



# Stacked central configurations for Newtonian $N + 2p$ -body problems



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## ABSTRACT

We show the existence of spatial central configurations for the  $N + 2p$ -body problems. In such a configuration,  $N$  bodies are at the vertices of a regular  $N$ -gon  $T$ ;  $2p$  bodies are symmetric with respect to the center of  $T$ , and located on the straight line which is perpendicular to the regular  $N$ -gon  $T$  and passes through the center of  $T$ . The masses located on the vertices of the regular  $N$ -gon are assumed to be equal; the masses located on the same line and symmetric with respect to the center of  $T$  are equal.

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## 1. Introduction and main results

The Newtonian  $n$ -body problems [1,20,24] are concerned with the motions of  $n$  particles with masses  $m_j \in \mathbb{R}^+$  and positions  $q_j \in \mathbb{R}^3$  ( $j = 1, \dots, n$ ), the motion is governed by Newton's second law and the Universal law:

$$m_j \ddot{q}_j = -\frac{\partial U(q)}{\partial q_j}, \quad (1.1)$$

where  $q = (q_1, q_2, \dots, q_n)$  with Newtonian potential:

$$U(q) = \sum_{1 \leq j < k \leq n} \frac{m_j m_k}{|q_j - q_k|}. \quad (1.2)$$

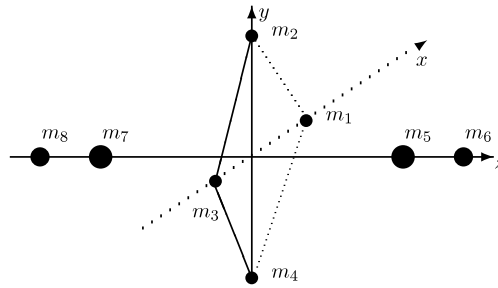
Consider the space

$$X = \left\{ q = (q_1, q_2, \dots, q_n) \in \mathbb{R}^{3n} : \sum_{j=1}^n m_j q_j = 0 \right\}, \quad (1.3)$$

that is, suppose that the center of mass is fixed at the origin of the space. Because the potential is singular when two particles have the same position, it is natural to assume that the configuration avoids the collision set  $\Delta = \{q = (q_1, \dots, q_n) : q_j = q_k \text{ for some } k \neq j\}$ . The set  $X \setminus \Delta$  is called the configuration space.

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Fig. 1.  $N + 2p$ -body.

**Definition 1.1** ([20,24]). A configuration  $q = (q_1, q_2, \dots, q_n) \in X \setminus \Delta$  is called a central configuration if there exists a constant  $\lambda$  such that

$$\sum_{j=1, j \neq k}^n \frac{m_j m_k}{|q_j - q_k|^3} (q_j - q_k) = -\lambda m_k q_k, \quad 1 \leq k \leq n. \quad (1.4)$$

The value of constant  $\lambda$  in (1.4) is uniquely determined by

$$\lambda = \frac{U}{I}, \quad (1.5)$$

where

$$I = \sum_{k=1}^n m_k |q_k|^2. \quad (1.6)$$

Since the general solution of the  $n$ -body problem cannot be given, great importance has been attached to search for particular solutions from the very beginning. A homographic solution is a configuration which is preserved for all time. Central configurations and homographic solutions are linked by the Laplace theorem [24]. Collapse orbits and parabolic orbits have relations with the central configurations [17–20], so finding central configurations becomes very important. The main general open problem for the central configurations is due to Wintner [24] and Smale [23]: Is the number of central configurations finite for any choice of positive masses  $m_1, \dots, m_n$ ? Hampton and Moeckel [7] have proved this conjecture for any given four positive masses.

For 5-body problems, Hampton [6] provided a new family of planar central configurations, called stacked central configurations. A stacked central configuration is one that has some proper subset of three or more points forming a central configuration. Ouyang et al. [15] studied pyramidal central configurations for Newtonian  $N + 1$ -body problems; Zhang and Zhou [26] considered double pyramidal central configurations for Newtonian  $N + 2$ -body problems; Mello and Fernandes [11] analyzed new classes of spatial central configurations for the  $N + 3$ -body problem. Llibre and Mello studied triple and quadruple nested central configurations for the planar  $n$ -body problem. There are many papers studying other central configuration problems such as [2,4,3,5,8–14,16,21,22,25].

In this paper we shall prove the following result.

**Theorem 1.1.** For  $N + 2p$ -body problems in  $R^3$  where  $N \geq 2$  and  $p \geq 1$ , there is at least one central configuration such that  $N$  bodies are at the vertices of a regular  $N$ -gon  $T$ , and  $2p$  bodies are symmetric with respect to the center of the regular  $N$ -gon  $T$ , and located on a line which is perpendicular to the regular  $N$ -gon  $T$  and passes through the center of  $T$ . The masses at the vertices of the regular  $N$ -gon are equal and the masses symmetric with respect to the center of  $T$  are equal. (See Fig. 1 for  $N = 4$  and  $p = 4$ .)

## 2. The proof of Theorem 1.1

To begin, we take a coordinate system which simplifies the analysis. The particles have positions given by

$$q_j = r_0 (\cos \alpha_j, \sin \alpha_j, 0), \quad \text{where } \alpha_j = \frac{(j-1)}{N} 2\pi, j = 1, \dots, N;$$

$$q_{N+j} = (0, 0, r_j), \quad q_{N+j+p} = (0, 0, -r_j), \quad \text{where } j = 1, \dots, p.$$

The masses are given by

$$m_1 = \dots = m_N = M_0, \quad m_{N+j} = m_{N+j+p} = M_j.$$

Notice that  $(q_1, \dots, q_N, q_{N+1}, \dots, q_{N+2p})$  is a central configuration if and only if

$$\sum_{j=1, j \neq k}^{N+2p} \frac{m_j m_k}{|q_j - q_k|^3} (q_j - q_k) = -\lambda m_k q_k, \quad 1 \leq k \leq N + 2p. \quad (2.1)$$

By the symmetries of the system, the Eq. (2.1) is equivalent to the following equations:

$$\sum_{j=1, j \neq k}^{N+2p} \frac{m_j m_k}{|q_j - q_k|^3} (q_j - q_k) = -\lambda m_k q_k, \quad k = 1, N+1, \dots, N+p. \quad (2.2)$$

That is,

$$-\lambda r_0 = -\frac{\beta M_0}{r_0^2} - \sum_{i=1}^p \frac{2M_i r_0}{|r_0^2 + r_i^2|^{3/2}}, \quad \text{where } \beta = \frac{1}{4} \sum_{j=1}^{N-1} \csc(\pi j/N) \quad (2.3)$$

$$-\lambda r_1 = -\frac{NM_0 r_1}{|r_0^2 + r_1^2|^{3/2}} - \frac{M_1}{4r_1^2} + \sum_{i=2}^p \left( \frac{M_i}{|r_i - r_1|^2} - \frac{M_i}{|r_i + r_1|^2} \right), \quad (2.4)$$

$$\begin{aligned} -\lambda r_j = & -\frac{NM_0 r_j}{|r_0^2 + r_j^2|^{3/2}} - \sum_{1 \leq i < j} \left( \frac{M_i}{|r_i - r_j|^2} + \frac{M_i}{|r_i + r_j|^2} \right) - \frac{M_j}{4r_j^2} \\ & - \sum_{j < i \leq p} \left( -\frac{M_i}{|r_i - r_j|^2} + \frac{M_i}{|r_i + r_j|^2} \right), \quad j = 2, \dots, p-1 \end{aligned} \quad (2.5)$$

$$-\lambda r_p = -\frac{NM_0 r_p}{|r_0^2 + r_p^2|^{3/2}} - \sum_{1 \leq i < p} \left( \frac{M_i}{|r_i - r_p|^2} + \frac{M_i}{|r_i + r_p|^2} \right) - \frac{M_p}{4r_p^2}. \quad (2.6)$$

By (2.3)–(2.6),  $(q_1, \dots, q_N, q_{N+1}, \dots, q_{N+2p})$  is a central configuration if and only if  $(r_0, r_1, \dots, r_p)$  is a critical point of  $\check{U}(r_0, r_1, \dots, r_p)$  on  $\check{C}$ , where:

$$\begin{aligned} \check{C} = & \left\{ (r_0, r_1, \dots, r_p) \mid r_0 > 0, 0 < r_1 < r_2 < \dots < r_p, \frac{1}{2} NM_0 r_0^2 + \sum_{i=1}^p M_i r_i^2 = 1 \right\}, \\ \check{U}(r_0, r_1, \dots, r_p) = & \frac{\beta M_0^2 N}{r_0} + \sum_{i=1}^p \frac{2NM_0 M_i}{|r_0^2 + r_i^2|^{1/2}} + \sum_{i=1}^p \frac{M_i^2}{2r_i} + \sum_{1 \leq i < j \leq p} \left( \frac{2M_i M_j}{|r_i - r_j|} + \frac{2M_i M_j}{|r_i + r_j|} \right). \end{aligned}$$

$\check{C}$  is an open, simple connected and bounded set.  $\check{U}(r_0, r_1, \dots, r_p) \rightarrow +\infty$  at the boundary of the set  $\check{C}$ . Now there is at least one critical point of  $\check{U}(r_0, r_1, \dots, r_p)$  on  $\check{C}$ . The proof of Theorem 1.1 is completed.

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