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L^p-solutions of singular integro-differential equations

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ABSTRACT

We study a variety of scalar integro-differential equations with singular kernels including linear, nonlinear, and resolvent equations. The first result involves a type of existence theorem which uses a fixed point mapping defined by the integro-differential equation itself and produces a unique solution with a continuous derivative in a very simple way. We then construct a Liapunov functional yielding qualitative properties of solutions. The work answers questions raised by Volterra in 1928, by Levin in 1963, and by Grimmer and Seifert in 1975. Previous results had produced bounded solutions from bounded perturbations. Our results mainly concern integrable solutions from integrable perturbations.

1. Introduction

We study a scalar nonlinear integro-differential equation of the form

$$x'(t) = f(t) - h\big(t, x(t)\big) - \int_0^t C(t,s)q\big(s, x(s)\big)\,ds, \tag{1}$$

together with its resolvent in the linear case. The objective is to determine qualitative properties of solutions when there is a $p \in [1,\infty)$ with

$$f \in L^p[0,\infty), \qquad xh(t,x) \geqslant 0, \qquad xq(t,x) \geqslant 0, \tag{2}$$

and C has a weak singularity at $t = s$ with properties to be described later.

Here, C is a convex kernel in the following sense. There is an $\epsilon > 0$ and for $0 \leqslant s \leqslant t - \epsilon$ we have

$$C(t,s) \geqslant 0, \qquad C_2(t,s) \geqslant 0, \qquad C_{2,1}(t,s) \leqslant 0, \qquad C_1(t,0) \leqslant 0, \tag{3}$$

where $C_1(t,s) = C_t(t,s)$, $C_2(t,s) = C_s(t,s)$ and $C_{2,1}(t,s) = C_{st}(t,s)$. This work contributes to the continuing solution of a major problem found in the work of Volterra [10] in 1928. He noted that many real-world problems were being modelled by integral and integro-differential equations with convex kernels. Then he conjectured that a Liapunov functional might be constructed which would yield some very precise qualitative properties of the solutions; moreover, he suggested how the Liapunov functional might be constructed. In 1963 Levin [9] constructed such a functional for an integro-differential equation of the form

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$$x'(t) = -\int_0^t C(t,s)g\big(x(s)\big)\,ds$$

when C is a nonsingular convex kernel with the notable property that the integral, itself, supplies the stability properties of the solution without relying on a uniformly asymptotically stable ordinary differential equation, as is found in

$$x'(t) = -Ax(t) - \int_0^t C(t,s)g\big(x(s)\big)\,ds, \quad A > 0.$$

Moreover, no bound was required on C or its integrals. Levin continued that work for many years. In 1993 [4] we constructed a parallel Liapunov functional for an integral equation, again with a nonsingular convex kernel. Very recently [5] we extended that work to integral equations with convex kernels and weak singularities, typified by a kernel $(t-s)^{-1/2}$ found in so many real world problems, such as heat equations. Our project here is to extend that work to integro-differential equations with convex kernels and weak singularities.

Unlike Levin, we find that the singularity demands a stable ODE part and we show why that is to be expected, both by an example and by analogy with parallel work with Razumikhin functions. In 1975 [7] Grimmer and Seifert developed a Razumikhin technique to deal with a vector equation

$$x'(t) = Ax(t) + \int_0^t B(t,s)x(s)\,ds + f(t) \tag{4}$$

where A is a constant matrix which is negative definite, B is a matrix satisfying

$$\lim_{h\to 0}\int_0^t \big|B(t,s) - B(t+h,s)\big|\,ds = 0$$

and

$$\lim_{h\to 0}\int_t^{t+h} \big|B(t+h,s)\big|\,ds = 0, \quad t \geqslant 0,$$

as well as a number of other conditions, some of which are listed below.

Here is the development of their question. They give conditions under which solutions of (4) will have certain qualitative properties in case f is bounded and continuous. All of that work is based on a Razumikhin technique which utilizes a Liapunov function instead of a Liapunov functional. Its central requirement is that for a constant matrix K satisfying

$$A^T K + KA = -I \quad \text{then} \quad \int_0^t \big|KB(t,s)\big|\,ds \leqslant M \tag{5}$$

where M is related to the eigenvalues of K and, generally, M is small. The Grimmer–Seifert results rest on smallness conditions, while ours rest on convexity.

On the last page of their paper, Grimmer and Seifert express the desire to show that the solution of (4) is in L^p when f is in L^q for some positive integers p and q. The conditions of [7] allow for B to have weak singularities. To the best of our knowledge, those desired results have never been obtained for equations with singular kernels.

2. Existence: Direct fixed point mappings

The first part of this section concerns the existence of a solution of (1) with continuous derivative when C has some discontinuities. In our subsequent work we will only allow discontinuities of C at $t = s$, typified by $C(t-s) = (t-s)^{-1/2}$ which occurs so often in the literature. Our existence result here will be more general and it will rest on ideas from Burton and Zhang [6] and later papers. Our terminology follows that of Becker [2] who studied integral equations, not integro-differential equations.

Definition 2.1. Let $\Omega_T := \{(t,s):\ 0 \leqslant s \leqslant t \leqslant T\}$. The kernel C of (1) is weakly singular on the set Ω_T if it is unbounded in Ω_T; but for each $t \in [0,T]$, $C(t,s)$ has at most finitely many discrete singularities in the interval $\{s:\ 0 \leqslant s \leqslant t\}$ and for every continuous function $\phi : [0,T] \to \Re^n$,

$$\int_0^t C(t,s)\phi(s)\,ds$$

and

$$\int_0^t \left|C(t,s)\right| ds$$

both exist and are continuous on $[0,T]$. If $C(t,s)$ is weakly singular on Ω_T for every $T>0$, then it is weakly singular on the set $\Omega := \{(t,s)\colon\ 0 \leqslant s \leqslant t < \infty\}$.

For (1) we suppose that $f:[0,\infty)\to\Re^n$ is continuous, $h,q:[0,\infty)\times\Re^n\to\Re^n$ are both continuous and both satisfy a global Lipschitz condition for the same constant K.

Theorem 2.2. *In addition to these continuity conditions, let $C(t,s)$ be weakly singular on Ω. Suppose also that for each $T>0$ and each $k\in(0,1)$, there is a constant $\gamma_1>0$ with*

$$\int_0^t e^{-\gamma_1(t-s)}\left|C(t,s)\right| ds \leqslant k$$

for $t\in[0,T]$. Then for every $x_0\in\Re^n$ (1) has a unique solution $x(t)$ with a continuous derivative and satisfying $x(0)=x_0$.

Proof. Let $T>0$ and $x_0\in\Re^n$ be given and let $(Y,\|\cdot\|)$ be the Banach space of continuous functions $\phi:[0,T]\to\Re^n$ with the supremum norm. Define $P:Y\to Y$ by $\phi\in Y$ implies that

$$(P\phi)(t) = f(t) - h\left(t, x_0 + \int_0^t \phi(s)\,ds\right) - \int_0^t C(t,s)q\left(s, x_0 + \int_0^s \phi(u)\,du\right) ds.$$

By the continuity assumptions and the weak singularity, $P\phi\in Y$. As the existence of γ_1 implies that for any $\gamma>\gamma_1$ we also have $\int_0^t e^{-\gamma(t-s)}|C(t,s)|\,ds\leqslant k$ (see Lemma 2.3 below), we will define a weighted norm $\|\cdot\|_T$ by $\phi\in Y$ implies that

$$\|\phi\|_T = \sup_{0\leqslant t\leqslant T} e^{-\gamma t}\left|\phi(t)\right|,$$

where $\gamma\geqslant\gamma_1$ is yet to be chosen. Note that $(Y,\|\cdot\|_T)$ is a Banach space.

If $\phi,\eta\in Y$ then

$$\begin{aligned}
\left|(P\phi)(t)-(P\eta)(t)\right|e^{-\gamma t} &\leqslant e^{-\gamma t}\left[\left|h\left(t,x_0+\int_0^t\phi(s)\,ds\right)-h\left(t,x_0+\int_0^t\eta(s)\right)ds\right|\right.\\
&\quad\left.+\int_0^t\left|C(t,s)\right|\left|q\left(s,x_0+\int_0^s\phi(u)\,du\right)-q\left(s,x_0+\int_0^s\eta(u)\,du\right)\right|ds\right]\\
&\leqslant e^{-\gamma t}K\int_0^t\left|\phi(s)-\eta(s)\right|ds+e^{-\gamma t}K\int_0^t\left|C(t,s)\right|\int_0^s\left|\phi(u)-\eta(u)\right|du\,ds\\
&=K\int_0^t e^{-\gamma(t-s)}e^{-\gamma s}\left|\phi(s)-\eta(s)\right|ds+K\int_0^t\left|C(t,s)\right|e^{-\gamma(t-s)}e^{-\gamma s}\int_0^s\left|\phi(u)-\eta(u)\right|du\,ds.
\end{aligned}$$

Now the last line yields

$$\begin{aligned}
&K\int_0^t\left|C(t,s)\right|e^{-\gamma(t-s)}e^{-\gamma s}\int_0^s\left|\phi(u)-\eta(u)\right|du\,ds\\
&\quad\leqslant K\int_0^t\left|C(t,s)\right|e^{-\gamma(t-s)}\int_0^s e^{-\gamma(s-u)}e^{-\gamma u}\left|\phi(u)-\eta(u)\right|du\,ds
\end{aligned}$$

$$\begin{aligned}
&\leqslant K\|\phi-\eta\|_T\int_0^t T\big|C(t,s)\big|e^{-\gamma(t-s)}\,ds\\
&=\|\phi-\eta\|_T KT\int_0^t \big|C(t,s)\big|e^{-\gamma(t-s)}\,ds\\
&\leqslant \|\phi-\eta\|_T KTk.
\end{aligned}$$

We then have

$$\begin{aligned}
\big|(P\phi)(t)-(P\eta)(t)\big|e^{-\gamma t} &\leqslant K\|\phi-\eta\|_T\int_0^t e^{-\gamma(t-s)}\,ds+\|\phi-\eta\|_T KTk\\
&\leqslant \|\phi-\eta\|_T\left[K\frac{e^{-\gamma(t-s)}}{\gamma}\bigg|_0^t+KTk\right]\\
&\leqslant \|\phi-\eta\|_T\big[(K/\gamma)+KTk\big].
\end{aligned}$$

Now, take k so small and γ so large that $(K/\gamma)+KTk\leqslant 1/2$. Thus, we have a contraction and a unique $\phi\in Y$ with $P\phi=\phi$ and, clearly, $[x_0+\int_0^t\phi(s)\,ds]'=\phi(t)$. That unique continuous ϕ is the continuous derivative of the unique solution $x(t)=x_0+\int_0^t\phi(s)\,ds$ of (1). □

Definition 2.1 is far more general than we will be needing here. We will allow a singularity only at $t=s$ and we will have a corresponding condition. Our next result offers a simple integral condition to ensure the existence of the constant γ_1 in Theorem 2.2. The proof is routine and will not be given here.

Lemma 2.3. *Let $C(t,s)$ be a weakly singular kernel on the set Ω and fix $T>0$. Moreover, suppose that for any $k\in(0,1)$ there exists an $\epsilon:=\epsilon(k,T)>0$ such that*

$$\int_{t-\epsilon}^t \big|C(t,s)\big|\,ds\leqslant k\quad \textit{for all } t\in[0,T],$$

where we have set $C(t,s)=0$, $(t,s)\in\Re^2-\Omega$. Then there always exists a $\gamma_{k,T}>0$ such that for any $\gamma\geqslant\gamma_{k,T}$ we have

$$\int_0^t e^{-\gamma(t-s)}\big|C(t,s)\big|\,ds\leqslant k\quad \textit{for all } t\in[0,T].$$

There are many other existence results and it would be a distraction to pursue more of them. Grimmer and Seifert [7] and Grossman and Miller [8] deal with some far more complicated ones. The result here is simple, general, and very instructive concerning existence ideas.

In the following material we will assume that the Liapunov results are being applied to problems in which existence has been established.

3. A simple result

Our next result does not contain a singularity, but it does introduce a new differential inequality relation and it is used primarily to show that the singularity causes us to add a term to the equation very much like the Ax of Grimmer and Seifert and for the same reason. Moreover, we streamline the proof so that the reader can see with ease exactly what techniques are involved. The results are extendable to vector equations, as may be verified by consulting [3] for the nonsingular case. However, the details are very lengthy.

We begin by showing that with a nonsingular convex kernel (i.e., the inequalities in (3) hold *for all* $\epsilon\geqslant 0$) then we can obtain $x\in L^\infty$ when $f\in L^1[0,\infty)$ and it allows $A=0$. The details of differentiation of V are not simple, but are parallel to those given in full in the proof of Theorem 4.1.

Theorem 3.1. *Let*

$$x'(t)=f(t)-\int_0^t C(t,s)x(s)\,ds$$

with C satisfying

$$C(t,s) \geqslant 0, \qquad C_2(t,s) \geqslant 0, \qquad C_{2,1}(t,s) \leqslant 0, \qquad C_1(t,0) \leqslant 0,$$

on $\{(s,t)\colon 0 \leqslant s \leqslant t\}$ *and with* $f \in L^1[0,\infty)$*. Then* $x \in L^\infty[0,\infty)$*.*

Proof. Define

$$V(t) = x^2(t) + \int_0^t C_2(t,s)\left(\int_s^t x(u)\,du\right)^2 ds + C(t,0)\left(\int_0^t x(s)\,ds\right)^2, \quad t \geqslant 0.$$

Using Leibnitz's rule and integration by parts we obtain

$$\begin{aligned} V'(t) &= 2x(t)f(t) + \int_0^t C_{2,1}(t,s)\left(\int_s^t x(u)\,du\right)^2 ds + C_1(t,0)\left(\int_0^t x(s)\,ds\right)^2 \\ &\leqslant 2x(t)f(t) \leqslant 2\sqrt{V(t)}\big|f(t)\big|. \end{aligned} \tag{6}$$

Separation of variables in (6) yields

$$V^{-1/2}(t)V'(t) \leqslant 2\big|f(t)\big|$$

and so

$$2\big|x(t)\big| \leqslant 2V^{1/2}(t) \leqslant 2V^{1/2}(0) + 2\int_0^t \big|f(s)\big|\,ds. \qquad \square$$

4. The singular convex case

There is a pleasant surprise as we proceed from integral equations to integro-differential equations. In parallel work on integral equations with weakly singular kernels [5] we required that $|q(t,x)| \leqslant |x|$ for integral equations, and that is not needed here. Our conclusion will be that $q(t,x(t)) \in L^2[0,\infty)$, as a result of $f \in L^2[0,\infty)$, a direct solution to the Grimmer–Seifert question.

Theorem 4.1. *Let x be a continuous solution of* (1) *on* $[0,\infty)$ *and let* (2) *and* (3) *be satisfied. If* $\epsilon > 0$ *is chosen so that* (3) *is satisfied and if* $V(t,\epsilon)$ *is defined for* $t \geqslant \epsilon$ *by*

$$V(t,\epsilon) = 2\int_0^{x(t)} q(t,s)\,ds + \int_0^{t-\epsilon} C_2(t,s)\left(\int_s^t q\big(u,x(u)\big)\,du\right)^2 ds + C(t,0)\left(\int_0^t q\big(u,x(u)\big)\,du\right)^2 \tag{7}$$

we have

$$\begin{aligned} \frac{dV(t,\epsilon)}{dt} \leqslant{}& 2\int_0^{x(t)} q_t(t,s)\,ds + 2q\big(t,x(t)\big)\left[f(t) - h\big(t,x(t)\big) + C(t,t-\epsilon)\int_{t-\epsilon}^t q\big(u,x(u)\big)\,du \right. \\ &\left. - \int_{t-\epsilon}^t C(t,s)q\big(s,x(s)\big)\,ds\right] + C_2(t,t-\epsilon)\left(\int_{t-\epsilon}^t q\big(u,x(u)\big)\,du\right)^2. \end{aligned} \tag{8}$$

Proof. Let x be a continuous solution of (1) on $[0,\infty)$. For any $t \geqslant \epsilon$ we have $C_1(t,0) \leqslant 0$ and $C_{2,1}(t,s) \leqslant 0$ when $0 \leqslant s \leqslant t-\epsilon$ so by Leibnitz's rule and the chain rule we have

$$\begin{aligned} V'(t,\epsilon) \leqslant{}& 2\int_0^{x(t)} q_t(t,s)\,ds + C_2(t,t-\epsilon)\left(\int_{t-\epsilon}^t q\big(u,x(u)\big)\,du\right)^2 \\ &+ 2q\big(t,x(t)\big)\left[f(t) - h\big(t,x(t)\big) - \int_0^t C(t,s)q\big(s,x(s)\big)\,ds\right] \end{aligned}$$

$$+ 2q\big(t, x(t)\big) \int_0^{t-\epsilon} C_2(t,s) \int_s^t q\big(u, x(u)\big)\,du\,ds$$

$$+ 2q\big(t, x(t)\big) C(t,0) \int_0^t q\big(u, x(u)\big)\,du.$$

Integrating the next-to-last term by parts yields

$$\begin{aligned}
&2q\big(t, x(t)\big) \int_0^{t-\epsilon} C_2(t,s) \int_s^t q\big(u, x(u)\big)\,du\,ds \\
&\quad = 2q\big(t, x(t)\big) \left[C(t,s) \int_s^t q\big(u, x(u)\big)\,du \Big|_0^{t-\epsilon} + \int_0^{t-\epsilon} C(t,s) q\big(s, x(s)\big)\,ds \right] \\
&\quad = 2q\big(t, x(t)\big) \left[C(t,t-\epsilon) \int_{t-\epsilon}^t q\big(u, x(u)\big)\,du - C(t,0) \int_0^t q\big(u, x(u)\big)\,du + \int_0^{t-\epsilon} C(t,s) q\big(s, x(s)\big)\,ds \right].
\end{aligned}$$

We cancel two terms and obtain

$$\begin{aligned}
V'(t,\epsilon) \leqslant{}& 2 \int_0^{x(t)} q_t(t,s)\,ds + 2q\big(t, x(t)\big) \left[f(t) - h\big(t, x(t)\big) - \int_0^t C(t,s) q\big(s, x(s)\big)\,ds \right] \\
&+ C_2(t, t-\epsilon) \left(\int_{t-\epsilon}^t q\big(u, x(u)\big)\,du \right)^2 \\
&+ 2q\big(t, x(t)\big) \left[C(t, t-\epsilon) \int_{t-\epsilon}^t q\big(u, x(u)\big)\,du + \int_0^{t-\epsilon} C(t,s) q\big(s, x(s)\big)\,ds \right].
\end{aligned}$$

Now, write that last integral as

$$\int_0^{t-\epsilon} C(t,s) q\big(s, x(s)\big)\,ds = \int_0^t C(t,s) q\big(s, x(s)\big)\,ds - \int_{t-\epsilon}^t C(t,s) q\big(s, x(s)\big)\,ds$$

and cancel two terms. This will yield

$$\begin{aligned}
V'(t,\epsilon) \leqslant{}& 2 \int_0^{x(t)} q_t(t,s)\,ds + C_2(t, t-\epsilon) \left(\int_{t-\epsilon}^t q\big(u, x(u)\big)\,du \right)^2 \\
&+ 2q\big(t, x(t)\big) \left[C(t, t-\epsilon) \int_{t-\epsilon}^t q\big(u, x(u)\big)\,du - \int_{t-\epsilon}^t C(t,s) q\big(s, x(s)\big)\,ds \right] \\
&+ 2q\big(t, x(t)\big) \big[f(t) - h\big(t, x(t)\big) \big],
\end{aligned}$$

as required. □

Three relations will be needed for us to parlay this Liapunov functional derivative into a qualitative result for a solution of (1).

In our opening theorem we saw that when the kernel is nonsingular, then we could take $h(t,x) \equiv 0$. Thus, we can expect that the larger the singularity, the more we will need from $h(t,x)$. Our assumption is that there is a $\gamma > 0$ with

$$\big|h(t,x)\big| \geqslant \gamma \big|q(t,x)\big|, \quad 0 \leqslant t < \infty,\ x \in \Re. \tag{9}$$

This condition bears some loose relation to the Grimmer–Seifert condition (5) and Theorem 3.1 shows that it is needed only because of the singularity. But notice the weakness of (2) in that $xq(t,x) \geqslant 0$ so that h can be zero for any value of x. We have not lost the essential properties of Theorem 3.1. Such latitude is missing in the Grimmer–Seifert result.

We now verify a certain relation which will be needed in the middle of the proof of the next theorem. The following claims will assist in the flow of logic of the argument.

Claim 1. *If* (9) *and the sign assumptions in* (2) *hold, then for* $(t,x) \in [0,\infty) \times \Re$ *we have*

$$2q\big(t,x(t)\big)\big[f(t)-h\big(t,x(t)\big)\big] \leqslant \frac{1}{\gamma}f^2(t) - \gamma q^2\big(t,x(t)\big).$$

Proof. From the sign properties in (2) we have

$$\begin{aligned}&\big|h\big(t,x(t)\big)\big| \geqslant \gamma\big|q\big(t,x(t)\big)\big| \quad\Rightarrow\quad \big|h\big(t,x(t)\big)\big|\big|q\big(t,x(t)\big)\big| \geqslant \gamma\big|q\big(t,x(t)\big)\big|\big|q\big(t,x(t)\big)\big|\\ &\quad\Rightarrow\quad h\big(t,x(t)\big)q\big(t,x(t)\big) \geqslant \gamma q^2\big(t,x(t)\big) \quad\Rightarrow\quad -h\big(t,x(t)\big)q\big(t,x(t)\big) \leqslant -\gamma q^2\big(t,x(t)\big)\end{aligned}$$

and so

$$\begin{aligned}2q\big(t,x(t)\big)\big[f(t)-h\big(t,x(t)\big)\big] &= 2q\big(t,x(t)\big)f(t) - 2q\big(t,x(t)\big)h\big(t,x(t)\big)\\ &\leqslant \gamma q^2\big(t,x(t)\big) + \frac{1}{\gamma}f^2(t) - 2\gamma q^2\big(t,x(t)\big)\\ &= \frac{1}{\gamma}f^2(t) - \gamma q^2\big(t,x(t)\big). \qquad \Box\end{aligned}$$

Next, we ask for positive constants α and β with $\alpha+\beta<\gamma$ where γ is the constant in (9) such that

$$\int_s^{s+\epsilon}\big[\epsilon C_2(u,u-\epsilon) + C(u,u-\epsilon) + \big|C(u,s)\big|\big]\,du < \alpha \tag{10}$$

for $0 \leqslant s < \infty$ and let

$$C(t,t-\epsilon)\epsilon + \int_{t-\epsilon}^{t}\big|C(t,s)\big|\,ds < \beta \tag{11}$$

for $\epsilon \leqslant t < \infty$. For technical reasons we ask $C(t,s)=0$ for $s<0$ and $t \geqslant 0$. Note in (2) and (3) that we do not specify the sign of $C(s,s)$ so the absolute value is needed in these relations.

Claim 2. *Using* (10) *and the Hobson–Tonelli test we can verify the relation*

$$\int_\epsilon^t\int_{u-\epsilon}^{u}\big[\epsilon C_2(u,u-\epsilon) + C(u,u-\epsilon) + \big|C(u,s)\big|\big]q^2\big(s,x(s)\big)\,ds\,du \leqslant \int_0^t \alpha q^2\big(s,x(s)\big)\,ds.$$

Proof. We have

$$\begin{aligned}&\int_\epsilon^t\int_{u-\epsilon}^{u}\big[\epsilon C_2(u,u-\epsilon) + C(u,u-\epsilon) + \big|C(u,s)\big|\big]q^2\big(s,x(s)\big)\,ds\,du\\ &\quad\leqslant \int_0^t\int_s^{s+\epsilon}\big[\epsilon C_2(u,u-\epsilon) + C(u,u-\epsilon) + \big|C(u,s)\big|\big]q^2\big(s,x(s)\big)\,du\,ds\\ &\quad\leqslant \int_0^t \alpha q^2\big(s,x(s)\big)\,ds. \qquad \Box\end{aligned}$$

When we treated the parallel problem for integral equations in [5] using Liapunov functionals and assumptions very much like the ones given here the function $q(t,x)$ was allowed to depend on t in a very natural way and caused no difficulty. However, investigators going all the way back to Levin [9] have been forced to require that q be independent of t. That is a definite defect and one which we rectify here. Several things need to be said about the term $\int_0^{x(t)} q(t,s)\,ds$ in the Liapunov functional $V(t,\epsilon)$ and, in order to not break the flow of ideas here, we will discuss this in Section 6. This will include a very instructive example of q.

Note. In this result, if $q(t,x)$ is independent of t and if $\int_0^x q(t,s)\,ds \to \infty$ as $|x| \to \infty$ then it does yield a bounded solution, just as was the case in Levin's original theorem. This requires explanation and the reader is referred to Section 6.

Theorem 4.2. *Let x be a continuous solution of* (1) *on* $[0,\infty)$ *and let* (2), (3), (9)–(11) *hold. Moreover, assume that*

$$xq_t(t,x) \leqslant 0 \quad \text{for } t \in [0,\infty),\ x \in \Re, \tag{12}$$

$$\left| \int_0^{\pm\infty} q_t(t,s)\,ds \right| < \infty \quad \text{for each fixed } t \in [0,\infty), \tag{13}$$

and that for the function

$$Q(t) = \max\left\{ \left| \int_0^{\pm\infty} q_t(t,s)\,ds \right| \right\}, \quad t \in [0,\infty) \tag{14}$$

we have $Q \in L^1(0,\infty)$. If, in addition, $f \in L^2[0,\infty)$ so are $q(t,x(t))$ and $q(t,x(t)) - f(t)$.

Proof. We begin by organizing the terms of the derivative of V which we applied to (8). First, by the Schwarz inequality we have

$$C_2(t,t-\epsilon)\left(\int_{t-\epsilon}^{t} q\big(u,x(u)\big)\,du \right)^2 \leqslant \epsilon C_2(t,t-\epsilon) \int_{t-\epsilon}^{t} q^2\big(u,x(u)\big)\,du.$$

Next,

$$\left| 2q\big(t,x(t)\big)C(t,t-\epsilon) \int_{t-\epsilon}^{t} q\big(u,x(u)\big)\,du \right| \leqslant C(t,t-\epsilon) \int_{t-\epsilon}^{t} \big[q^2\big(t,x(t)\big) + q^2\big(u,x(u)\big)\big]\,du$$

and

$$\left| 2q\big(t,x(t)\big) \int_{t-\epsilon}^{t} C(t,s)q\big(s,x(s)\big)\,ds \right| \leqslant \int_{t-\epsilon}^{t} \big|C(t,s)\big|\big[q^2\big(t,x(t)\big) + q^2\big(s,x(s)\big)\big]\,ds.$$

These three relations along with the result of Claim 1 in (8) yield

$$\begin{aligned}
V'(t,\epsilon) &\leqslant 2\left| \int_0^{x(t)} q_t(t,s)\,ds \right| + C_2(t,t-\epsilon)\left(\int_{t-\epsilon}^{t} q\big(u,x(u)\big)\,du \right)^2 \\
&\quad + 2q\big(t,x(t)\big)\left[C(t,t-\epsilon)\left(\int_{t-\epsilon}^{t} q\big(u,x(u)\big)\,du \right) - \int_{t-\epsilon}^{t} C(t,s)q\big(s,x(s)\big)\,ds \right] \\
&\quad + 2q\big(t,x(t)\big)\big[f(t) - h\big(t,x(t)\big)\big] \\
&\leqslant 2Q(t) + \epsilon C_2(t,t-\epsilon)\left(\int_{t-\epsilon}^{t} q^2\big(u,x(u)\big)\,du \right) + C(t,t-\epsilon) \int_{t-\epsilon}^{t} \big[q^2\big(t,x(t)\big) + q^2\big(s,x(s)\big)\big]\,ds \\
&\quad + \int_{t-\epsilon}^{t} \big|C(t,s)\big|\big[q^2\big(t,x(t)\big) + q^2\big(s,x(s)\big)\big]\,ds + \frac{1}{\gamma} f^2(t) - \gamma q^2\big(t,x(t)\big) \\
&= 2Q(t) + q^2\big(t,x(t)\big) \int_{t-\epsilon}^{t} \big[C(t,t-\epsilon) + \big|C(t,s)\big|\big]\,ds \\
&\quad + \int_{t-\epsilon}^{t} \big[\epsilon C_2(t,t-\epsilon) + C(t,t-\epsilon) + \big|C(t,s)\big|\big]q^2\big(s,x(s)\big)\,ds + \frac{1}{\gamma} f^2(t) - \gamma q^2\big(t,x(t)\big).
\end{aligned}$$

That is,

$$V'(t,\epsilon) \leqslant 2Q(t) + q^2\big(t,x(t)\big)\beta + \frac{1}{\gamma} f^2(t) - \gamma q^2\big(t,x(t)\big) + \int_{t-\epsilon}^{t} \big[\epsilon C_2(t,t-\epsilon) + C(t,t-\epsilon) + \big|C(t,s)\big|\big] q^2\big(s,x(s)\big)\,ds.$$

Integrating from ϵ to t and invoking Claim 2 yields

$$\begin{aligned} V(t,\epsilon) - V(\epsilon,\epsilon) &\leqslant 2\int_\epsilon^t Q(s)\,ds + \frac{1}{\gamma}\int_\epsilon^t f^2(s)\,ds - [\gamma-\beta]\int_\epsilon^t q^2\big(s,x(s)\big)\,ds + \int_0^t \alpha q^2\big(s,x(s)\big)\,ds \\ &= 2\int_\epsilon^t Q(s)\,ds + \frac{1}{\gamma}\int_\epsilon^t f^2(s)\,ds - [\gamma-\beta-\alpha]\int_\epsilon^t q^2\big(s,x(s)\big)\,ds + \alpha\int_0^\epsilon q^2\big(s,x(s)\big)\,ds, \end{aligned}$$

from which we have

$$V(t,\epsilon) + [\gamma-\beta-\alpha]\int_\epsilon^t q^2\big(s,x(s)\big)\,ds \leqslant V(\epsilon,\epsilon) + 2\int_\epsilon^t Q(s)\,ds + \frac{1}{\gamma}\int_\epsilon^t f^2(s)\,ds + \alpha\int_0^\epsilon q^2\big(s,x(s)\big)\,ds.$$

As x is continuous and ϵ is a positive number, it follows that $x(\epsilon)$ is finite so by the continuity of $q(t,s)$ we see that $\int_0^{x(\epsilon)} q(\epsilon,s)\,ds < \infty$, thus

$$V(\epsilon,\epsilon) = 2\int_0^{x(\epsilon)} q(\epsilon,s)\,ds + C(\epsilon,0)\left(\int_0^\epsilon q\big(u,x(u)\big)\,du\right)^2 < \infty.$$

Hence, for any $t \geqslant \epsilon$ we have

$$\int_\epsilon^t q^2\big(s,x(s)\big)\,ds \leqslant \frac{1}{\gamma-\beta-\alpha}\left[V(\epsilon,\epsilon) + 2\int_\epsilon^t Q(s)\,ds + \frac{1}{\gamma}\int_0^\infty f^2(s)\,ds + \alpha\int_0^\epsilon q^2\big(s,x(s)\big)\,ds\right],$$

which proves our assertion. □

5. The resolvent

Let C be a scalar function with weak singularities and consider

$$x'(t) = Ax(t) - \int_0^t C(t,s)x(s)\,ds + f(t), \qquad x(0) = x_0, \tag{15}$$

where A is a negative constant.

Associated with (15) is the resolvent equation

$$\frac{d}{dt}Z(t,s) = AZ(t,s) - \int_s^t C(t,u)Z(u,s)\,du, \qquad Z(s,s) = 1,$$

the principal solution of

$$z'(t,s) = Az(t,s) - \int_s^t C(t,u)z(u,s)\,du. \tag{16}$$

Becker [1] obtained the following variation of parameters formula for continuous kernels, but the proof extends to this case:

$$x(t) = Z(t,0)x_0 + \int_0^t Z(t,s)f(s)\,ds.$$

Theorem 5.1. *Let* (15) *be a scalar equation, A be a negative constant, and C have a singularity at $t = s$. We assume that there exists an $\epsilon > 0$ such that for $0 \leqslant s \leqslant t - \epsilon$ then* (3) *holds. Suppose there exist $\alpha^* > 0$ and $\beta^* > 0$ with*

$$\int_s^{s+\epsilon} \left[\epsilon C_2(u, u-\epsilon) + \left|C(u, u-\epsilon) - C(u, s)\right|\right] du < \alpha^*, \quad 0 \leqslant s < \infty, \tag{17}$$

$$\int_{t-\epsilon}^{t} \left|C(t, t-\epsilon) - C(t, s)\right| ds < \beta^*, \quad \epsilon \leqslant t < \infty, \tag{18}$$

and

$$A < -\frac{\alpha^* + \beta^*}{2}. \tag{19}$$

If $V(t, \epsilon)$ is defined by

$$V(t, \epsilon) = z^2(t, s) + \int_s^{t-\epsilon} C_2(t, u)\left(\int_u^t z(v, s)\, dv\right)^2 du + C(t, s)\left(\int_s^t z(v, s)\, dv\right)^2$$

for $t \geqslant \epsilon$ and $s \leqslant t - \epsilon$ and if $z(t, s)$ solves (16)*, then*

$$z^2(t, s) + \mu \int_0^t z^2(u, s)\, du \leqslant V(s + \epsilon, \epsilon) + \left|2A + \beta^*\right| \int_0^s z^2(u, s)\, du,$$

where $\mu = -(2A + \beta^* + \alpha^*) > 0$. That is, $z(t, s) \in L_t^2[0, \infty) \cap L_t^\infty[0, \infty)$ (where the subscript t denotes the variable of integration).

Proof. We have

$$\begin{aligned}
V'(t, \epsilon) = {} & 2z(t, s)\left[Az(t, s) - \int_s^t C(t, u) z(u, s)\, du\right] + C_2(t, t-\epsilon)\left(\int_{t-\epsilon}^t z(v, s)\, dv\right)^2 \\
& + \int_s^{t-\epsilon} C_{2,1}(t, u)\left(\int_u^t z(v, s)\, dv\right)^2 du + C_1(t, s)\left(\int_s^t z(v, s)\, dv\right)^2 \\
& + 2z(t, s) C(t, s) \int_s^t z(v, s)\, dv + 2z(t, s) \int_s^{t-\epsilon} C_2(t, u) \int_u^t z(v, s)\, dv\, du.
\end{aligned}$$

Integration of the last term by parts yields

$$\begin{aligned}
& 2z(t, s)\left[C(t, u) \int_u^t z(v, s)\, dv \Big|_s^{t-\epsilon} + \int_s^{t-\epsilon} C(t, u) z(u, s)\, du\right] \\
& \quad = 2z(t, s)\left[C(t, t-\epsilon) \int_{t-\epsilon}^t z(v, s)\, dv - C(t, s) \int_s^t z(v, s)\, dv + \int_s^{t-\epsilon} C(t, u) z(u, s)\, du\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
V'(t, \epsilon) \leqslant {} & C_2(t, t-\epsilon)\epsilon \int_{t-\epsilon}^t z^2(v, s)\, dv + 2z(t, s)\left[Az(t, s) - \int_s^t C(t, u) z(u, s)\, du\right] \\
& + 2z(t, s)\left[C(t, t-\epsilon) \int_{t-\epsilon}^t z(v, s)\, dv + \int_s^t C(t, u) z(u, s)\, du - \int_{t-\epsilon}^t C(t, u) z(u, s)\, du\right] \\
= {} & 2z(t, s)\left[Az(t, s) + C(t, t-\epsilon) \int_{t-\epsilon}^t z(v, s)\, dv - \int_{t-\epsilon}^t C(t, u) z(u, s)\, du\right] + C_2(t, t-\epsilon)\epsilon \int_{t-\epsilon}^t z^2(v, s)\, dv
\end{aligned}$$

$$\leqslant C_2(t,t-\epsilon)\epsilon\int_{t-\epsilon}^{t} z^2(v,s)\,dv + 2Az^2(t,s) + \int_{t-\epsilon}^{t}\big[\big|C(t,u)-C(t,t-\epsilon)\big|\big]\big[z^2(t,s)+z^2(u,s)\big]\,du$$

$$= \left[2A + \int_{t-\epsilon}^{t}\big|C(t,u)-C(t,t-\epsilon)\big|\,du\right]z^2(t,s)$$

$$+ \int_{t-\epsilon}^{t}\big[\big|C(t,t-\epsilon)-C(t,u)\big| + C_2(t,t-\epsilon)\epsilon\big]z^2(u,s)\,du.$$

Integrating the last term on the interval $[s+\epsilon, t]$, where $0 \leqslant s \leqslant t-\epsilon$, changing the order of integration, and taking (17) into consideration, we obtain

$$\int_{s+\epsilon}^{t}\int_{v-\epsilon}^{v}\big[\big|C(v,u)-C(v,v-\epsilon)\big| + C_2(v,v-\epsilon)\epsilon\big]z^2(u,s)\,du\,dv$$

$$\leqslant \int_{s}^{t}\int_{u}^{u+\epsilon}\big[\big|C(v,u)-C(v,v-\epsilon)\big| + C_2(v,v-\epsilon)\epsilon\big]\,dv\, z^2(u,s)\,du$$

$$\leqslant \int_{0}^{t}\int_{u}^{u+\epsilon}\big[\big|C(v,u)-C(v,v-\epsilon)\big| + C_2(v,v-\epsilon)\epsilon\big]\,dv\, z^2(u,s)\,du$$

$$\leqslant \int_{0}^{t}\alpha^* z^2(u,s)\,du.$$

Using the above and taking into account (18), we have for $s+\epsilon \leqslant t$

$$V(t,\epsilon) - V(s+\epsilon,\epsilon) \leqslant \int_{s+\epsilon}^{t}\left[2A + \int_{v-\epsilon}^{v}\big|C(v,u)-C(v,v-\epsilon)\big|\,du\right]z^2(v,s)\,dv$$

$$+ \int_{s+\epsilon}^{t}\left[\int_{v-\epsilon}^{v}\big[\big|C(v,v-\epsilon)-C(v,u)\big| + C_2(v,v-\epsilon)\epsilon\big]z^2(u,s)\,du\right]dv$$

$$\leqslant \int_{s+\epsilon}^{t}\big(2A+\beta^*\big)z^2(v,s)\,dv + \int_{0}^{t}\alpha^* z^2(u,s)\,du$$

and so

$$V(t,\epsilon) - V(s+\epsilon,\epsilon) \leqslant \int_{0}^{t}\big(2A+\beta^*+\alpha^*\big)z^2(v,s)\,dv - \int_{0}^{s+\epsilon}\big(2A+\beta^*\big)z^2(u,s)\,du.$$

Hence for $t \geqslant s+\epsilon$ we obtain

$$z^2(t,s) - \big(2A+\beta^*+\alpha^*\big)\int_{0}^{t} z^2(u,s)\,du \leqslant V(s+\epsilon,\epsilon) - \big(2A+\beta^*\big)\int_{0}^{s} z^2(u,s)\,du.$$

By (19) we have $2A+\beta^*+\alpha^* < 0$ and $2A+\beta^* < 0$.

Since for any fixed s the right-hand side of the above inequality is a positive constant which does not depend on t, taking into consideration the fact that the solution $z(u,s)$ is continuous on $[0, s+\epsilon]\times\{s\}$ for all $s \geqslant 0$, it follows that for any $s \geqslant 0$ there exists an $M_z(s) > 0$ with

$$z^2(t,s) \leqslant M_z(s), \quad t \geqslant 0$$

and

$$\int_0^t z^2(u,s)\,du \leqslant M_z(s), \quad \text{for all } t \geqslant 0,$$

as required. □

6. Discussion

Allowing $q(t,x)$ to depend on t raises two important issues. Under the convexity conditions of Levin [9], a Liapunov functional parallel to (7) was employed and singularities were not allowed; thus, $h(t,x)$ was not required, as considered in Theorem 3.1 (see, also, the discussion before Claim 2). But Levin only allowed q to be independent of t and he required that $\int_0^x q(s)\,ds \to \infty$ as $|x| \to \infty$. That is, his qualitative results depended on the Liapunov functional being radially unbounded and $V'(t) \leqslant 0$.

When we examine (13) in Theorem 4.2 and note that when q_t does not vanish, then we cannot even allow the Liapunov functional to be radially unbounded. An example will clarify this. Suppose that

$$q(t,x) = \operatorname{sgn}(x) r(x) m(t)$$

where $m : [0,\infty) \to \Re$ and $r : \Re \to [0,\infty)$ with

$$r(x) \geqslant 0, \qquad m(t) \geqslant 0, \qquad m'(t) < 0, \qquad m'(t) \in L^1[0,\infty), \qquad \int_0^\infty r(x)\,dx < \infty.$$

This will satisfy (12), (13), and (14). Note that $\int_0^\infty r(x)\,dx < \infty$ is not compatible with the Levin condition that when $q(t,x) = q(x)$ then $\int_0^x q(s)\,ds \to \infty$ as $|x| \to \infty$. Our result goes beyond the Levin result in that we consider $q(t,x)$ instead of $q(x)$ and treat the case where V is not radially unbounded. On the other hand, if $q_t(t,x) \equiv 0$ then (12), (13), and (14) of Theorem 4.2 are trivially satisfied and we allow $\int_0^x q(s)\,ds \to \infty$ so that the Levin case is also covered. An example is

$$q(t,x) = \frac{(t+2)x^{2n+1}}{(t+1)(|x|+1)(|x|+2)(|x|^{2n+1}+1)}, \quad t \geqslant 0,\ x \in \Re.$$

Thus, this paper deals with three issues not allowed in the Levin work: singularities, $q(t,x)$ instead of $q(x)$, and V not radially unbounded. The work of Grimmer–Seifert also requires $q(x)$, not $q(t,x)$.

References

[1] L.C. Becker, Principal matrix solutions and variation of parameters for a Volterra integro-differential equation and its adjoint, Electron. J. Qual. Theory Differ. Equ. 14 (2006) 1–22, www.math.u-szeged.hu/ejqtde/2006/200614.html.
[2] L.C. Becker, Resolvents and solutions of weakly singular linear Volterra integral equations, Nonlinear Anal. 74 (2011) 1892–1912.
[3] T.A. Burton, An integrodifferential equation, Proc. Amer. Math. Soc. 79 (1980) 393–399.
[4] T.A. Burton, Boundedness and periodicity in integral and integro-differential equations, Differ. Equ. Dyn. Syst. 1 (1993) 161–172.
[5] T.A. Burton, A Liapunov functional for a singular integral equation, Nonlinear Anal. 73 (2010) 3873–3882.
[6] T.A. Burton, Bo Zhang, Periodicity in delay equations by direct fixed point mappings, Differ. Equ. Dyn. Syst. 6 (1998) 413–424.
[7] R. Grimmer, G. Seifert, Stability properties of Volterra integrodifferential equations, J. Differential Equations 19 (1975) 142–166.
[8] S.I. Grossman, R.K. Miller, Perturbation theory for Volterra integrodifferential systems, J. Differential Equations 8 (1970) 457–474.
[9] J.J. Levin, The asymptotic behavior of the solution of a Volterra equation, Proc. Amer. Math. Soc. 14 (1963) 534–541.
[10] V. Volterra, Sur la théorie mathématique des phénomès héréditaires, J. Math. Pures Appl. 7 (1928) 249–298.