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On stability of two degenerate reaction–diffusion systems ☆

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ABSTRACT

In this paper, we construct a partially degenerate reaction–diffusion equation subject to the Neumann boundary condition and show that the zero solution is asymptotically stable but not exponentially asymptotically stable. In this way, we solve an open problem proposed by Casten and Holland (1977) [4]. Moreover, we give the exponential asymptotic stability of the zero solution to a totally degenerate system with cross-diffusion effects, which cannot be determined by a simple spectral analysis based on the well developed semigroup theory.
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1. Introduction

In this paper, we consider two reaction systems having the following form

$$\frac{\partial z}{\partial t} = D\Delta z + f(z), \tag{1.1}$$

in $[0,\infty)\times\Omega$ subject to the Neumann boundary data $(\partial z/\partial\nu)(t,x)=0$ for $(t,x)\in[0,\infty)\times\partial\Omega$ with initial data $z(0,x)=\alpha(x)$, where $z=(z_1,z_2,\ldots,z_k)^T\in\mathbb{R}^k$, $\Omega\subset\mathbb{R}^q$ is some open bounded domain with smooth boundary, D – a square matrix, ν – the outward unit normal vector on $\partial\Omega$ and $f\in C^3(\mathbb{R}^k,\mathbb{R}^k)$ with

$$f(z)=\big(f_1(z),f_2(z),\ldots,f_k(z)\big)^T.$$

Since the pioneering work by Turing [35], many mathematicians are devoted to studying the diffusion-driven instability phenomenon, or in other words, the Turing bifurcation phenomenon [20,21,25,42]. And since Shigesada, Kawasaki and Teramoto [30] introduced the cross diffusion in the dispersion terms in their study of spatial segregation of interacting species, lots of mathematicians have flooded into the research on PDE models with cross-diffusion effects, such as cross-diffusion induced instability and stability, existence of positive steady-state solutions, existence of global solutions, cross-diffusion driven instability, existence of traveling wave solutions, Lyapunov stability of solutions, etc. (See [3,6,9,12,14,15,17,19,22,23,26,27,29,31,33,34,39,40,43].)

In this paper, we call a system partially degenerate if the diffusion matrix has eigenvalues with both positive real parts and zero real parts [11,12,37,41,45]. Moreover, we call a system totally degenerate if all the eigenvalues of the nonzero

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diffusion matrix have zero real parts, for instance, the diffusion matrix has a pair of purely imaginary eigenvalues [24]. We call a system degenerate if it is partially degenerate or totally degenerate. If a system is not degenerate, we call it non-degenerate.

There are many papers addressing partially degenerate reaction–diffusion systems with one of the diffusion coefficients being zero [10,11,28,41], and investigating the instability of the cross-diffusion systems with the diffusion matrix having eigenvalues with zero real parts as well [24].

However, it seems that there is no enough theory to guarantee the correctness of work related to the stability analysis of such degenerate systems, though there have been gigantic papers already [8,10,28,38,44]. Indeed, there has been inadequate work on this topic, i.e., the related spectral stability criterion with respect to degenerate PDEs, since the classical work by Casten and Holland [4], Smoller's famous monograph [32] and the celebrated work on cross-diffusion equations by H. Amann using semigroup theory [1,2].

In [4], Casten and Holland gave stability properties of reaction–diffusion systems with all the diffusion coefficients being positive as well as reaction–diffusion systems with some of the diffusion coefficients being zero. Also they gave a generalized version of Lyapunov function method in determining the asymptotic stability of a constant equilibrium solution. In [5], they also addressed the instability of nonconstant equilibrium solutions.

In [36], Wang and Li gave some sufficient and necessary conditions (minors condition) for the stability and instability of related matrices involved in reaction–diffusion systems, i.e., the diffusion matrix and the Jacobian of the reaction function. They gave conditions on the maximum of the real parts of all the eigenvalues of matrices in order for the matrices being stable. Also conditions were presented to make sure that all the eigenvalues of A_n have negative real parts with a uniform negative upper bound (for the definition of A_n, we refer the readers to page 129 in the present paper). Their method is based on compound matrix and Lozinskiĭ measure. However, they did not further investigate reaction system with cross-diffusion effects, let alone the degenerate case.

In fact, local stability has various fundamental uses. For instance, detection of Hopf bifurcation phenomenon strongly depends on it. Global stability is also based on local asymptotic stability. Hence it is of great significance to present some general technique in establishing stability properties based on the spectrum for some degenerate PDEs which may have vast scientific background [8,10,28,38,44].

By solving the following elliptic equation

$$D\Delta z + f(z) = 0, \tag{1.2}$$

we can obtain the steady states of system (1.1). According to [4], we usually call the solutions of system (1.2) equilibrium solutions of system (1.1); in particular, we call such solution a constant equilibrium solution to system (1.1) if the equilibrium solution is a constant. Throughout this paper, we assume $f(0) = 0$ without loss of generality so that the zero solution $z \equiv 0$ is an equilibrium solution. Let $f(z) = Az + g(z)$ with $g(0) = 0$, $\nabla g(0) = 0$, and define the linearized system of (1.1) as follows

$$\frac{\partial z}{\partial t} = D\Delta z + Az. \tag{1.3}$$

This paper is motivated by an open problem (see Section 3) proposed in [4]. We mainly focus on two reaction systems. By studying a partially degenerate reaction system, we give a negative answer to the open problem and show the non-equivalence of asymptotic stability and exponential asymptotic stability. Also we give the exponential asymptotic stability of the zero solution to a totally degenerate reaction system with cross-diffusion effects.

This paper is organized as follows. In Section 2, we first give the definitions of stability of an equilibrium solution, show that stability properties are preserved under nonsingular linear transformation and give the relationship between system (1.1) and its linearized system (1.3) with respect to exponential asymptotic stability. Then, we solve an open problem proposed in [4] by giving a linear reaction–diffusion system and show that the system is globally asymptotically stable but not exponentially stable in Section 3. In Section 4, we give exponential asymptotic stability of the zero solution to a totally degenerate reaction system with cross-diffusion effects, Lugiato–Lefever equation [24], arising from the complex dynamical systems. Finally, a brief conclusion is presented in Section 5.

2. Definitions of stability and basic properties of stability with respect to systems (1.1) and (1.3)

In this section, we give the definitions of stability and some basic properties of stability.

Throughout this paper, we use l_1 vector norm for $z = (z_1, z_2, \dots, z_k) \in \mathbb{C}^k$, i.e., $\|z\|_1 = |z_1| + |z_2| + \dots + |z_k|$. And we use the maximum column sum matrix norm for $P = (p_{ij})_{m\times n} \in M_{m,n}(\mathbb{C})$, i.e., $\|P\|_1 = \max_{1\leqslant j\leqslant n} \sum_{i=1}^{m} |p_{ij}|$. Here $M_{m,n}(\mathbb{C})$ is the set of $m \times n$ matrices defined over $\mathbb{C}$ and $M_{k,k}(\mathbb{C})$ is abbreviated to $M_k(\mathbb{C})$ in the following. For simplicity, we use $|\cdot|$ instead of $\|\cdot\|_1$ for vectors in $\mathbb{C}^k$ and $\|\cdot\|$ instead of $\|\cdot\|_1$ for matrices in $M_{m,n}(\mathbb{C})$ (note that $|\cdot|$ is also used to represent the norm for complex numbers and $\|\cdot\|$ for a function norm later in this paper). Then we know that such matrix norm $\|\cdot\|$ is induced by the vector norm $|\cdot|$ (see [13]). Hence we have the following lemma according to [13].

Lemma 2.1. *$|Pz| \leqslant \|P\| \cdot |z|$, for all $P \in M_k(\mathbb{C})$ and $z \in \mathbb{C}^k$.*

Similarly, we further give the following inequalities without proofs.

Lemma 2.2. *$|Pz| \leqslant \|P\| \cdot |z|$, for all $P \in M_{m,n}(\mathbb{C})$ and $z \in \mathbb{C}^n$.*

Lemma 2.3. *$\|PQ\| \leqslant \|P\| \cdot \|Q\|$, for all $P \in M_{m,n}(\mathbb{C})$ and $Q \in M_{n,p}(\mathbb{C})$.*

These lemmas can be viewed as a basis for the inequalities appearing in the proofs of the main results, e.g., Theorem 2.6.

We use the supreme norm for a continuous vector-valued function, i.e., for a continuous vector-valued function g defined on $\bar{\Omega}$, $\|g\| = \sup_{x\in\bar{\Omega}} |g(x)|$. Now we are ready to give the definitions of stability. An equilibrium solution $\beta(x)$ is said to be stable if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|z(0,\cdot) - \beta\| < \delta$, then $\|z(t,\cdot) - \beta\| < \varepsilon$ for all $t \geqslant 0$. The solution is said to be asymptotically stable if it is stable and there exists $\delta > 0$ such that $\|z(t,\cdot) - \beta\| \to 0$ as $t \to \infty$ for $\|z(0,\cdot) - \beta\| < \delta$. Further more, the solution is said to be exponentially asymptotically stable if there exist three positive constants δ, K and ω such that if $\|z(0,\cdot) - \beta\| < \delta$, then

$$\left\|z(t,\cdot) - \beta\right\| \leqslant Ke^{-\omega t}\left\|z(0,\cdot) - \beta\right\| \tag{2.1}$$

for all $t \geqslant 0$. If δ can be chosen arbitrarily large, then the solution is said to be globally (exponentially) asymptotically stable. The solution is unstable if it is not stable.

It is easy to see that if a solution is (globally) exponentially asymptotically stable, then the solution is (globally) asymptotically stable.

Remark 2.4. The definitions with respect to stability of a constant equilibrium solution are equivalent to those defined in [4].

Remark 2.5. By definition, if a solution to a system is (globally) (exponentially) asymptotically stable (unstable) with respect to l_1 vector norm, then it is (globally) (exponentially) asymptotically stable (unstable) with respect to any other vector norm, e.g., the Euclidean norm (l_2 vector norm) due to the norm-equivalence for finite-dimensional linear vector spaces. This implies that in fact if we choose other vector norm, the conclusions in this paper still hold.

Now we give a fundamental theorem that will be used extensively in the sequel.

Theorem 2.6. *Nonsingular linear transformation preserves the stability property.*

Proof. Suppose that the equilibrium solution $\beta(x)$ of system (1.1) is stable. According to the definition of stability, we know for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|z(0,\cdot) - \beta\| < \delta$, then $\|z(t,\cdot) - \beta\| < \varepsilon$ for all $t \geqslant 0$. Let $P \in M_k(\mathbb{C})$ be nonsingular and make the transformation $w = P^{-1}z$. Consider the equilibrium solution $P^{-1}\beta(x)$ of the transformed system

$$\frac{\partial w}{\partial t} = P^{-1}D\Delta Pw + P^{-1}f(Pw). \tag{2.2}$$

For any $\varepsilon_1 > 0$, we take $\varepsilon = \|P^{-1}\|^{-1}\varepsilon_1/2$ and $\delta_1 = \|P\|^{-1}\delta/2 > 0$. For $\|w(0,\cdot) - P^{-1}\beta\| < \delta_1$, we have $|z(0,x) - \beta(x)| = |P(w(0,x) - P^{-1}\beta(x))| \leqslant \|P\| \cdot |w(0,x) - P^{-1}\beta(x)| < \|P\|\delta_1 = \delta/2$, for any $x \in \bar{\Omega}$, which implies $\|z(0,\cdot) - \beta\| \leqslant \delta/2 < \delta$. Hence we have $\|z(t,\cdot) - \beta\| < \varepsilon$, for all $t \geqslant 0$. And therefore for any $x \in \bar{\Omega}$, we have $|z(t,x) - \beta(x)| < \varepsilon$, for all $t \geqslant 0$. So for any $x \in \bar{\Omega}$, $|w(t,x) - P^{-1}\beta(x)| = |P^{-1}P(w(t,x) - P^{-1}\beta(x))| = |P^{-1}(z(t,x) - \beta(x))| \leqslant \|P^{-1}\||z(t,x) - \beta(x)| < \|P^{-1}\|\varepsilon = \varepsilon_1/2$, for all $t \geqslant 0$, from which follows $\|w(t,\cdot) - P^{-1}\beta\| \leqslant \varepsilon_1/2 < \varepsilon_1$, for all $t \geqslant 0$.

By following a similar argument, (global) (exponential) asymptotic stability as well as instability can also be proved to be preserved under nonsingular linear transformation. $\square$

Remark 2.7. Note that for any square matrix $P = (p_{ij})_{k\times k}$, $\Delta P = P\Delta$, where

$$\Delta = \mathrm{diag}\left\{\sum_{i=1}^{q} \frac{\partial^2}{\partial x_i^2}, \ldots, \sum_{i=1}^{q} \frac{\partial^2}{\partial x_i^2}\right\}.$$

In fact,

$$\begin{aligned}
\Delta Pz &= \left(\sum_{i=1}^{q} \frac{\partial^2}{\partial x_i^2} \sum_{j=1}^{k} p_{1j}z_j, \ldots, \sum_{i=1}^{q} \frac{\partial^2}{\partial x_i^2} \sum_{j=1}^{k} p_{kj}z_j\right)^T \\
&= \left(\sum_{j=1}^{k} p_{1j} \sum_{i=1}^{q} \frac{\partial^2}{\partial x_i^2} z_j, \ldots, \sum_{j=1}^{k} p_{kj} \sum_{i=1}^{q} \frac{\partial^2}{\partial x_i^2} z_j\right)^T \\
&= P\Delta z.
\end{aligned}$$

This means that if the diffusion matrix is D for some reaction system cross-diffusion effects, then we can study an equivalent system with the diffusion matrix $P^{-1}DP$, where P is a suitable nonsingular matrix (in order for $P^{-1}DP$ being the Jordan canonical form of D). Here the 'equivalent' means that the corresponding equilibrium solutions of the two systems share the same stability properties by Theorem 2.6.

We use the technique of eigenfunction expansions throughout this paper. The justification of this approach for system (1.3) follows from [18]. Let $0 = \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$ denote the eigenvalues and $\phi_0, \phi_1, \dots, \phi_n, \dots,$ the corresponding normalized eigenfunctions of Laplace's equation in Ω with the Neumann boundary condition; that is λ_n, ϕ_n satisfy $-\Delta\phi_n = \lambda_n\phi_n$ in Ω, with $\partial\phi_n/\partial\nu = 0$ on $\partial\Omega$, and $\int_\Omega \phi_n^2(x)\,dx = 1$. For each nonnegative integer n, let z_{0n} be the k-vector

$$z_{0n} = \int_\Omega \alpha(x)\phi_n(x)\,dx \tag{2.3}$$

and let the $k \times k$ matrix exponential $e^{A_n t}$ be the matrix solution to the differential equation

$$\frac{dz}{dt} = A_n z, \tag{2.4}$$

where $A_n = A - \lambda_n D$, and the initial condition is $e^{A_n 0} = I$. Then the solution to the linearized system (1.3) can be written in the following form

$$z(t,x) = \sum_{n=0}^{\infty} \phi_n(x) e^{A_n t} z_{0n}. \tag{2.5}$$

Note that (2.5) defines a semigroup T_t through the definition

$$z(t,x) = (T_t\alpha)(x). \tag{2.6}$$

Next we will use the representation (2.6) to establish a relationship between system (1.1) and system (1.3) with respect to exponential asymptotic stability.

Beforehand we give the following preliminary inequalities according to [4]:

$$|z_{0n}| = \left|\int_\Omega \phi_n(x)\alpha(x)\,dx\right| \leqslant \sqrt{|\Omega|}\,\|\alpha\| \tag{2.7}$$

and

$$\|\phi_n\| \leqslant C\lambda_n, \tag{2.8}$$

where $|\Omega| = \int_\Omega 1\,dx$ and C is a positive constant possibly depending on Ω. For more results on eigenvalues and eigenfunctions, we refer the readers to [7].

Theorem 2.8. *The zero solution to* (1.1) *is exponentially asymptotically stable if the zero solution to* (1.3) *is exponentially asymptotically stable.*

Proof. Actually the proof is the same as in [4] but a small typo. For the readers' convenience, we copy it as follows. We write the solution in the form

$$z(t,x) = (T_t\alpha)(x) + \int_0^t (T_{t-s}g)\big(z(s,x)\big)\,ds, \tag{2.9}$$

where T_t was defined in (2.6). Let δ, K and ω be as in (2.1). Then we have $\|T_t\alpha\| \leqslant Ke^{-\omega t}\|\alpha\|$. By $g(0) = 0$, $\nabla g(0) = 0$ and $g \in C^2(\mathbb{R}^k, \mathbb{R}^k)$, we know that there exists $\gamma > 0$ such that $|g(z)| \leqslant (\omega/(2K))|z|$ if $|z| \leqslant 2\gamma$. Let $\|\alpha\| < \min\{2\gamma/K, \delta\}$. Then there is a time $0 < T \leqslant \infty$ (possibly depending on α) such that $\|z(t,\cdot)\| < 2\gamma$ for $0 \leqslant t < T$. Then on $[0,T)$, we have

$$\big\|z(t,\cdot)\big\| \leqslant Ke^{-\omega t}\|\alpha\| + \int_0^t \frac{\omega}{2} e^{-\omega(t-s)} \big\|z(s,\cdot)\big\|\,ds. \tag{2.10}$$

Define $R(t) = \|z(t,x)\|e^{\omega t}$. Then Gronwall's inequality yields

$$R(t) \leqslant R(0)Ke^{\frac{\omega}{2}t}.$$

Substituting we obtain that

$$\left\|z(t,\cdot)\right\| \leqslant K e^{-\frac{\omega}{2}t}\|\alpha\| < 2\gamma e^{-\frac{\omega}{2}t}. \tag{2.11}$$

Since the right-hand side is less than 2γ for all $t \geqslant 0$, we have $T = \infty$ and therefore (2.11) is valid for all $t \geqslant 0$. The desired conclusion follows from the estimate (2.11). □

Remark 2.9. We would like to mention that a careful examination of the proofs of all the main results obtained in the previous subsections shows that when system (1.1) is subject to the zero Dirichlet boundary conditions, i.e.,

$$z(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \tag{2.12}$$

all the conclusions obtained in the previous subsections still hold.

3. A response to an open problem

In this section, we consider a partially degenerate reaction–diffusion system as a response to Remark 10 in [4]. For the readers' convenience, we copy the remark as follows.

Remark 10. The estimate (12) is a sufficient condition for asymptotic stability. It is not clear whether it is a necessary condition. Without the assumption on C, it is possible that the real parts of the eigenvalues of A_n may not be bounded below zero even though they are negative.

The estimate (12) here mentioned in this remark is the exponential estimate, i.e., there exist positive constants ω and K such that for all $t \geqslant 0$,

$$\|T_t\alpha\| \leqslant K e^{-\omega t}\|\alpha\|; \tag{3.1}$$

where

$$D = \begin{pmatrix} \tilde{D} & 0 \\ 0 & 0 \end{pmatrix} \tag{3.2}$$

and

$$A = \begin{pmatrix} \tilde{A} & B_1 \\ B_2 & C \end{pmatrix}, \tag{3.3}$$

with $\tilde{D}$ a diagonal matrix with positive entries and $A_n = A - \lambda_n D$.

There are partially degenerate cases where the assumption made in Theorem 4 in [4] is satisfied, for instance [16,45]. Thus the asymptotic stability for these cases can be treated according to [4]. However, now we are ready to give one case where the assumption fails. In the following, we concentrate on system (1.3) subject to the Neumann boundary data with

$$D = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix},$$

where $d > 0$. For further convenience, we rewrite the system as follows

$$\begin{pmatrix} \frac{\partial z_1}{\partial t} \\ \frac{\partial z_2}{\partial t} \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial^2 z_1}{\partial x^2} \\ \frac{\partial^2 z_2}{\partial x^2} \end{pmatrix} + \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{3.4}$$

In fact, this is the case where the assumption in Theorem 4 in [4] fails, because in this case, the real parts of the eigenvalues of A_n are not bounded below and away from zero though they are negative. More accurately, there is a sequence of eigenvalues approaching to $-\infty$ and the other approaching to 0 from the left on the real axis. Now we will show that the open problem whether exponential asymptotic stability is equivalent to asymptotic stability proposed in [4] by giving a negative answer through this example.

Theorem 3.1. *The zero solution to system* (3.4) *is not exponentially asymptotically stable.*

Proof. First, according to (2.5), the solution to (3.4) can be represented as follows:

$$z(t,x) = \sum_{n=0}^{\infty} \phi_n(x) e^{A_n t} z_{0n},$$

where $A_n = A - \lambda_n D = \begin{pmatrix} -3-d\lambda_n & 1 \\ -1 & 0 \end{pmatrix}$, λ_n, ϕ_n, α and z_{0n} were defined in Section 2. Let $\mu_{1,n} < \mu_{2,n}$ be the two real eigenvalues of A_n. Simple calculations lead to $\mu_{1,n}\mu_{2,n} = 1$, $\mu_{2,n} < 0$, $\lim_{n\to\infty} \mu_{1,n} = -\infty$ and $\lim_{n\to\infty} \mu_{2,n} = 0$. More precisely,

$$\lim_{n\to\infty} \mu_{1,n}/(-d\lambda_n) = 1 \quad \text{and} \quad \lim_{n\to\infty} \mu_{2,n}(-d\lambda_n) = 1. \tag{3.5}$$

Let $P_n = \begin{pmatrix} 1 & 1 \\ -1/\mu_{1,n} & -1/\mu_{2,n} \end{pmatrix}$. Then $P_n^{-1} A_n P_n = \begin{pmatrix} \mu_{1,n} & 0 \\ 0 & \mu_{2,n} \end{pmatrix}$. Hence we have

$$e^{A_n t} = \begin{pmatrix} \frac{\mu_{2,n} e^{\mu_{2,n} t} - \mu_{1,n} e^{\mu_{1,n} t}}{\mu_{2,n} - \mu_{1,n}} & \frac{\mu_{1,n}\mu_{2,n}(e^{\mu_{2,n} t} - e^{\mu_{1,n} t})}{\mu_{2,n} - \mu_{1,n}} \\ \frac{e^{\mu_{1,n} t} - e^{\mu_{2,n} t}}{\mu_{2,n} - \mu_{1,n}} & \frac{\mu_{2,n} e^{\mu_{1,n} t} - \mu_{1,n} e^{\mu_{2,n} t}}{\mu_{2,n} - \mu_{1,n}} \end{pmatrix}. \tag{3.6}$$

Thus

$$\phi_n(x) e^{A_n t} z_{0n} = \phi_n(x) \begin{pmatrix} \frac{\mu_{2,n} e^{\mu_{2,n} t} - \mu_{1,n} e^{\mu_{1,n} t}}{\mu_{2,n} - \mu_{1,n}} z_{0n}^{(1)} + \frac{e^{\mu_{2,n} t} - e^{\mu_{1,n} t}}{\mu_{2,n} - \mu_{1,n}} z_{0n}^{(2)} \\ \frac{e^{\mu_{1,n} t} - e^{\mu_{2,n} t}}{\mu_{2,n} - \mu_{1,n}} z_{0n}^{(1)} + \frac{\mu_{2,n} e^{\mu_{1,n} t} - \mu_{1,n} e^{\mu_{2,n} t}}{\mu_{2,n} - \mu_{1,n}} z_{0n}^{(2)} \end{pmatrix}.$$

Note here we use $\mu_{1,n}\mu_{2,n} = 1$. Hence for any $\epsilon > 0$, there exists $N \in \mathbb{N}^+$ such that $\epsilon/2 + \mu_{2,n} > 0$, for any $n \geqslant N$. Choose $\alpha(x) = (0, \phi_N(x))^T$.

$$\|z(t,\cdot)\| = \|\phi_N(\cdot) e^{A_N t} z_{0N}\| \geqslant \frac{\mu_{2,n} e^{\mu_{1,n} t} - \mu_{1,n} e^{\mu_{2,n} t}}{\mu_{2,n} - \mu_{1,n}} \geqslant e^{\mu_{2,n} t},$$

which indicates

$$\|z(t,\cdot) e^{\epsilon t}\| > e^{\epsilon t/2} \to \infty$$

as $t \to \infty$. Notice that according to (2.8), $\|\alpha\| \leqslant C\lambda_N$, which shows that the zero solution to (3.4) is not exponentially asymptotically stable. $\square$

In the following, we will show that the zero solution to (3.4) is globally asymptotically stable with the spatial domain $\Omega = (0, \pi)$, under some circumstance.

Theorem 3.2. *If the initial function satisfies*

$$(H) \quad \int_0^\pi \alpha(x) \cos nx \, dx \sim \mathcal{O}(1/n^2) \quad \textit{as } n \to \infty,$$

then the zero solution to (3.4) *is globally asymptotically stable.*

Proof. We divide the proof into two steps. In the first step, we will show that the global asymptotic profile of the solution to (1.1) with initial data α. In the second step, we show the stability of the zero solution.

Step 1. Note that the eigenvalues and the corresponding normalized eigenfunctions of $-\Delta\phi_n = \lambda_n \phi_n$ in $(0, \pi)$ with the Neumann boundary conditions are n^2 and $\sqrt{\frac{2}{\pi}} \cos nx$, respectively, for $n \in \mathbb{N}$. According to (3.5), we know

$$\lim_{n\to\infty} \mu_{1,n}/\left(-dn^2\right) = 1 \quad \text{and} \quad \lim_{n\to\infty} \mu_{2,n}\left(-dn^2\right) = 1. \tag{3.7}$$

Furthermore, $\mu_{1,n} \sim \mathcal{O}(n^2)$ and $\mu_{2,n} \sim \mathcal{O}(1/n^2)$ as $n \to \infty$. According to (3.6), we have

$$\|e^{A_n t}\| \sim \mathcal{O}\left(e^{-t/(dn^2)}\right).$$

Thus there exist constants $N \in \mathbb{N}^+$ and $C > 0$, such that

$$\left|\sum_{n=N+1}^{\infty} \phi_n(x) e^{A_n t} z_{0n}\right| = \left|\sum_{n=N+1}^{\infty} \sqrt{\frac{2}{\pi}} \cos nx e^{A_n t} z_{0n}\right| \leqslant C \sum_{n=N+1}^{\infty} 1/n^2 e^{-t/(dn^2)}. \tag{3.8}$$

Before proceeding the remainders of Step 1, we give the following lemma, which is useful for the forthcoming estimates.

Lemma 3.3. *$h(x) = x^{1/4} - e^x < 0$ for $x > 0$.*

Proof. Note $h'(x) = \frac{1}{4}x^{-3/4} - e^x$. Hence there is an $x_0 > 0$ such that $h'(x) > 0$ when $x \in (0, x_0)$ while $h'(x) < 0$ when $x \in (x_0, \infty)$. Moreover, $h'(x_0) = 0$. This is easily seen because $h'(x)$ is strictly decreasing in x for $x \in (0, \infty)$. This means that $h(x_0) = \max_{x\in(0,\infty)} h(x)$. Moreover, notice that $x_0^{-3/4}/4 = e^{x_0}$, hence we have $h(x_0) = x_0^{1/4} - x_0^{-3/4}/4 = x_0^{-3/4}(x_0 - 1/4)$. Hence it suffices to show that $x_0 < 1/4$. In fact, $h'(1/4) = 1/\sqrt{2} - e^{1/4} < 0$, which indicates $x_0 < 1/4$ by the monotonicity of $h'(x)$. $\square$

By this lemma, we see that

$$\sum_{n=N+1}^{\infty} 1/n^2 e^{-t/(dn^2)} \leqslant d^{1/4} t^{-1/4} \sum_{n=N+1}^{\infty} 1/n^{3/2}. \tag{3.9}$$

Also, there exists $K > 0$ and $\omega > 0$ such that

$$\left| \sum_{n=0}^{N} \phi_n(x) e^{A_n t} z_{0n} \right| \leqslant K e^{-\omega t} \|\alpha\|. \tag{3.10}$$

Then it follows from inequalities (3.8)–(3.10) that $\lim_{t\to\infty} |z(t,x)| = 0$, i.e.,

$$\lim_{t\to\infty} z(t,x) = 0. \tag{3.11}$$

Step 2. First note that according to [4], for any fixed $T > 0$, there exists a positive constant K_1 such that

$$\sup_{0\leqslant t\leqslant T} \left\| z(t,\cdot) \right\| \leqslant K_1 \|\alpha\|. \tag{3.12}$$

From (3.11) in Step 1, we know that for any $\epsilon > 0$, there exists $T > 0$ such that $\|z(t,\cdot)\| < \epsilon$ when $t > T$, for $\|\alpha\| < 1$ and then choose $\delta > 0$ such that $0 < \delta < \min\{1, \epsilon/K_1\}$. Hence by (3.12), for $\|\alpha\| < \delta$, we have $\|z(t,\cdot)\| \leqslant K_1\|\alpha\| < K_1\delta < K_1 \times \epsilon/K_1 = \epsilon$, for $t \in [0, T]$. This implies the stability of the zero solution, and therefore the global asymptotic stability follows from (3.11). $\square$

From Theorem 3.1 and Remark 3.5, we can see that in this example, such asymptotic stability is not the exponential asymptotic stability as we often encounter. And by Theorems 2.8, 3.1 and 3.2, we still do not know whether the zero solution to a system with (3.4) as its linearized system is asymptotically stable or not. This is quite different from ODE systems and contrary to what we have taken for granted for years [8,10,28,36,38,44].

Now we give a specific condition when the assumption (H) is satisfied.

Corollary 3.4. *If the derivative of the initial function α is of bounded variation, then the solution to system* (1.1) *satisfies $\lim_{t\to\infty} z(t,x) = 0$.*

Proof. We only need to verify that the hypothesis (H) holds. In fact,

$$\left| \int_0^{\pi} \alpha(x) \cos nx \, dx \right| = \left| -\frac{1}{n} \int_0^{\pi} \alpha'(x) \sin nx \, dx \right| = \left| -1/n^2 \int_0^{\pi} \cos nx \, d\alpha'(x) \right| \leqslant 1/n^2 V_{\alpha'}(0,\pi), \tag{3.13}$$

which implies that the hypothesis (H) holds, where $V_{\alpha'}(0,\pi)$ is the total variation of α' on $[0,\pi]$. $\square$

Remark 3.5. If we are concerned with sufficient smooth solutions (for instance, classical solutions), then we can restrict the initial functions in the space $\{u \in C^2(0,\pi) \cap C^1[0,\pi]:\ u'(0) = u'(\pi) = 0\}$, then by Corollary 3.4, we know that the zero to system (3.4) is globally asymptotically stable.

In practice, we are prone to have faith in that the assumption (H) always holds for initial functions we are concerned with or interested in. Moreover, for arbitrary bounded spatial domain Ω, we guess that the conclusions of Theorem 3.2 still hold.

4. A totally degenerate reaction system with cross diffusion effect

In this section, we consider a totally degenerate reaction system with cross-diffusion effects. There is one example arising from the complex dynamical systems, e.g., Lugiato–Lefever equation with the diffusion matrix having a pair of purely imaginary eigenvalues, which implies that the equation is totally degenerate.

Lugiato–Lefever equation is as follows [24]:

$$\begin{pmatrix} \frac{\partial u_1}{\partial t} \\ \frac{\partial u_2}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & -b^2 \\ b^2 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial^2 u_1}{\partial x^2} \\ \frac{\partial^2 u_2}{\partial x^2} \end{pmatrix} + \begin{pmatrix} -u_1 + (\theta - c)u_2 - c(2u_1u_2 + u_2(u_1^2 + u_2^2)) \\ (3c-\theta)u_1 - u_2 + c(3u_1^2 + 2u_2^2 + u_1(u_1^2+u_2^2)) \end{pmatrix}, \tag{4.1}$$

where $c > 0$, $b^2 \neq 0$ and $\theta \in \mathbb{R}$. In [24], the spatial domain is $(-1/2, 1/2)$ and periodic boundary conditions are imposed. For simplicity, we only consider system (4.1) subject to the Dirichlet boundary condition when the spatial domain is $(0, \pi)$. In fact, if Neumann boundary condition is taken into consideration, similar results will be obtained by substituting assumption (H') by (H).

Theorem 4.1. *If*

$$(P) \quad 3c \leqslant \theta \quad \text{or} \quad c \geqslant \theta$$

and the initial function α satisfies

$$(H') \quad \int_0^\pi \alpha(x)\sin nx\,dx \sim \mathcal{O}(1/n^2) \quad \text{as } n \to \infty,$$

then the zero solution to system (4.1) *is exponentially asymptotically stable.*

Proof. We only need to show that the linearized system of (4.1) is exponentially asymptotically stable. The proof in the following is similar to that of Theorem 3.1 in Section 3 in this paper. Note that

$$D = \begin{pmatrix} 0 & -b^2 \\ b^2 & 0 \end{pmatrix}$$

and

$$A = \begin{pmatrix} -1 & \theta - c \\ 3c - \theta & -1 \end{pmatrix}.$$

Straightforward calculations lead to that both of the eigenvalues of A_n are $-1 \pm ib_n$, where i is the imaginary unit and $b_n = \sqrt{b^4\lambda_n^2 + 2cb^2\lambda_n + (3c-\theta)(c-\theta)} > 0$, for all $n \in \mathbb{N}^+$. It is easy to see that $\lim_{n\to\infty} \frac{b_n}{b^2\lambda_n + \theta - c} = 1$. Moreover,

$$z(t,x) = \sum_{n=0}^{\infty} \phi_n(x) e^{A_n t} z_{0n},$$

where $\phi_n(x) = \sin nx$ and

$$e^{A_n t} = \begin{pmatrix} \frac{e^{(-1+ib_n)t} + e^{(-1-ib_n)t}}{2} & \frac{(b^2\lambda_n + \theta - c)(e^{(-1+ib_n)t} - e^{(-1-ib_n)t})}{2ib_n} \\ \frac{b_n(e^{(-1+ib_n)t} + e^{(-1-ib_n)t})}{-2i(b^2\lambda_n - \theta + c)} & \frac{e^{(-1+ib_n)t} + e^{(-1-ib_n)t}}{2} \end{pmatrix}. \tag{4.2}$$

It follows that there exists $N_0 \in \mathbb{N}$ such that $\|e^{A_n t}\| \leqslant 2e^{-t}$ for all $t \geqslant 0$, $n \geqslant N_0$ (in fact, we only need $\frac{b_n}{b^2\lambda_n + \theta - c} \in [1/\sqrt{2}, \sqrt{2}]$ for $n \geqslant N_0$). There exist constants $N \geqslant N_0$, $C > 0$ and $C' > 0$, such that

$$\left| \sum_{n=N+1}^{\infty} \phi_n(x) e^{A_n t} z_{0n} \right| \leqslant Ce^{-t} \sum_{n=N+1}^{\infty} 1/n^2 \tag{4.3}$$

and

$$\left| \sum_{n=0}^{N} \phi_n(x) e^{A_n t} z_{0n} \right| \leqslant C' e^{-t}. \tag{4.4}$$

So far we have obtained the asymptotic profile, i.e., estimate (2.1) holds with $\omega = 1$, $\beta = 0$ and $z(0, \cdot) = \alpha$.

The rest of the proof of the local stability can be derived by the same argument as that in Step 2 in the proof of Theorem 3.1. □

From this example, we can see that even all the eigenvalues of the linear system have negative real parts uniformly bounded above by a negative constant, we still have to impose some other assumption (like (H') in this model) in order to obtain the exponential asymptotic stability. In fact system (4.1) is no longer a parabolic system, it is easy to see that system (4.1) can be written as a nonlinear Schrödinger system, to which the analytic semigroup theories cannot be applied directly, however C_0 semigroup theories can be still applied to. According to the abstract linear or nonlinear exponential stability based on C_0 semigroup theories, the linear exponential stability of zero cannot be derived directly from the spectral stability of zero solutions. Some additional assumptions (like some stronger spectral estimates) are needed to guarantee the linear exponential stability, which may explain why some additional assumptions on initial values are required to get the linear exponential stability by further verifying the related spectral assumptions. Conditions in order for a negative common uniform upper bound can be found in [36] when D is a diagonal matrix. In practice, we believe in the fact that, the method and technique used in [36] can be used to deal with the case when cross-diffusion effects are considered, i.e., when D is not a diagonal matrix. If such conditions are derived, it is easy for one to judge whether solutions to some specific system is stable or not by applying the method in the proof of Theorem 4.1.

5. Conclusion

In this paper, we first define stability, (global) (exponential) asymptotic stability and instability of an equilibrium solution. Then we demonstrate that nonsingular linear transformation preserves the stability properties. Also we prove that the exponential asymptotic stability of the zero solution to a system can be induced by that of the zero solution to its linearized system.

Then we give a negative answer to an open problem proposed by Casten and Holland [4] by proving the zero solution to a partially degenerate reaction diffusion system is asymptotically stable but not exponentially asymptotically stable, which indicates the non-equivalence of asymptotic stability and exponential asymptotic stability.

Finally, we give exponential asymptotic stability of the zero solution to a totally degenerate system with cross diffusion effects.

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