



Atomic decomposition of Besov spaces with variable smoothness and integrability

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ABSTRACT

The aim of this paper is twofold. First we characterize the Besov spaces with variable smoothness and integrability by so-called Peetre maximal functions. Secondly we use these results to prove the atomic decomposition for these spaces.

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1. Introduction

Besov spaces of variable smoothness and integrability, $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$, initially appeared in the paper of A. Almeida and P. Hästö [1]. Several basic properties were established, such as the Fourier analytical characterization. When p, q, α are constants they coincide with the usual function spaces $B_{p,q}^s$. Also Sobolev type embeddings and the characterization by approximations of these function spaces were obtained. Some properties of such a type are well known with variable p , but fixed q and/or α . J. Vybiral [22] proved Sobolev type embeddings in these spaces. H. Kempka [12,13] has studied so-called micro-local versions of variable index Besov spaces, when local means characterizations, atomic, molecular and wavelet decomposition of these spaces are given. This setting includes also some range of weights as well as slightly more general smoothness. These studies were all restricted to variable p , but fixed q . Also J.-S. Xu [23,24] has studied Besov spaces with variable p , but fixed q and α .

The interest in these spaces comes not only from theoretical reasons but also from their applications to several classical problems in analysis. For the range of parameters $p = q = 2$, the spaces $B_{2,2}^{m(\cdot)}(\mathbb{R})$ have been considered in the analysis of certain Black–Scholes equations, see Schneider, Reichmann and Schwab [16]. For further considerations of PDEs, we refer to [9] and references therein.

The main aim of this paper is to present a decomposition by atoms for $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$. All these results generalize the existing classical results on Besov spaces by taking p, q and α are constants.

The structure of the article is as follows. After recalling some preliminaries and notation in Section 2 we define the Besov spaces $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ and repeat some results from [1] in Section 3. Some necessary tools are given in Section 4. In Section 5 we prove an useful characterization of these spaces based on the so-called local means. The theorem on local means that proved for Besov spaces of variable smoothness and integrability is highly technical and its proved required (based on maximal functions and the classical situation) new techniques and ideas. Using the results from Section 5, we prove in Section 6 the atomic decomposition for $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$.

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2. Preliminaries

As usual, \mathbb{R}^n the n -dimensional real Euclidean space, \mathbb{N} the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter \mathbb{Z} stands for the set of all integer numbers. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, we write $|\alpha| = \alpha_1 + \dots + \alpha_n$. The Euclidean scalar product of $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ is given by $x \cdot y = x_1 y_1 + \dots + x_n y_n$.

For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x, r)$ the open ball in \mathbb{R}^n with center x and radius r . By $\text{supp } f$ we denote the support of the function f , i.e., the closure of its non-zero set.

By $\mathcal{S}(\mathbb{R}^n)$ we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on \mathbb{R}^n and by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on \mathbb{R}^n . We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx.$$

Its inverse is denoted by $\mathcal{F}^{-1}f$. Both \mathcal{F} and \mathcal{F}^{-1} are extended to the dual Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ in the usual way.

The Hardy-Littlewood maximal operator \mathcal{M} is defined on L^1_{loc} by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| dy,$$

and $\mathcal{M}_t f = \mathcal{M}|f|^t$ for any $0 < t \leq 1$. The variable exponents that we consider are always measurable functions on \mathbb{R}^n with range in $[c, \infty]$ for some $c > 0$. We denote the set of such functions by \mathcal{P}_0 . The subset of variable exponents with range $[1, \infty]$ is denoted by \mathcal{P} . We use the standard notation

$$p^- = \text{ess-inf}_{x \in \mathbb{R}^n} p(x), \quad p^+ = \text{ess-sup}_{x \in \mathbb{R}^n} p(x).$$

We define

$$\rho_p(t) = \begin{cases} t^p & \text{if } p \in (0, \infty), \\ 0 & \text{if } p = \infty \text{ and } t \leq 1, \\ \infty & \text{if } p = \infty \text{ and } t > 1. \end{cases}$$

The convention $1^\infty = 0$ is adopted in order that ρ_p be left-continuous. The variable exponent modular is defined by

$$\mathcal{Q}_{p(\cdot)}(f) = \int_{\mathbb{R}^n} \rho_{p(x)}(|f(x)|) dx.$$

The variable exponent Lebesgue space $L^{p(\cdot)}$ consists of measurable functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ with $\mathcal{Q}_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$. We define the Luxemburg (quasi)-norm on this space by the formula

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \mathcal{Q}_{p(\cdot)} \left(\frac{f}{\lambda} \right) \leq 1 \right\}.$$

As is known, the following inequalities hold (see [9, Lemma 2.1.14 and Corollary 2.1.15])

$$\|f\|_{p(\cdot)} \leq 1 \quad \Leftrightarrow \quad \mathcal{Q}_{p(\cdot)}(f) \leq 1 \tag{1}$$

and

$$\|f\|_{p(\cdot)} \leq \mathcal{Q}_{p(\cdot)}(f) + 1. \tag{2}$$

Let $p, q \in \mathcal{P}_0$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$ -functions by the modular

$$\mathcal{Q}_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \inf \left\{ \lambda_v > 0 : \mathcal{Q}_{p(\cdot)} \left(\frac{f_v}{\lambda_v^{1/q(\cdot)}} \right) \leq 1 \right\}.$$

The norm is defined from this as usual:

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = \inf \left\{ \mu > 0 : \mathcal{Q}_{\ell^{q(\cdot)}(L^{p(\cdot)})} \left(\frac{1}{\mu} (f_v)_v \right) \leq 1 \right\}.$$

We will use the notation

$$\mathcal{Q}_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \| |f_v|^{q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}}$$

for the modular. In (1) and (2) $\|f\|_{p(\cdot)}$ and $\mathcal{Q}_{p(\cdot)}(f)$ can be replaced by $\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}$ and $\mathcal{Q}_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v)$, respectively.

We say that $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is *locally log-Hölder continuous*, abbreviated $g \in C_{\text{loc}}^{\log}$, if there exists $c_1 > 0$ such that

$$|g(x) - g(y)| \leq \frac{c_1}{\log(e + 1/|x - y|)}$$

for all $x, y \in \mathbb{R}^n$. We say that g satisfies the *log-Hölder decay condition*, if there exists $g_\infty > 0$ and a constant $c_2 > 0$ such that

$$|g(x) - g_\infty| \leq \frac{c_2}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. We say that g is *globally-log-Hölder continuous*, abbreviated $g \in C^{\log}$, if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. The constants c_1 and c_2 are called the *locally log-Hölder constant* and the *log-Hölder decay constant*, respectively. The maximum $\max\{c_1, c_2\}$ is just called the *log-Hölder constant* of g and it is denoted by $c_{\log}(g)$.

We note that all functions $g \in C_{\text{loc}}^{\log}$ always belong to L^∞ .

We define the following class of variable exponents

$$\mathcal{P}^{\log} = \left\{ p \in \mathcal{P} : \frac{1}{p} \text{ is globally-log-Hölder continuous} \right\}.$$

The class \mathcal{P}_0^{\log} is defined analogously. It was shown in [9], Theorem 4.3.8 that $\mathcal{M} : L^{p(\cdot)} \rightarrow L^{p(\cdot)}$ is bounded if $p \in \mathcal{P}^{\log}$ and $p^- > 1$. We also refer to the papers [6] and [7], where various results on maximal function in variable Lebesgue spaces were obtained.

Recall that $\eta_{v,m}(x) = 2^{nv} (1 + 2^v |x|)^{-m}$, for any $x \in \mathbb{R}^n$, $v \in \mathbb{N}_0$ and $m > 0$. Note that $\eta_{v,m} \in L^1$ when $m > n$ and that $\|\eta_{v,m}\|_1 = c_m$ is independent of v .

By c we denote generic positive constants, which may have different values at different occurrences. Although the exact values of the constants are usually irrelevant for our purposes, sometimes we emphasize their dependence on certain parameters (e.g. $c(p)$ means that c depends on p , etc.). Further notation will be properly introduced whenever needed.

3. Some technical lemmas

In this section we present some results which are useful for us. The first two lemmas are from [8], Lemma 6.1 and Lemma A.7 respectively.

Lemma 1. *If $\alpha \in C_{\text{loc}}^{\log}$, then there exists $d \in (n, \infty)$ such that if $m > d$, then*

$$2^{v\alpha(x)} \eta_{v,2m}(x - y) \leq c 2^{v\alpha(y)} \eta_{v,m}(x - y)$$

with $c > 0$ independent of $x, y \in \mathbb{R}^n$ and $v \in \mathbb{N}_0$.

The previous lemma allows us to treat the variable smoothness in many cases as if it were not variable at all, namely we can move the term inside the convolution as follows:

$$2^{v\alpha(x)} \eta_{v,2m} * f(x) \leq c \eta_{v,m} * (2^{v\alpha(\cdot)} f)(x).$$

The next lemma often allows us to deal with exponents which are smaller than 1.

Lemma 2. *Let $r > 0$, $v \in \mathbb{N}_0$ and $m > n$. Then there exists $c = c(r, m, n) > 0$ such that for all $g \in \mathcal{S}'(\mathbb{R}^n)$ with $\text{supp } \mathcal{F}g \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2^{v+1}\}$, we have*

$$|g(x)| \leq c (\eta_{v,m} * |g|^r(x))^{1/r}, \quad x \in \mathbb{R}^n.$$

The next lemma is a Hardy-type inequality which is easy to prove.

Lemma 3. *Let $0 < a < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_k\}_{k \in \mathbb{N}_0}$ be a sequence of positive real numbers, such that*

$$\|\{\varepsilon_k\}_{k \in \mathbb{N}_0}\|_{\ell^q} = I < \infty.$$

The sequence $\{\delta_k\} : \delta_k = \sum_{j=0}^{\infty} a^{|k-j|} \varepsilon_j\}_{k \in \mathbb{N}_0}$ is in ℓ^q with

$$\|\{\delta_k\}_{k \in \mathbb{N}_0}\|_{\ell^q} \leq cI.$$

c depends only on a and q .

The following lemmas are from [15].

Lemma 4. Let $\omega, \mu \in \mathcal{S}(\mathbb{R}^n)$ and $M \geq -1$, an integer such that $\int_{\mathbb{R}^n} x^\alpha \mu(x) dx = 0$ for all $|\alpha| \leq M$. Then for any $N > 0$, there is a constant $c_N > 0$ so that

$$\sup_{z \in \mathbb{R}^n} |t^{-n} \mu(t^{-1} \cdot) * \omega(z)| (1 + |z|)^N \leq c_N t^{M+1}.$$

Lemma 5. Let $0 < r \leq 1$, and let $\{b_j\}_{j \in \mathbb{N}_0}, \{d_j\}_{j \in \mathbb{N}_0}$ be two sequences taking values in $(0, +\infty)$. Assume that for some $N_0 > 0$

$$d_j = O(2^{jN_0}), \quad j \longrightarrow \infty, \quad (3)$$

and that for any $N > 0$

$$d_j \leq C_N \sum_{k=j}^{\infty} 2^{(j-k)N} b_k d_k^{1-r}, \quad j \in \mathbb{N}_0.$$

Then for any $N > 0$

$$d_j^r \leq C_N \sum_{k=j}^{\infty} 2^{(j-k)Nr} b_k, \quad j \in \mathbb{N}_0$$

with the same constants C_N .

4. Spaces $B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$

In this section we present the Fourier analytical definition of the spaces $B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$ and recall their basic properties. We first need the concept of a smooth dyadic resolution of unity.

Definition 1. Let Ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\Psi(x) = 1$ for $|x| \leq 1$ and $\Psi(x) = 0$ for $|x| \geq 2$. We put $\mathcal{F}\varphi_0(x) = \Psi(x)$, $\mathcal{F}\varphi_1(x) = \Psi(\frac{x}{2}) - \Psi(x)$ and

$$\mathcal{F}\varphi_v(x) = \mathcal{F}\varphi_1(2^{-v+1}x) \quad \text{for } v = 2, 3, \dots$$

Then $\{\mathcal{F}\varphi_v\}_{v \in \mathbb{N}_0}$ is a smooth dyadic resolution of unity,

$$\sum_{v=0}^{\infty} \mathcal{F}\varphi_v(x) = 1$$

for all $x \in \mathbb{R}^n$. Thus we obtain the Littlewood–Paley decomposition

$$f = \sum_{v=0}^{\infty} \varphi_v * f$$

of all $f \in \mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$).

We are now in a position to state the definition of the spaces $B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$.

Definition 2. Let φ_v be as in Definition 1. For $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ and $p, q \in \mathcal{P}_0$, the Besov space $B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$ consists of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} = \|(2^{v\alpha(\cdot)} \varphi_v * f)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty. \quad (4)$$

For any $p, q \in \mathcal{P}_0^{\log}$ and $\alpha \in C_{\text{loc}}^{\log}$, the space $B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$ does not depend on the chosen smooth dyadic resolution of unity $\{\mathcal{F}\varphi_v\}_{v \in \mathbb{N}_0}$ (in the sense of equivalent quasi-norms). They are quasi-Banach spaces, and

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)} \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Moreover, if p, q, α are constants, we re-obtain the usual Besov spaces $B_{p, q}^{\alpha}$, studied in detail by H. Triebel in [18–21].

The full treatment of the spaces $B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$ can be found in [1] and [9]. We refer to the papers [12,13,22,23], for further results on the variable Besov spaces $B_{p(\cdot), q}^{\alpha(\cdot)}$ (only the case of constant q was considered, see also [4,5]). We also mention the papers [2,3,11], for further results on the variable Bessel potentials spaces and variable Sobolev spaces.

5. Equivalent quasi-norms

In this section we characterize the spaces $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$ by so-called local means. We follow closely the method presented by V.S. Rychkov [15]. Therefore, we define for $a > 0$, $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$, the Peetre maximal function

$$\varphi_v^{*,a} 2^{v\alpha(\cdot)} f(x) = \sup_{y \in \mathbb{R}^n} \frac{2^{v\alpha(y)} |\varphi_v * f(y)|}{(1 + 2^v |x - y|)^a}, \quad v \in \mathbb{N}_0.$$

We now present a fundamental characterization of spaces under consideration.

Theorem 1. Let $\alpha \in C_{\text{loc}}^{\log}$, $p, q \in \mathcal{P}_0^{\log}$ and $a > \frac{n}{p^-}$. Then

$$\|(\varphi_v^{*,a} 2^{v\alpha(\cdot)} f)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \quad (5)$$

is an equivalent quasi-norm in $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$.

Proof. We will do the proof in two steps.

Step 1. It is easy to see that for any $f \in \mathcal{S}'(\mathbb{R}^n)$ and any $x \in \mathbb{R}^n$ we have

$$2^{v\alpha(x)} |\varphi_v * f(x)| \leq \varphi_v^{*,a} 2^{v\alpha(\cdot)} f(x).$$

This shows that the right-hand side in (4) is less than or equal (5).

Step 2. We will prove in this step that there is a constant $c > 0$ such that for every $f \in \mathcal{S}'(\mathbb{R}^n)$

$$\|(\varphi_v^{*,a} 2^{v\alpha(\cdot)} f)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq c \|(2^{v\alpha(\cdot)} \varphi_v * f)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

By a λ -argument, we see that it suffices to consider the case

$$\|(2^{v\alpha(\cdot)} \varphi_v * f)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = 1$$

and show that the modular of a constant times the function on the left-hand side is bounded. In particular, we will show that

$$\sum_{v=0}^{\infty} \|c \varphi_v^{*,a} 2^{v\alpha(\cdot)} f\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^{q(\cdot)} \leq C \quad \text{whenever} \quad \sum_{v=0}^{\infty} \|2^{v\alpha(\cdot)} \varphi_v * f\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^{q(\cdot)} = 1.$$

This clearly follows from the inequality

$$\|c \varphi_v^{*,a} 2^{v\alpha(\cdot)} f\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^{q(\cdot)} \leq \|2^{v\alpha(\cdot)} \varphi_v * f\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^{q(\cdot)} + 2^{-\sigma v} = \delta,$$

for some $\sigma > 0$. This claim can be reformulated as showing that

$$\|\delta^{-1} c \varphi_v^{*,a} 2^{v\alpha(\cdot)} f\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^{q(\cdot)} \leq 1,$$

which is equivalent to

$$\|c \delta^{-\frac{1}{q(\cdot)}} \varphi_v^{*,a} 2^{v\alpha(\cdot)} f\|_{p(\cdot)} \leq 1.$$

We choose $t > 0$ such that $a > \frac{n}{t} > \frac{n}{p^-}$. By Lemmas 2 and 1 the estimates

$$2^{v\alpha(y)} |\varphi_v * f(y)| \leq C 2^{v\alpha(y)} (\eta_{v,2m} * |\varphi_v * f|^t(y))^{1/t} \leq C (\eta_{v,m} * (2^{v\alpha(\cdot)} |\varphi_v * f|)^t(y))^{1/t} \quad (6)$$

are true for any $y \in \mathbb{R}^n$, $v \in \mathbb{N}_0$ and any $m > d$ (with d as in Lemma 1). Divide both sides of (6) by $(1 + 2^v |x - y|)^a$, in the right-hand side we use the inequality

$$(1 + 2^v |x - y|)^{-a} \leq (1 + 2^v |x - z|)^{-a} (1 + 2^v |y - z|)^a, \quad x, y, z \in \mathbb{R}^n,$$

in the left-hand side take the supremum over $y \in \mathbb{R}^n$ and get for all $f \in \mathcal{S}'(\mathbb{R}^n)$, any $x \in \mathbb{R}^n$, $m > \max(d, at)$ and any $v \in \mathbb{N}_0$

$$\begin{aligned} (\varphi_v^{*,a} 2^{v\alpha(\cdot)} f(x))^t &\leq C 2^{vn} \int_{\mathbb{R}^n} \frac{2^{v\alpha(z)t} |\varphi_v * f(z)|^t}{(1 + 2^v |x - z|)^{at}} dz \\ &= C \int_{B(x, 2^{-v/2})} \dots dz + C \sum_{i=0}^{\infty} \int_{B(x, 2^{-v/2+i+1}) \setminus B(x, 2^{-v/2+i})} \dots dz \\ &= J_v^1(x) + \sum_{i \geq 0} J_{v-i}^2(x), \end{aligned}$$

where $C > 0$ is independent of x, v and f . We choose $\sigma > 0$ such that

$$0 < \sigma < \frac{a - n/t}{4(\frac{1}{q^-} - \frac{1}{q^+})}.$$

Since $1/q$ is log-Hölder continuous and $\delta \in [2^{-\sigma v}, 1 + 2^{-\sigma v}]$, we have

$$\delta^{(\frac{1}{q(z)} - \frac{1}{q(x)})} \leq (2^{\sigma v} \delta)^{|\frac{1}{q(z)} - \frac{1}{q(x)}|} 2^{v\sigma |\frac{1}{q(z)} - \frac{1}{q(x)}|} \leq c 2^{2c_{\log}(q)\sigma v / \log(e+1/|x-z|)} \leq c \quad (7)$$

for any $z \in B(x, 2^{-v/2})$. Hence

$$\delta^{-\frac{t}{q(x)}} J_v^1(x) \leq C 2^{vn} \int_{\mathbb{R}^n} \frac{\delta^{-\frac{t}{q(z)}} 2^{v\alpha(z)t} |\varphi_v * f(z)|^t}{(1 + 2^v |x - z|)^{at}} dz.$$

Now the function $z \mapsto \frac{1}{(1+|z|)^{at}}$ is in L^1 (since $a > \frac{n}{t}$), then using the majorant property for the Hardy–Littlewood maximal operator \mathcal{M} , see E.M. Stein and G. Weiss [17, Chapter 2, (3.9)],

$$\left(|g|^t * \frac{1}{(1 + |\cdot|)^{at}} \right)(x) \leq C \left\| \frac{1}{(1 + |\cdot|)^{at}} \right\|_1 \mathcal{M}_t(g)(x),$$

it follows that for any $x \in \mathbb{R}^n$

$$\delta^{-\frac{t}{q(x)}} J_v^1(x) \leq C \mathcal{M}_t(\delta^{-\frac{1}{q(\cdot)}} 2^{v\alpha(\cdot)} \varphi_v * f)(x),$$

where the constant $C > 0$ is independent of x and v . Since $|x - z| \geq 2^{-v/2+i}$ and the right-hand side of (7) can be estimated by $c 2^{2v\sigma(1/q^- - 1/q^+)}$, then for any $x \in \mathbb{R}^n$ and any $v \in \mathbb{N}_0$, $\delta^{-\frac{t}{q(x)}} J_{v-i}^2(x)$ is bounded by

$$\begin{aligned} & C 2^{vt(2\sigma(1/q^- - 1/q^+) - a/2 + n/t)} 2^{-iat} \int_{B(x, 2^{-v/2+i+1})} \delta^{-\frac{t}{q(z)}} 2^{v\alpha(z)t} |\varphi_v * f(z)|^t dz \\ & \leq C 2^{vt(2\sigma(1/q^- - 1/q^+) - a/2 + n/2t)} 2^{i(n-at)} \mathcal{M}_t(\delta^{-\frac{1}{q(\cdot)}} 2^{v\alpha(\cdot)} \varphi_v * f)(x) \\ & \leq C 2^{i(n-at)} \mathcal{M}_t(\delta^{-\frac{1}{q(\cdot)}} 2^{v\alpha(\cdot)} \varphi_v * f)(x), \end{aligned}$$

due to our choice of σ . Hence,

$$\sum_{i=0}^{\infty} \delta^{-\frac{t}{q(x)}} J_{v-i}^2(x) \leq C \sum_{i=0}^{\infty} 2^{i(n-at)} \mathcal{M}_t(\delta^{-\frac{1}{q(\cdot)}} 2^{v\alpha(\cdot)} \varphi_v * f)(x) \leq C \mathcal{M}_t(\delta^{-\frac{1}{q(\cdot)}} 2^{v\alpha(\cdot)} \varphi_v * f)(x),$$

since again $a > n/t$. Consequently we have proved that

$$(\delta^{-\frac{1}{q(x)}} \varphi_v^{*,a} 2^{v\alpha(\cdot)} f(x))^t \leq C \mathcal{M}_t(\delta^{-\frac{1}{q(\cdot)}} 2^{v\alpha(\cdot)} \varphi_v * f)(x),$$

for all $x \in \mathbb{R}^n$. Taking the $L^{\frac{p(\cdot)}{t}}$ -norm and using the fact that $\mathcal{M} : L^{\frac{p(\cdot)}{t}} \rightarrow L^{\frac{p(\cdot)}{t}}$ is bounded we obtain that

$$\begin{aligned} \|c \delta^{-\frac{1}{q(\cdot)}} \varphi_v^{*,a} 2^{v\alpha(\cdot)} f\|_{p(\cdot)}^t &= \|c \delta^{-\frac{1}{q(\cdot)}} \varphi_v^{*,a} 2^{v\alpha(\cdot)} f\|_{\frac{p(\cdot)}{t}}^t \\ &\leq \|\delta^{-\frac{t}{q(\cdot)}} 2^{v\alpha(\cdot)t} |\varphi_v * f|^t\|_{\frac{p(\cdot)}{t}} \\ &= \|\delta^{-\frac{1}{q(\cdot)}} 2^{v\alpha(\cdot)} \varphi_v * f\|_{p(\cdot)}^t, \end{aligned}$$

with an appropriate choice of $c > 0$. Now the right-hand side is less than or equal to one if and only if

$$\|\delta^{-\frac{1}{q(\cdot)}} 2^{v\alpha(\cdot)} \varphi_v * f\|_{\frac{p(\cdot)}{q(\cdot)}}^{q(\cdot)} \leq 1,$$

which follows immediately from the definition of δ .

The proof is completed. \square

In order to formulate the main result of this section, let us consider $k_0, k \in S(\mathbb{R}^n)$ and $S \geq -1$ an integer such that for an $\varepsilon > 0$

$$|\mathcal{F}k_0(\xi)| > 0 \quad \text{for } |\xi| < 2\varepsilon, \quad (8)$$

$$|\mathcal{F}k(\xi)| > 0 \quad \text{for } \frac{\varepsilon}{2} < |\xi| < 2\varepsilon \quad (9)$$

and

$$\int_{\mathbb{R}^n} x^\alpha k(x) dx = 0 \quad \text{for any } |\alpha| \leq S. \quad (10)$$

Here (8) and (9) are Tauberian conditions, while (10) are moment conditions on k . We recall the notation

$$k_t(x) = t^{-n} k(t^{-1}x), \quad k_j(x) = k_{2^{-j}}(x), \quad \text{for } t > 0 \text{ and } j \in \mathbb{N}.$$

For any $a > 0$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we denote

$$k_j^{*,a} 2^{j\alpha(\cdot)} f(x) = \sup_{y \in \mathbb{R}^n} \frac{2^{j\alpha(y)} |k_j * f(y)|}{(1 + 2^j |x - y|)^a}, \quad j \in \mathbb{N}_0. \quad (11)$$

Usually $k_j * f$ is called local mean.

We are able now to state the main result of this section.

Theorem 2. Let $\alpha \in C_{\text{loc}}^{\log}$ and $p, q \in \mathcal{P}_0^{\log}$ with $q^+ < \infty$. Let $a > \frac{n}{p^-}$ and $\alpha^+ < S + 1$. Then

$$\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}' = \|(k_j^{*,a} 2^{j\alpha(\cdot)} f)_j\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \quad (12)$$

and

$$\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}'' = \|(2^{j\alpha(\cdot)} k_j * f)_j\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}, \quad (13)$$

are equivalent quasi-norms on $B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}$.

Proof. First H. Kempka [12] proved this result, but only the case of constant q was included. J.-S. Xu [23] has proved this result with variable p , but fixed q and α . H. Kempka and J. Vybřal [14, Theorem 14], independently, proved this result with $2^{j\alpha(\cdot)} k_j^{*,a} f$, $\frac{n+c_{\log}(1/q)}{p^-} + c_{\log}(\alpha)$ in place of $k_j^{*,a} 2^{j\alpha(\cdot)} f$, $\frac{n}{p^-}$, respectively. The idea of the proof is from V.S. Rychkov [15].

Step 1. Take any pair of functions ϕ_0 and $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\begin{aligned} |\mathcal{F}\phi_0(\xi)| &> 0 \quad \text{for } |\xi| < 2\varepsilon, \\ |\mathcal{F}\phi(\xi)| &> 0 \quad \text{for } \frac{\varepsilon}{2} < |\xi| < 2\varepsilon. \end{aligned}$$

We will prove that there is a constant $c > 0$ such that for any $f \in \mathcal{S}'(\mathbb{R}^n)$

$$\|f\|_{B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}}' \leq c \|(\phi_j^{*,a} 2^{j\alpha(\cdot)} f)_j\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}. \quad (14)$$

By a scaling argument, we see that it suffices to consider the case

$$\|(\phi_j^{*,a} 2^{j\alpha(\cdot)} f)_j\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} = 1$$

and show that the modular of a constant times the function on the left-hand side is bounded. In particular, we will show that

$$\sum_{j=0}^{\infty} \|ck_j^{*,a} 2^{j\alpha(\cdot)} f\|_{\frac{p(\cdot)}{q(\cdot)}}^{q(\cdot)} \leq C \quad \text{whenever} \quad \sum_{j=0}^{\infty} \|\phi_j^{*,a} 2^{j\alpha(\cdot)} f\|_{\frac{p(\cdot)}{q(\cdot)}}^{q(\cdot)} = 1.$$

Let $\Lambda, \lambda \in \mathcal{S}(\mathbb{R}^n)$ so that

$$\begin{aligned} \text{supp } \mathcal{F}\Lambda &\subset \{\xi \in \mathbb{R}^n: |\xi| < 2\varepsilon\}, \quad \text{supp } \mathcal{F}\lambda \subset \{\xi \in \mathbb{R}^n: \varepsilon/2 < |\xi| < 2\varepsilon\}, \\ \mathcal{F}\Lambda(\xi) \mathcal{F}\phi_0(\xi) + \sum_{v=1}^{\infty} \mathcal{F}\lambda(2^{-v}\xi) \mathcal{F}\phi(2^{-v}\xi) &= 1, \quad \xi \in \mathbb{R}^n. \end{aligned} \quad (15)$$

In particular, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ the identity is true

$$f = \Lambda * \phi_0 * f + \sum_{v=1}^{\infty} \lambda_v * \phi_v * f. \quad (16)$$

Hence we can write

$$k_j * f = k_j * \Lambda * \phi_0 * f + \sum_{v=1}^{\infty} k_j * \lambda_v * \phi_v * f.$$

We have

$$2^{j\alpha(y)} |k_j * \lambda_v * \phi_v * f(y)| \leq 2^{j\alpha(y)} \int_{\mathbb{R}^n} |k_j * \lambda_v(z)| |\phi_v * f(y-z)| dz. \quad (17)$$

First let $v \leq j$. Writing for any $z \in \mathbb{R}^n$

$$k_j * \lambda_v(z) = 2^{vn} k_{2^{v-j}} * \lambda(2^v z),$$

we get by Lemma 4, that for any integer $S \geq -1$ and any $N > 0$ there is a constant $c > 0$ independent of j and v

$$|k_j * \lambda_v(z)| \leq c \frac{2^{(v-j)(S+1)+vn}}{(1+2^v|z|)^{2N}}, \quad z \in \mathbb{R}^n.$$

So the right-hand side of (17) can be estimated from above by

$$c 2^{j\alpha(y)+(v-j)(S+1)+vn} \int_{\mathbb{R}^n} (1+2^v|z|)^{-2N} |\phi_v * f(y-z)| dz = c 2^{(v-j)(S+1)} 2^{j\alpha(y)} \eta_{v,2N} * |\phi_v * f|(y).$$

By Lemma 1 the estimates

$$\begin{aligned} 2^{j\alpha(y)} \eta_{v,2N} * |\phi_v * f(y)| &\leq 2^{(j-v)\alpha^+} \eta_{v,N} * (2^{v\alpha(\cdot)} |\phi_v * f|)(y) \\ &\leq 2^{(j-v)\alpha^+} \phi_v^{*,a} 2^{v\alpha(\cdot)} f(y) \|\eta_{v,N-a}\|_1 \\ &\leq c 2^{(j-v)\alpha^+} \phi_v^{*,a} 2^{v\alpha(\cdot)} f(y), \end{aligned}$$

are true for any $N > \max(d, n+a)$ and any $v \leq j$ (with d as in Lemma 1).

Let now $v \geq j$. Then, again by Lemma 4 we have for any $z \in \mathbb{R}^n$ and any $L > 0$

$$|k_j * \lambda_v(z)| = 2^{jn} |k * \lambda_{2^{j-v}}(2^j z)| \leq c \frac{2^{(j-v)(M+1)+jn}}{(1+2^j|z|)^{2L}},$$

where $M \geq -1$ an integer can be taken arbitrarily large, since $D^\alpha \mathcal{F}\lambda(0) = 0$ for all α . Therefore, for $v \geq j$, the right-hand side of (17) can be estimated from above by

$$c 2^{j\alpha(y)+(j-v)(M+1)+jn} \int_{\mathbb{R}^n} (1+2^j|z|)^{-2L} |\phi_v * f(y-z)| dz = c 2^{j\alpha(y)+(j-v)(M+1)} \eta_{j,2L} * |\phi_v * f|(y).$$

We have for any $v \geq j$

$$(1+2^j|z|)^{-2L} \leq 2^{2(v-j)L} (1+2^v|z|)^{-2L}.$$

Then, again, the right-hand side of (17) is dominated by

$$\begin{aligned} c 2^{j\alpha(y)+(j-v)(M-2L+1+n)} \eta_{v,2L} * |\phi_v * f|(y) &\leq c 2^{(j-v)(M-2L+1+\alpha^-+n)} \eta_{v,L} * (2^{v\alpha(\cdot)} |\phi_v * f|)(y) \\ &\leq c 2^{(j-v)(M-2L+1+\alpha^-+n)} \phi_v^{*,a} 2^{v\alpha(\cdot)} f(y) \|\eta_{v,L-a}\|_1 \\ &\leq c 2^{(j-v)(M-2L+1+\alpha^-+n)} \phi_v^{*,a} 2^{v\alpha(\cdot)} f(y), \end{aligned}$$

where in the first inequality we have used Lemma 1 (by taking $L > \max(d, n+a)$). Taking $M > 2L - \alpha^- + a - n$ to estimate the last expression by

$$c 2^{(j-v)(a+1)} \phi_v^{*,a} 2^{v\alpha(\cdot)} f(y),$$

where $c > 0$ is independent of j, v and f . Further, note that for all $x, y \in \mathbb{R}^n$ and all $j, v \in \mathbb{N}$

$$\phi_v^{*,a} 2^{v\alpha(\cdot)} f(y) \leq \phi_v^{*,a} 2^{v\alpha(\cdot)} f(x) (1+2^v|x-y|)^a \leq \phi_v^{*,a} 2^{v\alpha(\cdot)} f(x) \max(1, 2^{(v-j)a}) (1+2^j|x-y|)^a.$$

Hence

$$\sup_{y \in \mathbb{R}^n} \frac{2^{j\alpha(y)} |k_j * \lambda_v * \phi_v * f(y)|}{(1+2^j|x-y|)^a} \leq C \phi_v^{*,a} 2^{v\alpha(\cdot)} f(x) \times \begin{cases} 2^{(v-j)(S+1-\alpha^+)} & \text{if } v \leq j, \\ 2^{j-v} & \text{if } v \geq j. \end{cases}$$

Using the fact that for any $z \in \mathbb{R}^n$, any $N > 0$ and any integer $S \geq -1$

$$|k_j * \Lambda(z)| = |k_{2^{-j}} * \Lambda(z)| \leq c \frac{2^{-j(S+1)}}{(1+|z|)^{2N}},$$

we obtain by the similar arguments that for any $j \in \mathbb{N}$

$$\sup_{y \in \mathbb{R}^n} \frac{2^{j\alpha(y)} |k_j * \Lambda * \phi_0 * f(y)|}{(1 + 2^j |x - y|)^a} \leq C 2^{-j(S+1-\alpha^+)} \phi_0^{*,a} f(x).$$

Hence with $\delta = \min(1, S + 1 - \alpha^+) > 0$ for all $f \in \mathcal{S}'(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $j \in \mathbb{N}$

$$k_j^{*,a} 2^{j\alpha(\cdot)} f(x) \leq C 2^{-j\delta} \phi_0^{*,a} f(x) + C \sum_{v=1}^{\infty} 2^{-|j-v|\delta} \phi_v^{*,a} 2^{v\alpha(\cdot)} f(x) = C \sum_{v=0}^{\infty} 2^{-|j-v|\delta} \phi_v^{*,a} 2^{v\alpha(\cdot)} f(x).$$

Also for $j = 0$, we use the fact that for $v \geq 1$, any $z \in \mathbb{R}^n$, any $N > 0$ and any integer $M \geq -1$

$$|k_0 * \lambda_v(z)| = |k_0 * \lambda_{2^{-v}}(z)| \leq c \frac{2^{-v(M+1)}}{(1 + |z|)^{2N}}$$

and

$$|k_0 * \Lambda(z)| \leq c \frac{1}{(1 + |z|)^{2N}}$$

to get for any $x \in \mathbb{R}^n$

$$k_0^{*,a} f(x) \leq C \phi_0^{*,a} f(x) + C \sum_{v=1}^{\infty} 2^{-v\delta} \phi_v^{*,a} 2^{v\alpha(\cdot)} f(x) = C \sum_{v=0}^{\infty} 2^{-v\delta} \phi_v^{*,a} 2^{v\alpha(\cdot)} f(x).$$

Let $\tau > \max(q^+, \frac{q^+}{p^-})$. Then by Lemma 3

$$\begin{aligned} \sum_{j=0}^{\infty} \| |ck_j^{*,a} 2^{j\alpha(\cdot)} f|^{q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}} &= \sum_{j=0}^{\infty} \| |ck_j^{*,a} 2^{j\alpha(\cdot)} f|^{q(\cdot)/\tau} \|_{\frac{\tau p(\cdot)}{q(\cdot)}}^{\tau} \\ &\leq \sum_{j=0}^{\infty} \left(\sum_{v=0}^{\infty} 2^{-|j-v|\delta} \| |\phi_v^{*,a} 2^{v\alpha(\cdot)} f|^{q(\cdot)/\tau} \|_{\frac{\tau p(\cdot)}{q(\cdot)}} \right)^{\tau} \\ &\leq C \sum_{j=0}^{\infty} \| |\phi_j^{*,a} 2^{j\alpha(\cdot)} f|^{q(\cdot)/\tau} \|_{\frac{\tau p(\cdot)}{q(\cdot)}}^{\tau} = C \sum_{j=0}^{\infty} \| |\phi_j^{*,a} 2^{j\alpha(\cdot)} f|^{q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}} \leq C, \end{aligned}$$

with an appropriate choice of $c > 0$.

Step 2. We will prove in this step that there is a constant $c > 0$ such that for any $f \in \mathcal{S}'(\mathbb{R}^n)$

$$\|f \mid B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}\|' \leq c \|f \mid B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}\|''. \quad (18)$$

Analogously to (15), (16) find two functions $\Lambda, \psi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\text{supp } \mathcal{F}\Lambda \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\varepsilon\}, \quad \text{supp } \mathcal{F}\psi \subset \{\xi \in \mathbb{R}^n : \varepsilon/2 < |\xi| < 2\varepsilon\}$$

and for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $j \in \mathbb{N}_0$

$$f = \Lambda_j * (k_0)_j * f + \sum_{m=j+1}^{\infty} \psi_m * k_m * f.$$

Hence

$$k_j * f = \Lambda_j * (k_0)_j * k_j * f + \sum_{m=j+1}^{\infty} k_j * \psi_m * k_m * f.$$

By a scaling argument, we see that it suffices to consider the case

$$\|f \mid B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}\|'' = 1$$

and show that the modular of a constant times the function on the left-hand side is bounded. In particular, we will show that

$$\sum_{j=0}^{\infty} \| |ck_j^{*,a} 2^{j\alpha(\cdot)} f|^{q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}} \leq C \quad \text{whenever} \quad \sum_{j=0}^{\infty} \| |2^{j\alpha(\cdot)} k_j * f|^{q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}} = 1.$$

Writing for any $z \in \mathbb{R}^n$

$$k_j * \psi_m(z) = 2^{jn} (k * \psi_{2^{j-m}})(2^j z),$$

we get by Lemma 4, that for any integer $K \geq -1$ and any $M > 0$ there is a constant $c > 0$ independent of j and m

$$|k_j * \psi_m(z)| \leq c \frac{2^{(j-m)(K+1)+jn}}{(1+2^j|z|)^{2M}}, \quad z \in \mathbb{R}^n.$$

Analogous estimate

$$|\Lambda_j * (k_0)_j(z)| \leq c \frac{2^{jn}}{(1+2^j|z|)^{2M}}, \quad z \in \mathbb{R}^n,$$

is obvious. From this it follows that

$$\begin{aligned} 2^{j\alpha(y)} |k_j * f(y)| &\leq c \sum_{m=j}^{\infty} 2^{(j-m)(K+1+\alpha^-)} 2^{m\alpha(y)} \eta_{j,2M} * |k_m * f|(y) \\ &= c \sum_{m=j}^{\infty} 2^{(j-m)(K+1+\alpha^-)+jn} \int_{\mathbb{R}^n} \frac{2^{m\alpha(y)} |k_m * f(z)|}{(1+2^j|y-z|)^{2M}} dz. \end{aligned}$$

Since

$$(1+2^j|y-z|)^{-2M} \leq 2^{2(m-j)M} (1+2^m|y-z|)^{-2M},$$

then by Lemma 1 we have

$$\begin{aligned} 2^{j\alpha(y)} |k_j * f(y)| &\leq c \sum_{m=j}^{\infty} 2^{(j-m)(K+1+\alpha^- - 2M+n)} 2^{m\alpha(y)} \eta_{m,2M} * |k_m * f|(y) \\ &\leq c \sum_{m=j}^{\infty} 2^{(j-m)(K+1+\alpha^- - 2M+n)} \eta_{m,a} * (2^{m\alpha(\cdot)} |k_m * f|)(y), \end{aligned} \quad (19)$$

by taking $M > \max(d, a)$. Using the elementary estimates

$$\begin{aligned} (1+2^j|x-y|)^{-a} &\leq (1+2^j|x-z|)^{-a} (1+2^j|y-z|)^a \\ &\leq 2^{(m-j)a} (1+2^m|x-z|)^{-a} (1+2^m|y-z|)^a, \end{aligned} \quad (20)$$

to get

$$k_j^{*,a} 2^{j\alpha(\cdot)} f(x) \leq c \sum_{m=j}^{\infty} 2^{(j-m)(K+1-a-\alpha^- - 2M+n)+mn} \int_{\mathbb{R}^n} \frac{2^{m\alpha(z)} |k_m * f(z)|}{(1+2^m|x-z|)^a} dz.$$

Fix any $r \in (0, 1]$. We have

$$\begin{aligned} 2^{m\alpha(z)} |k_m * f(z)| &= (2^{m\alpha(z)} |k_m * f(z)|)^r (2^{m\alpha(z)} |k_m * f(z)|)^{1-r} \\ &= (2^{m\alpha(z)} |k_m * f(z)|)^r \left(\frac{2^{m\alpha(z)} |k_m * f(z)|}{(1+2^m|x-z|)^a} \right)^{1-r} (1+2^m|x-z|)^{a(1-r)} \\ &\leq (2^{m\alpha(z)} |k_m * f(z)|)^r (k_m^{*,a} 2^{m\alpha(\cdot)} f(x))^{1-r} (1+2^m|x-z|)^{a(1-r)}. \end{aligned}$$

Then

$$k_j^{*,a} 2^{j\alpha(\cdot)} f(x) \leq c \sum_{m=j}^{\infty} 2^{(j-m)N'+mn} \int_{\mathbb{R}^n} \frac{2^{mr\alpha(z)} |k_m * f(z)|^r}{(1+2^m|x-z|)^{ar}} dz (k_m^{*,a} 2^{m\alpha(\cdot)} f(x))^{1-r},$$

where $N' = K + 1 - a + n - \alpha^- - 2M$ can be still be taken arbitrarily large. Quite analogously one proves for all $f \in \mathcal{S}'(\mathbb{R}^n)$ the estimate

$$k_0^{*,a} f(x) \leq c \sum_{m=0}^{\infty} 2^{-mN'+mn} \int_{\mathbb{R}^n} \frac{2^{mr\alpha(z)} |k_m * f(z)|^r}{(1+2^m|x-z|)^{ar}} dz (k_m^{*,a} 2^{m\alpha(\cdot)} f(x))^{1-r}.$$

We now fix any $x \in \mathbb{R}^n$ and apply Lemma 5 with

$$d_j = k_j^{*,a} 2^{j\alpha(\cdot)} f(x), \quad j \in \mathbb{N}_0,$$

$$b_m = \int_{\mathbb{R}^n} \frac{2^{mr\alpha(z)+mn} |k_m * f(z)|^r}{(1 + 2^m |x - z|)^{ar}} dz, \quad m \in \mathbb{N}_0.$$

The assumption (3) is satisfied with $N_0 = N_1 + n + [\max(0, \alpha^+)] + 1$, where N_1 is the order of the distribution $f \in S'(\mathbb{R}^n)$ ([a] the integer part of the real number a). We conclude that for any $f \in S'(\mathbb{R}^n)$, any $N > 0$ and any $j \in \mathbb{N}_0$

$$(k_j^{*,a} 2^{j\alpha(\cdot)} f(x))^r \leq c \sum_{m=j}^{\infty} 2^{(j-m)Nr+mn} \int_{\mathbb{R}^n} \frac{2^{mr\alpha(z)} |k_m * f(z)|^r}{(1 + 2^m |x - z|)^{ar}} dz.$$

This estimate is also true for $r > 1$, with much simpler proof. It suffices to take (19) with $a + n$ instead of a , apply Hölder's inequalities in m and z , and finally the inequality (20). We omit the details.

Since $a > \frac{n}{p^-}$, it is possible to take $\frac{n}{a} < r < p^-$. Let $\tau > \frac{q^+}{r}$. We see that

$$\begin{aligned} \| |ck_j^{*,a} 2^{j\alpha(\cdot)} f|^{q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}} &= \| |ck_j^{*,a} 2^{j\alpha(\cdot)} f|^{rq(\cdot)/r} \|_{\frac{p(\cdot)}{q(\cdot)}} \\ &= \| |ck_j^{*,a} 2^{j\alpha(\cdot)} f|^{rq(\cdot)/r\tau} \|_{\frac{\tau p(\cdot)}{q(\cdot)}}^{\tau} \\ &\leq c \left(\sum_{m=j}^{\infty} 2^{(j-m)Nq^-/\tau} \| |c\eta_{m,ar} * (2^{m\alpha(\cdot)} |k_m * f|)^r |^{q(\cdot)/r\tau} \|_{\frac{\tau p(\cdot)}{q(\cdot)}} \right)^{\tau}. \end{aligned}$$

By the same method given in the proof of Theorem 1 (with $m, q(\cdot)/\tau, r$ in place of $v, q(\cdot), t$ respectively) we can prove that

$$\| |c\eta_{m,ar} * (2^{m\alpha(\cdot)} |k_m * f|)^r |^{q(\cdot)/r\tau} \|_{\frac{\tau p(\cdot)}{q(\cdot)}} \leq \| |2^{m\alpha(\cdot)} k_m * f|^{q(\cdot)/\tau} \|_{\frac{p(\cdot)}{q(\cdot)}} + 2^{-m\sigma} = \| |2^{m\alpha(\cdot)} k_m * f|^{q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}}^{1/\tau} + 2^{-m\sigma},$$

with an appropriate choice of $c > 0$ and here $0 < \sigma < \frac{a-n/r}{4r(\frac{1}{q^-} - \frac{1}{q^+})}$. Then for any $f \in S'(\mathbb{R}^n)$ and any $j \in \mathbb{N}_0$

$$\| |ck_j^{*,a} 2^{j\alpha(\cdot)} f|^{q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}} \leq \left(\sum_{m=j}^{\infty} 2^{(j-m)Nq^-/\tau} (\| |2^{m\alpha(\cdot)} k_m * f|^{q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}}^{1/\tau} + 2^{-m\sigma}) \right)^{\tau}.$$

By Lemma 3 we get

$$\begin{aligned} \sum_{j=0}^{\infty} \| |ck_j^{*,a} 2^{j\alpha(\cdot)} f|^{q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}} &\leq c \sum_{j=0}^{\infty} (\| |2^{j\alpha(\cdot)} k_j * f|^{q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}}^{1/\tau} + 2^{-j\sigma})^{\tau} \\ &\leq c \sum_{j=0}^{\infty} \| |2^{j\alpha(\cdot)} k_j * f|^{q(\cdot)} \|_{\frac{p(\cdot)}{q(\cdot)}} + c \sum_{j=0}^{\infty} 2^{-j\sigma\tau} \\ &\leq C. \end{aligned}$$

Step 3. We will prove in this step that for all $f \in S'(\mathbb{R}^n)$ the following estimates are true:

$$\| f | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \|' \leq c \| f | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \| \leq c \| f | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \|''.$$

Let $\{\varphi_j\}_{j \in \mathbb{N}_0}$ be as in Definition 1 and let $\phi_j = \varphi_j$. The first inequality is proved by the chain of the estimates

$$\| f | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \|' \leq c \| (\varphi_j^{*,a} 2^{j\alpha(\cdot)} f)_j \|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq c \| (2^{j\alpha(\cdot)} \phi_j * f)_j \|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq c \| f | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \|,$$

where the first inequality is (14), see Step 1, the second inequality is (18) (with ϕ and ϕ_0 instead of k and k_0), see Step 2, and finally the third inequality is obvious. Now the second inequality can be obtained by the following chain

$$\| f | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \| \leq c \| (\varphi_j^{*,a} 2^{j\alpha(\cdot)} f)_j \|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq c \| f | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \|' \leq c \| f | B_{p(\cdot),q(\cdot)}^{\alpha(\cdot)} \|'',$$

where the first inequality is obvious, the second inequality is (14), see Step 1, with the roles of k_0 and k respectively φ_0 and φ interchanged, and finally the last inequality is (18), see Step 2. Hence the theorem is proved. \square

6. Decomposition by atoms

Let \mathbb{Z}^n be the lattice of all points in \mathbb{R}^n with integer-valued components. If $v \in \mathbb{N}_0$ and $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$ we denote Q_{vm} the dyadic cube in \mathbb{R}^n centred at $2^{-v}m$ which has sides parallel to the axes and side length 2^{-v} . If Q_{vm} is such a cube in \mathbb{R}^n and $c > 0$ then cQ_{vm} is the cube in \mathbb{R}^n concentric with Q_{vm} and with side length $c \cdot 2^{-v}$. By χ_{vm} we denote the characteristic function of the cube Q_{vm} . The main goal of this section is to prove an atomic decomposition result for $B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$. First we introduce the basic notation.

Definition 3. Let $p, q \in \mathcal{P}_0(\mathbb{R}^n)$ and let $\alpha \in C_{\text{loc}}^{\log}$. Then for all complex-valued sequences $\lambda = \{\lambda_{vm} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ we define

$$b_{p(\cdot), q(\cdot)}^{\alpha(\cdot)} = \{\lambda : \|\lambda\|_{b_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} < \infty\}$$

where

$$\|\lambda\|_{b_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}} = \left\| \left(\sum_{m \in \mathbb{Z}^n} 2^{v\alpha(\cdot)} \lambda_{vm} \chi_{vm} \right)_v \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

We define atoms which are the building blocks for atomic decompositions.

Definition 4. Let $K, L \in \mathbb{N}_0$ and let $\gamma > 1$. A K -times continuous differentiable function $a \in C^K(\mathbb{R}^n)$ is called $[K, L]$ -atom centered at Q_{vm} , $v \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, if

$$\text{supp } a \subseteq \gamma Q_{vm}, \quad (21)$$

$$|D^\beta a(x)| \leq 2^{v|\beta|}, \quad \text{for } 0 \leq |\beta| \leq K, x \in \mathbb{R}^n \quad (22)$$

and if

$$\int_{\mathbb{R}^n} x^\beta a(x) dx = 0, \quad \text{for } 0 \leq |\beta| < L \text{ and } v \geq 1. \quad (23)$$

If the atom a located at Q_{vm} , that means if it fulfills (21), then we will denote it by a_{vm} . For $v = 0$ or $L = 0$ there are no moment conditions (23) required.

For proving the decomposition by atoms we need two basic lemmas. The next lemma go back to M. Frazier and B. Jawerth [10, Lemma 3.3].

Lemma 6. Let $\{\mathcal{F}\varphi_j\}$, $j \in \mathbb{N}_0$ be a resolution of unity and let ρ_{vm} be an $[K, L]$ -atom. Then

$$|\varphi_j * \rho_{vm}(x)| \leq c 2^{(v-j)K} (1 + 2^v |x - 2^{-v}m|)^{-M}$$

if $v \leq j$, and

$$|\varphi_j * \rho_{vm}(x)| \leq c 2^{(j-v)(L+n+1)} (1 + 2^j |x - 2^{-v}m|)^{-M}$$

if $v \geq j$, where M is sufficiently large.

We also need a partition of unity of Calderón type; a proof can be found in [10, Theorem 2.6].

Lemma 7. Let $\{\mathcal{F}\varphi_j\}$, $j \in \mathbb{N}_0$ be a resolution of unity and let $R \in \mathbb{N}$. Then there exist functions $\theta_0, \theta \in \mathcal{S}(\mathbb{R}^n)$ with:

$$\text{supp } \theta_0, \text{supp } \theta \subset \{x \in \mathbb{R}^n : |x| \leq 1\}, \quad (24)$$

$$|\mathcal{F}\theta_0(\xi)| > c_0 \quad \text{for } |\xi| \leq 2, \quad |\mathcal{F}\theta(\xi)| > c \quad \text{for } \frac{1}{2} \leq |\xi| \leq 2,$$

$$\int_{\mathbb{R}^n} x^\beta \theta(x) dx = 0, \quad \text{for } 0 \leq |\beta| \leq R, \quad (25)$$

such that

$$\mathcal{F}\theta_0(\xi)\mathcal{F}\psi_0(\xi) + \sum_{j \geq 1} \mathcal{F}\theta(2^{-j}\xi)\mathcal{F}\psi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^n,$$

where the functions $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ are defined via

$$\mathcal{F}\psi_0(\xi) = \frac{\mathcal{F}\varphi_0(\xi)}{\mathcal{F}\theta_0(\xi)} \quad \text{and} \quad \mathcal{F}\psi(\xi) = \frac{\mathcal{F}\varphi_1(2\xi)}{\mathcal{F}\theta(\xi)}.$$

Now we come to the atomic decomposition theorem.

Theorem 3. Let $\alpha \in C_{\text{loc}}^{\log}$ and $p, q \in \mathcal{P}_0^{\log}$ with $q^+ < \infty$. Further, let $K, L \in \mathbb{N}_0$ such that

$$K > \alpha^+, \quad L > n \left(\frac{1}{\min(1, p^-)} - 1 \right) - 1 - \alpha^-. \quad (26)$$

Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$, if and only if, it can be represented as

$$f = \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{vm} \rho_{vm}, \quad (27)$$

convergence being in $\mathcal{S}'(\mathbb{R}^n)$, where ρ_{vm} are $[K, L]$ -atoms and $\lambda = \{\lambda_{vm} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$. Furthermore, $\inf \|\lambda\| b_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$, where the infimum is taken over admissible representations (27), is an equivalent quasi-norm in $B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$.

Before giving the proof let us make here some remarks. H. Kempka [13] proved this result (note that only the case of constant q was included). The convergence in $\mathcal{S}'(\mathbb{R}^n)$ can be obtained as a by-product of the proof using the same method as in [13, Lemma 6] (combined with Corollary 13.9 in [20]), so we will not stress this point.

If p, q , and α are constants, then the restrictions (26), and their counterparts, in the atomic decomposition theorem are $K > \alpha$ and $L > n \left(\frac{1}{\min(1, p)} - 1 \right) - 1 - \alpha$, which are essentially the restrictions from the works of M. Frazier and B. Jawerth [10, Theorem 6], see also the formulation in [20, Theorem 13.8].

Proof. The proof follows the ideas in [10, Theorem 6].

Step 1. Assume that $f \in B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$ and let θ_0, θ, ψ_0 and ψ be the functions introduced in Lemma 7. We have

$$f = \theta_0 * \psi_0 * f + \sum_{v=1}^{\infty} \theta_v * \psi_v * f$$

and using the definition of the cubes Q_{vm} we obtain

$$f(x) = \sum_{m \in \mathbb{Z}^n} \int_{Q_{0m}} \theta_0(x-y) \psi_0 * f(y) dy + \sum_{v=1}^{\infty} 2^{vn} \sum_{m \in \mathbb{Z}^n} \int_{Q_{vm}} \theta(2^v(x-y)) \psi_v * f(y) dy,$$

with convergence in $\mathcal{S}'(\mathbb{R}^n)$. We define for every $v \in \mathbb{N}$ and all $m \in \mathbb{Z}^n$

$$\lambda_{vm} = C_{\theta} \sup_{y \in Q_{vm}} |\psi_v * f(y)| \quad (28)$$

where

$$C_{\theta} = \max \left\{ \sup_{|y| \leq 1} |D^{\alpha} \theta(y)| : |\alpha| \leq K \right\}.$$

Define also

$$\rho_{vm}(x) = \frac{1}{\lambda_{vm}} 2^{vn} \int_{Q_{vm}} \theta(2^v(x-y)) \psi_v * f(y) dy. \quad (29)$$

Similarly we define for every $m \in \mathbb{Z}^n$ the numbers λ_{0m} and the functions ρ_{0m} taking in (28) and (29) $v = 0$ and replacing ψ_v and θ by ψ_0 and θ_0 , respectively. Let us now check that such ρ_{vm} are atoms in the sense of Definition 4. Note that the support and moment conditions are clear by (24) and (25), respectively. It thus remains to check (22) in Definition 4. We have

$$\begin{aligned}
|D^\beta \rho_{vm}(x)| &\leq \frac{2^{v(n+|\beta|)}}{C_\theta} \int_{Q_{vm}} |(D^\beta \theta)(2^v(x-y))| |\psi_v * f(y)| dy \left(\sup_{y \in Q_{vm}} |\psi_v * f(y)| \right)^{-1} \\
&\leq \frac{2^{v(n+|\beta|)}}{C_\theta} \int_{Q_{vm}} |(D^\beta \theta)(2^v(x-y))| dy \leq 2^{v(n+|\beta|)} |Q_{vm}| \leq 2^{v|\beta|}.
\end{aligned}$$

The modifications for the terms with $v = 0$ are obvious.

Step 2. Next we show that there is a constant $c > 0$ such that

$$\|\lambda \mid b_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}\| \leq c \|f \mid B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}\|.$$

For that reason we exploit the equivalent quasi-norms given in Theorem 2 involving Peetre's maximal function. Let $v \in \mathbb{N}$. Taking into account that $|x - y| \leq c \cdot 2^{-v}$ for $x, y \in Q_{vm}$ we obtain

$$2^{v(\alpha(x) - \alpha(y))} \leq 2^{\frac{c \log(\alpha)v}{\log(e+1/|x-y|)}} \leq 2^{\frac{c \log(\alpha)v}{\log(e+2^{-v}/c)}} \leq c$$

if $v \geq [\log_2 c] + 2$. If $0 < v < [\log_2 c] + 2$, then $2^{v(\alpha(x) - \alpha(y))} \leq 2^{v(\alpha^+ - \alpha^-)} \leq c$. Therefore,

$$2^{v\alpha(x)} |\psi_v * f(y)| \leq c 2^{v\alpha(y)} |\psi_v * f(y)|$$

for any $x, y \in Q_{vm}$ and any $v \in \mathbb{N}$. Hence,

$$\begin{aligned}
\sum_{m \in \mathbb{Z}^n} \lambda_{vm} 2^{v\alpha(x)} \chi_{vm}(x) &= C_\theta \sum_{m \in \mathbb{Z}^n} 2^{v\alpha(x)} \sup_{y \in Q_{vm}} |\psi_v * f(y)| \chi_{vm}(x) \\
&\leq c \sum_{m \in \mathbb{Z}^n} \sup_{|z| \leq c 2^{-v}} \frac{2^{v\alpha(x-z)} |\psi_v * f(x-z)|}{(1 + 2^v |z|)^a} (1 + 2^v |z|)^a \chi_{vm}(x) \\
&\leq c \psi_v^{*,a} 2^{v\alpha(\cdot)} f(x) \sum_{m \in \mathbb{Z}^n} \chi_{vm}(x) = c \psi_v^{*,a} 2^{v\alpha(\cdot)} f(x),
\end{aligned}$$

where we have used $\sum_{m \in \mathbb{Z}^n} \chi_{vm}(x) = 1$. This estimate and its counterpart for $v = 0$ (which can be obtained by a similar calculation) give

$$\|\lambda \mid b_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}\| \leq c \|(\psi_v^{*,a} 2^{v\alpha(\cdot)} f)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq c \|f \mid B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}\|,$$

by Theorem 2 (since $\psi_0 \in \mathcal{S}(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ are two kernels which fulfill Tauberian conditions (8) and (9) and the moment conditions (10)).

Step 3. Assume that f can be represented by (27), with K and L satisfying (26). We will show that $f \in B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}$ and that for some $c > 0$

$$\|f \mid B_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}\| \leq c \|\lambda \mid b_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}\|.$$

By a scaling argument, we see that it suffices to consider the case $\|\lambda \mid b_{p(\cdot), q(\cdot)}^{\alpha(\cdot)}\| = 1$ and show that the modular of a constant times the function on the left-hand side is bounded. In particular, we will show that

$$\sum_{j=0}^{\infty} \|c 2^{j\alpha(\cdot)} \varphi_j * f\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leq C \quad \text{whenever} \quad \sum_{j=0}^{\infty} \left\| \sum_{m \in \mathbb{Z}^n} 2^{j\alpha(\cdot)} \lambda_{jm} \chi_{jm} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^{q(\cdot)} = 1, \quad (30)$$

where $\{\mathcal{F}\varphi_j\}_{j \in \mathbb{N}_0}$ is the resolution of unity. We write

$$f = \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{vm} \rho_{vm} = \sum_{v=0}^j + \cdots + \sum_{v=j+1}^{\infty} \cdots.$$

Let $0 < r < \max(1/q^+, p^-/q^+)$. We have

$$\begin{aligned}
\sum_{j=0}^{\infty} \|c 2^{j\alpha(\cdot)} \varphi_j * f\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^{p(\cdot)/q(\cdot)} &\leq \sum_{j=0}^{\infty} \left(\sum_{v=0}^{\infty} \left\| c \sum_{m \in \mathbb{Z}^n} 2^{j\alpha(\cdot)} \lambda_{vm} \varphi_j * \rho_{vm} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^{rq(\cdot)} \right)^{1/r} \\
&\leq c \sum_{j=0}^{\infty} \left(\sum_{v=0}^j \left\| c \sum_{m \in \mathbb{Z}^n} 2^{j\alpha(\cdot)} \lambda_{vm} \varphi_j * \rho_{vm} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}^{rq(\cdot)} \right)^{1/r}
\end{aligned}$$

$$\begin{aligned}
& + c \sum_{j=0}^{\infty} \left(\sum_{v=j+1}^{\infty} \left\| c \sum_{m \in \mathbb{Z}^n} 2^{j\alpha(\cdot)} \lambda_{vm} \varphi_j * \rho_{vm} \right\|_{\frac{p(\cdot)}{rq(\cdot)}}^{rq(\cdot)} \right)^{1/r} \\
& = I + II.
\end{aligned}$$

For each $k \in \mathbb{N}$ we define $\Omega_k = \{m \in \mathbb{Z}^n : 2^{k-1} < 2^{\min(v,j)} |x - 2^{-v}m| \leq 2^k\}$ and $\Omega_0 = \{m \in \mathbb{Z}^n : 2^{\min(v,j)} |x - 2^{-v}m| \leq 1\}$.

Estimate of I. From Lemma 6, we have for any M sufficiently large

$$\sum_{m \in \mathbb{Z}^n} 2^{j\alpha(x)} |\lambda_{vm}| |\varphi_j * \rho_{vm}(x)| \leq c 2^{(v-j)(K-\alpha^+)} \sum_{m \in \mathbb{Z}^n} 2^{v\alpha(x)} |\lambda_{vm}| (1 + 2^v |x - 2^{-v}m|)^{-M}.$$

We claim that there exists $c > 0$ such that

$$\left\| c \sum_{m \in \mathbb{Z}^n} 2^{v\alpha(\cdot)} \lambda_{vm} (1 + 2^v |\cdot - 2^{-v}m|)^{-M} \right\|_{\frac{p(\cdot)}{rq(\cdot)}}^{rq(\cdot)} \leq \left\| \sum_{m \in \mathbb{Z}^n} 2^{v\alpha(\cdot)} \lambda_{vm} \chi_{vm} \right\|_{\frac{p(\cdot)}{rq(\cdot)}}^{rq(\cdot)} + 2^{-v} = \delta. \quad (31)$$

Therefore, by Lemma 3 (with the help of (26)) we obtain

$$\begin{aligned}
I & \leq c \sum_{j=0}^{\infty} \left(\left\| \sum_{m \in \mathbb{Z}^n} 2^{j\alpha(\cdot)} \lambda_{jm} \chi_{jm} \right\|_{\frac{p(\cdot)}{rq(\cdot)}}^{rq(\cdot)} + 2^{-j} \right)^{1/r} \\
& \leq c \sum_{j=0}^{\infty} \left\| \sum_{m \in \mathbb{Z}^n} 2^{j\alpha(\cdot)} \lambda_{jm} \chi_{jm} \right\|_{\frac{p(\cdot)}{rq(\cdot)}}^{1/r} + c \sum_{j \geq 0} 2^{-j/r} \\
& = c \sum_{j=0}^{\infty} \left\| \sum_{m \in \mathbb{Z}^n} 2^{j\alpha(\cdot)} \lambda_{jm} \chi_{jm} \right\|_{\frac{p(\cdot)}{q(\cdot)}}^{q(\cdot)} + c \leq C.
\end{aligned}$$

Let us prove (31). This claim can be reformulated as showing that

$$\left\| \delta^{-1} \left\| c \sum_{m \in \mathbb{Z}^n} 2^{v\alpha(\cdot)} \lambda_{vm} (1 + 2^v |\cdot - 2^{-v}m|)^{-M} \right\|_{\frac{p(\cdot)}{rq(\cdot)}}^{rq(\cdot)} \right\|_{\frac{p(\cdot)}{rq(\cdot)}} \leq 1,$$

which is equivalent to

$$\left\| c \delta^{-\frac{1}{rq(\cdot)}} \sum_{m \in \mathbb{Z}^n} 2^{v\alpha(\cdot)} \lambda_{vm} (1 + 2^v |\cdot - 2^{-v}m|)^{-M} \right\|_{p(\cdot)} \leq 1.$$

We have, with $M = R + T$,

$$\begin{aligned}
& \sum_{m \in \mathbb{Z}^n} \delta^{-\frac{1}{rq(x)}} 2^{v\alpha(x)} |\lambda_{vm}| (1 + 2^v |x - 2^{-v}m|)^{-M} \\
& = \sum_{k=0}^{\infty} \sum_{m \in \Omega_k} \delta^{-\frac{1}{rq(x)}} 2^{v\alpha(x)} |\lambda_{vm}| (1 + 2^v |x - 2^{-v}m|)^{-M} \\
& \leq c \sum_{k=0}^{\infty} \sum_{m \in \Omega_k} \delta^{-\frac{1}{rq(x)}} 2^{v\alpha(x)} 2^{-Mk} |\lambda_{vm}| \\
& = \sum_{k=0}^{\infty} 2^{-(T-n/t)k} \sum_{m \in \Omega_k} \delta^{-\frac{1}{rq(x)}} 2^{v\alpha(x)} 2^{-(R+n/t)k} |\lambda_{vm}| \\
& \leq \sup_{k \in \mathbb{N}_0} \sum_{m \in \Omega_k} \delta^{-\frac{1}{rq(x)}} 2^{v\alpha(x)} |\lambda_{vm}| 2^{-(R+n/t)k},
\end{aligned}$$

for any T sufficiently large such that $T > n/t$. For any $0 < t \leq 1$, the last expression is bounded by

$$\begin{aligned}
& \sup_{k \in \mathbb{N}_0} \left(\sum_{m \in \Omega_k} \delta^{-\frac{t}{rq(x)}} 2^{v\alpha(x)t} |\lambda_{vm}|^t 2^{-(Rt+n/t)k} \right)^{1/t} \\
& = \left(\sup_{k \in \mathbb{N}_0} 2^{-Rtk + (v-k)n} \int_{\bigcup_{m \in \Omega_k} Q_{vm}} \left(\sum_{m \in \Omega_k} \delta^{-\frac{1}{rq(x)}} 2^{v\alpha(x)} |\lambda_{vm}| \chi_{vm}(y) \right)^t dy \right)^{1/t}. \quad (32)
\end{aligned}$$

Let $y \in \bigcup_{m \in \Omega_k} Q_{vm}$ then $y \in Q_{vm}$ for some $m \in \Omega_k$ and $2^{k-1} < 2^v |x - 2^{-v}m| \leq 2^k$. From this it follows that

$$|y - x| \leq |y - 2^{-v}m| + |x - 2^{-v}m| \leq \sqrt{n}2^{-v} + 2^{k-v} \leq 2^{k-v+h_n}, \quad h_n \in \mathbb{N},$$

which implies that y is located in some ball $B(x, 2^{k-v+h_n})$. Therefore, (32) does not exceed

$$c \left(\sup_{k \in \mathbb{N}_0} \frac{2^{-Rtk}}{|B(x, 2^{k-v+h_n})|} \int_{B(x, 2^{k-v+h_n})} \left(\sum_{m \in \Omega_k} \delta^{-\frac{1}{rq(x)}} 2^{v\alpha(x)} |\lambda_{vm}| \chi_{vm}(y) \right)^t dy \right)^{1/t}. \quad (33)$$

Since $1/q$ is log-Hölder continuous and $\delta \in [2^{-v}, 1 + 2^{-v}]$, we have

$$\delta^{\frac{1}{q(x)} - \frac{1}{q(y)}} = (2^v \delta)^{\frac{1}{q(x)} - \frac{1}{q(y)}} 2^{(\frac{1}{q(x)} - \frac{1}{q(y)})v} \leq 2^{|\frac{1}{q(x)} - \frac{1}{q(y)}|(2v+1)} \leq 2^{\frac{c_{\log}(q)(2v+1)}{\log(e + \frac{1}{|x-y|})}} \leq 2^{\frac{c_{\log}(q)(2v+1)}{v-k-h_n}} \leq c 2^{2c_{\log}(q)k},$$

for any $k < \max(0, v - h_n)$ and any $y \in B(x, 2^{k-v+h_n})$. If $k \geq \max(0, v - h_n)$ then since again $\delta \in [2^{-v}, 1 + 2^{-v}]$,

$$\delta^{\frac{1}{q(x)} - \frac{1}{q(y)}} \leq c 2^{|\frac{1}{q(x)} - \frac{1}{q(y)}|(2v+1)} \leq c 2^{2(\frac{1}{q^-} - \frac{1}{q^+})k}.$$

Also since α is log-Hölder continuous we can prove that

$$2^{v(\alpha(x) - \alpha(y))} \leq c \times \begin{cases} 2^{c_{\log}(\alpha)k} & \text{if } k < \max(0, v - h_n), \\ 2^{(\alpha^+ - \alpha^-)k} & \text{if } k \geq \max(0, v - h_n), \end{cases}$$

where $c > 0$ not depending on v and k . Hence with R sufficiently large such that

$$R > \max\left(\frac{2}{r} c_{\log}(q) + c_{\log}(\alpha), \frac{2}{r} \left(\frac{1}{q^-} - \frac{1}{q^+}\right) + \alpha^+ - \alpha^-\right),$$

we get that (33) is bounded by

$$c \left(\mathcal{M}_t \left(\sum_{m \in \Omega_k} \delta^{-\frac{1}{rq(\cdot)}} 2^{v\alpha(\cdot)} |\lambda_{vm}| \chi_{vm} \right)(x) \right)^{1/t}, \quad x \in \mathbb{R}^n.$$

Now taking $0 < t < \min(1, p^-)$ and using the fact that $\mathcal{M} : L^{\frac{p(\cdot)}{t}} \rightarrow L^{\frac{p(\cdot)}{t}}$ is bounded we obtain

$$\begin{aligned} \left\| c \sum_{m \in \mathbb{Z}^n} \delta^{-\frac{1}{rq(\cdot)}} 2^{v\alpha(\cdot)} |\lambda_{vm}| (1 + 2^v |\cdot - 2^{-v}m|)^{-L} \right\|_{p(\cdot)} &\leq c \left\| \mathcal{M}_t \left(\sum_{m \in \Omega_k} \delta^{-\frac{1}{rq(\cdot)}} 2^{v\alpha(\cdot)} |\lambda_{vm}| \chi_{vm} \right) \right\|_{p(\cdot)/t}^{1/t} \\ &\leq \left\| \sum_{m \in \mathbb{Z}^n} \delta^{-\frac{1}{rq(\cdot)}} 2^{v\alpha(\cdot)} |\lambda_{vm}| \chi_{vm} \right\|_{p(\cdot)}, \end{aligned}$$

with an appropriate choice of $c > 0$. Now this expression is less than or equal to one if and only if

$$\left\| \sum_{m \in \mathbb{Z}^n} \delta^{-\frac{1}{rq(\cdot)}} 2^{v\alpha(\cdot)} |\lambda_{vm}| \chi_{vm} \right\|_{\frac{p(\cdot)}{rq(\cdot)}}^{rq(\cdot)} \leq 1,$$

which follows immediately from the definition of δ .

Estimate of II. From Lemma 6, we have for any M sufficiently large

$$\sum_{m \in \mathbb{Z}^n} 2^{j\alpha(x)} |\lambda_{vm}| |\varphi_j * \rho_{vm}(x)| \leq c 2^{(j-v)(L+n+1)} \sum_{m \in \mathbb{Z}^n} 2^{j\alpha(x)} |\lambda_{vm}| (1 + 2^j |x - 2^{-v}m|)^{-M}.$$

Let $0 < t < \min(1, p^-)$ be a real number such that $L > n/t - 1 - n - \alpha^-$. Using a combination of the arguments used in the estimate of I, we arrive at the inequality

$$\left\| c 2^{(j-v)(n/t - \alpha^-)} \sum_{m \in \mathbb{Z}^n} 2^{j\alpha(\cdot)} |\lambda_{vm}| (1 + 2^j |\cdot - 2^{-v}m|)^{-M} \right\|_{\frac{p(\cdot)}{rq(\cdot)}}^{rq(\cdot)} \leq \left\| \sum_{m \in \mathbb{Z}^n} 2^{v\alpha(\cdot)} |\lambda_{vm}| \chi_{vm} \right\|_{\frac{p(\cdot)}{rq(\cdot)}}^{rq(\cdot)} + 2^{-j},$$

with some positive constant c . Hence II can be estimated by

$$\sum_{j=0}^{\infty} \left(\sum_{v=j+1}^{\infty} 2^{(j-v)(L+n+1-n/t+\alpha^-)rq^-} \left(\left\| \sum_{m \in \mathbb{Z}^n} 2^{v\alpha(\cdot)} |\lambda_{vm}| \chi_{vm} \right\|_{\frac{p(\cdot)}{rq(\cdot)}}^{rq(\cdot)} + 2^{-j} \right) \right)^{1/r}.$$

Observing that $L > n/t - 1 - n - \alpha^-$, an application of Lemma 3 yields the desired inequality, i.e.

$$\begin{aligned}
II &\leq \sum_{j=0}^{\infty} \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{j\alpha(\cdot)} \lambda_{jm} \chi_{jm} \right\|^{rq(\cdot)} \right\|_{\frac{p(\cdot)}{rq(\cdot)}}^{1/r} + c \\
&= c \sum_{j=0}^{\infty} \left\| \left\| \sum_{m \in \mathbb{Z}^n} 2^{j\alpha(\cdot)} \lambda_{jm} \chi_{jm} \right\|^{q(\cdot)} \right\|_{\frac{p(\cdot)}{q(\cdot)}} + c \\
&\leq C.
\end{aligned}$$

The proof is completed. \square

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