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Existence and multiplicity of solutions for a class of fourth-order elliptic equations in R^N ☆

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ABSTRACT

In this paper, we study the existence and multiplicity of nontrivial solutions for the following fourth-order elliptic equations

$$\begin{cases} \triangle^2 u - \triangle u + \lambda V(x)u = f(x,u), & \text{in } R^N \\ u \in H^2(R^N). \end{cases}$$

via variational methods. Two main theorems are obtained.

1. Introduction and preliminaries

Consider the following nonlinear fourth-order elliptic equations

$$\begin{cases} \triangle^2 u - \triangle u + \lambda V(x)u = f(x,u), & \text{in } R^N \\ u \in H^2(R^N). \end{cases} \tag{1.1}$$

There are a number of papers concerned with the Eqs. (1.1). For example, see [1–12]. In [2], An and Liu use the Mountain Pass Theorem to get the existence results for the following problem

$$\begin{cases} \triangle^2 u + c\triangle u = g(x,u), & \text{in } \Omega, \\ u = \triangle u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset R^N (N > 4)$ is a smooth bounded domain, $c \in R$. In [10], Wang et al. use linking approaches to obtain at least three nontrivial solutions for (1.2). In [11], Yang and Zhang consider the existence of positive, negative and sign-changing solutions for (1.2). In [6], Chabrowski and Marcos do Ó studied the existence of two solutions for the following fourth-order elliptic problems

$$\begin{cases} \triangle^2 u - \lambda g(x)u = f(x)|u|^{p-2}u, & \text{in } R^N \\ u \in D^{2,2}(R^N) \setminus \{0\}, \end{cases} \tag{1.3}$$

where $\lambda > 0$, $p = \frac{2N}{N-4}$. In [12], Yin and Wu use variational methods to get the high energy solutions and nontrivial solutions for Eqs. (1.1), where $\lambda = 1$, $V(x)$ is satisfying the following condition

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(V_1) $V \in C(R^N, R)$, $\inf_{x\in R^N} V(x) \geq a > 0$ and for each $M > 0$, $meas\{x \in R^N : V(x) \leq M\} < \infty$, where a is a constant and *meas* denote Lebesgue measure in R^N.

In the present paper, we will research the existence and multiplicity of nontrivial solutions for problem (1.1) under large $\lambda > 0$ and the condition

(V_2) $V \in C(R^N, R)$, $\inf_{x\in R^N} V(x) \geq a_0 > 0$, where a_0 is a constant. Moreover, there exists a constant $b > 0$ such that the set $\{x \in R^N : V(x) \leq b\}$ is nonempty and $meas\{x \in R^N : V(x) \leq b\} < \infty$, where *meas* denote the Lebesgue measure in R^N.

Remark 1.1. It is obvious that (V_2) is weaker than (V_1). There is a function $V(x) = \arctan|x| + \frac{\pi}{2}$, which satisfies (V_2) but does not satisfy (V_1).

We need the following preliminaries. Let

$$H = H^2(R^N) := \{u \in L^2(R^N) : |\nabla u|, \triangle u \in L^2(R^N)\}$$

with the inner product and norm

$$\langle u, v\rangle_H = \int_{R^N} (\triangle u \triangle v + \nabla u \cdot \nabla v + uv)dx, \quad \|u\|_H^2 = \langle u, u\rangle_H.$$

By $\|\cdot\|_p$ we denote L^p-norm. Set

$$E_\lambda = \left\{u \in H : \int_{R^N} (|\Delta u|^2 + |\nabla u|^2 + \lambda V(x)u^2)dx < \infty\right\}.$$

Then E_λ is a Hilbert space with the following inner product and the norm

$$\langle u, v\rangle_{E_\lambda} = \int_{R^N} (\Delta u \Delta v + \nabla u \cdot \nabla v + \lambda V(x)uv)dx, \quad \|u\|_{E_\lambda}^2 = \langle u, u\rangle_{E_\lambda}.$$

We use C to denote various positive constants. In order to deduce our statements, we need the following assumptions

(f_1) $f \in C(R^N \times R, R)$, $F(x, u) = \int_0^u f(x, s)ds \geq 0$ and $|f(x, u)| \leq C(1 + |u|^{p-1})$ for some $p \in (2, 2_*)$, $2_* = \frac{2N}{N-4}$ if $N > 4$; $2_* = \infty$ if $N \leq 4$.
(f_2) $f(x, u) = o(|u|)$ as $|u| \to 0$ uniformly for $x \in R^N$.
(f_3) There exists $\mu > 2$ such that $\mu F(x, u) \leq uf(x, u)$, $\forall (x, u) \in R^N \times R$.
(f_4) $c_1 = \inf_{u\in R, |u|=1} F(x, u) > 0$.
(f_5) There exists $\mu > 2$ such that $u \to \frac{f(x,u)}{|u|^{\mu-1}}$ is increasing on $(-\infty, 0)$ and $(0, +\infty)$.
(f_6) $f(x, -u) = -f(x, u)$, $\forall (x, u) \in R^N \times R$.

Now we are ready to state our main result.

Theorem 1.1. *If (V_2) and (f_1)–(f_4) hold, then problem (1.1) has at least one nontrivial solution for large $\lambda > 0$. Further, if the condition (f_6) is added, then the problem (1.1) has infinitely many distinct pairs of nontrivial solutions.*

Theorem 1.2. *If (V_2), (f_1)–(f_2) and (f_4)–(f_5) hold, then problem (1.1) has at least one nontrivial solution for large $\lambda > 0$. Further, if the condition (f_6) is added, then the problem (1.1) has infinitely many distinct pairs of nontrivial solutions.*

Remark 1.2. Under the assumption (V_1), motivated by Lemma 3.4 in [13], we can prove the embedding $E_1 \hookrightarrow L^s(R^N)$ is compact for any $s \in [2, 2_*)$, where $E_1 = \{u \in H : \int_{R^N} (|\Delta u|^2 + |\nabla u|^2 + V(x)u^2)dx < \infty\}$. Hence, the corresponding results have been obtained by using the variational techniques in a standard way. But (V_2) is weaker than (V_1), the embedding lacks the compactness, we have to overcome the difficulty.

Remark 1.3. Obviously, it follows from (V_2) that the embedding $E_\lambda \hookrightarrow L^s(R^N)$ is continuous for each $s \in [2, 2_*)$ if $\lambda \geq 1$. Hence, for any $s \in [2, 2_*)$, there is a constant $a_s > 0$ independent on λ such that $\|u\|_s \leq a_s\|u\|_{E_\lambda}$ for all $u \in E_\lambda$.

It is well known that a weak of problem (1.1) is a critical point of the following functional

$$I(u) = \frac{1}{2}\int_{R^N} (|\triangle u|^2 + |\nabla u|^2 + \lambda V(x)u^2)dx - \int_{R^N} F(x, u)dx.$$

Under the above assumptions, it is easy to know that $I \in C^1(E_\lambda, R)$ and

$$\langle I'(u), v\rangle = \int_{R^N} (\triangle u \triangle v + \nabla u \cdot \nabla v + \lambda V(x)uv)dx - \int_{R^N} f(x, u)v dx, \quad \forall u, v \in E_\lambda.$$

The following lemmas are our main tools.

Lemma 1.1 *([14, Theorem 2.2]). Let E be a real Banach space and $I \in C^1(E, R)$ satisfying (PS) condition. Suppose $I(0) = 0$ and*

(I_1) there are constants $\rho, \alpha > 0$ such that $I_{\partial B_\rho(0)} \geq \alpha$, and

(I_2) there is an $e \in E \setminus \overline{B_\rho(0)}$ such that $I(e) \leq 0$,

then I possesses a critical value $\eta \geq \alpha$.

Lemma 1.2 *([14, Theorem 9.12]). Let E be an infinite dimensional Banach space, $I(0) = 0$, $I \in C^1(E, R)$ be even and satisfy (PS) condition. Assume that $E = V \bigoplus X$, where V is finite dimensional, and I satisfies*

(I_3) there are constants $\rho, \alpha > 0$ such that $I_{\partial B_\rho(0) \bigcap X} \geq \alpha$

and

(I_4) for each finite dimensional subspace $\widetilde{E} \subset E$, there is an $R = R(\widetilde{E}) > 0$ such that $I_{\widetilde{E} \setminus B_R(0)} \leq 0$.

Then I possess an unbounded sequence of critical values.

2. Proof of main results

To complete the proof of our main theorems, we need following lemmas.

Lemma 2.1 *((See [15] or [16])). Let Ω be an open set in R^N and $f \in C(\Omega \times R, R)$ a function such that $|f(x, u)| \leq C_1(|u|^r + |u|^s)$ for some $C_1 > 0$ and $1 \leq r < s < \infty$. Suppose that $s \leq p < \infty, r \leq q < \infty, q > 1$, $\{u_n\}$ is a bounded sequence in $L^p(\Omega) \cap L^q(\Omega)$, $u_n \to u$ in $L^p(\Omega \cap B_R) \cap L^q(\Omega \cap B_R)$ for all $R > 0$ and $u_n(x) \to u(x)$ a.e. in $x \in \Omega$. Then, passing to a subsequence, there exists a sequence $\{v_n\}$ such that $v_n \to u$ in $L^p(\Omega) \cap L^q(\Omega)$ and*

$$f(x, u_n) - f(x, u_n - v_n) - f(x, u) \to 0,$$

in $L^{\frac{p}{s}}(\Omega) + L^{\frac{q}{r}}(\Omega)$, where $v_n(x) = \chi\left(\frac{2|x|}{R_n}\right) u(x)$, $\chi \in C^\infty(R, [0, 1])$ be such that $\chi(t) = 1$ for $t \leq 1$, $\chi(t) = 0$ for $t \geq 2$, $R_n > 0$ is a sequence of constants with $R_n \to \infty$, as $n \to \infty$, the space $L^p(\Omega) \cap L^q(\Omega)$ with the norm

$$\|u\|_{p\wedge q} := \|u\|_p + \|u\|_q$$

and the space $L^p(\Omega) + L^q(\Omega)$ with the norm

$$\|u\|_{p\vee q} := \inf\{\|v\|_p + \|w\|_q : v \in L^p(\Omega), w \in L^q(\Omega), u = v + w\}.$$

Lemma 2.2. *If $u_n \rightharpoonup u$ in E_λ, then, passing to a subsequence, there exists a sequence $\{v_n\}$ such that $v_n \to u$ in E_λ and*

$$I(u_n) = I(u_n - v_n) + I(u) + o(1) \tag{2.1}$$

and

$$I'(u_n) = I'(u_n - v_n) + I'(u) + o(1). \tag{2.2}$$

Particularly, if $\{u_n\}$ is a $(PS)_c$ sequence, then, passing to a subsequence, one has $I(u_n - v_n) \to c - I(u)$ and $I'(u_n - v_n) \to 0$ as $n \to \infty$.

Proof. Let $\chi \in C^\infty(R, [0, 1])$ be a cut-off function such that $\chi(t) = 1$ for $t \leq 1, \chi(t) = 0$ for $t \geq 2$ and set $v_n(x) = \chi\left(\frac{2|x|}{R_n}\right) u(x)$, where $R_n > 0$ is a sequence of constants with $R_n \to \infty$ as $n \to \infty$. Then $v_n \to u$ in E_λ. Indeed, $u \in E_\lambda$ implies that for any $\varepsilon > 0$, there is a corresponding $\rho_1 = \rho_1(\varepsilon) > 0$ such that

$$\int_{R^N \setminus B_{\rho_1}} |\triangle u|^2 dx < \varepsilon, \qquad \int_{R^N \setminus B_{\rho_1}} \lambda V(x) u^2 dx < \varepsilon, \qquad \int_{R^N \setminus B_{\rho_1}} |\nabla u|^2 dx < \varepsilon,$$

where $B_{\rho_1} := B_{\rho_1}(0)$. Thus,

$$\begin{aligned} \|v_n - u\|_{E_\lambda}^2 &= \int_{R^N} (|\triangle(v_n - u)|^2 + |\nabla(v_n - u)|^2 + \lambda V(x)|v_n - u|^2) dx \\ &= \int_{R^N} \left(\left| \triangle \left(\chi\left(\frac{2|x|}{R_n}\right) u - u \right) \right|^2 + \left| \nabla \left(\chi\left(\frac{2|x|}{R_n}\right) u - u \right) \right|^2 + \lambda V(x) \left| \chi\left(\frac{2|x|}{R_n}\right) u - u \right|^2 \right) dx \\ &\leq 4 \int_{R^N} \left| \chi\left(\frac{2|x|}{R_n}\right) - 1 \right|^2 |\triangle u|^2 dx + \frac{64}{R_n^2} \int_{R^N} \left| \chi'\left(\frac{2|x|}{R_n}\right) \right|^2 |\nabla u|^2 dx \end{aligned}$$

$$
\begin{aligned}
&+\frac{16(N-1)^2}{R_n^2}\int_{R^N}\frac{\left|\chi'\left(\frac{2|x|}{R_n}\right)\right|^2}{|x|^2}u^2dx+\frac{64}{R_n^4}\int_{R^N}\left|\chi''\left(\frac{2|x|}{R_n}\right)\right|^2u^2dx\\
&+2\int_{R^N}\left|\chi\left(\frac{2|x|}{R_n}\right)-1\right|^2|\nabla u|^2dx+\frac{8}{R_n^2}\int_{R^N}\left|\chi'\left(\frac{2|x|}{R_n}\right)\right|^2u^2dx\\
&+\int_{R^N}\lambda V(x)\left|\chi\left(\frac{2|x|}{R_n}\right)-1\right|^2u^2dx\\
\le\ &4\int_{B_{\rho_1}}\left|\chi\left(\frac{2|x|}{R_n}\right)-1\right|^2|\triangle u|^2dx+\frac{64}{R_n^2}\int_{R^N}\left|\chi'\left(\frac{2|x|}{R_n}\right)\right|^2|\nabla u|^2dx\\
&+\frac{16(N-1)^2}{R_n^2}\int_{R^N}\frac{\left|\chi'\left(\frac{2|x|}{R_n}\right)\right|^2}{|x|^2}u^2dx+\frac{64}{R_n^4}\int_{R^N}\left|\chi''\left(\frac{2|x|}{R_n}\right)\right|^2u^2dx\\
&+2\int_{B_{\rho_1}}\left|\chi\left(\frac{2|x|}{R_n}\right)-1\right|^2|\nabla u|^2dx+\frac{8}{R_n^2}\int_{R^N}\left|\chi'\left(\frac{2|x|}{R_n}\right)\right|^2u^2dx\\
&+\int_{B_{\rho_1}}\lambda V(x)\left|\chi\left(\frac{2|x|}{R_n}\right)-1\right|^2u^2dx+C\varepsilon\\
\le\ &4\int_{B_{\rho_1}}\left|\chi\left(\frac{2|x|}{R_n}\right)-1\right|^2|\triangle u|^2dx+\frac{64\alpha^2}{R_n^2}\int_{R^N}|\nabla u|^2dx\\
&+\frac{64\alpha^2(N-1)^2}{R_n^4}\int_{R^N}u^2dx+\frac{64\beta^2}{R_n^4}\int_{R^N}u^2dx\\
&+2\int_{B_{\rho_1}}\left|\chi\left(\frac{2|x|}{R_n}\right)-1\right|^2|\nabla u|^2dx+\frac{8\alpha^2}{R_n^2}\int_{R^N}u^2dx\\
&+\int_{B_{\rho_1}}\lambda V(x)\left|\chi\left(\frac{2|x|}{R_n}\right)-1\right|^2u^2dx+C\varepsilon,
\end{aligned}
$$

where $\alpha=\max_{1\le t\le 2}|\chi'(t)|$, $\beta=\max_{1\le t\le 2}|\chi''(t)|$. Hence, using the Lebesgue dominated converge theorem, we obtain

$$\|v_n-u\|_{E_\lambda}\to 0$$

as $n\to\infty$. Now, set $L:=\triangle^2-\triangle+\lambda V(x)$. Then

$$I(u)=\frac{1}{2}\langle Lu,u\rangle_2-\int_{R^N}F(x,u)dx,$$

where $\langle\cdot,\cdot\rangle_2$ denote inner product in $L^2(R^N)$. By $v_n\to u$ in E_λ and $u_n\rightharpoonup u$ in E_λ, we have

$$\langle Lu_n,v_n\rangle_2=\langle Lv_n,u_n\rangle_2\to\langle Lu,u\rangle_2$$

and

$$\langle Lu_n,u_n\rangle_2=\langle L(u_n-v_n),u_n-v_n\rangle_2+\langle Lu,u\rangle_2+o(1)$$

and

$$\langle u_n,\varphi\rangle_{E_\lambda}=\langle u_n-v_n,\varphi\rangle_{E_\lambda}+\langle u,\varphi\rangle_{E_\lambda}+o(1)$$

for each $\varphi\in E_\lambda$. Note that

$$\langle I'(u),\varphi\rangle=\langle u,\varphi\rangle_{E_\lambda}-\int_{R^N}f(x,u)\varphi dx,\quad\forall\varphi\in E_\lambda.$$

Hence, in order to prove (2.1) and (2.2), we only need to prove

A. $\int_{R^N}F(x,u_n)dx=\int_{R^N}F(x,u_n-v_n)dx+\int_{R^N}F(x,u)dx+o(1)$

and

B. $\sup_{\|\varphi\|_{E_\lambda}=1}\int_{R^N}(f(x,u_n)-f(x,u_n-v_n)-f(x,u))\varphi dx=o(1)$.

The proof of A: Since $u_n \rightharpoonup u$ in E_λ, passing to a subsequence, we can assume that $u_n \to u$ in $L^t_{loc}(R^N)$ for each $t \in [2, 2_*)$ and $u_n(x) \to u(x)$ a.e. in R^N. By (f_1) and (f_2) we know that for any $\varepsilon_1 > 0$ there is a constant $C(\varepsilon_1) > 0$ such that

$$|f(x,u)| \le \varepsilon_1|u| + C(\varepsilon_1)|u|^{p-1}, \quad \forall (x,u) \in R^N \times R, \tag{2.3}$$

and

$$|F(x,u)| \le \frac{\varepsilon_1}{2}|u|^2 + \frac{C(\varepsilon_1)}{p}|u|^p, \quad \forall (x,u) \in R^N \times R. \tag{2.4}$$

Taking $r = 2$, $s = p$, and $q = 2$ in Lemma 2.1, we know that

$$\int_{R^N} F(x,u_n)dx = \int_{R^N} F(x,u_n - v_n)dx + \int_{R^N} F(x,u)dx + o(1).$$

The proof of B: Taking $r = 1$, $s = p - 1$, and $q = 2$ in Lemma 2.1, we know that

$$g_n(x) := f(x,u_n) - f(x,u_n - v_n) - f(x,u) \to 0, \quad \text{in } L^2(R^N) + L^{\frac{p}{p-1}}(R^N).$$

Hence, for each $\varphi \in E_\lambda$ with $\|\varphi\|_{E_\lambda} = 1$, by Hölder's inequality and Sobolev's embedding, one has

$$\left|\int_{R^N} g_n\varphi dx\right| \le \|g_n\|_{2\vee p'}\|\varphi\|_{2\wedge p} \le C\|g_n\|_{2\vee p'},$$

where $p' = \frac{p}{p-1}$. Consequently,

$$\sup_{\|\varphi\|_{E_\lambda}=1} \int_{R^N} (f(x,u_n) - f(x,u_n - v_n) - f(x,u))\varphi dx = o(1).$$

This completes the proof of (2.1) and (2.2).

Moreover, if $\{u_n\}$ is a $(PS)_c$ sequence, that is, $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to \infty$, then

$$I(u_n - v_n) = c - I(u) + o(1).$$

Now, we prove $I'(u_n - v_n) \to 0$ as $n \to \infty$.

By (2.2) and $I'(u_n) \to 0$, it is sufficient to prove that $\langle I'(u), \varphi\rangle = 0$ for all $\varphi \in E_\lambda$. Since $u_n \to u$ in $L^t_{loc}(R^N)$ for $t \in [2, 2_*)$, by (f_1) and Theorem A.2 in [17], we have

$$f(x,u_n) \to f(x,u), \tag{2.5}$$

in $L^{\frac{p}{p-1}}_{loc}(R^N)$. Further, for any $\varepsilon_2 > 0$, there exists $\rho_2 > 0$ such that

$$\left(\int_{R^N\setminus B_{\rho_2}} |\varphi|^p dx\right)^{\frac{1}{p}} < \varepsilon_2. \tag{2.6}$$

It follows from (2.3), (2.5), (2.6) and Hölder's inequality that

$$\begin{aligned}
\left|\int_{R^N} (f(x,u_n) - f(x,u))\varphi dx\right| &\le \left(\int_{B_{\rho_2}} |f(x,u_n) - f(x,u)|^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \left(\int_{B_{\rho_2}} |\varphi|^p dx\right)^{\frac{1}{p}} \\
&\quad + \int_{R^N\setminus B_{\rho_2}} (\varepsilon_1(|u_n| + |u|) + C(\varepsilon_1)(|u_n|^{p-1} + |u|^{p-1}))\varphi dx \\
&\le \left(\int_{B_{\rho_2}} |f(x,u_n) - f(x,u)|^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \left(\int_{B_{\rho_2}} |\varphi|^p dx\right)^{\frac{1}{p}} \\
&\quad + \varepsilon_1\left(\int_{R^N\setminus B_{\rho_2}} |\varphi|^2 dx\right)^{\frac{1}{2}} (\|u_n\|_2 + \|u\|_2) + C(\varepsilon_1)\varepsilon_2(\|u_n\|_p^{p-1} + \|u\|_p^{p-1}).
\end{aligned}$$

Hence $\int_{R^N}(f(x,u_n) - f(x,u))\varphi dx \to 0$ as $n \to \infty$, and hence, for each $\varphi \in E_\lambda$, we have

$$\langle I'(u_n) - I'(u), \varphi\rangle = \langle u_n - u, \varphi\rangle_{E_\lambda} - \int_{R^N} (f(x,u_n) - f(x,u))\varphi dx \to 0.$$

Therefore, $\langle I'(u), \varphi\rangle = 0$ for all $\varphi \in E_\lambda$. This completes the proof. □

Lemma 2.3. *Let* (f_1) *and* (f_2) *be satisfied. Then* $\frac{|f(x,u)|^\tau}{|u|^\tau} \le \frac{1}{2}uf(x,u) - F(x,u) := g(x,u)$ *for some* $\tau \in \left(\max\left\{1, \frac{N}{4}\right\}, \frac{p}{p-2}\right)$ *and all* (x,u) *with* $|u|$ *large enough.*

Proof. By (f_1) and (f_2), we know that we have $d_i > 0$, $i = 1, 2$ such that

$$|f(x,u)| \le d_1|u| + d_2|u|^{p-1}$$

for all $(x,u) \in R^N \times R$. We note that $\frac{p}{p-2} > \max\left\{1, \frac{N}{4}\right\}$ because $p \in (2, 2_*)$. Fix $\tau \in \left(\max\left\{1, \frac{N}{4}\right\}, \frac{p}{p-2}\right)$, if $|u| \ge 1$, then $|f(x,u)| \le (d_1+d_2)|u|^{p-1}$. Choose $R \ge 1$ so large that $\frac{1}{\mu} \le \frac{1}{2} - \frac{(d_1+d_2)^{\tau-1}}{|u|^{p-(p-2)\tau}}$ whenever $|u| \ge R$. Then, for $|u|$ large enough,

$$\begin{aligned} 0 &\le F(x,u) \le \frac{1}{\mu}uf(x,u) \le \left(\frac{1}{2} - \frac{(d_1+d_2)^{\tau-1}}{|u|^{p-(p-2)\tau}}\right) uf(x,u) \\ &\le \left(\frac{1}{2} - \frac{|f(x,u)|^{\tau-1}}{|u|^{\tau+1}}\right) uf(x,u) \end{aligned}$$

and it follows that $\frac{|f(x,u)|^\tau}{|u|^\tau} \le \frac{1}{2}uf(x,u) - F(x,u) = g(x,u)$. This completes the proof. □

Lemma 2.4. *Let* (V_2) *and* $(f_1) - (f_3)$ *be satisfied. Then there is* $\Lambda > 0$ *such that for each* $c \in R$, I *satisfies* $(PS)_c$ *condition for all* $\lambda > \Lambda$.

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence. Then, by (f_3), we have

$$\begin{aligned} 1 + c + \|u_n\|_{E_\lambda} &\ge I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_{E_\lambda}^2 + \int_{R^N}\left(\frac{1}{\mu}u_n f(x,u_n) - F(x,u_n)\right)dx \\ &\ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_n\|_{E_\lambda}^2 \end{aligned}$$

for large n. This implies $\{u_n\}$ is bounded in E_λ. Hence, passing to a subsequence, we may assume that $u_n \rightharpoonup u$ in E_λ. Furthermore, Lemma 2.2 implies that, passing to a subsequence, there exists a sequence $v_n \to u$ in E_λ such that

$$I(u_n - v_n) \to c - I(u)$$

and

$$I'(u_n - v_n) \to 0$$

as $n \to \infty$. It follows from (V_2) and $w_n := u_n - v_n \rightharpoonup 0$ in E_λ that

$$\begin{aligned} \|w_n\|_2^2 &= \int_{\{x\in R^N : V(x) > b\}} w_n^2 dx + \int_{\{x \in R^N : V(x) \le b\}} w_n^2 dx \\ &\le \frac{1}{\lambda b}\int_{R^N} \lambda V(x) w_n^2 dx + \int_{\{x\in R^N : V(x)\le b\}} w_n^2 dx \\ &\le \frac{1}{\lambda b}\|w_n\|_{E_\lambda}^2 + o(1). \end{aligned} \tag{2.7}$$

Moreover, for τ in Lemma 2.3, set $s = \frac{2\tau}{\tau-1}$, then $2 < p < s < 2_*$. Given $q \in (s, 2_*)$, then by (2.7) and Hölder and Sobolev inequalities,

$$\|w_n\|_s^s \le \|w_n\|_2^{\frac{2(q-s)}{q-2}} \|w_n\|_q^{\frac{q(s-2)}{q-2}} \le a_q^{\frac{q(s-2)}{q-2}} \left(\frac{1}{\lambda b}\right)^{\frac{q-s}{q-2}} \|w_n\|_{E_\lambda}^s + o(1). \tag{2.8}$$

By Lemma 2.3, we know that for large $R > 0$, $\frac{|f(x,u)|^\tau}{|u|^\tau} \le g(x,u)$ holds for all (x,u) with $|u| \ge R$. By (2.3), we have $|f(x,u)| \le (\varepsilon_1 + C(\varepsilon_1)R^{p-2})|u| = C|u|$ for all (x,u) with $|u| \le R$. It follows from (2.7) that

$$\int_{|w_n|\le R} f(x,w_n)w_n dx \le C\int_{|w_n|\le R} w_n^2 dx \le \frac{C}{\lambda b}\|w_n\|_{E_\lambda}^2 + o(1).$$

By $I(w_n) \to c - I(u)$ and $I'(w_n) \to 0$, we have

$$I(w_n) - \frac{1}{2}\langle I'(w_n), w_n\rangle = \int_{R^N} g(x,w_n)dx \to c - I(u).$$

Therefore, there is $M > 0$ independent of λ such that $\left|\int_{R^N} g(x, w_n)dx\right| \le M$. Using (2.8) and Hölder's inequality, we obtain

$$\begin{aligned}\int_{|w_n|>R} f(x, w_n)w_n dx &\le \left(\int_{|w_n|>R} \frac{|f(x, w_n)|^\tau}{|w_n|^\tau} dx\right)^{\frac{1}{\tau}} \left(\int_{R^N} |w_n|^s dx\right)^{\frac{1}{s}} \left(\int_{R^N} |w_n|^s dx\right)^{\frac{1}{s}} \\ &\le \left(\int_{|w_n|>R} g(x, w_n)dx\right)^{\frac{1}{\tau}} \|w_n\|_s^2 \\ &\le M^{\frac{1}{\tau}} \|w_n\|_s^2 \\ &\le C\left(\frac{1}{\lambda b}\right)^\theta \|w_n\|_{E_\lambda}^2 + o(1),\end{aligned}$$

where $\theta = \frac{2(q-s)}{s(q-2)} > 0$. Therefore,

$$\begin{aligned}o(1) &= \langle I'(w_n), w_n\rangle = \|w_n\|_{E_\lambda}^2 - \int_{R^N} f(x, w_n)w_n dx \\ &\ge \|w_n\|_{E_\lambda}^2 - \frac{C}{\lambda b}\|w_n\|_{E_\lambda}^2 - C\left(\frac{1}{\lambda b}\right)^\theta \|w_n\|_{E_\lambda}^2 + o(1) \\ &= \left(1 - \frac{C}{\lambda b} - C\left(\frac{1}{\lambda b}\right)^\theta\right) \|w_n\|_{E_\lambda}^2 + o(1).\end{aligned}$$

Set $\Lambda > 0$ be so large that the term in the brackets above is positive for all $\lambda \ge \Lambda$. Then $w_n \to 0$ in E_λ for all $\lambda \ge \Lambda$. Since again $w_n = u_n - v_n$ and $v_n \to u$ in E_λ, $u_n \to u$ in E_λ. This completes the proof. □

Proof of Theorem 1.1. By (2.4) and Sobolev's embedding theorem, for $0 < \varepsilon_1 < \frac{1}{a_2^2}$ and small $\rho > 0$, we have

$$\begin{aligned}I(u) &\ge \frac{1}{2}\|u\|_{E_\lambda}^2 - \frac{\varepsilon_1}{2}\|u\|_2^2 - \frac{C(\varepsilon_1)}{p}\|u\|_p^p \\ &\ge \frac{1}{2}\|u\|_{E_\lambda}^2 - \frac{\varepsilon_1}{2}a_2^2\|u\|_{E_\lambda}^2 - \frac{C(\varepsilon_1)}{p}a_p^p\|u\|_{E_\lambda}^p \\ &\ge \frac{1}{5}(1 - \varepsilon_1 a_2^2)\|u\|_{E_\lambda}^2\end{aligned}$$

for all $u \in \overline{B_\rho(0)}$, where $B_\rho(0) = \{u \in E_\lambda : \|u\|_{E_\lambda} < \rho\}$. Therefore

$$I_{\partial B_\rho(0)} \ge \frac{1}{5}(1 - \varepsilon_1 a_2^2)\rho^2 := \alpha > 0.$$

Since $E_\lambda \hookrightarrow L^2(R^N)$ and $L^2(R^N)$ is a separable Hilbert space, E_λ has a countable orthogonal basis $\{e_j\}$. Set $E_\lambda^k = span\{e_1, e_2, \ldots, e_k\}$ and $Z_k = (E_\lambda^k)^\perp$. Then $E_\lambda = E_\lambda^k \bigoplus Z_k$, E_λ^k is finite-dimensional and $I_{\partial B_\rho(0)\cap Z_k} \ge \alpha > 0$. Moreover, for any finite-dimensional subspace $\widetilde{E} \subset E_\lambda$, there is a positive integral number m such that $\widetilde{E} \subset E_\lambda^m$. Since all norms are equivalent in a finite-dimensional space, there is a constant $\beta > 0$ such that $\|u\|_\mu \ge \beta\|u\|_{E_\lambda}$ for all $u \in E_\lambda^m$. By (2.4) and (f_3), (f_4), there exists $C(\varepsilon_1) > 0$ such that

$$F(x, u) \ge C|u|^\mu - C(\varepsilon_1)u^2, \quad \forall (x, u) \in R^N \times R.$$

Hence

$$\begin{aligned}I(u) &\le \frac{1}{2}\|u\|_{E_\lambda}^2 - C\|u\|_\mu^\mu + C(\varepsilon_1)\|u\|_2^2 \\ &\le \frac{1}{2}\|u\|_{E_\lambda}^2 - C\beta^\mu\|u\|_{E_\lambda}^\mu + C(\varepsilon_1)a_2^2\|u\|_{E_\lambda}^2\end{aligned}$$

for all $u \in E_\lambda^m$. Consequently, there is a large $R > 0$ such that $I < 0$ on $\widetilde{E} \setminus B_R$. Thus, there is an $e \in E_\lambda$ with $\|e\|_{E_\lambda} > R$ such that $I(e) < 0$. Finally, obviously, $I(0) = 0$ and $P.S.$ condition was proved in Lemma 2.4. Hence I possesses a critical value $\eta \ge \alpha$ by Lemma 1.1, i.e. the problem (1.1) has a nontrivial solution in E_λ. Moreover, obviously, (f_6) implies I is even. Hence, the second conclusion follows from Lemma 1.2. This completes the proof. □

Proof of Theorem 1.2. It is sufficient to prove that (f_5) implies (f_3). In fact, when $u > 0$,

$$F(x, u) = \int_0^1 f(x, ut)udt = \int_0^1 \frac{f(x, ut)}{(ut)^{\mu-1}}u^\mu t^{\mu-1}dt \le \int_0^1 \frac{f(x, u)}{u^{\mu-1}}u^\mu t^{\mu-1}dt = \frac{1}{\mu}f(x, u)u.$$

When $u < 0$,

$$\begin{aligned} F(x,u) &= \int_0^1 f(x,ut)u\,dt \\ &= -\int_0^1 \frac{f(x,ut)}{(-ut)^{\mu-1}}(-u)^{\mu}t^{\mu-1}dt \\ &= -\int_0^1 \frac{f(x,ut)}{|ut|^{\mu-1}}|u|^{\mu}t^{\mu-1}dt \\ &\le -\int_0^1 \frac{f(x,u)}{|u|^{\mu-1}}|u|^{\mu}t^{\mu-1}dt \\ &= \frac{1}{\mu}f(x,u)u. \end{aligned}$$

It shows that (f_3) holds. This completes the proof. □

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