\ u|_{\partial\Omega} = 0\}$. As is well known in the theory of Hilbert spaces, $L^2(\Omega)$ is decomposed into $L^2(\Omega) = H \oplus H^\perp$, where $H^\perp$ is the orthogonal complement of H. Let P be the orthogonal projection of $L^2(\Omega)$ onto H. The norm in $C(\overline{\Omega})$ is denoted by $\|\cdot\|_{C(\overline{\Omega})}$. $C^{0,\gamma}(\overline{\Omega})$ $(0 < \gamma \leqslant 1)$ is the Banach space of all functions which are uniformly Hölder continuous with the exponent γ on $\overline{\Omega}$. The norm in $C^{0,\gamma}(\overline{\Omega})$ is denoted by $\|\cdot\|_{C^{0,\gamma}(\overline{\Omega})}$, i.e.,

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} := \|u\|_{C(\overline{\Omega})} + [[u]]_{C^{0,\gamma}(\overline{\Omega})}, \qquad [[u]]_{C^{0,\gamma}(\overline{\Omega})} := \sup_{x,y \in \overline{\Omega}, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\gamma}.$$

Let I be an open interval in $\mathbb{R}$, $(X, \|\cdot\|_X)$ be a Banach space. $L^p(I;X)$ $(1 \leqslant p < \infty)$ is the Banach space of all X-valued functions u which u is strongly measurable and $\|u\|_X^p$ is integrable in I. $L^\infty(I;X)$ is the Banach space of all X-valued functions u which u is strongly measurable and $\|u\|_X$ is essentially bounded in I. The norm in $L^p(I;X)$ and in $L^\infty(I;X)$ are denoted by $\|\cdot\|_{L^p(I;X)}$ and $\|\cdot\|_{L^\infty(I;X)}$ respectively. In the case where I is a bounded closed interval in $\mathbb{R}$, $C(I;X)$ is the Banach space of all X-valued functions which are continuous on I. If I is not bounded or closed, $C_b(I;X)$ is the Banach space of all X-valued functions which are bounded and continuous in I. The norm in $C(I;X)$ and in $C_b(I;X)$ is denoted by $\|\cdot\|_{C(I;X)}$ and $\|\cdot\|_{C_b(I;X)}$ respectively.

2.2. Strong solutions to (1.1) and (1.2)

In this subsection, we will make the properties of A and B which appeared in (1.1). First, A is the densely defined closed linear operator from $D(A) := H^2(\Omega) \cap V$ to H defined as

$$Au = -P\left\{\sum_{i,j=1}^n \partial_{x_j}(a_{ij}\partial_{x_i} u)\right\}.$$

It is required throughout this paper that A has the following properties (A.1)–(A.4):

(A.1) $a_{ij} \in C^{0,1}(\overline{\Omega})$ for any $i, j = 1, \dots, n$.
(A.2) $a_{ij} = a_{ji}$ on $\overline{\Omega}$ for any $i, j = 1, \dots, n$.
(A.3) There exists a positive constant a such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant a|\xi|^2$$

for any $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$.
(A.4) Set $(u, v)_{D(A)} = (Au, Av)_{L^2(\Omega)}$, $\|u\|_{D(A)} = ((u, u)_{D(A)})^{1/2}$. Then $\|\cdot\|_{D(A)}$ is equivalent to the standard norm in $H^2(\Omega)$. Therefore, there exist two positive constants a_1 and a_2 such that

$$a_1\|u\|_{H^2(\Omega)} \leqslant \|u\|_{D(A)} \leqslant a_2\|u\|_{H^2(\Omega)}$$

for any $u \in D(A)$.

Note that $A = -\kappa\Delta$ is a typical example of A. The norm $\|-\kappa\Delta\cdot\|_{L^2(\Omega)}$ induced by $-\kappa\Delta$ is equivalent to the standard norm in $H^2(\Omega)$, which follows from [5, Theorem 8.12]. Second, B is the nonlinear operator from $D(B) := H^2(\Omega) \cap V$ to H satisfying the following properties (B.1), (B.2):

(B.1) $B0 = 0$.
(B.2) There exist two constants $C_B > 0$ and $p > 1$ such that

$$\|Bu - Bv\|_{L^2(\Omega)} \leqslant C_B\big(\|u\|_{D(A)}^{p-1} + \|v\|_{D(A)}^{p-1}\big)\|u - v\|_{H^1(\Omega)}$$

for any $u, v \in D(B)$.

It is important for our main results that $Bu = -|u|^{p-1}u$ and $Bu = P(u \cdot \nabla)u$ can be considered. By virtue of (A.1)–(A.4), the scalar product and the norm in V can be introduced as follows:

$$(u, v)_a = \sum_{i,j=1}^{n} (a_{ij}\partial_{x_i}u, \partial_{x_j}v)_{L^2(\Omega)}, \qquad \|u\|_a = \big((u, u)_a\big)^{1/2}.$$

It follows easily from (A.3) and the Schwarz inequality that $\|\cdot\|_a$ and the standard norm in $H^1(\Omega)$ are equivalent norms in V. Consequently, there exist two positive constants a_3 and a_4 such that

$$a_3\|u\|_{H^1(\Omega)} \leqslant \|u\|_a \leqslant a_4\|u\|_{H^1(\Omega)}$$

for any $u \in V$. Finally, strong solutions to (1.1) and (1.2) are defined as follows:

Definition 2.1. Let $u_0 \in V$, $f \in L^2((0, \infty); H)$. Then u is called a strong solution to (1.1) if it satisfies

$$u \in L^2\big((0, \infty); D(A)\big) \cap C_b\big([0, \infty); V\big), \qquad d_t u \in L^2\big((0, \infty); H\big)$$

and (1.1). Let $\mathcal{S}(u_0, f)$ be the set of all functions which are strong solutions to (1.1).

Definition 2.2. Let $\bar{f} \in H$. Then $\bar{u}$ is called a strong solution to (1.2) if it satisfies

$$\bar{u} \in D(A)$$

and (1.2). Let $\mathcal{S}(\bar{f})$ be the set of all functions which are strong solutions to (1.2).

2.3. Main results

Our main results of this paper will be stated in this subsection. We begin by formulation of determining nodes. For any $N \in \mathbb{Z}$, $N \geqslant 1$, $x \in \overline{\Omega}$, $u \in D(A)$, set

$$E_N = \{x_1, \dots, x_N;\ x_i \in \overline{\Omega},\ i = 1, \dots, N\},$$
$$d_N(x) = \min_{i=1,\dots,N} |x - x_i|,$$
$$d_N = \max_{x\in\overline{\Omega}} d_N(x),$$
$$\eta_N(u) = \max_{i=1,\dots,N} \big|u(x_i)\big|.$$

Note that E_N and d_N can be considered as determining nodes and the density of E_N in Ω respectively. As for strong solutions to (1.1) and (1.2), the following assumptions (H.1)–(H.4) are essentially required for our main results.

(H.1) $\mathcal{S}(\bar{f}) \neq \emptyset$ for any $\bar{f} \in H$.
(H.2) There exists a positive constant $M(\bar{f})$ for any $\bar{f} \in H$ such that

$$\|\bar{u}\|_{D(A)} \leqslant M(\bar{f})$$

for any $\bar{u} \in \mathcal{S}(\bar{f})$.
(H.3) $\mathcal{S}(u_0, f) \neq \emptyset$ for any $u_0 \in V$, $f \in L^\infty((0,\infty); H)$.
(H.4) There exists a positive constant $M(f, t_0)$ for any $R > 0$, $f \in L^\infty((0,\infty); H)$, $t_0 > 0$ such that

$$\|u\|_{C_b([t_0,\infty); D(A))} \leqslant M(f, t_0)$$

for any $u \in \mathcal{S}(V(R), f)$, where

$$\mathcal{S}\big(V(R), f\big) := \bigcup_{u_0 \in V(R)} \mathcal{S}(u_0, f), \qquad V(R) := \big\{u_0 \in V;\ \|u_0\|_a \leqslant R\big\}.$$

Our main results are given by the following theorems on the existence of determining nodes for (1.1) and (1.2):

Theorem 2.1. *Let $n = 2, 3$, $\bar{f} \in H$, and assume* (H.1), (H.2). *Then there exists a positive constant δ_1 depending only on Ω, A, B and $M(\bar{f})$ such that if $0 < d_N \leqslant \delta_1$ and if $\bar{u}, \bar{v} \in \mathcal{S}(\bar{f})$ satisfy*

$$\bar{u}(x_i) = \bar{v}(x_i)$$

for any $i = 1, \ldots, N$, then

$$\bar{u} = \bar{v} \quad \text{in } \Omega.$$

Theorem 2.2. *Let $n = 2, 3$, $R > 0$, $f \in L^\infty((0,\infty); H)$, $t_0 > 0$, and assume* (H.2)–(H.4),

$$f(t) \to f_\infty \in H \quad \text{in } H \text{ as } t \to \infty.$$

Then there exists a positive constant δ_2 depending only on Ω, A, B, $M(f, t_0)$ and $M(f_\infty)$ such that if $0 < d_N \leqslant \delta_2$ and if $u \in \mathcal{S}(V(R), f)$ satisfies

$$u(x_i, t) \to \xi_i \in \mathbb{R} \quad \text{as } t \to \infty$$

for any $i = 1, \ldots, N$, then (1.2) *has uniquely a strong solution $u_\infty \in \mathcal{S}(f_\infty)$ satisfying*

$$u(t) \to u_\infty \quad \text{in } V \cap C^{0,\gamma}(\overline{\Omega}) \text{ as } t \to \infty$$

for any $0 < \gamma < 1/2$ and $u_\infty(x_i) = \xi_i$ for any $i = 1, \ldots, N$.

Theorem 2.3. *Let $n = 2, 3$, $R > 0$, $f, g \in L^\infty((0,\infty); H)$, $t_0 > 0$, and assume* (H.3), (H.4),

$$f(t) - g(t) \to 0 \quad \text{in } H \text{ as } t \to \infty.$$

Then there exists a positive constant δ_3 depending only on Ω, A, B, $M(f, t_0)$ and $M(g, t_0)$ such that if $0 < d_N \leqslant \delta_3$ and if $u \in \mathcal{S}(V(R), f)$, $v \in \mathcal{S}(V(R), g)$ satisfy

$$u(x_i, t) - v(x_i, t) \to 0$$

for any $i = 1, \ldots, N$, then

$$u(t) - v(t) \to 0 \quad \text{in } V \cap C^{0,\gamma}(\overline{\Omega}) \text{ as } t \to \infty$$

for any $0 < \gamma < 1/2$.

2.4. Auxiliary lemma

In this subsection, we will state three interpolation inequalities concerning the density of determining nodes. The following lemma yields that the standard norms in $C(\overline{\Omega})$, in $L^2(\Omega)$ and in $H^1(\Omega)$ are connected with d_N.

Lemma 2.1. *Let $n = 2, 3$. Then we have the following inequalities*:

(i) *There exists a positive constant C_1 depending only on Ω such that*

$$\|u\|_{C(\overline{\Omega})} \leqslant \eta_N(u) + C_1 d_N^{1/2} \|u\|_{D(A)} \tag{2.1}$$

for any $u \in D(A)$.

(ii) *There exist two positive constants C_2 and C_3 depending only on Ω such that*

$$\|u\|_{L^2(\Omega)} \leqslant C_2 \eta_N(u) + C_3 d_N^{1/2} \|u\|_{D(A)} \tag{2.2}$$

for any $u \in D(A)$.

(iii) *There exist two positive constants C_4 and C_5 depending only on Ω such that*

$$\|u\|_{H^1(\Omega)} \leqslant C_4 d_N^{-1/4} \eta_N(u) + C_5 d_N^{1/4} \|u\|_{D(A)} \tag{2.3}$$

for any $u \in D(A)$.

Proof. It is [3, Lemma 2.1]. □

3. Existence of determining nodes for (1.1) and (1.2)

Theorems 2.1–2.3 will be proved in this subsection. The proofs of Theorems 2.1–2.3 are based on the energy method with the aid of Lemma 2.1.

3.1. Proof of Theorem 2.1

Recall that $\bar{v}$ satisfies

$$A\bar{v} + B\bar{v} = \bar{f}. \tag{3.1}$$

Then it follows from (1.2), (3.1) that

$$A(\bar{u} - \bar{v}) + B\bar{u} - B\bar{v} = 0. \tag{3.2}$$

By taking the H-norm of (3.2) and (B.2), we obtain

$$\|\bar{u} - \bar{v}\|_{D(A)} \leqslant 2C_B M(\bar{f})^{p-1} \|\bar{u} - \bar{v}\|_{H^1(\Omega)}. \tag{3.3}$$

Notice that $\eta(\bar{u} - \bar{v}) = 0$, which follows from $\bar{u}(x_i) = \bar{v}(x_i)$ for any $i = 1, \dots, N$. Then (2.3) yields

$$\|\bar{u} - \bar{v}\|_{H^1(\Omega)} \leqslant C_5 d_N^{1/4} \|\bar{u} - \bar{v}\|_{D(A)}. \tag{3.4}$$

Therefore, by (3.3), (3.4), we have

$$\begin{aligned} &\|\bar{u} - \bar{v}\|_{D(A)} \leqslant 2C_B C_5 M(\bar{f})^{p-1} d_N^{1/4} \|\bar{u} - \bar{v}\|_{D(A)}, \\ &\big(1 - 2C_B C_5 M(\bar{f})^{p-1} d_N^{1/4}\big) \|\bar{u} - \bar{v}\|_{D(A)} \leqslant 0. \end{aligned} \tag{3.5}$$

Assume that

$$\begin{aligned} &1 - 2C_B C_5 M(\bar{f})^{p-1} d_N^{1/4} > 0, \\ &0 < d_N < \frac{1}{(2C_B C_5 M(\bar{f})^{p-1})^4}. \end{aligned} \tag{3.6}$$

Then (3.5) implies $\bar{u} = \bar{v}$ in Ω. Consequently, the sufficient condition for (3.6) is

$$0 < \delta_1 < \frac{1}{(2C_B C_5 M(\bar{f})^{p-1})^4}.$$

This completes the proof of Theorem 2.1.

3.2. Proof of Theorem 2.2

We begin with the energy-type estimate for strong solutions to (1.1). Consider two times t and s satisfying $t < s$, write $s = t + \tau$ for any $\tau > 0$, and set $v(t) = u(t+\tau)$, $g(t) = f(t+\tau)$. Then (1.1) implies that v satisfies

$$d_t v + Av + Bv = g. \tag{3.7}$$

It is easy to see from (1.1), (3.7) that

$$d_t(u-v) + A(u-v) + Bu - Bv = f - g. \tag{3.8}$$

By taking the H-scalar product of (3.8) with $A(u-v)$ and (B.2), we get

$$\frac{1}{2}d_t\big(\|u-v\|_a^2\big) + \|u-v\|_{D(A)}^2 \leqslant 2C_B M(f,t_0)^{p-1}\|u-v\|_{H^1(\Omega)}\|u-v\|_{D(A)} + \|u-v\|_{D(A)}\|f-g\|_{L^2(\Omega)}. \tag{3.9}$$

Notice that

$$\|u-v\|_{H^1(\Omega)} \leqslant C_4 d_N^{-1/4}\eta_N(u-v) + C_5 d_N^{1/4}\|u-v\|_{D(A)}, \tag{3.10}$$

which follows from (2.3). Then, by (3.9), (3.10) and the Cauchy inequality, we have

$$\begin{aligned}
&d_t\big(\|u-v\|_a^2\big) + \big(1 - 4C_B C_5 M(f,t_0)^{p-1} d_N^{1/4}\big)\|u-v\|_{D(A)}^2 \\
&\quad \leqslant 8C_B^2 C_4^2 M(f,t_0)^{2(p-1)} d_N^{-1/2}\eta_N(u-v)^2 + 2\|f-g\|_{L^2(\Omega)}^2.
\end{aligned} \tag{3.11}$$

Assume that

$$\begin{aligned}
&1 - 4C_B C_5 M(f,t_0)^{p-1} d_N^{1/4} > 0, \\
&0 < d_N < \frac{1}{(4C_B C_5 M(f,t_0)^{p-1})^4},
\end{aligned} \tag{3.12}$$

and set

$$\begin{aligned}
&\lambda = \frac{a_1^2}{a_4^2}\big(1 - 4C_B C_5 M(f,t_0)^{p-1} d_N^{1/4}\big) > 0, \\
&h(t) = 8C_B^2 C_4^2 M(f,t_0)^{2(p-1)} d_N^{-1/2}\eta_N\big((u-v)(t)\big)^2 + 2\big\|(f-g)(t)\big\|_{L^2(\Omega)}^2.
\end{aligned}$$

Then (3.11) yields

$$d_t\big(\big\|(u-v)(t)\big\|_a^2\big) + \lambda\big\|(u-v)(t)\big\|_a^2 \leqslant h(t) \tag{3.13}$$

for any $t \geqslant t_0$. We shall show that $\{u(t)\}_{t\geqslant t_0}$ is a Cauchy sequence in V with the aid of (3.13). Since $f(t) \to f_\infty$ in H as $t \to \infty$ and $u(x_i,t) \to \xi_i$ as $t \to \infty$ for any $i = 1,\dots,N$, we have $h(t) \to 0$ as $t \to \infty$. Therefore, there exists a positive constant t_ε for any positive constant ε such that $|h(t)| \leqslant \varepsilon$ for any $t \geqslant t_\varepsilon$. It is easy to see from (3.13) that

$$d_t\big(\big\|(u-v)(t)\big\|_a^2\big) + \lambda\big\|(u-v)(t)\big\|_a^2 \leqslant \varepsilon \tag{3.14}$$

for any $t \geqslant t_\varepsilon$. The Gronwall lemma and (3.14) imply

$$\big\|u(t)-u(s)\big\|_a^2 \leqslant \big\|(u-v)(t_\varepsilon)\big\|_a^2 e^{-\lambda(t-t_\varepsilon)} + \frac{\varepsilon}{\lambda}\big(1 - e^{-\lambda(t-t_\varepsilon)}\big) \tag{3.15}$$

for any $t \geqslant t_\varepsilon$. By taking t and s to infinity in (3.15), we have

$$\limsup_{t,s\to\infty}\big\|u(t)-u(s)\big\|_a^2 \leqslant \frac{\varepsilon}{\lambda}.$$

Since ε is an arbitrary positive constant, we conclude that $u(t) - v(t) \to 0$ in V as $t,s \to \infty$, i.e., $\{u(t)\}_{t\geqslant t_0}$ is a Cauchy sequence in V. The completeness of V yields that there exists a function $u_\infty \in V$ satisfying

$$u(t) \to u_\infty \quad \text{in } V \text{ as } t \to \infty. \tag{3.16}$$

As for the function u_∞ which is obtained above, we shall prove that $u_\infty \in \mathcal{S}(f_\infty)$ and $u_\infty(x_i) = \xi_i$ for any $i = 1,\dots,N$. Notice that $\{u(t)\}_{t\geqslant t_0}$ is bounded in $D(A)$ by virtue of (H.4). Then (3.16) implies

$$u(t) \to u_\infty \quad \text{in } C^{0,\gamma}(\overline{\Omega}) \text{ as } t \to \infty \tag{3.17}$$

for any $0 < \gamma < 1/2$, which follows from the Rellich–Kondrachov theorem [1, Theorem 6.3]. Since $u(x_i, t) \to \xi_i$ as $t \to \infty$ for any $i = 1, \ldots, N$, it follows from (3.17) that $u_\infty(x_i) = \xi_i$ for any $i = 1, \ldots, N$. By taking t to infinity in $(1.1)_1$, a straightforward argument shows that $u_\infty \in \mathcal{S}(f_\infty)$. Assume that $\delta_2 \leqslant \delta_1(M(f_\infty))$. Then (1.2) has uniquely a strong solution $u_\infty \in \mathcal{S}(f_\infty)$ satisfying $u_\infty(x_i) = \xi_i$ for any $i = 1, \ldots, N$, which follows from Theorem 2.1. Therefore, the sufficient condition for (3.12) and desired properties of u_∞ is

$$0 < \delta_2 < \min\left\{\delta_1\bigl(M(f_\infty)\bigr), \frac{1}{(4C_B C_5 M(f, t_0)^{p-1})^4}\right\},$$

which completes the proof of Theorem 2.2.

3.3. Proof of Theorem 2.3

In the same manner as in Subsection 3.2, we shall establish the energy-type estimate for strong solution to (1.1). Recall that v satisfies

$$d_t v + Av + Bv = g. \tag{3.18}$$

It follows easily from (1.1), (3.18) that

$$d_t(u - v) + A(u - v) + Bu - Bv = f - g. \tag{3.19}$$

By taking the H-scalar product of (3.19) with $A(u - v)$ and (B.2), we have

$$\begin{aligned} &\frac{1}{2} d_t\bigl(\|u - v\|_a^2\bigr) + \|u - v\|_{D(A)}^2 \\ &\quad \leqslant C_B\bigl(M(f, t_0)^{p-1} + M(g, t_0)^{p-1}\bigr)\|u - v\|_{H^1(\Omega)}\|u - v\|_{D(A)} + \|u - v\|_{D(A)}\|f - g\|_{L^2(\Omega)}. \end{aligned} \tag{3.20}$$

Notice that

$$\|u - v\|_{H^1(\Omega)} \leqslant C_4 d_N^{-1/4}\eta_N(u - v) + C_5 d_N^{1/4}\|u - v\|_{D(A)}, \tag{3.21}$$

which follows from (2.3). Then, by (3.20), (3.21) and the Cauchy inequality, we get

$$\begin{aligned} &d_t\bigl(\|u - v\|_a^2\bigr) + \bigl\{1 - 2C_B C_5\bigl(M(f, t_0)^{p-1} + M(g, t_0)^{p-1}\bigr)d_N^{1/4}\bigr\}\|u - v\|_{D(A)}^2 \\ &\quad \leqslant 2C_B^2 C_4^2\bigl(M(f, t_0)^{p-1} + M(g, t_0)^{p-1}\bigr)^2 d_N^{-1/2}\eta_N(u - v)^2 + 2\|f - g\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.22}$$

Assume that

$$\begin{aligned} &1 - 2C_B C_5\bigl(M(f, t_0)^{p-1} + M(g, t_0)^{p-1}\bigr)d_N^{1/4} > 0, \\ &0 < d_N < \frac{1}{\{2C_B C_5(M(f, t_0)^{p-1} + M(g, t_0)^{p-1})\}^4}, \end{aligned} \tag{3.23}$$

and set

$$\begin{aligned} &\lambda = \frac{a_1^2}{a_4^2}\bigl\{1 - 2C_B C_5\bigl(M(f, t_0)^{p-1} + M(g, t_0)^{p-1}\bigr)d_N^{1/4}\bigr\} > 0, \\ &h(t) = 2C_B^2 C_4^2\bigl(M(f, t_0)^{p-1} + M(g, t_0)^{p-1}\bigr)^2 d_N^{-1/2}\eta_N\bigl((u - v)(t)\bigr)^2 + \bigl\|(f - g)(t)\bigr\|_{L^2(\Omega)}^2. \end{aligned}$$

Then (3.22) gives

$$d_t\bigl(\bigl\|(u - v)(t)\bigr\|_a^2\bigr) + \lambda\bigl\|(u - v)(t)\bigr\|_a^2 \leqslant h(t) \tag{3.24}$$

for any $t \geqslant t_0$. By (3.24), we shall prove that $u(t) - v(t) \to 0$ in V as $t \to \infty$. Notice that $f(t) - g(t) \to 0$ in H as $t \to \infty$ and $u(x_j, t) - v(x_j, t) \to 0$ as $t \to \infty$ for any $j = 1, \ldots, N$. Then we have $h(t) \to 0$ as $t \to \infty$. Hence, there exists a positive constant t_ε for any positive constant ε such that $|h(t)| \leqslant \varepsilon$ for any $t \geqslant t_\varepsilon$. We can see easily from (3.24) that

$$d_t\bigl(\bigl\|(u - v)(t)\bigr\|_a^2\bigr) + \lambda\bigl\|(u - v)(t)\bigr\|_a^2 \leqslant \varepsilon \tag{3.25}$$

for any $t \geqslant t_\varepsilon$. It follows from the Gronwall lemma and (3.25) that

$$\bigl\|(u - v)(t)\bigr\|_a^2 \leqslant \bigl\|(u - v)(t_\varepsilon)\bigr\|_a^2 e^{-\lambda(t - t_\varepsilon)} + \frac{\varepsilon}{\lambda}\bigl(1 - e^{-\lambda(t - t_\varepsilon)}\bigr) \tag{3.26}$$

for any $t \geqslant t_\varepsilon$. By taking t to infinity in (3.26), we have

$$\limsup_{t\to\infty}\big\|(u-v)(t)\big\|_a^2 \leqslant \frac{\varepsilon}{\lambda}.$$

Since ε is an arbitrary positive constant, we conclude that

$$u(t)-v(t)\to 0 \quad \text{in } V \text{ as } t\to\infty. \tag{3.27}$$

It remains to prove that $u(t)-v(t)\to 0$ in $C^{0,\gamma}(\overline{\Omega})$ as $t\to\infty$ for any $0<\gamma<1/2$. We can see easily from (H.4) that $\{(u-v)(t)\}_{t\geqslant t_0}$ is bounded in $D(A)$. By the Rellich–Kondrachov theorem, (3.27) yields

$$u(t)-v(t)\to 0 \quad \text{in } C^{0,\gamma}(\overline{\Omega}) \text{ as } t\to\infty.$$

Consequently, the sufficient condition for (3.23) is

$$0<\delta_3<\frac{1}{\{2C_BC_5(M(f,t_0)^{p-1}+M(g,t_0)^{p-1})\}^4}.$$

This completes the proof of Theorem 2.3.

4. Applications

We will apply our main results to the semilinear heat equation and the Navier–Stokes equations in Subsections 4.2 and 4.3 respectively after some preliminaries in Subsection 4.1. Let Ω be, throughout this section, a bounded domain in $\mathbb{R}^n$ with its $C^{1,1}$-boundary $\partial\Omega$.

4.1. Sectorial operators in L^2 and analytic semigroups on L^2

The theory of analytic semigroups on $L^2(\Omega)$ and fractional powers of sectorial operators are introduced as follows: Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be Banach spaces. $\mathcal{B}(X;Y)$ is the Banach space of all bounded linear operators from X to Y, $\mathcal{B}(X):=\mathcal{B}(X;X)$. The norm in $\mathcal{B}(X;Y)$ is denoted by $\|\cdot\|_{\mathcal{B}(X;Y)}$, i.e.,

$$\|A\|_{\mathcal{B}(X;Y)} := \sup_{x\in X\setminus\{0\}}\frac{\|Ax\|_Y}{\|x\|_X}.$$

Let A be a sectorial operator in $L^2(\Omega)$ defined as in [6, Definition 1.3.1], $D(A)\subset H^2(\Omega)$. Then the spectrum of A is denoted by $\sigma(A)$, $\operatorname{Re}\sigma(A):=\{\operatorname{Re}\lambda;\ \lambda\in\sigma(A)\}$. Assume that $\operatorname{Re}\sigma(A)>0$, where $\operatorname{Re}\sigma(A)>0$ means that $\operatorname{Re}\lambda>0$ for any $\lambda\in\sigma(A)$. As is well known in [6, Theorem 1.3.4 and Definition 1.4.1], [11, Theorem 2.5.2 and Definition 2.6.7], $-A$ generates a uniformly bounded analytic semigroup $\{e^{-tA}\}_{t\geqslant 0}$ on $L^2(\Omega)$, fractional powers A^α of A can be defined for any $\alpha\geqslant 0$, $A^0=I_2$, where I_2 is the identity operator in $L^2(\Omega)$. Let us introduce the Hilbert space derived from A, i.e., $D(A^\alpha)$ with the scalar product $(u,v)_{D(A^\alpha)}=(A^\alpha u,A^\alpha v)_{L^2(\Omega)}$ and the norm $\|u\|_{D(A^\alpha)}=((u,u)_{D(A^\alpha)})^{1/2}$ for any $0\leqslant\alpha\leqslant 1$.

We state some lemmas concerning sectorial operators in $L^2(\Omega)$. See, for example, [6, Chapter 1], [11, Chapter 2] on the theory of analytic semigroups on Banach spaces and fractional powers of sectorial operators.

Lemma 4.1. *Let $\alpha\geqslant 0$, $0<\lambda<\Lambda_1$, where $\Lambda_1:=\min\{\lambda>0;\ \lambda\in\operatorname{Re}\sigma(A)\}$. Then there exists a positive constant $C_{\alpha,\lambda}$ depending only on n, Ω, A, α and λ such that*

$$\big\|A^\alpha e^{-tA}\big\|_{\mathcal{B}(L^2(\Omega))}\leqslant C_{\alpha,\lambda}t^{-\alpha}e^{-\lambda t}. \tag{4.1}$$

Proof. It is [6, Theorem 1.4.3]. □

Lemma 4.2. *Let $0\leqslant\alpha\leqslant 1$. Then*

$$D\big(A^\alpha\big)\hookrightarrow L^q(\Omega)\quad \text{if } 1<q<\infty,\qquad \frac{1}{2}-\frac{2\alpha}{n}\leqslant\frac{1}{q}\leqslant\frac{1}{2}, \tag{4.2}$$

$$D\big(A^\alpha\big)\hookrightarrow C^{0,\gamma}(\overline{\Omega})\quad \text{if } 0<\gamma<1,\qquad \frac{1}{2}-\frac{2\alpha-\gamma}{n}\leqslant 0, \tag{4.3}$$

where $\hookrightarrow$ is the continuous inclusion.

Proof. It is [6, Theorem 1.6.1]. □

4.2. Semilinear heat equation

The initial-boundary value problem for the semilinear heat equation is described as follows:

$$\begin{aligned} &\partial_t u - \kappa \Delta u - |u|^{p-1}u = f \quad \text{in } \Omega \times (0,\infty), \\ &u|_{t=0} = u_0 \quad \text{in } \Omega, \\ &u|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega \times (0,\infty), \end{aligned} \tag{4.4}$$

where u is the absolute temperature, $\kappa > 0$ is the coefficient of heat conductivity, $p > 1$, f is the external force, u_0 is the initial temperature.

Set $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, $P = I_2$. Then we have the following strong formulation of (4.4):

$$\begin{aligned} &d_t u + Au + b(u) = f \quad \text{in } L^2\big((0,\infty); L^2(\Omega)\big), \\ &u(0) = u_0 \quad \text{in } H_0^1(\Omega), \end{aligned} \tag{4.5}$$

where $Au = -\kappa\Delta u$, $b(u) = -|u|^{p-1}u$. It is well known in [5, Theorem 8.12] that A satisfies (A.1)–(A.4). Moreover, an elementary calculation shows that b has the following properties (b.1), (b.2):

(b.1) $b(0) = 0$.
(b.2) There exists a positive constant C_b depending only on p such that

$$\big|b(u) - b(v)\big| \leqslant C_b\big(|u|^{p-1} + |v|^{p-1}\big)|u - v|$$

for any $u, v \in \mathbb{R}$.

It is assured by the following lemma that $Bu = b(u)$ satisfies (B.1), (B.2).

Lemma 4.3. *Let $n = 2, 3$, $1 < p \leqslant n/(n-2)$. Then there exists a positive constant C_B depending only on Ω and p such that*

$$\big\|b(u) - b(v)\big\|_{L^2(\Omega)} \leqslant C_B\big(\|u\|_{H^1(\Omega)}^{p-1} + \|v\|_{H^1(\Omega)}^{p-1}\big)\|u - v\|_{H^1(\Omega)} \tag{4.6}$$

for any $u, v \in H^1(\Omega)$.

Proof. After taking the L^2-norm of (b.2), the Hölder and Minkowski inequalities imply

$$\big\|b(u) - b(v)\big\|_{L^2(\Omega)} \leqslant C_b\big(\|u\|_{L^{2p}(\Omega)}^{p-1} + \|v\|_{L^{2p}(\Omega)}^{p-1}\big)\|u - v\|_{L^{2p}(\Omega)} \tag{4.7}$$

for any $u, v \in H^1(\Omega)$. It is easy to see from (4.7) and the Sobolev embedding theorem that we have (4.6). □

The following theorems yield that (H.3), (H.4) hold for (4.5) under suitable assumptions for p, u_0 and f.

Theorem 4.1. *Let $n = 2, 3$, $1 < p \leqslant n/(n-2)$, $u_0 \in H_0^1(\Omega)$, $f \in L^2((0,\infty); L^2(\Omega))$. Then there exist two positive constants ε_1 and ε_2 depending only on Ω, κ and p such that* (4.5) *has uniquely a strong solution satisfying*

$$\|u\|_{C_b([0,\infty); H_0^1(\Omega))} \leqslant \varepsilon_1$$

provided that

$$\|u_0\|_a \leqslant \varepsilon_1, \qquad \|f\|_{L^\infty((0,\infty); L^2(\Omega))} \leqslant \varepsilon_2.$$

Proof. Let $\tilde{u}_0 \in H_0^1(\Omega)$, $\tilde{f} \in L^2((0,T); L^2(\Omega))$, $T > 0$. Then

$$\begin{aligned} &d_t u + Au = \tilde{f} \quad \text{in } L^2\big((0,T); L^2(\Omega)\big), \\ &u(0) = \tilde{u}_0 \quad \text{in } H_0^1(\Omega) \end{aligned} \tag{4.8}$$

has uniquely a strong solution u satisfying

$$\begin{aligned} &u \in L^2\big((0,T); D(A)\big) \cap C\big([0,T]; H_0^1(\Omega)\big), \qquad d_t u \in L^2\big((0,T); L^2(\Omega)\big), \\ &\kappa\|\nabla u\|_{C([0,T]; L^2(\Omega))}^2 + \|d_t u\|_{L^2((0,T); L^2(\Omega))}^2 + \|u\|_{L^2((0,T); D(A))}^2 \leqslant \kappa\|\nabla\tilde{u}_0\|_{L^2(\Omega)}^2 + \|\tilde{f}\|_{L^2((0,T); L^2(\Omega))}^2, \end{aligned} \tag{4.9}$$

which is well known in [8, Theorem 3.2.1]. A fixed point argument with the aid of (4.8), (4.9) and the Banach fixed point theorem shows that there exists a positive constant $T_* \leqslant T$ depending only on Ω, κ, p, u_0 and f such that

$$\begin{aligned} &d_t u + Au = -b(u) + f \quad \text{in } L^2\big((0,T); L^2(\Omega)\big), \\ &u(0) = u_0 \quad \text{in } H_0^1(\Omega) \end{aligned} \tag{4.10}$$

has uniquely a strong solution u satisfying

$$u \in L^2\big((0,T_*); D(A)\big) \cap C\big([0,T_*]; H_0^1(\Omega)\big), \qquad d_t u \in L^2\big((0,T_*); L^2(\Omega)\big).$$

By taking the L^2-scalar product of (4.10) with Au and the Poincaré inequality, a priori estimate for strong solutions to (4.5) is established as follows:

$$d_t\big(\big\|\nabla u(t)\big\|^2_{L^2(\Omega)}\big) \leqslant -\kappa\lambda_1\big\|\nabla u(t)\big\|^2_{L^2(\Omega)} + 2\kappa^{-1}C^{2p}\big\|\nabla u(t)\big\|^{2p}_{L^2(\Omega)} + 2\kappa^{-1}\big\|f(t)\big\|^2_{L^2(\Omega)} \tag{4.11}$$

for any $t > 0$, where λ_1 is the first eigenvalue of $-\Delta$ with the zero Dirichlet boundary condition, C is a positive constant depending only on Ω. Assume that

$$\|\nabla u_0\|^2_{L^2(\Omega)} \leqslant \left(\frac{\kappa^2\lambda_1}{4C^{2p}}\right)^{1/(p-1)}, \qquad \|f\|^2_{L^\infty((0,\infty);L^2(\Omega))} \leqslant \left(\frac{\kappa^2\lambda_1}{4C^{2p}}\right)^{p/(p-1)}. \tag{4.12}$$

Then (4.11), (4.12) give $d_t(\|\nabla u(t)\|^2_{L^2(\Omega)}) \leqslant 0$ for any $t > 0$, consequently,

$$\|\nabla u\|^2_{C_b([0,\infty);L^2(\Omega))} \leqslant \left(\frac{\kappa^2\lambda_1}{4C^{2p}}\right)^{1/(p-1)}. \tag{4.13}$$

By applying (4.13) to the existence and uniqueness of solutions to (4.10), therefore, (4.5) has uniquely a strong solution satisfying (4.13) provided that u_0 and f satisfy (4.12). □

Theorem 4.2. *Let $n = 2, 3$, $1 < p \leqslant n/(n-2)$, $0 < \alpha \leqslant 1$, $R > 0$, $f \in L^\infty((0,\infty); D(A^\alpha))$, $t_0 > 0$. Then there exists a positive constant $M_\alpha(f, t_0)$ depending only on Ω, κ, p, R, f, t_0 and α such that*

$$\|u\|_{C_b([t_0,\infty);D(A))} \leqslant M_\alpha(f, t_0)$$

for any $u \in \mathcal{S}(H_0^1(\Omega)(R), f)$ satisfying $\|u\|_{C_b([0,\infty);H_0^1(\Omega))} \leqslant R$.

Proof. By virtue of [11, Theorems 2.5.2 and 7.3.6], A is a sectorial operator in $L^2(\Omega)$ satisfying $\operatorname{Re}\sigma(A) > 0$. Since $u \in C_b([0,\infty); H_0^1(\Omega))$, it follows from [6, Lemma 3.3.2] that

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-s)A} b(u)(s)\,ds + \int_0^t e^{-(t-s)A} f(s)\,ds \tag{4.14}$$

for any $t \geqslant 0$,

$$u(t) = e^{-tA}u(t_0) - \int_{t_0}^t e^{-(t-s)A} b(u)(s)\,ds + \int_{t_0}^t e^{-(t-s)A} f(s)\,ds \tag{4.15}$$

for any $t \geqslant t_0$. In the case where $1/2 < \beta < 1$, we can see easily from (4.1), (4.14) that

$$\begin{aligned} \big\|u(t)\big\|_{D(A^\beta)} \leqslant{}& C_{\beta-1/2,\lambda} t^{-\beta+1/2} e^{-\lambda t}\|u_0\|_{D(A^{1/2})} + C_{\beta,\lambda}\int_0^t (t-s)^{-\beta} e^{-\lambda(t-s)}\big\|b(u)(s)\big\|_{L^2(\Omega)}\,ds \\ &+ C_{\beta,\lambda}\int_0^t (t-s)^{-\beta} e^{-\lambda(t-s)}\big\|f(s)\big\|_{L^2(\Omega)}\,ds \end{aligned} \tag{4.16}$$

for any $t > 0$. Notice that $D(A^{1/2}) = H_0^1(\Omega)$ and $(u,v)_{D(A^{1/2})} = (u,v)_a$. Then, by (4.6), (4.16), we obtain

$$\|u\|_{C_b([t_0,\infty);D(A^\beta))} \leqslant C_{\beta-1/2,\lambda} t_0^{-\beta+1/2} R + C_{\beta,\lambda}\lambda^{1+\beta}\Gamma(1-\beta)\big(C_B R^p + \|f\|_{L^\infty((0,\infty);L^2(\Omega))}\big), \tag{4.17}$$

where $\Gamma(x)$ is the gamma function. Let $n/4 < \beta < 1$, and set

$$M(f, t_0) = C_{\beta-1/2,\lambda} t_0^{-\beta+1/2} R + C_{\beta,\lambda} \lambda^{1+\beta} \Gamma(1-\beta)\big(C_B R^p + \|f\|_{L^\infty((0,\infty);L^2(\Omega))}\big).$$

Then it follows from (4.3), (4.17) that

$$\big\|b(u)(t)\big\|_{D(A^{1/2})} = \big\|b(u)(t)\big\|_a \leqslant \kappa^{1/2} p C^p M(f, t_0)^p \tag{4.18}$$

for any $t \geqslant t_0$, where C is a positive constant depending only on Ω. In the case where $\beta = 1$, it is easy to see from (4.1), (4.15) that

$$\begin{aligned}\big\|u(t)\big\|_{D(A)} \leqslant{}& C_{1/2,\lambda}(t-t_0)^{-1/2} e^{-\lambda(t-t_0)} \big\|u(t_0)\big\|_{D(A^{1/2})} + C_{1/2,\lambda} \int_{t_0}^{t} (t-s)^{-1/2} e^{-\lambda(t-s)} \big\|b(u)(s)\big\|_{D(A^{1/2})}\, ds \\ &+ C_{1-\alpha,\lambda} \int_{t_0}^{t} (t-s)^{-1+\alpha} e^{-\lambda(t-s)} \big\|f(s)\big\|_{D(A^\alpha)}\, ds\end{aligned} \tag{4.19}$$

for any $t > t_0$. By (4.18), (4.19) and the same argument as in (4.17), we have

$$\begin{aligned}\|u\|_{C_b([2t_0,\infty);D(A))} \leqslant{}& C_{1/2,\lambda} t_0^{-1/2} R + C_{1/2,\lambda} \lambda^{3/2} \pi^{1/2} \kappa^{1/2} p C^p M(f, t_0)^p \\ &+ C_{1-\alpha,\lambda} \lambda^{2-\alpha} \Gamma(\alpha) \|f\|_{L^\infty((0,\infty);D(A^\alpha))}.\end{aligned} \tag{4.20}$$

Set

$$M_\alpha(f, 2t_0) = C_{1/2,\lambda} t_0^{-1/2} R + C_{1/2,\lambda} \lambda^{3/2} \pi^{1/2} \kappa^{1/2} p C^p M(f, t_0)^p + C_{1-\alpha,\lambda} \lambda^{2-\alpha} \Gamma(\alpha) \|f\|_{L^\infty((0,\infty);D(A^\alpha))}.$$

Then the conclusion follows immediately from (4.20). □

4.3. Navier–Stokes equations

The initial-boundary value problem for the Navier–Stokes equations is described as follows:

$$\begin{aligned}&\operatorname{div} u = 0 \quad \text{in } \Omega \times (0,\infty), \\ &\partial_t u + (u \cdot \nabla) u + \nabla p - \mu \Delta u = f \quad \text{in } \Omega \times (0,\infty), \\ &u|_{t=0} = u_0 \quad \text{in } \Omega, \\ &u|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega \times (0,\infty),\end{aligned} \tag{4.21}$$

where $u = (u_1, \dots, u_n)^T$ is the fluid velocity, p is the pressure, $\mu > 0$ is the coefficient of viscosity, $f = (f_1, \dots, f_n)^T$ is the external force field, u_0 is the initial velocity, $\cdot^T$ is the transposition.

In order to utilize the strong formulation of (4.21), we introduce the solenoidal function spaces as follows: $C_{0,\sigma}^\infty(\Omega) := \{u \in (C_0^\infty(\Omega))^n;\ \operatorname{div} u = 0\}$. $L_\sigma^2(\Omega)$ is the completion of $C_{0,\sigma}^\infty(\Omega)$ in $(L^2(\Omega))^n$. Note that $L_\sigma^2(\Omega)$ is characterized as $L_\sigma^2(\Omega) = \{u \in (L^2(\Omega))^n;\ \operatorname{div} u = 0,\ \nu \cdot u|_{\partial\Omega} = 0\}$, where ν is the outward normal vector on $\partial\Omega$. It follows from the Helmholtz decomposition that $(L^2(\Omega))^n$ is decomposed into $(L^2(\Omega))^n = L_\sigma^2(\Omega) \oplus L_\pi^2(\Omega)$, where $L_\pi^2(\Omega) := \{\nabla p;\ p \in H^1(\Omega)\}$. Let P_2 be the orthogonal projection of $(L^2(\Omega))^n$ onto $L_\sigma^2(\Omega)$. See, for example, [13, Chapter 1] on the basic properties of the Helmholtz decomposition.

Set $H^n = L_\sigma^2(\Omega)$, $V^n = H_{0,\sigma}^1(\Omega)$, $P = P_2$, where $H_{0,\sigma}^1(\Omega) := (H_0^1(\Omega))^n \cap L_\sigma^2(\Omega)$. Then the strong formulation of (4.21) is given by

$$\begin{aligned}&d_t u + Au + B(u) = f \quad \text{in } L^2\big((0,\infty); L_\sigma^2(\Omega)\big), \\ &u(0) = u_0 \quad \text{in } H_{0,\sigma}^1(\Omega),\end{aligned} \tag{4.22}$$

where $Au = -P_2(\mu\Delta u)$, $B(u) = P_2(u \cdot \nabla)u$. It follows from [13, Lemma 3.3.7] that A satisfies (A.1)–(A.4). The following lemma admits that $Bu = B(u)$ satisfies (B.1), (B.2).

Lemma 4.4. *Let $n = 2, 3$. Then there exists a positive constant C_B depending only on Ω such that*

$$\big\|B(u) - B(v)\big\|_{(L^2(\Omega))^n} \leqslant C_B\big(\|u\|_{(H^2(\Omega))^n} + \|v\|_{(H^2(\Omega))^n}\big)\|u - v\|_{(H^1(\Omega))^n} \tag{4.23}$$

for any $u, v \in (H^2(\Omega))^n$.

Proof. It is easy to see that

$$B(u) - B(v) = P_2(u \cdot \nabla)(u - v) + P_2\big((u - v) \cdot \nabla\big)v$$

for any $u, v \in (H^2(\Omega))^n$. Notice that $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, which follows from the Sobolev embedding theorem. Then we have

$$\big\| P_2(u \cdot \nabla)(u - v) \big\|_{(L^2(\Omega))^n} \leqslant C_1 \|u\|_{(H^2(\Omega))^n} \|u - v\|_{(H^1(\Omega))^n} \tag{4.24}$$

for any $u, v \in (H^2(\Omega))^n$, where C_1 is a positive constant depending only on Ω. Since $H^1(\Omega) \hookrightarrow L^6(\Omega) \hookrightarrow L^3(\Omega)$, which follows from the Sobolev embedding theorem, we can see easily from the Hölder inequality that

$$\big\| P_2\big((u - v) \cdot \nabla\big)v \big\|_{(L^2(\Omega))^n} \leqslant C_2 \|v\|_{(H^2(\Omega))^n} \|u - v\|_{(H^1(\Omega))^n} \tag{4.25}$$

for any $u, v \in (H^2(\Omega))^n$, where C_2 is a positive constant depending only on Ω. Consequently, (4.24), (4.25) yield clearly that we have (4.23). □

As is mentioned in [3] and the references given there, the existence and boundedness of strong solutions to (4.22) are partially known. Roughly speaking, (H.3), (H.4) hold for the case where $n = 2$. In the case where $n = 3$, (H.3) implies (H.4).

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