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Regions of meromorphy and value distribution of geometrically converging rational functions [☆]

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ABSTRACT

Let D be a region, $\{r_n\}_{n\in\mathbb{N}}$ a sequence of rational functions of degree at most n and let each r_n have at most m poles in D, for $m \in \mathbb{N}$ fixed. We prove that if $\{r_n\}_{n\in\mathbb{N}}$ converges geometrically to a function f on some continuum $S \subset D$ and if the number of zeros of r_n in any compact subset of D is of growth $o(n)$ as $n \to \infty$, then the sequence $\{r_n\}_{n\in\mathbb{N}}$ converges m_1-almost uniformly to a meromorphic function in D. This result about meromorphic continuation is used to obtain Picard-type theorems for the value distribution of m_1-maximally convergent rational functions, especially in Padé approximation and Chebyshev rational approximation.

1. Introduction

For a subset B in $\mathbb{C}$, we denote by $\mathcal{A}(B)$ (resp. $\mathcal{M}(B)$) the class of functions that are holomorphic (resp. meromorphic) on B. Moreover, $\mathcal{M}_m(B)$ is the subset of functions in $\mathcal{M}(B)$ that have at most m poles in B, each pole counted with its multiplicity. Furthermore, we denote by $\|\cdot\|_B$ the supremum norm on B. A compact set B in $\mathbb{C}$ is called a *continuum* if B is connected and consists of more than a single point.

For $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we denote by $\mathcal{P}_n$ the collection of all polynomials of degree at most n, and let

$$\mathcal{R}_{n,m} := \{r = p/q\colon\ p \in \mathcal{P}_n,\ q \in \mathcal{P}_m,\ q \not\equiv 0\}$$

be the rational functions of numerator degree $\leqslant n$ and denominator degree $\leqslant m$.

Let E be a compact set in $\mathbb{C}$ with regular and connected complement $\Omega = \overline{\mathbb{C}} \setminus E$. We denote by $G(z,t)$ the Green function of Ω with pole at $t \in \Omega$. Since Ω is regular, $G_E(z,t)$, $t \in \Omega$, can be continuously extended to $\mathbb{C}$ by defining $G_E(z,t) = 0$ for $z \in E$.

For $\rho > 1$ we define the Green domains E_ρ of $G_E(z,\infty)$ by

$$E_\rho := \left\{z \in \Omega\colon\ G_E(z,\infty) < \log\rho\right\} \cup E.$$

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For $f \in \mathcal{M}_m(E)$, we define $\rho_m(f)$ as the *maximal* parameter $\rho \in (1, \infty]$ such that f can be continued to a function in $\mathcal{M}_m(E_\rho)$. Analogously, $\rho(f)$ is the maximal parameter $\rho > 1$ such that f can be continued to a function in $\mathcal{M}(E_\rho)$. In both cases, we use the notation $f \in \mathcal{M}_m(E_{\rho_m(f)})$, resp. $f \in \mathcal{M}(E_{\rho(f)})$.

Given $n, m \in \mathbb{N}$, let $r^*_{n,m} = r^*_{n,m}(f) \in \mathcal{R}_{n,m}$ denote a rational function of best uniform approximation to $f \in \mathcal{M}(E)$ on E out of $\mathcal{R}_{n,m}$, i.e.,

$$e_{n,m}(f) := \inf_{r \in \mathcal{R}_{n,m}} \|f - r\|_E = \|f - r^*_{n,m}\|_E.$$

It is well known that

$$\limsup_{n\to\infty} e^{1/n}_{n,m}(f) \leqslant 1/\rho_m(f) \tag{1.1}$$

(cf. Walsh [14, Theorem 1, p. 378]).

Saff and Gončar have investigated the regions of meromorphy of f if the rational approximants $\{r_{n,m}\}_{n\in\mathbb{N}}$, with $m \in \mathbb{N}_0$ fixed, satisfy

$$\limsup_{n\to\infty} \|f - r_{n,m}\|^{1/n}_E \leqslant \frac{1}{\rho} < 1. \tag{1.2}$$

It was proved by Saff [10] for closed bounded Jordan regions E and by Gončar [7] for general compact sets E with regular, connected complement that (1.2) implies $f \in \mathcal{M}_m(E_\rho)$. Hence, using best rational approximants $r_{n,m} = r^*_{n,m}$ in (1.2) the inequality (1.1) yields $f \in \mathcal{M}_m(E_{\rho_m(f)})$.

Concerning the distribution of the zeros of $r^*_{n,m}$, it is known in the case $\rho_m(f) < \infty$ that the normalized zero counting measures ν_n of the numerators of $r^*_{n,m}$ converge weakly to the equilibrium distribution of the closure $\overline{E}_{\rho_m(f)}$, at least for a subsequence $\Lambda \subset \mathbb{N}$ as $n \to \infty$, $n \in \Lambda$. Hence, there exists a close connection between the zeros of $r^*_{n,m}$ and the maximal Green region $E_{\rho_m(f)}$ of m-meromorphy of f (cf. Theorem 4.1 in [5]).

If $\{m_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathbb{N}$ with $\lim_{n\to\infty} m_n = \infty$, then

$$\limsup_{n\to\infty} e^{1/n}_{n,m_n}(f) \leqslant 1/\rho(f).$$

Moreover, let $1 < \rho(f) < \infty$ and $m_n = o(n/\log n)$ as $n \to \infty$. If the sequence $\{r^*_{n,m_n}\}_{n\in\mathbb{N}}$ of best approximants to f converges m_1-maximally to f inside $E_{\rho(f)}$ and if there exists a point $z_0 \in \partial E_{\rho(f)}$ that is a singularity of multivalued character to f, then the normalized zero counting measures ν_n of r^*_{n,m_n} converge weakly to the equilibrium distribution of $\overline{E}_{\rho(f)}$, at least for a subsequence $\Lambda \subset \mathbb{N}$ as $n \to \infty$, $n \in \Lambda$ (Theorem 4 together with Theorem 2 in [4]). For the definition of m_1-maximally convergent sequences we refer to Section 4 or [4]. Examples of m_1-maximally convergent sequences are Chebyshev rational approximants on $E = [-1, 1]$ and classical Padé approximants.

It is well known that such m_1-maximally convergent sequences converge geometrically inside $E_{\rho(f)}$ on appropriate continua. On the other hand, it was proved in [9] that geometric convergence of $\{r_n\}_{n\in\mathbb{N}}$ to f on a continuum S in a region D implies, together with conditions on the distribution of the zeros and poles of r_n, that f is holomorphic in D.

Henceforth, let us consider the number of a-values of a rational function $r \in \mathcal{R}_{n,n}$, i.e., for a subset $B \subset \mathbb{C}$ and $a \in \overline{\mathbb{C}}$, let

$$N_a(r, B) := \#\{z \in B\colon r(z) = a\},$$

each a-value is counted with its multiplicity. Then the result of [9] can be stated as follows.

Theorem A. *Let S be a continuum and D a region in $\mathbb{C}$ with $S \subset D$. Let $f \in \mathcal{A}(S)$ and let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence of rational functions $r_n \in \mathcal{R}_{n,n}$ with no poles in D converging geometrically to f on S, i.e.,*

$$\limsup_{n\to\infty} \|f - r_n\|^{1/n}_S < 1.$$

Let f be not identically 0 ($f \not\equiv 0$) on S. Furthermore, for each compact set K in D we assume

$$N_0(r_n, K) = o(n) \quad \text{as } n \to \infty.$$

Then f can be continued to a holomorphic function in D.

The preceding theorem can be applied to m_1-maximally convergent sequences $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ to f on E: Let $\{m_n\}_{n\in\mathbb{N}}$ be a sequence with $\lim_{n\to\infty} m_n = \infty$, and let $1 < \rho(f) < \infty$. Then any boundary point z_0 of $E_{\rho(f)}$ is an accumulation point of poles *or* zeros of $\{r_{n,m_n}\}_{n\in\mathbb{N}}$ as $n \to \infty$ if z_0 is not a removable singularity.

One of the objectives of this paper is to describe more precisely the behavior of zeros and poles of m_1-maximally convergent sequences in the neighborhood of a point $z_0 \in \partial E_{\rho(f)}$ that is neither a removable singularity nor a pole of f. Moreover, the distribution of the a-values for any $a \in \overline{\mathbb{C}}$ is investigated. To obtain such results we first generalize Theorem A to the case that poles of r_n in D are allowed.

2. Main results

In our considerations, we will use the concept of m_1-measure [8]. For $B \subset \mathbb{C}$, we set

$$m_1(B) := \inf \sum_{\nu} |U_\nu|$$

where the infinimum is taken over all coverings $\{U_\nu\}$ of B by disks U_ν and $|U_\nu|$ is the radius of the disk U_ν.

Let D be a region in $\mathbb{C}$, and φ a function defined in D with values in $\overline{\mathbb{C}}$. A sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of meromorphic functions in D is said to *converge to a function φ with respect to the m_1-measure inside D* if for every $\varepsilon > 0$ and any compact set $K \subset D$ we have

$$m_1\bigl(\bigl\{z \in K:\ \bigl|(\varphi - \varphi_n)(z)\bigr| \geqslant \varepsilon\bigr\}\bigr) \to 0 \quad \text{as } n \to \infty.$$

The sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is said to converge to φ *m_1-almost uniformly inside D* if for any compact set $K \subset D$ and any $\varepsilon > 0$ there exists a set $K_\varepsilon \subset K$ such that $m_1(K \setminus K_\varepsilon) < \varepsilon$ and $\{\varphi_n\}$ converges uniformly to φ on K_ε.

The sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is said to converge to φ *m_1-almost geometrically inside D* if for any $\varepsilon > 0$ there exists a set $\Omega(\varepsilon)$ in $\mathbb{C}$ with $m_1(\Omega(\varepsilon)) < \varepsilon$ such that

$$\limsup_{n\to\infty} \|\varphi - \varphi_n\|_{K\setminus\Omega(\varepsilon)}^{1/n} < 1$$

for any compact set $K \subset D$.

Hence, the m_1-almost geometric convergence inside D implies m_1-almost uniform convergence inside D, which again implies m_1-convergence inside D.

Finally, two functions are said to be *m_1-equivalent* in D if they coincide on a set $\widetilde{D} \subset D$ such that $m_1(D \setminus \widetilde{D}) = 0$.

Our main result can now be stated as follows.

Theorem 1. *Let S be a continuum and D a region in $\mathbb{C}$ with $S \subset D$. Let $\{r_n\}_{n\in\mathbb{N}}$, $r_n \in \mathcal{R}_{n,n}$, be a sequence of rational functions converging geometrically to a continuous function f on S, i.e.,*

$$\limsup_{n\to\infty} \|f - r_n\|_S^{1/n} < 1 \tag{2.1}$$

and assume that $f \not\equiv 0$ on S. If there exists a fixed integer $m \in \mathbb{N}$ such that $r_n \in \mathcal{M}_m(D)$ for all n, and

$$N_0(r_n, K) = o(n) \quad \text{as } n \to \infty$$

for each compact subset $K \subset D$, then the sequence $\{r_n\}_{n\in\mathbb{N}}$ converges m_1-almost geometrically inside D to a meromorphic function $f \in \mathcal{M}_m(D)$.

We remark that Theorem 1 remains valid if $\mathbb{N}$ is replaced by a monotone subsequence $\Lambda = \{n_k\}_{k\in\mathbb{N}}$ with the additional property

$$\frac{n_{k+1}}{n_k} \leqslant c$$

for some constant $c \in \mathbb{R}$, since we can complete the sequence $\{r_{n_k}\}_{k\in\mathbb{N}}$ by setting

$$r_m = r_{n_k} \quad \text{for } n_k \leqslant m < n_{k+1}.$$

The sequence $\{r_m\}_{m\in\mathbb{N}}$ then still converges geometrically to f, and thus satisfies the hypothesis of Theorem 1.

We also remark that the condition $f \not\equiv 0$ on S is necessary. E.g., z^n converges to 0 geometrically on any compact subset of the open unit disk, but not outside the unit circle. The conclusion of the theorem fails if D contains points inside and outside the unit circle, but not the point 0 where all the zeros of z^n are. On the other hand, $z^n + 1$ converges to 1 geometrically on any compact subset of the open unit disk, and its zeros accumulate to every point $|z| = 1$. In this case, Theorem 1 can be applied to D equal to the open unit disk.

Next, we want to characterize boundary points z_0 of the region D where the function f of Theorem 1 cannot be continued meromorphically into some neighborhood U of z_0. The main tool will be the distribution of the a-values of r_n in U.

Theorem 2. *Let D be a region in $\mathbb{C}$, and $\{r_n\}_{n\in\mathbb{N}}$, $r_n \in \mathcal{R}_{n,n}$, a sequence that converges m_1-almost geometrically to a function f inside D. Let z_0 be a boundary point of D such that the function f cannot be continued m_1-equivalently to a meromorphic function in z_0. Moreover, let $a, b \in \overline{\mathbb{C}}$, $a \neq b$, then the following distribution result holds for the a-values and b-values in any neighborhood U of z_0 in $\overline{\mathbb{C}}$:*

$$\text{If } N_a(r_n, U) = o(n) \quad \text{as } n \to \infty, \quad \text{then } \limsup_{n\to\infty} N_b(r_n, U) = \infty. \tag{2.2}$$

Finally, we can formulate a result of Picard type as a direct consequence of Theorem 2.

Corollary 1. *Under the conditions of Theorem 2 for D, f and $\{r_n\}_{n\in\mathbb{N}}$, let z_0 be a finite boundary point of D such that the function cannot be continued m_1-equivalently to a meromorphic function in z_0. Then for any neighborhood U of z_0 and for all $a \in \overline{\mathbb{C}}$, with at most one exception,*

$$\limsup_{n\to\infty} N_a(r_n, U) = \infty.$$

As in Theorem 1, the statements of Theorem 2 and Corollary 1 remain valid if $\mathbb{N}$ is replaced by a monotone subsequence $\Lambda = \{n_k\}_{k\in\mathbb{N}}$ with the property that n_{k+1}/n_k, $k \in \mathbb{N}$, is bounded.

3. Proofs

For a subset B in $\mathbb{C}$, we denote by B° the set of interior points of B, by $\overline{B}$ the closure of B and by ∂B the boundary of B. Before we proceed to the proof of Theorem 1, we quote some well-known notions and results.

Lemma 1. *(See Gončar [6, Lemma 1, p. 153].) Let E be compact in $\mathbb{C}$ with regular, connected complement, D be any open set, and K compact with $K \subset D$. Suppose that $r_n \in \mathcal{R}_{n,n}$ has no poles in D, i.e., $r_n \in \mathcal{A}(D)$. Then there exists a constant $\lambda = \lambda(E, D, K) \geqslant 1$ such that*

$$\|r_n\|_K \leqslant \lambda^n \|r_n\|_E.$$

Especially, if $E \subset D$ then $\lambda(E, D, K)$ can be chosen as

$$\lambda(E, D, K) := \max_{z\in K} \sup_{t\in\overline{\mathbb{C}}\setminus D} \exp\bigl(G_E(z, t)\bigr).$$

We remark that

$$\lim_{\rho\to 1} \lambda(\overline{E_\rho}) = 1$$

for any fixed open set D with $E \subset D$.

Furthermore, we use the notion of an "exact harmonic majorant". Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of subharmonic functions in a region D of $\overline{\mathbb{C}}$. Then the harmonic function V in D is called *harmonic majorant* of $\{F_n\}_{n\in\mathbb{N}}$ if for any continuum $S \subset D$ we have

$$\limsup_{n\to\infty} \max_{z\in S} F_n(z) \leqslant \max_{z\in S} V(z). \tag{3.1}$$

The harmonic majorant V of $\{F_n\}_{n\in\mathbb{N}}$ is called *exact* if there exists a compact set $S' \subset D$ with

$$\limsup_{n\to\infty} \max_{z\in S'} F_n(z) = \max_{z\in S'} V(z).$$

We remark that V is an exact harmonic majorant if and only if (3.1) holds for any continuum $S' \subset D$ [13].

If the subharmonic sequence $\{F_n\}_{n\in\mathbb{N}}$ has the harmonic function V as a harmonic majorant in D and if there exists a continuum $B \subset D$ with

$$\limsup_{n\to\infty} \max_{z\in B} F_n(z) < \max_{z\in B} V(z),$$

then the strict inequality

$$\limsup_{n\to\infty} \max_{z\in B'} F_n(z) < \max_{z\in B'} V(z)$$

holds for any compact subset $B' \subset D$.

Finally, we quote a result of Gončar connecting m_1-almost uniform convergence with uniform convergence and meromorphic continuation.

Lemma 2. *(See Gončar [8, Lemma 1, p. 507].) Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence of meromorphic functions in a region D converging to a function φ with respect to the m_1-measure inside D. Then the following assertions are true.*

(1) *If $\varphi_n \in \mathcal{A}(D)$, $n \in \mathbb{N}$, then the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ converges uniformly inside D on compact subsets.*
(2) *If $\varphi_n \in \mathcal{M}_m(D)$, $n \in \mathbb{N}$, and $m \in \mathbb{N}$ is fixed, then the limit function φ is m_1-equivalent to a meromorphic function in $\mathcal{M}_m(D)$.*

Proof of Theorem 1. Since $f \not\equiv 0$ on S and since f is continuous on S, there exists a continuum $S' \subset S$ such that $f(z) \neq 0$ for all $z \in S'$.

Thus we may assume that $f(z) \neq 0$ on S, that the complement $\Omega = \overline{\mathbb{C}} \setminus S$ is simply connected and regular, and that Ω has a Green function $G_S(z,t)$ with pole at $t \in \Omega$ for $z \in \Omega$ such that

$$\lim_{z \to z_0 \in \partial\Omega} G_S(z,t) = 0.$$

Furthermore, we may assume that D is bounded.

Let l_n be the number of poles $\xi_{n,i}$ of r_n in D listed with their multiplicities, $1 \leqslant i \leqslant l_n$. Then we define

$$q_n(z) := \begin{cases} \prod_{i=1}^{l_n}(z - \xi_{n,i}), & l_n > 1, \\ 1, & l_n = 0. \end{cases}$$

Since $r_n \in \mathcal{M}_m(D)$, we have $\deg q_n = l_n \leqslant m$ and

$$\limsup_{n \to \infty} \|q_n\|_K^{1/n} \leqslant 1 \tag{3.2}$$

for all compact $K \subset D$. Because of (2.1) and (3.2),

$$\limsup_{n \to \infty} \bigl\|(r_{n+1} - r_n) q_n q_{n+1}\bigr\|_S^{1/n} < 1. \tag{3.3}$$

Observe that

$$h_n := (r_{n+1} - r_n) q_n q_{n+1} \in \mathcal{A}(D) \cap \mathcal{R}_{n+m+1,n+1}.$$

Using Lemma 1 and (3.3) we can choose $\sigma > 1$ and $\rho < 1$ such that

$$\|h_n\|_{S_\sigma} \leqslant c_1 \rho^n \quad \text{for all } n \in \mathbb{N}$$

with some constant $c_1 > 0$. Here, as above, S_σ denotes the Green domain of S to the parameter $\sigma > 1$. Consequently,

$$\bigl|(r_{n+1} - r_n)(z)\bigr| \leqslant c_1 \frac{\rho^n}{|q_n(z) q_{n+1}(z)|} \tag{3.4}$$

for all $z \in S_\sigma$ which are not zeros of q_n and q_{n+1}. Fix an arbitrary $\varepsilon > 0$ and introduce the open sets

$$\Omega_n(\varepsilon) := \bigcup_{i=1}^{l_n} \left\{ z \in \mathbb{C}\colon\ |z - \xi_{n,i}| < \frac{\varepsilon}{2n^3} \right\}$$

and

$$\Omega(\varepsilon) := \bigcup_{n=1}^{\infty} \Omega_n(\varepsilon). \tag{3.5}$$

Because of the sub-additivity of the m_1-measure we get

$$m_1\bigl(\Omega(\varepsilon)\bigr) \leqslant \sum_{n=1}^{\infty} m_1\bigl(\Omega_n(\varepsilon)\bigr) < \frac{1}{2} \sum_{n=1}^{\infty} \frac{\varepsilon}{n^2} < \varepsilon.$$

Let K be a compact subset of S_σ. Then by (3.4) we get for $z \in S_\sigma \setminus \Omega(\varepsilon)$

$$\begin{aligned} \bigl|r_{n+1}(z) - r_n(z)\bigr| &\leqslant c_1 \rho^n \left[\frac{2}{\varepsilon} n^3\right]^m \left[\frac{2}{\varepsilon}(n+1)^3\right]^m \\ &\leqslant c_2 \left(\frac{1}{\varepsilon}\right)^{2m} \rho^n n^{6m} \end{aligned}$$

for all $n \in \mathbb{N}$, with some constant $c_2 > 0$ not depending on n.

Hence, the sequence $\{r_n\}_{n \in \mathbb{N}}$ converges uniformly on the compact set $K \setminus \Omega(\varepsilon)$ for any $\varepsilon > 0$. So this sequence converges m_1-almost uniformly inside of S_σ. Since m_1-almost uniform convergence implies m_1-convergence inside S_σ, we obtain by Lemma 2 that $\{r_n\}_{n \in \mathbb{N}}$ converges to a function, which is m_1-equivalent to a meromorphic function $f \in \mathcal{M}_m(S_\sigma)$ in m_1-measure. This function f is a meromorphic continuation of the function f given in (2.1) on S.

We want to show next that for any compact set K in D

$$\limsup_{n \to \infty} \max_{z \in K} \frac{1}{n} \log\bigl|r_n(z) q_n(z)\bigr| \leqslant 0. \tag{3.6}$$

Let us assume that (3.6) is false, i.e., there exists a compact set $K \subset D$ such that

$$\limsup_{n\to\infty} \max_{z\in K} \frac{1}{n} \log\left|r_n(z)q_n(z)\right| > 0.$$

Then we choose a subsequence $\Lambda \subset \mathbb{N}$ such that

$$\lim_{n\in\Lambda,\, n\to\infty} \max_{z\in K} \frac{1}{n} \log\left|r_n(z)q_n(z)\right| = \delta > 0. \tag{3.7}$$

Since all $q_n \in \mathcal{P}_m$, we can assume that the subsequence Λ is chosen in such a way that

$$\lim_{n\in\Lambda,\, n\to\infty} q_n = q \in \mathcal{P}_m.$$

The meromorphic function $f \in \mathcal{M}_m(S_\sigma)$ can have poles only at the zeros of the polynomial q.

We now choose a bounded subregion W of D such that

$$S_\sigma \subset W, \quad K \subset W \quad \text{and} \quad \overline{W} \subset D.$$

Let k_n be the number of zeros (with multiplicities) of r_n in $\overline{W}$. If $k_n \geqslant 1$, let $\eta_{n,i}$, $1 \leqslant i \leqslant k_n$, be the zeros of r_n in $\overline{W}$ and set

$$\pi_n(z) := \begin{cases} \prod_{i=1}^{k_n}(z-\eta_{n,i}) & \text{if } k_n > 0, \\ 1 & \text{if } k_n = 0. \end{cases}$$

Since $k_n = o(n)$ as $n \to \infty$, we get

$$\limsup_{n\to\infty} \|\pi_n\|_{\overline{W}}^{1/n} \leqslant 1. \tag{3.8}$$

Next, we define for $z \in D$

$$h_n(z) := \frac{1}{n} \log\left|\phi_n(z)\right| \tag{3.9}$$

with

$$\phi_n(z) := \frac{r_n(z)q_n(z)}{\pi_n(z)}, \tag{3.10}$$

and we have $\phi_n \in \mathcal{A}(D) \cap \mathcal{R}_{n,n}$. h_n is harmonic in W and subharmonic in D.

Since f is a meromorphic continuation of the continuous function f on S with $f(z) \neq 0$ for all $z \in S$, we can choose a closed disk $E \subset S_\sigma$ with nonempty interior such that f is holomorphic on E, $f(z) \neq 0$ and $q(z) \neq 0$ for $z \in E$. Moreover, we can choose the closed disk E small enough such that the zeros $\eta_{n,i}$, $n \in \Lambda$ $(1 \leqslant i \leqslant k_n)$ have a positive distance to ∂E, i.e., there exists $d > 0$ such that

$$\operatorname{dist}(E, \eta_{n,i}) \geqslant d, \quad 1 \leqslant i \leqslant k_n,\ n \in \Lambda.$$

With ϕ_n as in (3.10),

$$\begin{aligned} \|\phi_n\|_E &\leqslant \frac{\|r_n\|_E \|q_n\|_E}{\min_{z\in E} |\pi_n(z)|} \\ &\leqslant \left(\frac{1}{d}\right)^{k_n} \|r_n\|_E \|q_n\|_E. \end{aligned} \tag{3.11}$$

Moreover, there exists $\kappa > 0$ such that for $n \in \Lambda$

$$\left|\phi_n(z)\right| \geqslant \kappa \quad \text{for all } z \in E. \tag{3.12}$$

Thus, (3.11) and (3.12) imply

$$\lim_{n\in\Lambda,\, n\to\infty} h_n(z) = 0 \quad \text{for } z \in E. \tag{3.13}$$

According to Lemma 1, we obtain

$$\begin{aligned} \|\phi_n\|_{\overline{W}} &\leqslant \lambda^n \|\phi_n\|_E \\ &\leqslant c_{3,n} \lambda^n \left(\frac{1}{d}\right)^{k_n} \end{aligned}$$

with $c_{3,n} = \|r_n\|_E \|q_n\|_E$ and

$$\lambda = \lambda(E, D, \overline{W}) := \max_{z\in\overline{W}} \sup_{t\in\overline{\mathbb{C}}\setminus D} \exp\bigl(G_E(z,t)\bigr) \geqslant 1.$$

Since

$$\lim_{n\to\infty} \|r_n\|_E = \|f\|_E$$

and

$$\lim_{n\in\Lambda, n\to\infty} \|q_n\|_E = \|q\|_E,$$

we finally have that

$$\|\phi_n\|_{\overline{W}} \leqslant c_4\,\lambda^n \left(\frac{1}{d}\right)^{k_n} \tag{3.14}$$

with some constant $c_4 > 0$ for $n \in \Lambda$.

Consider the functions $h_n(z)$ of (3.9). Because of (3.14),

$$h_n(z) \leqslant \frac{1}{n}\log c_4 + \log\lambda + \frac{k_n}{n}\log\frac{1}{d}$$

for $n \in \Lambda$ and $z \in \overline{W}$. Hence, there exists a constant $c_5 > 0$ such that

$$h_n(z) \leqslant c_5 \quad \text{for all } z \in \overline{W} \text{ and all } n \in \Lambda.$$

Therefore, the harmonic functions $h_n(z)$, $n \in \Lambda$, are bounded from above for all $z \in W$. By Harnack's theorem, either

$$h_n(z) \to -\infty \quad \text{locally uniformly as } n \to \infty,\ n \in \Lambda,$$

or there exists a subsequence $\Lambda_1 \subset \Lambda$ such that $\{h_n\}_{n\in\Lambda_1}$ converges locally uniformly to h as $n \to \infty$, $n \in \Lambda_1$, and h is harmonic in W. The first situation cannot occur because of (3.13). Since $E^\circ \neq \emptyset$ we finally obtain by the identity principle, that $h(z) \equiv 0$ in W and consequently

$$\lim_{n\in\Lambda_1,\, n\to\infty} \max_{z\in K} h_n(z) = \lim_{n\in\Lambda_1,\, n\to\infty} \max_{z\in K} \frac{1}{n}\log\left|\frac{r_n(z)q_n(z)}{\pi_n(z)}\right| = 0$$

which contradicts (3.7), since

$$\begin{aligned}\max_{z\in K}\frac{1}{n}\log\bigl|r_n(z)q_n(z)\bigr| &= \max_{z\in K}\frac{1}{n}\log\bigl|\phi_n(z)\pi_n(z)\bigr| \\ &= \max_{z\in K} h_n(z) + \frac{1}{n}\max_{z\in K}\log\bigl|\pi_n(z)\bigr|\end{aligned}$$

and therefore, by (3.8),

$$\lim_{n\in\Lambda_1,\, n\to\infty} \max_{z\in K}\frac{1}{n}\log\bigl|r_n(z)q_n(z)\bigr| = \lim_{n\in\Lambda_1,\, n\to\infty} \max_{z\in K} h_n(z) = 0$$

and this contradiction establishes (3.6).

Then, together with

$$\limsup_{n\to\infty} \|q_n\|_K^{1/n} \leqslant 1$$

we obtain

$$\limsup_{n\to\infty} \max_{z\in K}\frac{1}{n}\log\bigl|(r_{n+1}-r_n)(z)q_n(z)q_{n+1}(z)\bigr| \leqslant 0 \tag{3.15}$$

on any compact subset K of D. On the other hand, by (3.3) we have

$$\limsup_{n\to\infty} \max_{z\in S}\frac{1}{n}\log\bigl|(r_{n+1}-r_n)(z)q_n(z)q_{n+1}(z)\bigr| < 0. \tag{3.16}$$

Using the arguments of Walsh [13, Corollary of Theorem 1, p. 197], we conclude that in (3.16) the strict inequality holds for any continuum S' in D, S' replacing S.

By the same arguments as above, it follows that $\{r_n\}_{n\in\mathbb{N}}$ converges m_1-almost uniformly in D. Since all $r_n \in \mathcal{M}_m(D)$, then the limit function is m_1-equivalent to a meromorphic function in D with at most m poles. Hence, f can be continued from S to $f \in \mathcal{M}_m(D)$.

To prove that $\{r_n\}_{n\in\mathbb{N}}$ converges to f m_1-almost geometrically inside D, let $\varepsilon > 0$ and let $\Omega(\varepsilon)$ be defined as in (3.5). If K is a compact subset of D, then by the remark following (3.15) there exist $\rho < 1$ and $c_1 > 0$ such that

$$\bigl|r_{n+1}(z) - r_n(z)\bigr| \leqslant c_1\,\frac{\rho^n}{|q_n(z)q_{n+1}(z)|}$$

for all $z \in K$. Using the same arguments following (3.4), we obtain for $z \in K \setminus \Omega(\varepsilon)$

$$\left|r_{n+1}(z) - r_n(z)\right| \leqslant c_2 \left(\frac{1}{\varepsilon}\right)^{2m} \rho^n n^{6m}$$

for all $n \in \mathbb{N}$, with some constant $c_2 > 0$ independent of n. Hence, for $z \in K \setminus \Omega(\varepsilon)$

$$\left|f - r_n(z)\right| \leqslant c_2 \left(\frac{1}{\varepsilon}\right)^{2m} \sum_{\nu=0}^{\infty} \rho^{n+\nu}(n+\nu)^{6m}$$

and therefore

$$\limsup_{n\to\infty} \|f - r_n\|_{K\setminus\Omega(\varepsilon)}^{1/n} \leqslant \rho < 1. \qquad \square$$

Proof of Theorem 2. If the boundary point $z_0 = \infty$, the mapping $z \mapsto 1/z$ transforms the situation to the case where z_0 is finite. Hence, we may assume that z_0 is finite and U is open and connected, hence a region.

Let us assume that (2.2) is not true, i.e., there exist $a, b \in \overline{\mathbb{C}}$ such that

$$N_a(r_n, U) = o(n) \quad \text{as } n \to \infty \quad \text{and} \quad \limsup_{n\to\infty} N_b(r_n, U) < \infty. \tag{3.17}$$

We consider the linear transformation

$$w(z) = w^{a,b}(z) = \begin{cases} \frac{z-a}{z-b} & \text{for } a, b \neq \infty, \\ z - a & \text{for } b = \infty, \\ \frac{1}{z-b} & \text{for } a = \infty \end{cases}$$

which maps the points a and b to 0 and ∞, and we define for $z \in D$ the functions

$$f^{a,b} := w^{a,b} \circ f \quad \text{and} \quad r_n^{a,b} := w^{a,b} \circ r_n.$$

Then $r_n^{a,b} \in \mathcal{M}_m(U) \cap \mathcal{R}_{n,n}$ for some fixed $m \in \mathbb{N}$. Since $\{r_n\}_{n\in\mathbb{N}}$ is m_1-almost geometrically convergent to f inside D and f is not equivalent to a constant function on U, we can choose a continuum $S \subset U \cap D$ and $\alpha > 0$ such that f is continuous on S,

$$\limsup_{n\to\infty} \|f - r_n\|_S^{1/n} < 1$$

and

$$\left|f(z) - b\right| \geqslant \alpha \quad \text{for } z \in S.$$

Since

$$f^{a,b}(z) - r_n^{a,b}(z) = \begin{cases} \frac{(a-b)(f(z)-r_n(z))}{(f(z)-b)(r_n(z)-b)} & \text{for } a, b \neq \infty, \\ f(z) - r_n(z) & \text{for } b = \infty, \\ \frac{r_n(z)-f_n(z)}{(f(z)-b)(r_n(z)-b)} & \text{for } a = \infty \end{cases}$$

we have

$$\limsup_{n\to\infty} \left\|f^{a,b} - r_n^{a,b}\right\|_S^{1/n} < 1.$$

Moreover, by (3.17)

$$N_0\big(r_n^{a,b}\big) = o(n) \quad \text{as } n \to \infty.$$

Consequently, all conditions of Theorem 1 are satisfied for $S, U, \{r_n^{a,b}\}_{n\in\mathbb{N}}$, replacing $S, D, \{r_n\}_{n\in\mathbb{N}}$. Hence, the function $f^{a,b}$ can be continued from S m_1-equivalently to a meromorphic function in the region U.

By definition, for $z \in D$

$$f^{a,b}(z) = \begin{cases} 1 + \frac{b-a}{f(z)-b} & \text{for } a, b \neq \infty, \\ f(z) - a & \text{for } b = \infty, \\ \frac{1}{f(z)-b} & \text{for } a = \infty. \end{cases}$$

Consequently, the function f is m_1-equivalent to a meromorphic function in U, contradicting the assumptions of Theorem 2. $\square$

4. Maximal rational approximants

Let E be compact in $\mathbb{C}$ with regular, connected complement $\Omega = \overline{\mathbb{C}} \setminus E$. We consider the approximation of $f \in \mathcal{M}(E)$ with respect to rational functions. As in Section 1, $\rho(f)$ denotes the maximal parameter $\rho > 1$ such that f can be continued to $f \in \mathcal{M}(E_\rho)$.

If $\rho(f) < \infty$, then a sequence $\{r_n\}_{n\in\mathbb{N}}$, $r_{n,n} \in \mathcal{R}_{n,n}$, is called m_1-*maximally convergent* to f on E if

$$\limsup_{n\to\infty} \|f - r_n\|_{\partial E}^{1/n} \leqslant \frac{1}{\rho(f)},$$

and for any $\varepsilon > 0$ there exists a set $\Omega(\varepsilon)$ with $m_1(\Omega(\varepsilon)) < \varepsilon$ such that

$$\limsup_{n\to\infty} \|f - r_n\|_{E_\sigma \setminus \Omega(\varepsilon)}^{1/n} \leqslant \frac{\sigma}{\rho(f)}$$

for all σ, $1 < \sigma < \rho(f)$ (cf. [4]).

Hence, any sequence $\{r_n\}_{n\in\mathbb{N}}$, $r_n \in \mathcal{R}_{n,n}$, that is m_1-maximally convergent to f on E, is m_1-almost geometrically convergent inside $E_{\rho(f)}$. Therefore, we can use Theorem 2 and obtain for the a-values of r_n the following distribution result.

Corollary 2. *Let E be compact with regular, connected complement $\Omega = \overline{\mathbb{C}} \setminus E$, $f \in \mathcal{M}(E)$ and $\rho(f) < \infty$. Moreover, let $\{r_n\}_{n\in\mathbb{N}}$, $r_n \in \mathcal{R}_{n,n}$, be m_1-maximally convergent to f on E. If z_0 is a boundary point of $E_{\rho(f)}$, such that f cannot be continued meromorphically to z_0, then for any neighborhood U of z_0 and any $a \in \overline{\mathbb{C}}$, with at most one exception,*

$$\limsup_{n\to\infty} N_a(r_n, U) = \infty.$$

An example of such m_1-maximally convergent sequences are real best rational Chebyshev approximants $r^*_{n,m_n}(f)$, $n \in \mathbb{N}$, to a continuous, real-valued functions f on $E = [-1, 1]$, where the degrees m_n of the denominators of r^*_{n,m_n} satisfy

$$\lim_{n\to\infty} m_n = \infty \quad \text{and} \quad m_n = o(n/\log n) \quad \text{as } n \to \infty. \tag{4.1}$$

The proof of the m_1-maximal convergence of $r^*_{n,m_n}(f)$ to f as $n \to \infty$ is based on the asymptotic distribution of the alternation points of $f - r^*_{n,m_n}(f)$ on $[-1, 1]$ (cf. [2–4]).

Other examples are classical Padé approximants $\pi_{n,m_n}(f)$ to a function f holomorphic in some closed disk $E = D_r = \{z{:}\ |z| \leqslant r\}$, $r > 0$. Under the condition (4.1), $\pi_{n,m_n}(f)$ converges m_1-almost geometrically inside $D_{R(f)}$ to f, where $R(f)$ is the maximal radius such that $f \in \mathcal{M}(D_{R(f)})$, and we can choose for z_0 any boundary point of $D_{R(f)}$ such that f cannot be continued meromorphically to z_0 (cf. [4]).

Hence, for $\{\pi_{n,m_n}(f)\}_{n\in\mathbb{N}}$ and $\{r^*_{n,m_n}(f)\}_{n\in\mathbb{N}}$ we can apply the property (2.2) of Theorem 2 to sharpen Corollary 2 in the case that $\{m_n\}_{n\in\mathbb{N}}$ satisfies (4.1): If z_0 is a boundary point of $D_{R(f)}$ (resp. $E_{\rho(f)}$) that is not a meromorphic point of f, then for any $a \in \mathbb{C}$ and any neighborhood U of z_0

$$\limsup_{n\to\infty} N_a\big(\pi_{n,m_n}(f), U\big) = \infty \quad \Big(\text{resp. } \limsup_{n\to\infty} N_a\big(r^*_{n,m_n}(f), U\big) = \infty\Big).$$

Other examples for the application of Corollary 2 with $a \in \mathbb{C}$ are Fourier–Padé approximants and Faber–Padé approximants of a function f (Suetin [12]). In these examples the approximating sequences $\{r_{n,m_n}(f)\}_{n\in\mathbb{N}}$ converge again m_1-almost geometrically in $E_{\rho(f)}$ if the sequence $\{m_n\}_{n\in\mathbb{N}}$ satisfies (4.1).

The example of classical Padé approximants can be generalized to more general interpolating rational functions such that again Theorem 2 is applicable.

Let E be compact in $\mathbb{C}$ with connected complement, D a region in $\mathbb{C}$ with $E \subset D$ and $F := \overline{\mathbb{C}} \setminus D$. We assume that the condenser (E, F) is regular with respect to the Dirichlet problem, i.e., there exists a continuous function $h : \overline{\mathbb{C}} \to \mathbb{R}$ which is harmonic in $\overline{\mathbb{C}} \setminus (E \cup F)$ and satisfies

$$h(z) = \begin{cases} 0 & \text{for } z \in E, \\ 1 & \text{for } z \in F. \end{cases}$$

Let Γ be a smooth Jordan curve in D such that the set E is contained in the interior of Γ. Then the capacity $C(E, F)$ of the condenser is defined by

$$C(E, F) := \frac{1}{2\pi} \int_\Gamma \frac{\partial h}{\partial n}\, ds$$

where n denotes the exterior normal to Γ.

Next, let us consider tables $\{\alpha_{n,k}\} = \alpha$ in E and $\{\beta_{n,k}\} = \beta$ in F $(n = 1, 2, \ldots;\ k = 1, \ldots, n)$ such that the associated polynomials

$$\omega_n^\alpha(z) = \prod_{k=1}^n (z - \alpha_{n,k})$$

and

$$\omega_n^{\beta}(z) = \prod_{\substack{k=1 \\ \beta_{n,k}\neq\infty}}^{n} (z - \beta_{n,k})$$

satisfy

$$\lim_{n\to\infty}\left|\frac{\omega_n^{\alpha}(z)}{\omega_n^{\beta}(z)}\right|^{1/n} = \lambda \exp\left(\frac{h(z)}{C(E,F)}\right)$$

locally uniformly in $H := \overline{\mathbb{C}} \setminus (E \cup F)$, where λ is a positive constant. Examples for such $\alpha = \{\alpha_{n,k}\}$ and $\beta = \{\beta_{n,k}\}$ are rational Fekete points or rational Leja–Bagby points (cf. [1,14]).

Now, let f be holomorphic on E and fix $(n,m) \in \mathbb{N} \times \mathbb{N}$. Then there exist polynomials $P_n \in \mathcal{P}_n$ and $Q_m \in \mathcal{P}_m$, $Q_m \not\equiv 0$, such that

$$\frac{Q_m \omega_{n-m}^{\beta} f - P_n}{\omega_{n+m+1}^{\alpha}}$$

is holomorphic on E. The rational function

$$\pi_{n,m}^{\alpha,\beta}(f) = \frac{P_n}{Q_m \omega_{n-m}^{\beta}} \in \mathcal{R}_{n,n}$$

has m free poles and $n-m$ fixed poles at the points $\beta_{n-m,k}$, $k = 1, \ldots, n-m$, and $\pi_{n,m}^{\alpha,\beta}(f)$ has maximal contact to f at the points

$$\alpha_{n+m+1,k}, \quad 1 \leqslant k \leqslant n+m+1$$

(cf. Gončar [8]). Let us define for $1 \leqslant \rho \leqslant \exp(1/C(E,F))$ the region

$$D_\rho := \left\{ z \in \mathbb{C}\colon\ \frac{h(z)}{C(E,F)} < \log\rho \right\},$$

that always contains E. For $f \in \mathcal{A}(E)$ we denote by $\rho(f)$ the maximal parameter ρ in $[1, \exp(1/C(E,F))]$ such that $f \in \mathcal{M}(D_\rho)$. If $\rho(f) < \exp(1/C(E,F))$ and if $\{m_n\}_{n\in\mathbb{N}}$ satisfy (4.1), then for any $\varepsilon > 0$ there exists a subset $\Omega(\varepsilon)$ with $\Omega(\varepsilon) < \varepsilon$ such that

$$\limsup_{n\to\infty} \left\| f - \pi_{n,m_n}^{\alpha,\beta} \right\|_{K\setminus\Omega(\varepsilon)}^{1/n} \leqslant \frac{\sigma}{\rho(f)}$$

for any compact subset $K \subset \overline{D}_\sigma$ and $\sigma < \rho(f)$, i.e., the sequence $\{\pi_{n,m_n}^{\alpha,\beta}\}_{n\in\mathbb{N}}$ converges again m_1-almost geometrically to f inside $D_{\rho(f)}$. Hence, we can apply Theorem 2: If z_0 is a boundary point of $D_{\rho(f)}$ such that f cannot be meromorphically continued to z_0, then

$$\limsup_{n\to\infty} N_a\left(\pi_{n,m_n}^{\alpha,\beta}, U\right) = \infty$$

for any neighborhood U of z_0 and for any $a \in \overline{\mathbb{C}}$, with at most one exception.

The above construction covers the multipoint Padé approximants by Saff [11], as well as the generalized Padé approximants by Gončar [8]. In the first case of multipoint Padé approximation, the points $\alpha_{n,k} \in \mathbb{C}$ have no accumulation point exterior to E and can be chosen for example as uniformly distributed on ∂E with respect to equilibrium measure of E; moreover, all $\beta_{n,k} = \infty$ for all $n,k \in \mathbb{N}$ and we take for D a sufficiently large Green domain $D = E_\sigma$ with respect to the Green function $G_E(z,\infty)$ of E.

In the case of generalized Padé approximation in a region, D is a regular region with $0 \in D$ and E is the set

$$E := \left\{ z \in D\colon\ g(z,0) \geqslant \kappa \right\},$$

where $g(z,0)$ is the Green function of D with pole at 0 and $\kappa > 0$ chosen appropriately. All the points $\alpha_{n,k} = 0$ and the points $\beta_{n,k}$, $1 \leqslant k \leqslant n$, are chosen on the boundary of D such that the normalized counting measures μ_n of the point sets $\{1/\beta_{n,k}\}_{k=1}^{n}$ converge weakly to the equilibrium measure μ of $\overline{D^*}$, where D^* is the image of D under the mapping $z \mapsto 1/z$ (cf. Gončar [8]).

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