J. Math. Anal. Appl. 405 (2013) 349–361

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

The Cauchy problem of a periodic higher order KdV equation in analytic Gevrey spaces

J. Gorsky [a], A. Alexandrou Himonas [b,*], C. Holliman [c], G. Petronilho [d]

[a] Department of Math. & Comp. Science, University of San Diego, San Diego, CA 92110, United States

[b] Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, United States

[c] Department of Biostatistics, University of Alabama at Birmingham, Birmingham, AL 35294, United States

[d] Departamento de Matemática, Universidade Federal de São Carlos, São Carlos, SP 13565-905, Brazil

ARTICLE INFO

Article history:
Received 6 May 2012
Available online 10 April 2013
Submitted by Thomas P. Witelski

Keywords:
KdV equation
Higher dispersion
Initial value problem
Well-posedness
Analytic Gevrey spaces
Uniform radius of analyticity
Sobolev spaces
Bilinear estimates
Bourgain spaces

ABSTRACT

This paper studies the periodic Cauchy problem for a KdV equation whose dispersion is of order $m = 2j + 1$, where j is a positive integer, (KdVm). Using Bourgain–Gevrey type analytic spaces and appropriate bilinear estimates, it is shown that local in time well-posedness holds when the initial data belong to an analytic Gevrey spaces of order σ. This implies that in the space variable the regularity of the solution remains the same with that of the initial data. It also implies that the size of the uniform radius of analyticity is preserved. Moreover, the solution is not necessarily G^σ in time. However, it belongs to $G^{m\sigma}(\mathbb{R})$ near zero for every x on the circle.

1. Introduction and results

We consider the initial value problem (IVP) for the following KdV equation with dispersion of order $m = 2j + 1$, where j is a positive integer, (KdVm)

$$\partial_t u + \partial_x^{2j+1} u + u\partial_x u = 0, \tag{1.1}$$

$$u(x, 0) = \varphi(x), \quad x \in \mathbb{T} \text{ and } t \in \mathbb{R}, \tag{1.2}$$

and study its well-posedness in analytic Gevrey spaces on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. In [12] it has been shown that KdVm is well-posed in $H^s(\mathbb{T})$ for $s \geq -1/2$. For KdV ($j = 1$), this result has been proved by Kenig, Ponce and Vega [24]. Furthermore, Hirayama [18] extending the result in [12] has shown that the Cauchy problem (1.1)–(1.2) is locally well-posed in $H^s(\mathbb{T})$ for any $s \geq -j/2$.

The global well-posedness for the same range of Sobolev exponents was proved in [7] by Colliander, Keel, Staffilani, Takaoka and Tao. For $s \geq 0$, both local and global well-posedness for the KdV was established earlier by Bourgain in [3]. Furthermore, Kappeler and Topalov [19] have shown well-posedness for KdV in $H^s(\mathbb{T})$, $s \geq -1$, in a weaker sense, using

* Corresponding author.
E-mail addresses: jgorsky@sandiego.edu (J. Gorsky), himonas.1@nd.edu (A.A. Himonas), cholliman@ms.soph.uab.edu (C. Holliman), gerson@dm.ufscar.br (G. Petronilho).

0022-247X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jmaa.2013.04.015

inverse scattering techniques. Concerning the ill-posedness of KdV, and in particular the smoothness of its data-to-solution map, it has been studied by many authors including Bourgain [4], Kenig, Ponce and Vega [25], Christ, Colliander and Tao [6], and Molinet [27]. For example, in [4] it is shown that the solution map for the periodic KdV is not analytic in $H^s(\mathbb{T})$ if $s < -1/2$. In [6] this result has been refined to failure of uniform continuity for $s < -1/2$.

Our main result here is about the well-posedness of KdVm when the initial data $\varphi(x)$ belong to a class of periodic analytic Gevrey functions, which in the analytic case can be extended holomorphically in a symmetric strip of the complex plane around the x-axis. More precisely, we have the following result.

Theorem 1. *Let $\sigma \geq 1$, $\delta > 0$, and $s \geq -j/2$. For initial data φ in the space*

$$G^{\sigma,\delta,s}(\mathbb{T}) = \left\{ f \in D'(\mathbb{T}) : \|f\|^2_{G^{\sigma,\delta,s}(\mathbb{T})} = \sum_{n\in\mathbb{Z}} |n|^{2s} e^{2\delta|n|^{1/\sigma}} |\widehat{f}(n)|^2 < \infty \right\} \tag{1.3}$$

there exists $T > 0$, which depends on φ, such that the Cauchy problem (1.1)–(1.2) has a unique solution $u(x,t)$ in $C([-T,T]; G^{\sigma,\delta,s}(\mathbb{T}))$. Furthermore, the data-to-solution map is continuous. Moreover, the regularity of the solution in the time variable is Gevrey of order $m\sigma$, and this result is sharp in the sense that there exist initial data that are in the space $G^{\sigma,\delta,s}(\mathbb{T})$ but the corresponding solution to (1.1)–(1.2) does not belong to $G^r(\mathbb{R})$, $1 \leq r < m\sigma$, in time near zero.

For the periodic KdV equation, spatial analyticity was proved first by Trubowitz [31]. An alternative approach using Bourgain spaces can be found in [11]. Furthermore, in [15] this approach was extended to gKdV and for initial data in analytic Gevrey spaces G^σ for $\sigma \geq 1$. Well-posedness of the non-periodic gKdV in analytic spaces $G^{1,\delta,s}$ defined by the norm

$$\|\varphi\|^2_{G^{1,\delta,s}} = \int_{\mathbb{R}} (1+|\xi|)^{2s} e^{2\delta(1+|\xi|)} |\widehat{\varphi}(\xi)|^2 d\xi, \tag{1.4}$$

has been proved by Grujić and Kalisch [13]. If the initial data belong to $G^{1,\delta,s}$, which means that they are analytic in a symmetric strip $\{z = x + iy : |y| < \delta\}$ around the x-axis in the complex plane, then there exists a time $T > 0$ such that the corresponding gKdV solution is analytic in the same strip during the time period $[0, T]$. This means that the uniform radius of spatial analyticity does not shrink as time progresses. In the periodic case the analogous result has been proved in [17]. Further results on the uniform radius of spatial analyticity have been established by Bona, Grujić and Kalisch [1]. For the KdV, the regularity in the time variable stated in Theorem 1 follows from [15]. Also for KdV, non-analytic solutions in time with analytic initial data have been constructed in [5].

We mention that the spaces $G^{\sigma,\delta,s}$ defined by (1.3) are spatially periodic analogues of the spaces introduced in [13] for the case of the real line, and later (e.g., in [1]) referred to as "Bourgain–Gevrey" spaces. Essentially they are hybrid spaces between the Bourgain and Foias–Temam-type Gevrey spaces.

Finally, for additional results concerning well-posedness and regularity properties of KdV type equations we refer the reader to De Bouard, Hayashi and Kato [9], Kato [20], Kato and Masuda [21], Kato and Ogawa [22], Kenig, Ponce and Vega [23], Bona and Smith [2], Ginibre and Tsutsumi [10], Saut and Temam [28], Sjöberg [29], Craig, Kappeler and Strauss [8], Linares and Ponce [26], Tao [30], and the references therein.

The rest of the paper is structured as follows. In Section 2 we define the Foias–Temam–Bourgain type analytic Gevrey spaces and prove well-posedness in these spaces by proving the corresponding bilinear estimates (Lemmas 2 and 3). In Section 3, we restrict our attention to the case of analytic initial data and show that the uniform radius of analyticity remains the same during the lifespan of the solution. In Section 4, we show that for initial data in Gevrey spaces G^σ the corresponding solution belongs to $G^{m\sigma}(\mathbb{R})$ for every x in $\mathbb{T}$. In Section 5, we show that this is optimal, that is the solution may not belong to $G^r(\mathbb{R})$ for any $1 \leq r < m\sigma$. In Section 6, we conclude by extending the proof of the bilinear estimates to the new range of Sobolev exponents $s \geq -j/2$.

2. Proof of Theorem 1

We shall use the notation

$$w \doteq u\partial_x u, \quad \text{so that } \widehat{w}(n,\lambda) = \frac{i}{8\pi^2} n(\widehat{u} * \widehat{u})(n,\lambda) \tag{2.1}$$

and set $m = 2j+1$, for $j = 1, 2, 3, \ldots$. Taking the Fourier transform with respect to x of the IVP (1.1)–(1.2) gives $\partial_t \widehat{u}(n,t) + (in)^m \widehat{u}(n,t) = -\widehat{w}(n,t)$, and $\widehat{u}(n,0) = \widehat{\varphi}(n)$. Note that $i^m = -i$ if $m = 3, 7, 11, \ldots$ and $i^m = i$ if $m = 5, 9, 13, \ldots$. Without loss of generality we will assume that $m = 3, 7, 11, \ldots$, since otherwise we replace t with $-t$. Thus, we may assume that our initial value problem can be rewritten as $\partial_t \widehat{u}(n,t) - in^m \widehat{u}(n,t) = -\widehat{w}(n,t)$, and $\widehat{u}(n,0) = \widehat{\varphi}(n)$. Solving it and using the inverse Fourier transform we get

$$u(x,t) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} e^{i(nx+n^m t)} \widehat{\varphi}(n) - \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \int_0^t e^{i[nx+n^m(t-t')]} \widehat{w}(n,t') dt'. \tag{2.2}$$

Defining $W(t)f(x) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{i(nx+n^m t)}\widehat{f}(n)$ Eq. (2.2) reads as follows

$$u(x,t) = W(t)\varphi(x) - \int_0^t W(t-t')w(x,t')dt'. \tag{2.3}$$

In order to localize in t we multiply Eq. (2.3) by a cut-off function $\psi(t) \in C_0^\infty(-1,1)$ with $0 \le \psi \le 1$ and such that $\psi(t) \equiv 1$ for $|t| < 1/2$. We then define the map T by

$$Tu(x,t) = \psi(t)W(t)\varphi(x) - \psi(t)\int_0^t W(t-t')w(x,t')dt'. \tag{2.4}$$

Using Eq. (2.2) we see that the definition of T in (2.4) is equivalent to

$$Tu(x,t) = \frac{1}{2\pi}\psi(t)\sum_{n\in\mathbb{Z}} e^{i(nx+n^m t)}\widehat{\varphi}(n) - \frac{1}{2\pi}\psi(t)\sum_{n\in\mathbb{Z}}\int_0^t e^{i[nx+n^m(t-t')]}\widehat{w}(n,t')dt'. \tag{2.5}$$

Substituting the inverse Fourier transform $\widehat{w}(n,t') = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda t'}\widehat{w}(n,\lambda)d\lambda$ into the above expression for Tu and after some manipulation we write it as follows

$$Tu(x,t) = \frac{1}{2\pi}\psi(t)\sum_{n\in\mathbb{Z}} e^{i(nx+n^m t)}\widehat{\varphi}(n) \tag{2.6}$$

$$+ \frac{i}{(2\pi)^2}\psi(t)\sum_{n\in\mathbb{Z}}\int_{|\lambda-n^m|\ge 1}\frac{e^{i(nx+\lambda t)}}{\lambda-n^m}\widehat{w}(n,\lambda)\,d\lambda \tag{2.7}$$

$$- \frac{i}{(2\pi)^2}\psi(t)\sum_{n\in\mathbb{Z}}\int_{|\lambda-n^m|\ge 1}\frac{e^{i(nx+n^m t)}}{\lambda-n^m}\widehat{w}(n,\lambda)\,d\lambda \tag{2.8}$$

$$+ \frac{i}{(2\pi)^2}\sum_{k=1}^{\infty}\frac{i^k}{k!}t^k\psi(t)\sum_{n\in\mathbb{Z}} e^{i(nx+n^m t)}\int_{|\lambda-n^m|\le 1}(\lambda-n^m)^{k-1}\widehat{w}(n,\lambda)\,d\lambda. \tag{2.9}$$

Note that in (2.7) and (2.8) $\lambda - n^m \neq 0$ since $|\lambda - n^m| \ge 1$; however, in (2.9) we will have that $\lambda - n^m = 0$ since $|\lambda - n^m| \le 1$.

Mean-zero data. For simplicity, we shall assume mean-zero data:

$$\widehat{\varphi}(0) = \int_{\mathbb{T}}\varphi(x)dx = 0. \tag{2.10}$$

For the necessary adjustments for reducing arbitrary initial data to mean-zero data see, for example, [16,3]. By (2.10) and (2.1) we have that $\widehat{\varphi}(0) = 0$ and $\widehat{w}(0,\lambda) = 0$, and, therefore, we can replace $n \in \mathbb{Z}$ with $n \in \mathbb{Z}^*$ in (2.6)–(2.9), where $\mathbb{Z}^* = \mathbb{Z}\setminus\{0\}$.

Bourgain space. We shall need the space Y^s, which is defined as the completion of the space of all functions that are in $\mathcal{S}(\mathbb{R})$ in the time variable and in $C^\infty(\mathbb{T})$ in the space variable with respect to the norm

$$\|u\|_{Y^s} \doteq \|u\|_{X^s} + \left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\left(\int_{\mathbb{R}}|\widehat{u}(n,\lambda)|d\lambda\right)^2\right)^{\frac{1}{2}}, \quad \text{where} \tag{2.11}$$

$$\|u\|_{X^s} \doteq \left(\sum_{n\in\mathbb{Z}^*}|n|^{2s}\int_{\mathbb{R}}(1+|\lambda-n^m|)|\widehat{u}(n,\lambda)|^2 d\lambda\right)^{\frac{1}{2}}. \tag{2.12}$$

The spaces X^s were introduced in [3] and the spaces Y^s were introduced in [7]. To incorporate our mean-zero data assumption (2.10) into our space, we define

$$\dot{Y}^s \doteq \{u \in Y^s : \widehat{u}(0,t) = 0 \text{ for all } t\} \quad \text{and} \quad \dot{X}^s \doteq \{u \in X^s : \widehat{u}(0,t) = 0 \text{ for all } t\}. \tag{2.13}$$

In (2.13) it is used that if u solves the IVP (1.1)–(1.2), then

$$\widehat{\varphi}(0) = \int_{\mathbb{T}}\varphi(x)dx = 0 \Leftrightarrow \widehat{u}(0,t) = \int_{\mathbb{T}}u(x,t)dx = 0 \quad \text{for every } t.$$

Bilinear estimates. Also, we shall need the following bilinear estimates.

Proposition 1. *For $s \geq -j/2$, where $j = 1, 2, 3, \ldots$ is fixed, and for all $f, g \in \dot{X}^s$ we have*

$$\left(\sum_{n \in \mathbb{Z}^*} |n|^{2s} \int_{\mathbb{R}} \frac{|\widehat{w}_{fg}(n, \lambda)|^2}{1 + |\lambda - n^m|} d\lambda \right)^{\frac{1}{2}} \lesssim \|f\|_{X^s} \|g\|_{X^s}, \quad \text{and} \tag{2.14}$$

$$\left(\sum_{n \in \mathbb{Z}^*} |n|^{2s} \left[\int_{\mathbb{R}} \frac{|\widehat{w}_{fg}(n, \lambda)|}{1 + |\lambda - n^m|} d\lambda \right]^2 \right)^{\frac{1}{2}} \lesssim \|f\|_{X^s} \|g\|_{X^s}, \tag{2.15}$$

where $w_{fg} = \partial_x (f \cdot g)$ and hence $\widehat{w}_{fg}(n, \lambda) = \frac{i}{8\pi^2} n (\widehat{f} * \widehat{g})(n, \lambda)$.

Here we use the notation $A \lesssim B$ to mean that $A \leq cB$, where c is a positive constant. The proof of Proposition 1 is presented in Section 6.

2.1. Existence

We start by recalling that in order to solve our Cauchy problem (1.1)–(1.2) we will solve the problem $Tu = u$, where the operator T is given by (2.6)–(2.9).

The solution spaces. For $\sigma \geq 1$, $\delta > 0$, $s \in \mathbb{R}$ we begin by defining the Foias–Temam–Bourgain type analytic Gevrey spaces that we will work with. We shall need to introduce the space $Y_{\sigma,\delta,s}$ that is defined as the completion of the space $C^\infty(\mathbb{T}; S(\mathbb{R}))$ with respect to the norm

$$\|v\|_{Y_{\sigma,\delta,s}} = \|v\|_{X_{\sigma,\delta,s}} + \left(\sum_{n \in \mathbb{Z}^*} |n|^{2s} e^{2\delta |n|^{1/\sigma}} \left[\int_{\mathbb{R}} |\widehat{v}(n, \lambda)| d\lambda \right]^2 \right)^{\frac{1}{2}}, \quad \text{where}$$

$$\|v\|^2_{X_{\sigma,\delta,s}} = \sum_{n \in \mathbb{Z}^*} |n|^{2s} e^{2\delta |n|^{1/\sigma}} \int_{\mathbb{R}} (1 + |\lambda - n^m|) |\widehat{v}(n, \lambda)|^2 d\lambda.$$

To incorporate our mean-zero data assumption (2.10) into our space, as in (2.13), we define

$$\dot{Y}_{\sigma,\delta,s} \doteq \{u \in Y_{\sigma,\delta,s} : \widehat{u}(0, t) = 0\} \quad \text{and} \quad \dot{X}_{\sigma,\delta,s} \doteq \{u \in X_{\sigma,\delta,s} : \widehat{u}(0, t) = 0\}.$$

The spaces $Y_{\sigma,\delta,s}$ possess the following key property.

Lemma 1. *$Y_{\sigma,\delta,s}(\mathbb{T} \times \mathbb{R}) \hookrightarrow C([-T, T], G^{\sigma,\delta,s}(\mathbb{T}))$, where T is any positive constant.*

Proof. Let $T > 0$ be given. For $u \in C([-T, T], G^{\sigma,\delta,s}(\mathbb{T}))$ we recall its norm $|u|_{C_{T,\sigma,\delta,s}} = \sup_{|t| \leq T} \|u(\cdot, t)\|_{G^{\sigma,\delta,s}(\mathbb{T})}$. For any $t \in \mathbb{R}$ we have

$$\begin{aligned} \|u(\cdot, t)\|_{G^{\sigma,\delta,s}(\mathbb{T})} &= \frac{1}{2\pi} \left(\sum_{n \in \mathbb{Z}} |n|^{2s} e^{2\delta |n|^{1/\sigma}} \left| \int_{\mathbb{R}} e^{i\lambda t} \widehat{u}(n, \lambda) d\lambda \right|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2\pi} \left(\sum_{n \in \mathbb{Z}} |n|^{2s} e^{2\delta |n|^{1/\sigma}} \left(\int_{\mathbb{R}} |\widehat{u}(n, \lambda)| d\lambda \right)^2 \right)^{\frac{1}{2}} \leq \frac{1}{2\pi} \|u\|_{Y_{\sigma,\delta,s}}. \end{aligned}$$

The proof is complete. □

Computing the $Y_{\sigma,\delta,s}$-norm of each one of the terms (2.6)–(2.9) defining the map T one can prove the following.

Proposition 2. *If $s \geq -j/2$ where $j = 1, 2, \ldots$ is fixed, then there is a constant $c_\psi > 0$ such that*

$$\begin{aligned} \|Tu\|_{Y_{\sigma,\delta,s}} \leq{}& c_\psi \left(\sum_{n \in \mathbb{Z}^*} |n|^{2s} e^{2\delta |n|^{1/\sigma}} \int_{\mathbb{R}} \frac{|\widehat{w}(n, \lambda)|^2}{1 + |\lambda - n^m|} d\lambda \right)^{1/2} \\ &+ c_\psi \left(\sum_{n \in \mathbb{Z}^*} |n|^{2s} e^{2\delta |n|^{1/\sigma}} \left(\int_{\mathbb{R}} \frac{|\widehat{w}(n, \lambda)|}{1 + |\lambda - n^m|} d\lambda \right)^2 \right)^{1/2} + c_\psi \|\varphi\|_{G^{\sigma,\delta,s}(\mathbb{T})}, \end{aligned} \tag{2.16}$$

for all $u \in \dot{X}_{\sigma,\delta,s}$.

We now are going to analyze the terms given in Proposition 2.

Lemma 2. *For $s \geq -j/2$ where $j = 1, 2, \dots$ is fixed, and $u \in \dot{X}_{\sigma,\delta,s}$ we have*

$$\left(\sum_{n\in\mathbb{Z}^*} |n|^{2s} e^{2\delta|n|^{1/\sigma}} \int_{\mathbb{R}} \frac{|\widehat{w}(n,\lambda)|^2}{1+|\lambda-n^m|} d\lambda \right)^{1/2} \leq C\|u\|^2_{X_{\sigma,\delta,s}},$$

for some positive constant C.

Proof. First, we observe that the operator A defined by $\widehat{Au}(n,\lambda) = e^{\delta|n|^{1/\sigma}}\widehat{u}(n,\lambda)$ satisfies the relations

$$\|u\|_{Y_{\sigma,\delta,s}} = \|Au\|_{Y^s}, \qquad \|v\|_{X_{\sigma,\delta,s}} = \|Av\|_{X^s}, \quad \text{for all } u \in Y_{\sigma,\delta,s} \text{ and } v \in X_{\sigma,\delta,s}. \tag{2.17}$$

It follows that

$$\begin{aligned}\sum_{n\in\mathbb{Z}^*} |n|^{2s} e^{2\delta|n|^{1/\sigma}} \int_{\lambda} \frac{|\widehat{w}(n,\lambda)|^2}{1+|\lambda-n^m|} d\lambda &= \sum_{n\in\mathbb{Z}^*} |n|^{2s} e^{2\delta|n|^{1/\sigma}} \int_{\lambda} \frac{|n(\widehat{u}*\widehat{u})(n,\lambda)|^2}{1+|\lambda-n^m|} d\lambda \\ &= \sum_{n\in\mathbb{Z}^*} |n|^{2s} e^{2\delta|n|^{1/\sigma}} \int_{\lambda} \frac{1}{1+|\lambda-n^m|} \\ &\quad \times \left| \sum_{n_1\in\mathbb{Z}^*} \int_{\lambda_1} n_1 \widehat{u}(n_1,\lambda_1)\widehat{u}(n-n_1,\lambda-\lambda_1) d\lambda_1 \right|^2 d\lambda.\end{aligned}$$

By using the inequality $|n|^{1/\sigma} \leq |n_1|^{1/\sigma} + |n-n_1|^{1/\sigma}$ it follows from the above that

$$\begin{aligned}\sum_{n\in\mathbb{Z}^*} |n|^{2s} e^{2\delta|n|^{1/\sigma}} \int_{\lambda} \frac{|\widehat{w}(n,\lambda)|^2}{1+|\lambda-n^m|} d\lambda &\leq \sum_{n\in\mathbb{Z}^*} |n|^{2s} \int_{\lambda} \frac{1}{1+|\lambda-n^m|} \\ &\quad \times \left| \sum_{n_1\in\mathbb{Z}^*} \int_{\lambda_1} n_1 \widehat{Au}(n_1,\lambda_1)\widehat{Au}(n-n_1,\lambda-\lambda_1) d\lambda_1 \right|^2 d\lambda.\end{aligned}$$

By setting $w_A = (Au)\partial_x(Au)$ we may conclude that

$$\sum_{n\in\mathbb{Z}^*} |n|^{2s} e^{2\delta|n|^{1/\sigma}} \int_{\lambda} \frac{|\widehat{w}(n,\lambda)|^2}{1+|\lambda-n^m|} d\lambda \leq \sum_{n\in\mathbb{Z}^*} |n|^{2s} \int_{\lambda} \frac{|\widehat{w_A}(n,\lambda)|^2}{1+|\lambda-n^m|} d\lambda.$$

It follows from Proposition 1 that

$$\sum_{n\in\mathbb{Z}^*} |n|^{2s} e^{2\delta|n|^{1/\sigma}} \int_{\lambda} \frac{|\widehat{w}(n,\lambda)|^2}{1+|\lambda-n^m|} d\lambda \leq \|Au\|^2_{X^s} = \|u\|^2_{X_{\sigma,\delta,s}}.$$

The proof is complete. □

Similarly one can prove the following.

Lemma 3. *For $s \geq -j/2$ where $j = 1, 2, \dots$ is fixed, and $u \in \dot{X}_{\sigma,\delta,s}$ we have*

$$\left(\sum_{n\in\mathbb{Z}^*} |n|^{2s} e^{2\delta|n|^{1/\sigma}} \left(\int_{\mathbb{R}} \frac{|\widehat{w}(n,\lambda)|}{1+|\lambda-n^m|} d\lambda \right)^2 \right)^{1/2} \leq C\|u\|^2_{X_{\sigma,\delta,s}},$$

for some positive constant C.

Combining Proposition 2, Lemmas 2 and 3 we obtain the following result.

Proposition 3. *If $s \geq -j/2$ where $j = 1, 2, \dots$ is fixed, then there is a constant $c_\psi > 0$ such that*

$$\|Tu\|_{Y_{\sigma,\delta,s}} \leq c_\psi \|u\|^2_{Y_{\sigma,\delta,s}} + c_\psi \|\varphi\|_{G_{\sigma,\delta,s}(\mathbb{T})}, \quad u \in \dot{Y}_{\sigma,\delta,s}, \tag{2.18}$$

and

$$\|Tu - Tv\|_{Y_{\sigma,\delta,s}} \leq c_\psi \left(\|u\|_{Y_{\sigma,\delta,s}} + \|v\|_{Y_{\sigma,\delta,s}} \right) \|u-v\|_{Y_{\sigma,\delta,s}}, \quad u, v \in \dot{Y}_{\sigma,\delta,s}. \tag{2.19}$$

Proof. Estimate (2.18) follows from Proposition 2, Lemmas 2 and 3. To prove estimate (2.19) we notice that

$$Tu - Tv = \frac{i}{(2\pi)^2}\psi(t)\sum_{n\in\mathbb{Z}} e^{i(nx+n^m t)}\int_{\mathbb{R}} \frac{e^{(\lambda-n^m)t}-1}{\lambda-n^m}\widehat{w}(n,\lambda)\,d\lambda,$$

where now w is given by

$$w = \frac{1}{2}\partial_x(u^2-v^2) = \frac{1}{2}\partial_x[(u-v)(u+v)] = \frac{1}{2}[\partial_x(u-v)](u+v) + \frac{1}{2}(u-v)[\partial_x(u+v)]. \tag{2.20}$$

Applying Proposition 2 with $\varphi = 0$ and Lemmas 2 and 3 for each one of the two terms of the sum (2.20), we obtain (2.19), thus completing the proof of Proposition 3. □

The next proposition shows that our map T is in fact a contraction. Its proof is an easy consequence of Proposition 3.

Proposition 4. *Let c_ψ be the constant appearing in Proposition 3. If $s \ge -j/2$ where $j = 1, 2, \dots$ is fixed, and the initial data φ satisfies the smallness condition $\|\varphi\|_{G^{\sigma,\delta,s}(\mathbb{T})} \le 1/18c_\psi^2$, if we choose the ball $B(0,r) \doteq \left\{u \in \dot{Y}_{\sigma,\delta,s} : \|u\|_{Y_{\sigma,\delta,s}} \le r\right\}$ with radius $r = 1/6c_\psi$, then $T : B(0,r) \to B(0,r)$ is a contraction.*

End of the proof of existence.

By Proposition 4 we see that for $\|\varphi\|_{G^{\sigma,\delta,s}(\mathbb{T})}$ sufficiently small, the operator T is a contraction on a small ball centered at the origin in $\dot{Y}_{\sigma,\delta,s}$, and hence the transformation T has a unique fixed point u in a $\dot{Y}_{\sigma,\delta,s}$-neighborhood of 0. Since $\psi(t) \equiv 1$, $|t| < 1/2$ it follows that $u(x,t)$ solves the KdVm initial-value problem (1.1)–(1.2). Finally, thanks to Lemma 1, with $T = 1/2$, we have proved the existence of a solution to our Cauchy problem which belongs to the space $C([-\frac{1}{2},\frac{1}{2}], G^{\sigma,\delta,s}(\mathbb{T}))$.

2.2. Uniqueness

Uniqueness of the solution in $C([-\frac{1}{2},\frac{1}{2}], G^{\sigma,\delta,s}(\mathbb{T}))$ can be proved by the following standard argument.

Lemma 4. *Suppose that u and v are solutions to (1.1)–(1.2) in $C([-\frac{1}{2},\frac{1}{2}], G^{\sigma,\delta,s}(\mathbb{T}))$ with $u(\cdot,0) = v(\cdot,0)$ in $G^{\sigma,\delta,s}(\mathbb{T})$ and $s \ge -j/2$. Then $u = v$.*

Proof. Setting $w = u - v$, we see that w solves the Cauchy problem

$$\partial_t w + \partial_x^{2j+1} w + \frac{1}{2}\partial_x[(u+v)w] = 0, \qquad w(0) = 0. \tag{2.21}$$

Then using Eq. (2.21) we form the following identity for the L^2-energy of w

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\mathbb{T})} = -\int_{\mathbb{T}} w\partial_x^{2j+1}w dx - \frac{1}{2}\int_{\mathbb{T}} w\partial_x[(u+v)w]dx. \tag{2.22}$$

Integrating by parts we obtain that $\int_{\mathbb{T}} w\partial_x^{2j+1}w dx = 0$ since $2j+1$ is odd. Using this and again integrating by parts, from Eq. (2.22) we get $\frac{d}{dt}\|w(t)\|^2_{L^2(\mathbb{T})} = -\frac{1}{2}\int_{\mathbb{T}} \partial_x(u+v)\cdot w^2 dx$, from which we deduce the inequality

$$\left|\frac{d}{dt}\|w(t)\|^2_{L^2(\mathbb{T})}\right| \le \frac{1}{2}\|\partial_x(u+v)\|_{L^\infty}\|w(t)\|^2_{L^2(\mathbb{T})}. \tag{2.23}$$

Since $u, v \in C([-\frac{1}{2},\frac{1}{2}], G^{\sigma,\delta,s}(\mathbb{T}))$ we have that u and v are continuous in t on the compact set $[-\frac{1}{2},\frac{1}{2}]$ and are C^∞ in x on the compact torus. This implies that the L^∞ norm of $\partial_x(u+v)$ is finite, that is

$$\|\partial_x(u+v)(t)\|_{L^\infty} \le c_0 < \infty. \tag{2.24}$$

Therefore, from (2.23) and (2.24) we obtain the differential inequality

$$\left|\frac{d}{dt}\|w(t)\|^2_{L^2(\mathbb{T})}\right| \le c\|w(t)\|^2_{L^2(\mathbb{T})}, \quad |t| \le 1/2, \tag{2.25}$$

where $c = \frac{c_0}{2}$. Solving this gives

$$\|w(t)\|^2_{L^2(\mathbb{T})} \le e^c\|w(0)\|^2_{L^2(\mathbb{T})}, \quad |t| \le 1/2. \tag{2.26}$$

Since $\|w(0)\|_{L^2(\mathbb{T})} = 0$, from Eq. (2.26) we obtain that $w(t) = 0$ or $u = v$. □

2.3. Continuous dependence of the initial data

To prove continuous dependence of the initial data we will prove the following.

Lemma 5. *Let $s \geq -j/2,\ j = 1, 2, \ldots$. If u and v are solutions to (1.1)–(1.2) corresponding to initial data φ and θ respectively with the norms $\|\varphi\|_{G^{\sigma,\delta,s}(\mathbb{T})}$, $\|\theta\|_{G^{\sigma,\delta,s}(\mathbb{T})}$ small, then*

$$|u - v|_{C_{1/2},\sigma,\delta,s} \leq \frac{3}{2} c_\psi \|\varphi - \theta\|_{G^{\sigma,\delta,s}(\mathbb{T})}.$$

Proof. We have

$$|u - v|_{C_{1/2},\sigma,\delta,s} = \sup_{t\in[-\frac{1}{2},\frac{1}{2}]} \|(u - v)(\cdot, t)\|_{G^{\sigma,\delta,s}(\mathbb{T})} \leq \|u - v\|_{Y_{\sigma,\delta,s}} = \|Tu - Tv\|_{Y_{\sigma,\delta,s}}. \tag{2.27}$$

To estimate $\|Tu - Tv\|_{Y_{\sigma,\delta,s}}$ we notice that

$$Tu - Tv = \frac{1}{2\pi}\psi(t)\sum_{n\in\mathbb{Z}} e^{i(nx+n^m t)}\widehat{(\varphi - \theta)}(n) + \frac{i}{(2\pi)^2}\psi(t)\sum_{n\in\mathbb{Z}} e^{i(nx+n^m t)}\int_{\mathbb{R}} \frac{e^{i(\lambda-n^m)t} - 1}{\lambda - n^m}\widehat{w}(n, \lambda)\, d\lambda,$$

where w is given by $w = \frac{1}{2}\partial_x(u^2 - v^2) = \frac{1}{2}\partial_x[(u - v)(u + v)]$. Thus, applying (2.18) with φ replaced by $\varphi - \theta$ and u^2 replaced by $(u - v)(u + v)$ we obtain

$$\|Tu - Tv\|_{Y_{\sigma,\delta,s}} \leq c_\psi \|\varphi - \theta\|_{G^{\sigma,\delta,s}(\mathbb{T})} + c_\psi \|u - v\|_{Y_{\sigma,\delta,s}}(\|u\|_{Y_{\sigma,\delta,s}} + \|v\|_{Y_{\sigma,\delta,s}}).$$

By taking $u, v \in B(0, r)$, where $r = 1/6c_\psi$, it follows from the last inequality that

$$\|u - v\|_{Y_{\sigma,\delta,s}} = \|Tu - Tv\|_{Y_{\sigma,\delta,s}} \leq c_\psi \|\varphi - \theta\|_{G^{\sigma,\delta,s}(\mathbb{T})} + \frac{1}{3}\|u - v\|_{Y_{\sigma,\delta,s}}.$$

Thus, $|u - v|_{C_{1/2},\sigma,\delta,s} \leq \|u - v\|_{Y_{\sigma,\delta,s}} \leq \frac{3}{2} c_\psi \|\varphi - \theta\|_{G_{\sigma,\delta,s}(\mathbb{T})}$. The proof is complete. □

3. Uniform radius of analyticity

Next we show that the radius of analyticity of the solution $u(\cdot, t)$ does not change as time progresses.

If $\varphi \in G^{1,\delta,s}(\mathbb{T})$, where $s \in \mathbb{R}$, then it follows from the definition of the space $G^{1,\delta,s}(\mathbb{T})$, that there exists a positive constant $L(s)$, which may depend on s, such that the following inequality holds true

$$|\hat{\varphi}(n)| \leq L(s)e^{-(\delta-\epsilon)|n|}, \quad \forall\, n \in \mathbb{Z}^*, \tag{3.1}$$

for any $0 < \epsilon < \delta$. Then, from (3.1) it follows that $\psi(x) \doteq \sum_{n=0}^{\infty} \widehat{\varphi}(n)e^{inx} \in C^\omega(\mathbb{T})$. Furthermore, $\widehat{\psi}(n) = \widehat{\varphi}(n), \forall\, n \in \mathbb{Z}$. Since $\varphi \in D'(\mathbb{T})$, we conclude that $\varphi \in C^\omega(\mathbb{T})$.

We also have the following analytic continuation result.

Lemma 6. *If $\varphi \in G^{1,\delta,s}(\mathbb{T})$, with $s \in \mathbb{R}$, then φ has an analytic extension in a symmetric strip around the real axis and its width is equal to δ.*

Proof. Since we can write $\varphi(x) = \sum_{n\in\mathbb{Z}} e^{inx}\hat{\varphi}(n)$, we define

$$\tilde{\varphi}(x + iy) = \sum_{n\in\mathbb{Z}} e^{in(x+iy)}\hat{\varphi}(n) = \sum_{n\in\mathbb{Z}} e^{inx}e^{-yn}\hat{\varphi}(n),$$

which gives $\tilde{\varphi}(x + i0) = \varphi(x)$. Next we show that $\tilde{\varphi}$ is holomorphic in the strip $|y| < \delta$. In fact, given y such that $|y| < \delta$, we define $\epsilon = \frac{\delta-|y|}{2}$, and therefore, it follows from (3.1), that there exists $L(s) > 0$ such that

$$|\tilde{\varphi}(x + iy)| \leq \sum_{n\in\mathbb{Z}} e^{-yn}|\hat{\varphi}(n)| \leq \sum_{n\in\mathbb{Z}} e^{|y||n|}L(s)e^{-(\delta-\epsilon)|n|} = L(s)\sum_{n\in\mathbb{Z}} e^{-\frac{(\delta-|y|)|n|}{2}} < \infty.$$

Differentiating the series defining $\tilde{\varphi}(x+iy)$, we can show similarly that the resulting series converges absolutely. Therefore, we can apply the Cauchy–Riemann operator $\bar{\partial}$ term by term to see that $\bar{\partial}\tilde{\varphi} = 0$, which shows that $\tilde{\varphi}$ is analytic in $|y| < \delta$ and 2π-periodic in x. This completes the proof of Lemma 6. □

Lemma 7. *The solution, $u(\cdot, t)$, to the Cauchy problem (1.1)–(1.2) with initial data $\varphi \in G^{1,\delta,s}(\mathbb{T})$, for $s \in \mathbb{R}$, has an analytic extension to a symmetric strip around the real axis and its width is δ. Therefore, the uniform analyticity radius does not change as the time progresses.*

Proof. Let $u \in C([-\frac{1}{2}, \frac{1}{2}], G^{1,\delta,s}(\mathbb{T}))$, $s \in \mathbb{R}$. Recalling that we can write $u(x,t) = \sum_{k\in\mathbb{Z}} e^{ikx}\hat{u}(k,t)$, we define

$$\tilde{u}(x+iy,t) = \sum_{k\in\mathbb{Z}} e^{ik(x+iy)}\hat{u}(k,t) = \sum_{k\in\mathbb{Z}} e^{ikx}e^{-yk}\hat{u}(k,t).$$

Then, as the proof of Lemma 6, it follows that $\tilde{u}(\cdot,t)$ is holomorphic in the strip $|y| < \delta$. □

This completes the proof of the first part of Theorem 1.

4. Gevrey regularity in time

In this section we are going to prove that the solution to the Cauchy problem (1.1)–(1.2) has Gevrey regularity in the time variable. More precisely, we will prove the following result.

Theorem 2. *The solution $u(x,t) \in C([-\frac{1}{2}, \frac{1}{2}], G^{\sigma,\delta,s}(\mathbb{T}))$, to the KdVm Cauchy problem (1.1)–(1.2) belongs to $G^{m\sigma}(\mathbb{R})$ in the time variable t, for t near zero.*

Proof (*First Case: $s \geq 0$*)**.** We will follow the proof of Theorem 4.1 in [14]. For this it suffices to prove the following.

Lemma 8. *For $k = 0, 1, \ldots$ and $j = 0, 1, 2, \ldots$ the following inequality holds true*

$$\left|\partial_t^j \partial_x^k u(x,t)\right| \leq C^{k+j+1}((k+mj)!)^\sigma \left(C^{m-1} + \frac{C}{2^\sigma}\right)^j, \tag{4.1}$$

for $t \in [-1,1]$, $x \in \mathbb{T}$, for some positive constant C.

Proof of Lemma 8. We will prove it by using induction on j. For $j = 0$, inequality (4.1) follows from the following result.

Proposition 5. *Let $u \in C([-\frac{1}{2}, \frac{1}{2}], G^{\sigma,\delta,s}(\mathbb{T}))$ be the solution to the Cauchy problem (1.1)–(1.2) with initial data $\varphi \in G^{\sigma,\delta,s}(\mathbb{T})$, $s \geq 0$. Then the solution u, in x, belongs to $G^\sigma(\mathbb{T})$, i.e., there exists $C > 0$ such that*

$$|\partial_x^\ell u(x,t)| \leq C^{\ell+1}(\ell!)^\sigma, \quad \forall\, x \in \mathbb{T},\ t \in [-1,1],\ \forall\, \ell \in \{0,1,\ldots\}. \tag{4.2}$$

Proof of Proposition 5. Let u be as in the statement of Proposition 5. Thus, for any $t \in [-\frac{1}{2}, \frac{1}{2}]$, we have

$$\|\partial_x^j u(\cdot,t)\|^2_{H^s(\mathbb{T})} = \sum_{k\in\mathbb{Z}} |k|^{2s}|\widehat{\partial_x^j u}(k,t)|^2 = \sum_{k\in\mathbb{Z}} |k|^{2s}|k|^{2j}|\widehat{u}(k,t)|^2. \tag{4.3}$$

It is easily seen that $|k|^{2j}e^{-\epsilon_1|k|^{1/\sigma}} \leq (j!)^{2\sigma}\left(\frac{2\sigma}{\epsilon_1}\right)^{2j\sigma}$ where $\epsilon_1 > 0$. Thanks to this, it follows from (4.3) that

$$\|\partial_x^j u(\cdot,t)\|^2_{H^s(\mathbb{T})} \leq \sum_{k\in\mathbb{Z}} |k|^{2s}\left(\frac{\sigma}{\delta}\right)^{2j\sigma}(j!)^{2\sigma}e^{2\delta|k|^{1/\sigma}}|\widehat{u}(k,t)|^2 = \left(\frac{\sigma}{\delta}\right)^{2j\sigma}(j!)^{2\sigma}\sum_{k\in\mathbb{Z}} |k|^{2s}e^{2\delta|k|^{1/\sigma}}|\widehat{u}(k,t)|^2.$$

By using the definition of the norms $\|\cdot\|_{G^{\sigma,\delta,s}(\mathbb{T})}$ and $|u|_{C_{1/2,\sigma,\delta,s}}$ we can conclude that

$$\|\partial_x^j u(\cdot,t)\|^2_{H^s(\mathbb{T})} \leq \left(\frac{\sigma}{\delta}\right)^{2j\sigma}(j!)^{2\sigma}|u|^2_{C_{1/2,\sigma,\delta,s}}$$

and therefore we have $\|\partial_x^j u(\cdot,t)\|_{H^s(\mathbb{T})} \leq C^j_{\sigma,\delta}(j!)^\sigma|u|^2_{C_{1/2,\sigma,\delta,s}}$, $\forall\, |t| \leq 1/2$, where $C_{\sigma,\delta} = (\frac{\sigma}{\delta})^\sigma$. Setting $A_{\sigma,\delta,s} \doteq |u|^2_{C_{1/2,\sigma,\delta,s}}$ we obtain

$$\|\partial_x^j u(\cdot,t)\|_{H^s(\mathbb{T})} \leq A_{\sigma,\delta,s}C^j_{\sigma,\delta}(j!)^\sigma, \quad \text{for } s \in \mathbb{R} \text{ and for any } t \in \left[-\frac{1}{2}, \frac{1}{2}\right]. \tag{4.4}$$

Thanks to (4.4), we will prove, for $s \geq 0$, that the solution u, in x, belongs to $G^\sigma(\mathbb{T})$.

Case 1.1: Fix $s > 1/2$. By the Sobolev Lemma we have that for all $j \in \{1,2,\ldots\}$

$$\|\partial_x^j u(\cdot,t)\|_{L^\infty(\mathbb{T})} \leq C_s\|\partial_x^j u(\cdot,t)\|_{H^s(\mathbb{T})} \leq C_s A_{\sigma,\delta,s}C^j_{\sigma,\delta}(j!)^\sigma \leq C_1^{j+1}(j!)^\sigma, \tag{4.5}$$

where $C_1 = \max\{C_s A_{\sigma,\delta,s}, C_{\sigma,\delta}\}$.

Case 1.2: Fix $0 \leq s \leq 1/2$. Applying again the Sobolev Lemma and using the fact that $\|\cdot\|_0 \leq \|\cdot\|_s$ we obtain $\|\partial_x^j u(\cdot,t)\|^2_{L^\infty(\mathbb{T})} \leq C_2\|\partial_x^j u(\cdot,t)\|^2_{H^1(\mathbb{T})} = C_2\left(\|\partial_x^j u(\cdot,t)\|^2_{L^2(\mathbb{T})} + \|\partial_x^{j+1}u(\cdot,t)\|^2_{L^2(\mathbb{T})}\right) \leq C_2\left(\|\partial_x^j u(\cdot,t)\|^2_{H^s(\mathbb{T})} + \|\partial_x^{j+1}u(\cdot,t)\|^2_{H^s(\mathbb{T})}\right)$. By using (4.4) it follows from the above that

$$\|\partial_x^j u(\cdot,t)\|^2_{L^\infty(\mathbb{T})} \leq C_2\left(A_{\sigma,\delta,s}C^j_{\sigma,\delta}(j!)^\sigma + A_{\sigma,\delta,s}C^{j+1}_{\sigma,\delta}((j+1)!)^\sigma\right) = C_2A_{\sigma,\delta,s}C^j_{\sigma,\delta}(j!)^\sigma\left[1 + C_{\sigma,\delta}(j+1)^\sigma\right].$$

Thanks to the fact that $x \le e^x$, $x \ge 0$ we have $1 + (j+1)^\sigma \le 2(e^{j+1})^\sigma$ and therefore we obtain from this and from the last inequality that

$$\|\partial_x^j u(\cdot,t)\|^2_{L^\infty(\mathbb{T})} \le \tilde{C_{\sigma,\delta}} C_2 A_{\sigma,\delta,s} C^j_{\sigma,\delta} (j!)^\sigma 2(e^{j+1})^{2\sigma}, \quad \text{where } \tilde{C_{\sigma,\delta}} = \max\{1, C_{\sigma,\delta}\}.$$

Finally, by setting $L = 2e^{2\sigma}\tilde{C_{\sigma,\delta}} C_2 A_{\sigma,\delta,s}$ and $M = e^{2\sigma} C_{\sigma,\delta}$ and $C_2 = \max\{L, M\}$ we can conclude that

$$\|\partial_x^j u(\cdot,t)\|^2_{L^\infty(\mathbb{T})} \le C_2^{j+1} (j!)^\sigma. \tag{4.6}$$

We have shown that for each fixed $s \ge 0$ there exist constants that depend on σ, δ and s such that (4.5) and (4.6) hold and, therefore, $u(\cdot,t) \in G^\sigma(\mathbb{T})$. The proof of Proposition 5 is complete. □

Remark 1. In (4.1) the constant $C = \max\{C_1, C_2\}$, where C_1 and C_2 are given above.

We now suppose that (4.1) holds for all derivatives in t of order $\le j$ and $k \in \{0, 1, 2, \ldots\}$ and we shall prove that (4.1) holds for $j+1$ and $k \in \{0, 1, 2, \ldots\}$.

Replacing t with $-t$ we may write our KdVm equation as $\partial_t u = \partial_x^m u + u\partial_x u$. Differentiating this equation j times with respect to t and k times with respect to x gives

$$\partial_t^{j+1}\partial_x^k u = \partial_t^j \partial_x^{k+m} u + \partial_t^j \partial_x^k (u \cdot \partial_x u).$$

Using the Leibniz formula for the derivative with respect to x we obtain

$$\partial_t^{j+1}\partial_x^k u = \partial_t^j \partial_x^{k+m} u + \partial_t^j \left(\sum_{p=0}^{k} \binom{k}{p} \partial_x^{k-p} u \partial_x^{p+1} u \right).$$

We now use the Leibniz formula for the derivative with respect to t and we obtain

$$\begin{aligned} \partial_t^{j+1}\partial_x^k u &= \partial_t^j \partial_x^{k+m} u + \sum_{p=0}^{k} \binom{k}{p} (\partial_t^j \partial_x^{k-p} u)(\partial_x^{p+1} u) \\ &\quad + \sum_{p=0}^{k} \binom{k}{p} (\partial_x^{k-p} u)(\partial_t^j \partial_x^{p+1} u) + \sum_{\ell=1}^{j-1}\sum_{p=0}^{k} \binom{j}{\ell}\binom{k}{p} (\partial_t^{j-\ell}\partial_x^{k-p} u)(\partial_t^\ell \partial_x^{p+1} u). \end{aligned}$$

By using the induction hypotheses and following the lines of the proof of Lemma 4.2 in [14] one can conclude the proof of Lemma 8.

Second case: $s < 0$. We notice that there exists a positive constant C such that

$$\sum_{k\in\mathbb{Z}} e^{2(\delta-\epsilon)|k|^{1/\sigma}} |\widehat{u}(k,t)|^2 \le C \sum_{k\in\mathbb{Z}} \frac{1}{|k|^{-2s}} e^{2\epsilon|k|^{1/\sigma}} e^{2(\delta-\epsilon)|k|^{1/\sigma}} |\widehat{u}(k,t)|^2 = C \sum_{k\in\mathbb{Z}} |k|^{2s} e^{2\delta|k|^{1/\sigma}} |\widehat{u}(k,t)|^2.$$

It now follows from this inequality that if $u(x,t) \in C([-\frac{1}{2}, \frac{1}{2}], G^{\sigma,\delta,s}(\mathbb{T}))$ where $s < 0$, then $u(x,t) \in C([-\frac{1}{2}, \frac{1}{2}], G^{\sigma,\delta-\epsilon,0}(\mathbb{T}))$ and therefore thanks to the first case we can conclude that $u(x,\cdot) \in G^{m\sigma}(\mathbb{R})$ in the time variable t, for t near zero.

5. Failure of G^r-regularity in time if $1 \le r < m\sigma$

Replacing t with $-t$ we can write our KdVm initial value problem as follows

$$\partial_t u = \partial_x^m u + u\partial_x u, \tag{5.1}$$

$$u(x,0) = \varphi(x), \quad x \in \mathbb{T},\ t \in \mathbb{R}, \tag{5.2}$$

where φ is a *real*-valued function to be chosen appropriately in the space $G^{\sigma,\delta,s}(\mathbb{T})$. The following lemma is crucial in estimating the higher-order derivatives of a solution with respect to t. We will omit its proof here since it is a particular case of Lemma 2.2 in [14].

Lemma 9. *If u is a solution to (5.1), then for every $j \in \{1, 2, \ldots\}$ we have*

$$\partial_t^j u = \partial_x^{mj} u + \sum_{q=1}^{j} \sum_{|\alpha|+(m-1)q=mj} C_\alpha^q (\partial_x^{\alpha_1} u)\cdots(\partial_x^{\alpha_{q+1}} u), \quad \textit{where } C_\alpha^q \ge 0. \tag{5.3}$$

We will split our study into two cases.

Case 1: $m = 3, 7, 11, \ldots$. We shall prove the following result.

Theorem 3. *Assume that $m = 4p+3$, $p = 0, 1, 2, \ldots, \sigma \geq 1, \delta > 0$ and $s \in \mathbb{R}$. If we take $u(x, 0) = -\mathrm{Re}\left(\sum_{n=1}^{\infty} \widehat{\psi}(n) e^{inx}\right)$, with $\widehat{\psi}(n) = e^{-2\delta n^{1/\sigma}}$, then the real-valued solution u to the initial value problem (5.1)–(5.2) is not in $G^r(\mathbb{R})$, $1 \leq r < m\sigma$, in the t variable, for t near zero.*

Proof. First we notice that $u(x, 0) \in G^{\sigma,\delta,s}(\mathbb{T})$ and it is suffices to prove our Theorem just for $\sigma \in \{1, 2, 3, \ldots\}$. Now differentiating $u(x, 0)$ with respect to x we obtain that $\partial_x^q u(x, 0) = -\mathrm{Re}(i^q \sum_{n=1}^{\infty} \hat{\psi}(n) n^q e^{inx})$. Therefore, $\partial_x^q u(0, 0) = -\mathrm{Re}(i^q) A_q$, where $A_q = \sum_{n=1}^{\infty} \hat{\psi}(n) n^q > 0$. For $j \in \mathbb{N}$, by using Lemma 9, we obtain

$$\partial_t^j u(0, 0) = -\mathrm{Re}(i^{mj}) A_{mj} + \sum_{q=1}^{j} \sum_{|\alpha|+(m-1)q=mj} C_\alpha^q (-\mathrm{Re}(i^{\alpha_1})) A_{\alpha_1} \cdots (-\mathrm{Re}(i^{\alpha_{q+1}})) A_{\alpha_{q+1}}.$$

Since $\mathrm{Re}(i^{mj}) \neq 0$ only if j is even, and the terms in the sum that are non-zero only happen when all α_μ are even it follows from the hypotheses that

$$\partial_t^j u(0, 0) = (-1)^{\frac{mj+2}{2}} \left(A_{mj} + \sum_{q=1}^{j} \sum_{|\alpha|+(m-1)q=mj} C_\alpha^q A_{\alpha_1} \cdots A_{\alpha_{q+1}} \right).$$

It follows from this that for j even we have $|\partial_t^j u(0, 0)| \geq A_{mj}$. We notice that for $\sigma \in \{1, 2, 3, \ldots\}$ we have

$$A_{mj} = \sum_{n=1}^{\infty} \hat{\psi}(n) n^{mj} > \hat{\psi}(j^\sigma)(j^\sigma)^{mj} = e^{-2\delta (j^\sigma)^{1/\sigma}} (j^\sigma)^{mj} \geq e^{-2\delta j}((j)^j)^{m\sigma}.$$

It follows from the last inequality that $|\partial_t^j u(0, 0)| > e^{-2\delta j}(j^j)^{m\sigma} \geq (\frac{1}{e^{2\delta}})^j (j!)^{m\sigma}$, which shows that $u(0, \cdot)$ cannot be in $G^r(\mathbb{R})$, for t near to zero, with $1 \leq r < m\sigma$. $\square$

Case 2: $m = 5, 9, 13, \ldots$.

Theorem 4. *Let $m = 4p+1$, $p = 1, 2, \ldots, \sigma \geq 1, \delta > 0$ and $s \in \mathbb{R}$. If we take $u(x, 0) = \mathrm{Re}\left(\sum_{n=1}^{\infty} \widehat{\psi}(n) e^{inx}\right)$, with $\widehat{\psi}(n) = e^{-2\delta n^{1/\sigma}}$, then the real-valued solution u to the initial value problem (5.1)–(5.2) is not in $G^r(\mathbb{R})$, $1 \leq r < m\sigma$, in the t variable, for t near zero.*

Proof. It is easily seen that $u(x, 0) \in G^{\sigma,\delta,s}(\mathbb{T})$. Now, as in Theorem 3, we have

$$\partial_t^j u(0, 0) = \mathrm{Re}(i^{mj}) A_{mj} + \sum_{q=1}^{j} \sum_{|\alpha|+(m-1)q=mj} C_\alpha^q \mathrm{Re}(i^{\alpha_1}) A_{\alpha_1} \cdots \mathrm{Re}(i^{\alpha_{q+1}}) A_{\alpha_{q+1}}.$$

Since $\mathrm{Re}(i^{mj}) \neq 0$ only if j is even, and the terms in the sum that are non-zero only happen when all α_μ are even it follows from the hypotheses that

$$\partial_t^j u(0, 0) = (-1)^{\frac{mj}{2}} \left(A_{mj} + \sum_{q=1}^{j} \sum_{|\alpha|+(m-1)q=mj} C_\alpha^q A_{\alpha_1} \cdots A_{\alpha_{q+1}} \right).$$

It follows from this that for j even we have $|\partial_t^j u(0, 0)| \geq A_{mj}$, which shows, as in Theorem 3, that $u(0, \cdot)$ cannot be in $G^r(\mathbb{R})$, for t near to zero, with $1 \leq r < m\sigma$. $\square$

6. Bilinear estimates

The bilinear estimates for $s \geq -1/2$ has been proved in [12]. Here we shall provide the changes needed in the proof of the first bilinear estimate presented in [12] so that it holds in the sharper range of the indices

$$-j/2 \leq s \leq -1/2. \tag{6.1}$$

The changes required in the proof of the second bilinear estimate are similar. For $f, g \in \dot{X}^s$ we have

$$|n|^s |\widehat{w}_{fg}(n, \lambda)| \leq |n|^{s+1} \sum_{n_1 \in \mathbb{Z}^*, n_1 \neq n} \int_{\mathbb{R}} |\widehat{f}(n - n_1, \lambda - \lambda_1)||\widehat{g}(n_1, \lambda_1)| d\lambda_1. \tag{6.2}$$

Also, we have

$$\|h\|_{X^s} = \left(\sum_{n \in \mathbb{Z}^*} \int_{\mathbb{R}} \left(c_h(n, \lambda)\right)^2 d\lambda \right)^{1/2}, \quad \text{where } c_h(n, \lambda) = |n|^s (1 + |\lambda - n^m|)^{1/2} |\widehat{h}(n, \lambda)|. \tag{6.3}$$

Using (6.2) and (6.3), for bilinear estimate (2.14) we have that

$$\frac{|n|^s|\widehat{w}_{fg}(n,\lambda)|}{(1+|\lambda-n^m|)^{\frac{1}{2}}} \leq \sum_{n_1\in\mathbb{Z}^*,n_1\neq n}\int_{\mathbb{R}} Q_1(n,\lambda,n_1,\lambda_1)c_f(n-n_1,\lambda-\lambda_1)c_g(n_1,\lambda_1)\,d\lambda_1, \tag{6.4}$$

where

$$Q_1(n,\lambda,n_1,\lambda_1) \doteq \frac{|n|^{s+1}|n_1(n-n_1)|^{-s}}{(1+|\lambda-n^m|)^{\frac{1}{2}}(1+|\lambda_1-n_1^m|)^{\frac{1}{2}}(1+|\lambda-\lambda_1-(n-n_1)^m|)^{\frac{1}{2}}}. \tag{6.5}$$

Due to the mean zero initial data assumption, in what follows we always assume that

$$n\neq 0, \qquad n_1\neq 0 \quad \text{and} \quad n_1\neq n. \tag{6.6}$$

Using the set

$$A \doteq \{(n,\lambda,n_1,\lambda_1)\in\mathbb{Z}^*\times\mathbb{R}\times\mathbb{Z}^*\times\mathbb{R} : |\lambda-\lambda_1-(n-n_1)^m| \leq |\lambda_1-n_1^m| \text{ and } n_1\neq n\} \tag{6.7}$$

we observe that for bilinear estimate (2.14) it is enough to show

$$\left(\sum_n\int_\lambda\left[\sum_{n_1}\int_{\lambda_1}(\chi_A Q_1)(n,\lambda,n_1,\lambda_1)c_f(n-n_1,\lambda-\lambda_1)c_g(n_1,\lambda_1)\,d\lambda_1\right]^2 d\lambda\right)^{\frac{1}{2}} \lesssim \|f\|_{X^s}\|g\|_{X^s}.$$

Furthermore we split the set A via the following two cases.

Case I: $|\lambda_1-n_1^m| \leq |\lambda-n^m|$.

Case II: $|\lambda-n^m| < |\lambda_1-n_1^m|$.

In our proofs we also will make use of the quantity

$$d_m \doteq (\lambda-n^m)-[(\lambda_1-n_1^m)+\lambda-\lambda_1-(n-n_1)^m] = -n^m+n_1^m+(n-n_1)^m. \tag{6.8}$$

Also, observe that $d_m(n,n_1)$ has the following lower bounds.

Lemma 10. *If $m\geq 3$ is an odd positive integer, then there exists a positive constant c_m such that for any $n, n_1\in\mathbb{Z}^*$ with $n_1\neq n$ we have*

$$|d_m(n,n_1)| \geq c_m|n|^{m-3}|nn_1(n-n_1)|, \quad \textit{and} \quad |d_m(n,n_1)| \geq c_m|n_1|^{m-3}|nn_1(n-n_1)|. \tag{6.9}$$

Also, we shall need the following inequalities, which are valid for $n, n_1\in\mathbb{Z}^*$ with $n_1\neq n$:

$$|nn_1(n-n_1)| \geq \frac{1}{2}n^2, \tag{6.10}$$

and

$$\frac{|n-n_1|^r}{|n|^r|n_1|^r} \leq 2^r, \quad \text{for } r\geq 0. \tag{6.11}$$

Case I. When $-j/2 \leq s \leq -1/2$ the only change required in [12] is in the proof of Lemma 3.1. We must show that the quantity

$$q_I^2 \doteq \frac{|n|^{2s+2}|n_1(n-n_1)|^{-2s}}{1+|\lambda-n^m|} \tag{6.12}$$

which is part of the quantity Q_1 defined by (6.5) is bounded. In fact, in this case we have

$$\begin{aligned} 1+|\lambda-n^m| \geq \quad & |\lambda-n^m| \geq \frac{1}{3}\left|(\lambda-n^m)-(\lambda_1-n_1^m)-(\lambda-\lambda_1-(n-n_1)^m)\right| \\ & \stackrel{(6.8)}{=} \frac{1}{3}|d_m(n,n_1)| \stackrel{(6.9)}{\geq} \frac{1}{3}c_m|n_1|^{m-3}|nn_1(n-n_1)|, \end{aligned} \tag{6.13}$$

where c_m is the constant appearing in Lemma 10. Since $m=2j+1$ using (6.13) we get

$$q_I^2 \leq \frac{|n|^{2s+2}|n_1(n-n_1)|^{-2s}}{\frac{1}{3}c_m|n_1|^{2j-1}|n(n-n_1)|} = \frac{3}{c_m}\cdot\left(\frac{|n-n_1|}{|n||n_1|}\right)^{-2s-1}\cdot\frac{1}{|n_1|^{4s+2j}}. \tag{6.14}$$

For all $n_1 \in \mathbb{Z}^*$ we have that

$$\frac{1}{|n_1|^{4s+2j}} \leq 1 \Leftrightarrow 4s + 2j \geq 0 \Leftrightarrow s \geq -j/2.$$

Also, if $-2s - 1 \geq 0$ or $s \leq -1/2$, then by applying (6.11) we see that the middle factor in (6.14) is bounded above by 2^{j-1}. Therefore, by our hypothesis $-j/2 \leq s \leq -1/2$, we have that $q_I^2 \leq \frac{3}{c_m} \cdot 2^{j-1} \lesssim 1$, which implies that

$$Q_1^2(n, \lambda, n_1, \lambda_1) \lesssim \frac{1}{(1 + |\lambda_1 - n_1^m|)(1 + |\lambda - \lambda_1 - (n - n_1)^m|)}.$$

From now on the proof of Lemma 3.1 in [12] remains unchanged.

Case II. Here, the only change required in [12] is in the proof of Lemma 3.2. We must show that the quantity

$$q_{II}^2 \doteq \frac{|n|^{2s+2}|n_1(n - n_1)|^{-2s}}{1 + |\lambda_1 - n_1^m|} \tag{6.15}$$

is bounded. In this case we have

$$1 + |\lambda_1 - n_1^m| \geq |\lambda_1 - n_1^m| \geq \frac{1}{3}|d_m(n, n_1)| \overset{(6.9)}{\geq} \frac{1}{3}c_m|n_1|^{m-3}|nn_1(n - n_1)|.$$

Therefore,

$$q_{II}^2 \leq \frac{3}{c_m} \cdot \left(\frac{|n - n_1|}{|n||n_1|}\right)^{-2s-1} \cdot \frac{1}{|n_1|^{4s+2j}} \lesssim 1, \tag{6.16}$$

where the last inequality follows from the hypothesis $-j/2 \leq s \leq -1/2$. Now, using (6.15) we get

$$Q_1^2(n, \lambda, n_1, \lambda_1) \lesssim \frac{1}{(1 + |\lambda - n^m|)(1 + |\lambda - \lambda_1 - (n - n_1)^m|)}.$$

Again, from now on the proof of Lemma 3.2 in [12] remains unchanged. $\square$

Acknowledgments

Part of this work was done during July 23–August 20, 2011, at the Isaac Newton Institute for Mathematical Sciences, Cambridge, UK, when the second author was a participant in the ''Inverse Problems'' program. This author is thankful to the Newton Institute for its hospitality and support. Also, the authors would like to thank the referee(s) for constructive comments.

The fourth author was partially supported by CNPq and FAPESP.

References

[1] J. Bona, Z. Grujić, H. Kalisch, Algebraic lower bounds for the uniform radius of spatial analyticity for the generalized KdV equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (6) (2005) 783–797.
[2] J. Bona, R. Smith, The initial-value problem for the Korteweg–de Vries equation, Philos. Trans. R. Soc. Lond. Ser. A 278 (1287) (1975) 555–601.
[3] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. Part 2: KdV equation, Geom. Funct. Anal. 3 (1993) 209–262.
[4] J. Bourgain, Periodic Korteweg de Vries equation with measures as initial data, Selecta Math. (N.S.) 3 (2) (1997) 115–159.
[5] P. Byers, A. Himonas, Nonanalytic solutions of the KdV equation, Abstr. Appl. Anal. 2004 (6) (2004) 453–460.
[6] M. Christ, J. Colliander, T. Tao, Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations, Amer. J. Math. 125 (2003) 1235–1293.
[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Sharp global well-posedness for KdV and modified KdV on $\mathbb{R}$ and $\mathbb{T}$, J. Amer. Math. Soc. 16 (3) (2003) 705–749.
[8] W. Craig, T. Kappeler, W. Strauss, Gain of regularity for equations of KdV type, Ann. Inst. H. Poincaré Anal. Non Linéaire 9 (2) (1992) 147–186.
[9] A. De Bouard, N. Hayashi, K. Kato, Gevrey regularizing effect for the (generalized) Korteweg–de Vries equation and nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 12 (6) (1995) 673–725.
[10] J. Ginibre, Y. Tsutsumi, Uniqueness of solutions for the generalized Korteweg–de Vries equation, SIAM J. Math. Anal. 20 (6) (1989) 1388–1425.
[11] J. Gorsky, A. Himonas, On analyticity in space variable of solutions to the KdV equation, in: Geometric Analysis of PDE and Several Complex Variables, in: Contemp. Math., vol. 368, Amer. Math. Soc., 2005, pp. 233–247.
[12] J. Gorsky, A. Himonas, Well-posedness of KdV with higher dispersion, Math. Comput. Simul. 80 (1) (2009) 173–183.
[13] Z. Grujić, H. Kalisch, Local well-posedness of the generalized Korteweg–de Vries equation in spaces of analytic functions, Differential Integral Equations 15 (11) (2002) 1325–1334.
[14] H. Hannah, A. Himonas, G. Petronilho, Gevrey regularity in time for generalized KdV type equations, in: Recent Progress on Some Problems in Several Complex Variables and Partial Differential Equations, in: Contemp. Math., vol. 400, Amer. Math. Soc., Providence, RI, 2006, pp. 117–127.
[15] H. Hannah, A. Himonas, G. Petronilho, Gevrey regularity of the periodic gKdV equation, J. Differential Equations 250 (2011) 2581–2600.
[16] A. Himonas, G. Misiołek, Well-posedness of the Cauchy problem for a shallow water equation on the circle, J. Differential Equations 161 (2000) 479–495.
[17] A. Himonas, G. Petronilho, Analytic well-posedness of periodic gKdV, J. Differential Equations 253 (11) (2012) 3101–3112.
[18] H. Hirayama, Local well-posedness for the periodic higher order KdV type equations, Nonlinear Differential Equations Appl. 19 (6) (2012) 677–693.
[19] T. Kappeler, P. Topalov, Global well-posedness of KdV in $H^{-1}(\mathbb{T}, \mathbb{R})$, Duke Math. J. 135 (2) (2006) 327–360.

[20] T. Kato, On the Cauchy problem for the (generalized) Korteweg–de Vries equation, in: Adv. Math. Suppl. Studies, vol. 8, 1983, pp. 93–128.
[21] T. Kato, K. Masuda, Nonlinear evolution equations and analyticity I, Ann. Inst. H. Poincaré Anal. Non Linéaire 3 (6) (1986) 455–467.
[22] K. Kato, T. Ogawa, Analyticity and smoothing effect for the Korteweg–de Vries equation with a single point singularity, Math. Ann. 316 (3) (2000) 577–608.
[23] C.E. Kenig, G. Ponce, L. Vega, Well-posedness and scattering results for the generalized Korteweg–de Vries equation via the contraction principle, Comm. Pure Appl. Math. 46 (4) (1993) 527–620.
[24] C. Kenig, G. Ponce, L. Vega, A Bilinear estimate with applications to the KdV equation, J. Amer. Math. Soc. 9 (1996) 573–603.
[25] C. Kenig, G. Ponce, L. Vega, On the ill-posedness of some canonical dispersive equations, Duke Math. J. 106 (3) (2001) 617–633.
[26] F. Linares, G. Ponce, Introduction to Nonlinear Dispersive Equations, in: Universitext, Springer, New York, 2009.
[27] L. Molinet, A note on ill posedness for the KdV equation, Differential Integral Equations 24 (7–8) (2011) 759–765.
[28] J. Saut, R. Temam, Remarks on the Korteweg–de Vries equation, Israel J. Math. 24 (1) (1976) 78–87.
[29] A. Sjöberg, On the Korteweg–de Vries equation: existence and uniqueness, J. Math. Anal. Appl. 29 (1970) 569–579.
[30] T. Tao, Nonlinear Dispersive Equations: Local and Global Solutions, American Mathematical Society, Providence, RI, 2006.
[31] E. Trubowitz, The inverse problem for periodic potentials, Comm. Pure Appl. Math. 30 (3) (1977) 321–337.