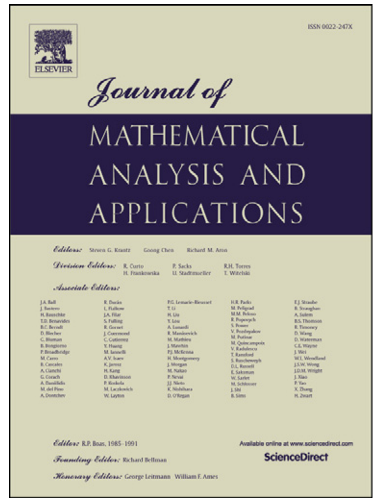
Accepted Manuscript



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PII: S0022-247X(13)01053-6
DOI: 10.1016/j.jmaa.2013.11.042
Reference: YJMAA 18078

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 22 April 2013

Uncited references

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NUMERICAL RANGES OF WEIGHTED COMPOSITION OPERATORS.

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Abstract. The operator that takes the function f to $\psi f \circ \varphi$ is a weighted composition operator. We study numerical ranges of some classes of weighted composition operators on H^2, the Hardy-Hilbert space of the unit disc. We consider the case where φ is a rotation of the unit disc and identify a class of convexoid operators. In the case of isometric weighted composition operators we give a complete classification of their numerical ranges. We also consider the inclusion of zero in the interior of the numerical range.

1. Introduction

Let φ be a holomorphic self map of the open unit disc $\mathbb{D}$ and ψ be a holomorphic map of $\mathbb{D}$. If f is holomorphic on $\mathbb{D}$, then the operator that takes f to $\psi \cdot f \circ \varphi$ is a weighted composition operator and is denoted by $C_{\psi,\varphi}$. If z is in $\mathbb{D}$, then

$$C_{\psi,\varphi}(f)(z) = \psi(z) f(\varphi(z)).$$

In case $\psi \equiv 1$, the operator is simply called a composition operator, and is denoted C_φ. In this work we investigate numerical ranges of weighted composition operators acting on the Hardy space H^2.

The numerical range of a bounded linear operator T on a Hilbert space $\mathcal{H}$ is the subset $W(T)$ of the complex plane given by

$$W(T) = \{\langle T(g), g\rangle : g \in \mathcal{H}, ||g|| = 1\}. \tag{1.1}$$

Numerical ranges of (unweighted) composition operators acting on H^2 are discussed in [1, 2, 16].

In section 3 we consider $C_{\psi,\varphi}$ with rotational composition maps, i.e. $\varphi(z) = e^{i\theta} z$. We identify a class of convexoid operators with rotational composition maps in Theorem 3.4. If V is an open convex set with n-fold symmetry about the origin, where $n > 1$, we prove in Theorem 3.5 that there is a weighted composition operator $C_{\psi,\varphi}$ where $\varphi(z) = e^{2\pi i/n} z$ such that $W(C_{\psi,\varphi}) = V$. In Theorem 3.13 we show that $W(C_{\psi,\varphi})$ contains such a convex, n-fold symmetric set whenever $\varphi(z) = e^{2\pi i/n} z$ and ψ is bounded.

Isometric weighted composition operators are studied in section 4. Isometries that are not unitary operators are studied using the Wold decomposition. We also compute the numerical ranges of unitary weighted composition operators.

2010 *Mathematics Subject Classification.* Primary: 47B32; Secondary: 47B33, 47B38.

Key words and phrases. weighted composition operator, numerical range, Hardy space.

The first author would like to thank the University of Hawaii at Manoa for its generosity in hosting him during the collaboration.

Inspired by Bourdon and Shapiro's work on numerical ranges of composition operators [2], in section 5 we consider the question of when zero is in the interior of $W(C_{\psi,\varphi})$. We provide the answer for different weighted composition operators and in some cases obtain the radius of a disc centered at the origin that lies in the numerical range.

Weighted composition operators naturally appear in studies of linear operators. For example, isometries on H^p for $p \neq 2$, are weighted composition operators [7]. A composition operator on the Hardy space of the upper-half plane is similar to a weighted composition operator on the Hardy space of the unit disc. Hermitian weighted composition operators are investigated in [5, 6] and normal weighted composition operators appear in [3]. Compact weighted composition operators are discussed in [8, 9] and invertibility in [10]. These operators also play an important role in adjoints of composition operators.

2. **Background material**

2.1. **The Hardy-Hilbert space.** The set of functions analytic on $\mathbb{D}$ for which

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty$$

is the Hardy-Hilbert space on the unit disc H^2. We refer to this space simply as the Hardy space. If f is in H^2, then f can be extended to the unit circle almost everywhere by taking radial limits [4, p. 10]. H^2 is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})} \frac{d\theta}{2\pi}.$$

If f is in H^2 and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $||f||^2 = \sum_{n=0}^{\infty} |a_n|^2$. The inner product on H^2 can also be expressed as

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{c_n},$$

where $g(z) = \sum_{n=0}^{\infty} c_n z^n$.

The *reproducing kernel* for $w \in \mathbb{D}$ is the function $K_w(z) = 1/(1 - \overline{w}z)$. Clearly $K_w \in H^2$, and if $f \in H^2$, then

$$\langle f, K_w \rangle = f(w).$$

In particular, $||K_w||^2 = \langle K_w, K_w \rangle = 1/(1 - |w|^2)$. Furthermore, if $C_{\psi,\varphi}$ is bounded, then

$$C^*_{\psi,\varphi}(K_w) = \overline{\psi(w)} K_{\varphi(w)}; \tag{2.1}$$

see, for example, [20, Lemma 3.2].

2.2. **Notation.** We use the following notation in this paper.

The closure of a subset A of the complex plane will be denoted by $Cl(A)$, the convex hull of A by $Hull(A)$, and $\overline{A}$ will be used to denote the set of complex conjugates of numbers in A.

The unit circle with center at the origin will be denoted by $\mathbb{T}$. The disc centered at a with radius r will be denoted by $D(a, r)$.

The pseudohyperbolic distance $|(a-b)/(1-\overline{a}b)|$ between points $a, b \in \mathbb{D}$ will be denoted by $\rho(a,b)$, and $\Delta(a,r)$ is our notation for the pseudohyperbolic disc centered at a with radius r.

The space of bounded analytic functions on $\mathbb{D}$ will be denoted by H^∞.

For an operator S on H^2 we use $\sigma(S)$ to denote the spectrum of S, and $\sigma_p(S)$ for the point spectrum of S.

If φ maps the disc into itself we use φ_n to denote the n^{th} iterate of φ i.e., φ_n is φ composed with itself n times. Also $\varphi_0(z) = z$.

2.3. **Weighted composition operators.** If $\psi \in H^\infty$, then it is elementary that the multiplication operator M_ψ defined by $M_\psi(f)(z) = \psi(z)f(z)$ is a bounded operator on H^2 with $||M_\psi(f)|| \le ||\psi||_\infty\ ||f||$. The composition operator C_φ, where φ is an analytic self-map of the open unit disk and $C_\varphi(f)(z) = f(\varphi(z))$, is also bounded [4, Ch. 3] on H^2. Thus, when $\psi \in H^\infty$ the operator $C_{\psi,\varphi}$ can be factored as a product of two bounded operators:

$$C_{\psi,\varphi} = M_\psi C_\varphi.$$

However, as shown by examples in [8, 20], it is possible for $C_{\psi,\varphi}$ to be bounded and even compact with an unbounded ψ.

Note that if $\varphi(z) = z$, then $C_{\psi,\varphi} = M_\psi$, and if $\psi \equiv k$ is constant, then $C_{\psi,\varphi} = kC_\varphi$. In this paper we are interested in weighted composition operators that do not simplify in this way. Hence we introduce the standing assumption that

$$\textit{φ is not the identity and ψ is not constant.} \tag{2.2}$$

If φ is an automorphism of $\mathbb{D}$, then $\varphi(z) = \lambda \dfrac{a-z}{1-\overline{a}z}$, where $|\lambda| = 1$ and $a \in \mathbb{D}$. Every automorphism not the identity belongs to one of three classes, depending upon the nature of its fixed points.

If φ has a fixed point inside $\mathbb{D}$, then φ is an *elliptic* automorphism.

If φ has only one fixed point on the unit circle $\mathbb{T}$, then φ is *parabolic*.

If φ has two distinct fixed points on $\mathbb{T}$, then φ is *hyperbolic*. One of the fixed points is attractive, and the other one is repulsive.

For every holomorphic self-map φ of the unit disc $\mathbb{D}$ that is not the identity or an elliptic automorphism of $\mathbb{D}$, there exists a unique point ζ in the closure of $\mathbb{D}$ so that the iterates φ_n of φ converge to ζ uniformly on compact subsets of $\mathbb{D}$. The point ζ is called the Denjoy-Wollff point of φ [4, p. 58].

We now record two results from the literature for later reference.

Lemma 2.1. [10, Lemma 1.4.1] *If k is a positive integer, then*

$$C^k_{\psi,\varphi} = C_{(\psi)(\psi\circ\varphi)(\psi\circ\varphi_2)\cdots(\psi\circ\varphi_{k-1}),\varphi_k}.$$

Lemma 2.2. [8, Theorem 1] *Suppose that $C_{\psi,\varphi}$ is a bounded operator on H^2. If φ has a fixed point ζ in the open unit disc then $\overline{\psi(\zeta)(\varphi'(\zeta))^j}$ is in the spectrum of the adjoint operator $C^*_{\psi,\varphi}$ for any nonnegative integer j.*

2.4. **Numerical range.** If T is a bounded operator on a Hilbert space H with the inner product $\langle \cdot, \cdot \rangle_H$, then the numerical range of T is the set

$$\{\langle T(x), x\rangle_H : x \in H,\ ||x||_H = 1\} \tag{2.3}$$

and is denoted by $W(T)$.

The following are various properties of the numerical range. Some are quite easy to prove; for others we have provided a reference.

- $W(T)$ is convex [13, p. 113].
- Similar operators may not share the same numerical range [2, p. 412].
- If $S = UTU^{-1}$ where U is unitary, then $W(S) = W(T)$.
- $W(P \oplus R) = Hull(W(P) \cup W(R))$.
- $\sigma_p(T) \subseteq W(T)$.
- $\sigma(T) \subseteq Cl(W(T))$ [13, p. 115].
- $W(T) \subseteq Cl(D(0, ||T||))$.
- If T' is the compression of T to the closed subspace M, then $W(T') \subseteq W(T)$.
- If S is a normal operator, then $Hull(\sigma(S)) = Cl(W(S))$ [13, p. 117].

Any operator with the property that the closure of its numerical range is equal to the convex hull of the spectrum is known as a *convexoid* operator.

An elementary fact about the Cauchy-Schwarz inequality yields the following result about certain boundary points of the numerical range.

Lemma 2.3. *Let T be a bounded operator on a Hilbert space H. If $\mu \in W(T)$ and $|\mu| = ||T||$, then μ is an eigenvalue of T.*

Proof. Under the hypotheses, there is a unit vector f_0 in H such that

$$\langle T(f_0), f_0 \rangle = \mu. \tag{2.4}$$

From the Cauchy-Schwarz inequality, $|\langle T(f), f \rangle| \leq ||T||$ whenever f is a unit vector. Since equality in the Cauchy-Schwarz inequality is achieved only when the two vectors are linearly dependent, $T(f_0) = \lambda f_0$ for some $\lambda \in \mathbb{C}$. From (2.4) it follows that $\mu = \lambda$ is an eigenvalue of T. □

3. Rotations

We begin with weighted composition operators whose composition map is $\varphi(z) = e^{i\theta} z$.

3.1. Rational rotations.

Let $\lambda = e^{2\pi i/n}$ and $\varphi(z) = \lambda z$. Suppose that j is a fixed integer and $0 \leq j < n$. If $f(z) = \sum_{k=0} a_k z^{kn+j}$ then $C_\varphi(f) = \lambda^j f$. Therefore it is natural to look at the subspaces H_j of functions that have such Maclaurin series.

Definition 3.1. *Let $2 \leq n$, and $0 \leq j < n$. Let H_j be the set defined by*

$$H_j = \left\{ f : f(z) = z^j g(z^n), g \in H^2 \right\}.$$

It is clear that H_j is a closed subspace of H^2. Let P_j denote the orthogonal projection onto it. When H_j is invariant under multiplication by a bounded function ψ, the compression of M_ψ to H_j is just the restriction $M_\psi|_{H_j}$ of M_ψ to H_j. Next we compute the numerical range of this operator.

Lemma 3.2. *Let $\psi \in H^\infty$. If H_j is an invariant subspace of M_ψ, then $W(M_\psi|_{H_j}) = Hull(\psi(\mathbb{D}))$.*

Proof. Let $K_{w,j} = P_j K_w$ where w is in $\mathbb{D}$. Then $K_{w,j}$ is the reproducing kernel for w in the space H_j, and

$$\left\langle M_\psi \frac{K_{w,j}}{||K_{w,j}||}, \frac{K_{w,j}}{||K_{w,j}||} \right\rangle = \frac{1}{||K_{w,j}||^2} \langle \psi K_{w,j}, K_{w,j} \rangle$$
$$= \frac{1}{||K_{w,j}||^2} \psi(w) K_{w,j}(w).$$

But $K_{w,j}(w) = \langle K_{w,j}, K_{w,j} \rangle = ||K_{w,j}||^2$, so $\psi(w) \in W(M_\psi|_{H_j})$ and hence

$$Hull(\psi(\mathbb{D})) \subseteq W(M_\psi|_{H_j}).$$

It is known that $W(M_\psi) = Hull(\psi(\mathbb{D}))$; see [15, Corollary 2]. Since $W(M_\psi|_{H_j}) \subseteq W(M_\psi)$ we get the desired result. □

Next we compute the numerical range of a weighted composition operator with a rotational composition map.

Lemma 3.3. *Let $\varphi(z) = \lambda z$ where $\lambda = e^{2\pi i/n}$. Suppose that ψ is a bounded function on $\mathbb{D}$. If $g \in H^\infty$ and $\psi(z) = g(z^n)$, then*

$$W(C_{\psi,\varphi}) = Hull(\psi(\mathbb{D}) \cup \lambda\psi(\mathbb{D}) \cup \cdots \cup \lambda^{n-1}\psi(\mathbb{D})).$$

Proof. Since the spaces $H_0, H_1, \cdots, H_{n-1}$ are orthogonal and together span H^2,

$$H^2 = H_0 \oplus H_1 \oplus \cdots \oplus H_{n-1}.$$

Let $f \in H_j$, so that $f(\lambda z) = \lambda^j f(z)$ and hence $C_\varphi(H_j) \subseteq H_j$. Since $\psi(z) = g(z^n)$ it is easy to see that $\psi f \in H_j$. Thus, $M_\psi(H_j) \subseteq H_j$ and

$$C_{\psi,\varphi}(H_j) \subseteq H_j.$$

Let $C_j = C_{\psi,\varphi}|_{H_j}$. Then

$$C_{\psi,\varphi} = C_0 \oplus C_1 \oplus \cdots \oplus C_{n-1}.$$

If $h \in H_j$ then

$$\langle C_j h, h \rangle = \langle \psi(z) h(\lambda z), h(z) \rangle = \lambda^j \langle \psi(z) h(z), h(z) \rangle.$$

It follows from Lemma 3.2 that

$$W(C_j) = Hull(\lambda^j \psi(\mathbb{D})).$$

Since $C_{\psi,\varphi}$ is the direct sum of $C_0, \cdots, C_{n-1}$

$$W(C_{\psi,\varphi}) = Hull(W(C_0) \cup \cdots \cup W(C_{n-1}))$$

(see [16, p. 64]). Thus,

$$W(C_{\psi,\varphi}) = Hull(\psi(\mathbb{D}) \cup \lambda\psi(\mathbb{D}) \cup \cdots \cup \lambda^{n-1}\psi(\mathbb{D})).$$

□

Next we classify some convexoid operators. To avoid interrupting of the flow of the arguments, we delay the computation of the spectra of the operators until the end of this section.

Theorem 3.4. *Let $\varphi(z) = e^{2\pi i/n} z$. Suppose that $\psi(z) = g(z^n)$, where $g \in H^\infty$. Then $C_{\psi,\varphi}$ is convexoid; that is*

$$Cl\left(W(C_{\psi,\varphi})\right) = Hull\left(\sigma(C_{\psi,\varphi})\right).$$

Proof. Let $\lambda = e^{2\pi i/n}$. From Lemma 3.3,

$$W(C_{\psi,\varphi}) = Hull(\psi(\mathbb{D}) \cup \lambda\psi(\mathbb{D}) \cup \cdots \cup \lambda^{n-1}\psi(\mathbb{D})).$$

Lemma 3.15 below identifies the spectrum of $C_{\psi,\varphi}$ as the closure of

$$\psi(\mathbb{D}) \cup \lambda\psi(\mathbb{D}) \cup \cdots \cup \lambda^{n-1}\psi(\mathbb{D}),$$

giving the desired result. We used that $Hull(Cl(A)) = Cl(Hull(A))$ for a bounded set $A \subset \mathbb{C}$. This follows from the fact that, in the complex plane, the convex hull of a compact set is compact and the closure of a convex set is convex [15]. □

A set $V \subset \mathbb{C}$ that satisfies $e^{2\pi i/n}V = V$ is said to have n-fold symmetry about the origin. We now apply the results above to a convex set that has n-fold symmetry.

Theorem 3.5. *Let V be an open, bounded, non-empty convex set. If V has n-fold symmetry about the origin, then there is a weighted composition operator $C_{\psi,\varphi}$ where φ is a rotation such that $W(C_{\psi,\varphi})$ is V.*

Proof. Let f be a Riemann map from $\mathbb{D}$ onto V and define $\psi(z) = f(z^n)$, for z in $\mathbb{D}$. Note that the $range(\psi) = range(f)$, and hence $\psi(\mathbb{D}) = V$. Let $\varphi(z) = \lambda z$, where $\lambda = e^{2\pi i/n}$. From Lemma 3.3 we have that

$$W(C_{\psi,\varphi}) = Hull(\psi(\mathbb{D}) \cup \lambda\psi(\mathbb{D}) \cup \cdots \cup \lambda^{n-1}\psi(\mathbb{D})).$$

Since $\psi(\mathbb{D})$ has n-fold symmetry about the origin, $\lambda^p\psi(\mathbb{D}) = \psi(\mathbb{D}) = V$, for $0 \le p \le n-1$. Since V is convex, $W(C_{\psi,\varphi}) = V$ as desired. □

Corollary 3.6. *Let $f \in H^\infty$ be nonconstant. If $n > 1$, then there is a weighted composition operator $C_{\psi,\varphi}$ such that $W(C_{\psi,\varphi})$ is the smallest convex set with n-fold symmetry about the origin that contains $f(\mathbb{D})$.*

We remark that the statement remains valid, with the same proof, for f constant. This was not included in the statement of the corollary, since in this case the weight ψ will be constant, contrary to our standing assumption (2.2).

Proof. Let $\psi(z) = f(z^n)$. Then $range(\psi) = range(f)$. Let $\varphi(z) = \lambda z$ where $\lambda = e^{2\pi i/n}$. From Lemma 3.3 it follows that

$$W(C_{\psi,\varphi}) = Hull(f(\mathbb{D}) \cup \lambda f(\mathbb{D}) \cup \cdots \cup \lambda^{n-1}f(\mathbb{D})).$$

It is easy to see that $\lambda^j W(C_{\psi,\varphi}) = W(C_{\psi,\varphi})$ for any $0 \le j < n$. Thus $W(C_{\psi,\varphi})$ has n-fold symmetry at the origin and is a convex set. Let Q be a convex set with n-fold symmetry at the origin that contains $f(\mathbb{D})$. Then Q also contains $\lambda^j f(\mathbb{D})$ for $0 \le j < n$. Thus Q contains $W(C_{\psi,\varphi})$. □

As the next examples show, if ψ is not of the form $\psi(z) = f(z^n)$, $f \in H^\infty$, then $W(C_{\psi,\varphi})$ may or may not be the convex hull of $\sigma(C_{\psi,\varphi})$.

Example 3.7. *Let $\psi(z) = e^z$ and $\varphi(z) = e^{\pi i}z$. Then $C_{\psi,\varphi}$ is not convexoid.*

It is easy to see that $C^2_{\psi,\varphi}$ is the identity on H^2 and thus $(\sigma(C_{\psi,\varphi}))^2 = \{1\}$. Hence $\sigma(C_{\psi,\varphi}) \subseteq \{-1, 1\}$ and $Hull(\sigma(C_{\psi,\varphi})) \subseteq [-1, 1]$. Let $g = \dfrac{K_{i\pi/4}}{||K_{i\pi/4}||}$. Then

$\langle C_{\psi,\varphi}(g), g\rangle = e^{i\pi/4}\dfrac{1-(\pi/4)^2}{1+(\pi/4)^2}$, and hence $W(C_{\psi,\varphi})$ is not contained in the real line.

Example 3.8. *Let $\psi(z) = z$ and $\varphi(z) = e^{\pi i}z$. Then $C_{\psi,\varphi}$ is convexoid.*

The operator $C_{\psi,\varphi}$ is an isometry and $Cl(W(C_{\psi,\varphi}))$ is the closed unit disc (see Corollary 4.5). The spectrum of $C_{\psi,\varphi}$ is also the closed unit disc [10, Theorem 3.1.1].

In following examples we compute the numerical range using the results above.

Example 3.9. *Let $\psi(z) = 1 + z^2/2$ and $\varphi(z) = e^{\pi i}z$.*

From Lemma 3.3 it follows that $W(C_{\psi,\varphi}) = Hull(\psi(\mathbb{D}) \cup (-\psi(\mathbb{D})))$, where $\psi(\mathbb{D})$ is the open disc of radius 1/2 centered at 1. The spectrum and the numerical range of $C_{\psi,\varphi}$ are illustrated in Figure 1.

Example 3.10. *Let $\psi(z) = 1 + z^4/2$ and $\varphi(z) = e^{\pi i/2}z$.*

From Lemma 3.3 it follows that

$$W(C_{\psi,\varphi}) = Hull(\psi(\mathbb{D}) \cup i\psi(\mathbb{D}) \cup (-\psi(\mathbb{D})) \cup (-i\psi(\mathbb{D}))).$$

The numerical range and the spectrum of $C_{\psi,\varphi}$ are depicted in Figure 2.

Lemma 3.3 does not reveal which unit vectors correspond to given points in the numerical range. Therefore we consider the following. Let $|\mu| = 1$ and n be a natural number. Assume p and m are integers such that $0 \le p \le 2$, $0 \le m \le n$, and $\lambda = i$. Let

$$p_n(z) = \frac{1}{\sqrt{n}}\left(\frac{\lambda^p}{\mu}z^{4+p} + \frac{\lambda^{2p}}{\mu^2}z^{8+p} + \frac{\lambda^{3p}}{\mu^3}z^{12+p} + \cdots + \frac{\lambda^{mp}}{\mu^m}z^{4m+p}\right)$$

and

$$q_n(z) = \frac{1}{\sqrt{n}}\left(\frac{\lambda^{p+1}}{\mu}z^{4+(p+1)} + \frac{\lambda^{2(p+1)}}{\mu^2}z^{8+(p+1)} + \cdots + \frac{\lambda^{(n-m)(p+1)}}{\mu^{n-m}}z^{4(n-m)+(p+1)}\right).$$

If $P_n(z) = p_n(z) + q_n(z)$, then $||P_n|| = 1$ and

$$\langle C_{\psi,\varphi}(P_n), P_n\rangle = \frac{\mu}{2}\left(1 - \frac{2}{n}\right) + \frac{m}{n}\lambda^p + \left(1 - \frac{m}{n}\right)\lambda^{p+1}.$$

The point $w_{m,n} = \dfrac{m}{n}\lambda^p + \left(1 - \dfrac{m}{n}\right)\lambda^{p+1}$ lies on the line segment with endpoints λ^p and λ^{p+1}. The set $\{w_{m,n} : 0 \le m \le n, n \in \mathbb{N}\}$ forms a dense set of the line segment from λ^p to λ^{p+1}. Therefore by choosing appropriate μ, m and n it is clear that for any given point w on the boundary of the numerical range, there is $P_n = p_n + q_n$ so that $\langle C_{\psi,\varphi}(P_n), P_n\rangle$ is arbitrarily close to w.

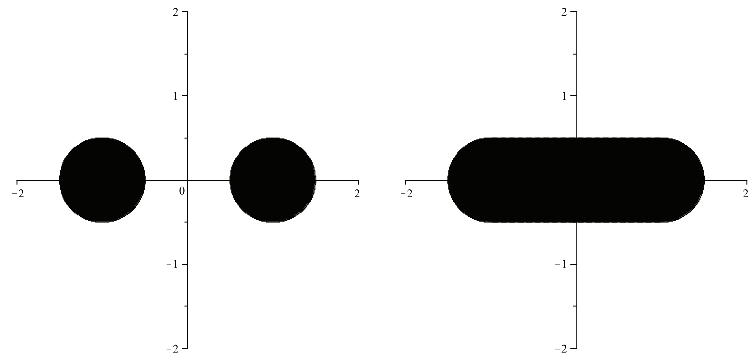

FIGURE 1. The figure on the left is the spectrum and the one on the right is the numerical range of the operator in Example 3.9.

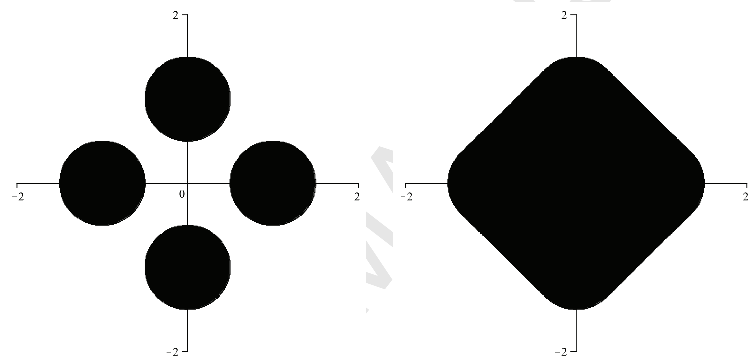

FIGURE 2. The figure on the left is the spectrum and the one on the right is the numerical range of the operator in Example 3.10.

It is not hard to see that if λ is any primitive n^{th} root of unity, our previous results still hold. Indeed, consider $\varphi(z) = e^{2\pi i m/n} z$, where m and n are relatively prime and $0 \leq m < n$. Let $\lambda = e^{2\pi i m/n}$ and $\mu = e^{2\pi i/n}$. If p_k denotes mj_k (mod n), then $\lambda^{j_k} = \mu^{p_k}$.

Suppose that $f \in H_j$ for some $0 \leq j < n$. Then $C_\varphi(f) = \mu^p f$ where $p \equiv mj$ (mod n). If ψ is in H^∞, then

$$\langle C_{\psi,\varphi}(f), f\rangle = \mu^p \langle \psi f, f\rangle.$$

If the Taylor series of ψ is of the form $\sum_{\ell=0}^{\infty} a_\ell z^{n\ell}$, then from Lemma 3.2 we have

$$W(C_{\psi,\varphi}|_{H_j}) = Hull(\mu^p \psi(\mathbb{D})).$$

Since $C_{\psi,\varphi} = C_{\psi,\varphi}|_{H_0} \oplus C_{\psi,\varphi}|_{H_1} \oplus \cdots \oplus C_{\psi,\varphi}|_{H_{n-1}}$, it follows that

$$W(C_{\psi,\varphi}) = Hull(\mu^{p_0}\psi(\mathbb{D}) \cup \mu^{p_1}\psi(\mathbb{D}) \cup \cdots \cup \mu^{p_{n-1}}\psi(\mathbb{D})). \tag{3.1}$$

If $0 \leq j_1 < j_2 < n$, then it is easy to see that $mj_1 \not\equiv mj_2$ (mod n) and thus $\{p_0, p_1, \cdots, p_{n-1}\} = \{0, 1, 2, \cdots, n-1\}$.

The normalizing assumption that $\psi(0) = 1$ is made in several of the following results. Since $C_{\alpha\psi,\varphi} = \alpha C_{\psi,\varphi}$ and $W(C_{\alpha\psi,\varphi}) = \alpha W(C_{\psi,\varphi})$ for any constant α, these results have easy extensions to any ψ with $\psi(0) \neq 0$.

The following propositions give more information about sets contained in $W(C_{\psi,\varphi})$.

Proposition 3.11. *Let $\psi(z) = 1 + \hat{\psi}_1 z + \hat{\psi}_2 z^2 + \cdots$ and $\varphi(z) = e^{2\pi i/n}z$. Assume that $\hat{\psi}_{np+j} \neq 0$ for some $0 < j < n$. Then $W(C_{\psi,\varphi})$ contains the ellipse with foci 1 and $e^{2\pi ij/n}$, and with major axis $\sqrt{|1 - e^{2\pi ij/n}|^2 + |\hat{\psi}_{np+j}|^2}$ and minor axis $|\hat{\psi}_{np+j}|$.*

Proof. Let $k = np + j, \lambda = e^{2\pi i/n}$. Define

$$Q_k = span\{e_1, e_2\}$$

where $e_1(z) = 1$ and $e_2(z) = z^k$. We next compute the matrix representation of the compression of $C_{\psi,\varphi}$ to Q_k. We have

$$C_{\psi,\varphi}(e_1)(z) = 1 + \hat{\psi}_1 z + \hat{\psi}_2 z^2 + \cdots + \hat{\psi}_k z^k + \cdots$$

and

$$C_{\psi,\varphi}(e_2)(z) = \lambda^k(z^k + \hat{\psi}_1 z^{k+1} + \hat{\psi}_2 z^{k+2} + \cdots)$$

Thus the compression of $C_{\psi,\varphi}$ to Q_k has the matrix representation

$$\begin{bmatrix} 1 & 0 \\ \hat{\psi}_k & \lambda^k \end{bmatrix}.$$

If $0 < j < n$, then $\lambda^k \neq 1$ and the numerical range of the compression of $C_{\psi,\varphi}$ to Q_k is the ellipse with foci at 1 and λ^k, and major axis $\sqrt{|1 - e^{2\pi ij/n}|^2 + |\hat{\psi}_{np+j}|^2}$ and minor axis $|\hat{\psi}_{np+j}|$ (see [12, p. 1]). Since the numerical range of the compression is contained in the numerical range of the operator, the stated result follows. □

Proposition 3.12. *Let $\varphi(z) = e^{2\pi i/n}z$ and $\psi(z) = 1 + \hat{\psi}_1 z + \cdots$. Suppose that m_1, m_2 are positive integers such that $m_2 > m_1$, and $\hat{\psi}_{nm_1}\hat{\psi}_{nm_2}\hat{\psi}_{n(m_2-m_1)} = 0$ but at least one of the 3 terms is nonzero. Then $W(C_{\psi,\varphi})$ contains the disc centered at 1 with radius $\frac{1}{2}(|\hat{\psi}_{nm_1}|^2 + |\hat{\psi}_{n(m_2-m_1)}|^2 + |\hat{\psi}_{nm_2}|^2)^{1/2}$.*

Proof. Let $e_1(z) = 1, e_2(z) = z^{nm_1}$ and $e_3(z) = z^{nm_2}$. Suppose that

$$Q = span\{e_1, e_2, e_3\}.$$

Let T be the compression of $C_{\psi,\varphi}$ to Q. We first compute the matrix of T. Now,

$$C_{\psi,\varphi}(e_1)(z) = \psi(z) = 1 + \hat{\psi}_1 z + \cdots + \hat{\psi}_{nm_1} z^{nm_1} + \cdots + \hat{\psi}_{nm_2} z^{nm_2} + \cdots,$$

and

$$\begin{aligned} C_{\psi,\varphi}(e_2)(z) &= \psi(z)(e^{2\pi i/n}z)^{nm_1} \\ &= z^{nm_1} + \hat{\psi}_1 z^{nm_1+1} + \cdots + \hat{\psi}_{n(m_2-m_1)} z^{nm_2} + \cdots. \end{aligned}$$

Also,

$$C_{\psi,\varphi}(e_3)(z) = \psi(z)(e^{2\pi i/n}z)^{nm_2} = z^{nm_2} + \hat{\psi}_1 z^{nm_2+1} + \cdots.$$

Thus, the matrix of T is

$$\begin{bmatrix} 1 & 0 & 0 \\ \hat{\psi}_{nm_1} & 1 & 0 \\ \hat{\psi}_{nm_2} & \hat{\psi}_{n(m_2-m_1)} & 1 \end{bmatrix}.$$

From Theorem 4.1 of [14] it follows that $W(T)$ is the disc centered at 1 with the radius

$$\frac{1}{2}\sqrt{|\hat{\psi}_{nm_1}|^2 + |\hat{\psi}_{n(m_2-m_1)}|^2 + |\hat{\psi}_{nm_2}|^2}.$$

Since $W(T) \subseteq W(C_{\psi,\varphi})$, the desired result follows. □

Theorem 3.13. *Let $\varphi(z) = \lambda z$ where $\lambda = e^{2\pi i/n}$. Let $\psi \in H^\infty$ and define $\widetilde{\psi}$ by the equation*

$$n\widetilde{\psi}(z) = \psi(z) + \psi(\lambda z) + \psi(\lambda^2 z) + \cdots + \psi(\lambda^{n-1} z).$$

Then

$$W(C_{\widetilde{\psi},\varphi}) = Hull\left(\widetilde{\psi}(\mathbb{D}) \cup \lambda\widetilde{\psi}(\mathbb{D}) \cup \cdots \cup \lambda^{n-1}\widetilde{\psi}(\mathbb{D})\right) \subset W(C_{\psi,\varphi}).$$

Proof. We first compute the compression of $C_{\psi,\varphi}$ to the subspace H_ℓ (see Definition 3.1), where $0 \le \ell \le n-1$. Let $f \in H_\ell$. Then $f(z) = \sum_{k=0}^{\infty} a_k z^{kn+\ell}$. Thus

$$C_{\psi,\varphi}(f)(z) = \psi(z) f(\lambda z) = \lambda^\ell(\psi_0(z) + \psi_1(z) + \cdots + \psi_{n-1}(z))(\sum_{k=0}^{\infty} a_k z^{kn+\ell})$$

where $\psi_r = P_r\psi$ and P_r is the projection onto H_r. Thus,

$$\psi_r(z) = \sum_{k=0}^{\infty} \widehat{\psi}_{nk+r} z^{nk+r}.$$

Hence $\psi_r f$ belongs to $H_\ell^\perp$ for $r = 1, \cdots, n-1$ and $\psi_0 f$ belongs to H_ℓ. If $C_\ell = P_\ell C_{\psi,\varphi}|_{H_\ell}$, then

$$C_\ell(f) = \lambda^\ell \psi_0 f.$$

Thus, C_ℓ is the multiplication operator on H_ℓ with multiplier $\lambda^\ell\psi_0$. Hence from Lemma 3.2 it follows that

$$W(C_\ell) = Hull(\lambda^\ell \psi_0(\mathbb{D})).$$

If $0 \le r, j \le n-1$, then $\psi_r(\lambda^j z) = \lambda^{jr}\psi_r(z)$. Therefore

$$\psi(\lambda^j z) = \psi_0(z) + \lambda^j\psi_1(z) + (\lambda^j)^2\psi_2(z) + \cdots + (\lambda^j)^{(n-1)}\psi_{n-1}(z).$$

Note that $1 + \lambda^p + (\lambda^p)^2 + \cdots + (\lambda^p)^{(n-1)} = 0$ for $0 < p \le n-1$, and so

$$\psi(z) + \psi(\lambda z) + \psi(\lambda^2 z) + \cdots + \psi(\lambda^{n-1} z) = n\psi_0(z).$$

Thus $\widetilde{\psi} = \psi_0$, and the equality relation in the statement to be proved is immediate from Lemma 3.3, while the subset relation follows from

$$W(C_\ell) \subseteq W(C_{\psi,\varphi}), \quad 0 \le \ell \le n-1.$$

□

Finally, we compute the spectrum when $\varphi(z) = e^{2\pi i/n}z$ and $\psi(z) = g(z^n)$, where $g \in H^\infty$. This result was used in the proof of Theorem 3.4. Spectra of weighted composition operators with rotational composition maps and general weights are computed in [10]. Since the weight functions take a special form in the operators that are studied here, we compute the spectrum in a different way.

Lemma 3.14. *Let $n > 1$. Suppose that $\psi(z) = g(z^n)$, where $g \in H^\infty$. If $0 \leq \ell < n$, then*

$$\sigma(M_\psi|_{H_\ell}) = Cl(\psi(\mathbb{D})).$$

Proof. Let $M_\ell = M_\psi|_{H_\ell}$. Since $M_\psi(H_\ell) \subseteq H_\ell$,

$$M_\psi = M_0 \oplus M_1 \oplus \cdots \oplus M_{n-1}.$$

It is well known that $Cl(\psi(\mathbb{D})) = \sigma(M_\psi)$, and hence

$$Cl(\psi(\mathbb{D})) = \sigma(M_0) \cup \cdots \cup \sigma(M_{n-1}).$$

Thus

$$\sigma(M_\ell) \subseteq Cl(\psi(\mathbb{D})). \tag{3.2}$$

Let $w \in \mathbb{D} \setminus \{0\}$ and let $\mu = \psi(w)$. If $h = (M_\ell - \mu)f$ for some $f \in H_\ell$, then

$$h(w) = \psi(w)f(w) - \mu f(w) = 0. \tag{3.3}$$

Hence all functions in the range of $M_\ell - \mu$ on H_ℓ vanish at w, and so $M_\ell - \mu$ is not invertible. Thus $\mu \in \sigma(M_\ell)$ and it follows that

$$\psi(\mathbb{D} \setminus \{0\}) \subseteq \sigma(M_\ell). \tag{3.4}$$

Since ψ is analytic on $\mathbb{D}$, $Cl(\psi(\mathbb{D} \setminus \{0\})) = Cl(\psi(\mathbb{D}))$, and from (3.2) and (3.4) we get that

$$Cl(\psi(\mathbb{D})) = \sigma(M_\ell).$$

□

Lemma 3.15. *Let $\varphi(z) = \lambda z$, where $\lambda = e^{2\pi i/n}$, and let $\psi(z) = g(z^n)$, where $g \in H^\infty$. Then*

$$\sigma(C_{\psi,\varphi}) = Cl\left(\psi(\mathbb{D}) \cup \lambda\psi(\mathbb{D}) \cup \cdots \cup \lambda^{n-1}\psi(\mathbb{D})\right).$$

Proof. Let $0 \leq \ell < n$. If $f \in H_\ell$, it is readily checked that $C_\varphi(f) = \lambda^\ell f$, and so

$$C_{\psi,\varphi}(f) = M_\psi C_\varphi(f) = \lambda^\ell \psi f = \lambda^\ell M_\ell(f) \in H_\ell, \tag{3.5}$$

where $M_\ell = M_\psi|_{H_\ell}$. Put $C_\ell = C_{\psi,\varphi}|_{H_\ell}$, so that $C_\ell = \lambda^\ell M_\ell$ from (3.5). Thus

$$C_{\psi,\varphi} = C_0 \oplus C_1 \oplus \cdots \oplus C_{n-1}, \tag{3.6}$$

and hence

$$\sigma(C_{\psi,\varphi}) = \sigma(C_0) \cup \sigma(C_1) \cup \cdots \cup \sigma(C_{n-1}). \tag{3.7}$$

Since $\sigma(C_k) = \sigma(\lambda^k M_k) = \lambda^k \sigma(M_k)$, the desired result follows from Lemma 3.14 and (3.7). □

3.2. **Irrational rotations.**

Lemma 3.16. *Let $\psi(z) = 1 + \hat{\psi}_1 z + \hat{\psi}_2 z^2 + \cdots$ and $\varphi(z) = \lambda z$, where $\lambda = e^{2\pi i\theta}$ for θ irrational. Assume that n is a nonnegative integer and m is a positive integer. Then $W(C_{\psi,\varphi})$ contains the ellipse with foci λ^n, λ^{n+m} whose major axis is $\sqrt{|\lambda^n - \lambda^{n+m}|^2 + |\hat{\psi}_m|^2}$ and minor axis is $|\hat{\psi}_m|$.*

Proof. Let
$$Q = span\{e_1, e_2\}$$
where $e_1(z) = z^n$ and $e_2(z) = z^{n+m}$. We have
$$\begin{aligned} C_{\psi,\varphi}(e_1)(z) &= (1 + \hat{\psi}_1 z + \hat{\psi}_2 z^2 + \cdots + \hat{\psi}_m z^m + \cdots)\lambda^n z^n \\ &= \lambda^n z^n + \lambda^n \hat{\psi}_1 z^{n+1} + \cdots + \lambda^n \hat{\psi}_m z^{n+m} + \cdots \end{aligned}$$
and
$$C_{\psi,\varphi}(e_2)(z) = \lambda^{n+m} z^{n+m} + \lambda^{n+m} \hat{\psi}_1 z^{n+m+1} + \cdots .$$
If T is the compression of $C_{\psi,\varphi}$ to Q, then the matrix representation of T is
$$\begin{bmatrix} \lambda^n & 0 \\ \lambda^n \hat{\psi}_m & \lambda^{n+m} \end{bmatrix}.$$
Since $W(T)$ is the ellipse described in the statement (see [12, p. 1]) and $W(T) \subseteq W(C_{\psi,\varphi})$, the desired result follows. □

For ψ and φ defined above, the set $\{\psi(0)\varphi'(0)^n, n \geq 0\}$ is in $\sigma(C_{\psi,\varphi})$ (see [8, Lemma 3]), and it forms a dense subset of the unit circle. Thus, it is easy to see that $W(C_{\psi,\varphi})$ contains the open unit disc. Using Lemma 3.16 the radius of the disc can be improved.

Theorem 3.17. *Suppose that $\psi(z) = 1 + \hat{\psi}_1 z + \hat{\psi}_2 z^2 + \cdots$ and $\varphi(z) = e^{2\pi i\theta} z$, where θ is irrational. If m is a positive integer, then $W(C_{\psi,\varphi})$ contains the open disc centered at 0 with radius $|\cos(\pi m\theta)| + |\hat{\psi}_m|$.*

Proof. Let $\lambda = e^{2\pi i\theta}$. From Lemma 3.16 it follows that the ellipse with foci λ^n, λ^{n+m} and minor axis of length $|\hat{\psi}_m|$ is contained in $W(C_{\psi,\varphi})$. With some simple trigonometry it is easy to see that there is a point on this ellipse whose distance from the origin is $|\cos(\pi m\theta)| + |\hat{\psi}_m|$. The result follows since the sequence $\{\lambda^n\}_{n=1}^{\infty}$ is dense on the unit circle. □

4. **Isometries**

The main tool in this section is the Wold decomposition, which states that every isometry is a direct sum of a shift (defined below) and a unitary operator. We will first discuss some essential facts about the Wold decomposition (see [17]).

4.1. **Shift operator.**

Definition 4.1. *A bounded operator S on a Hilbert space H is a shift operator if S is an isometry and $(S^*)^n \to 0$ strongly.*

Lemma 4.2. *An isometry S on a Hilbert space H is a shift operator if and only if $\bigcap_{j=0}^{\infty} S^j(H) = \{\mathbf{0}\}$.*

A proof can be found in [17, p.3].

Proposition 4.3. *If S is a shift operator on a Hilbert space H, then $W(S)$ is $\mathbb{D}$.*

Proof. Let k be a unit vector in $ker(S^*)$. For $|\lambda| < 1$, define f by

$$f = \sum_{j=0}^{\infty} \lambda^j S^j(k). \tag{4.1}$$

Since S is an isometry, this series converges in H. Note that $S^*S = I$, and hence

$$S^*(f) = \sum_{j=0}^{\infty} \lambda^j S^* S^j(k) = \sum_{j=1}^{\infty} \lambda^j S^{j-1}(k). \tag{4.2}$$

Thus $S^*(f) = \lambda f$, i.e. λ is in the point spectrum of S^*. Hence $\mathbb{D}$ is contained in $W(S)$. Since $||S|| = 1$, $W(S)$ is contained in $Cl(\mathbb{D})$.

If a unimodular λ is in $W(S)$, then it follows from Lemma 2.3 that $S(g) = \lambda g$ for some unit vector g. Thus for any nonnegative integer n

$$S^n(g) = \lambda^n g. \tag{4.3}$$

Hence g belongs to $\cap_{j=0}^{\infty} S^j(H)$ which contradicts Lemma 4.2. Therefore no point in the unit circle belongs to $W(S)$ and this yields the desired result. $\square$

In the following theorem, we identify certain ψ and φ that produce shift operators.

Theorem 4.4. *Suppose that $C_{\psi,\varphi}$ is an isometry on H^2. Further suppose that ζ is the fixed point of φ and $|\zeta| < 1$. If $|\psi(\zeta)| < 1$, then $C_{\psi,\varphi}$ is a shift operator, and hence $W(C_{\psi,\varphi}) = \mathbb{D}$.*

Proof. Let

$$H_m = C_{\psi,\varphi}^m(H^2),$$

where m is a nonnegative integer. Let $h \in \cap_{m=0}^{\infty} H_m$. Then, for each m, there is an $f_m \in H^2$ so that $h = C_{\psi,\varphi}^m(f_m)$. Since $C_{\psi,\varphi}$ is an isometry

$$||h|| = ||f_m||$$

for all m. From Lemma 2.1 it follows that

$$C_{\psi,\varphi}^m(f) = (\psi \circ \varphi_{m-1} \psi \circ \varphi_{m-2} \cdots \psi)(f \circ \varphi_m).$$

If $w \in \mathbb{D}$, then

$$h(w) = \psi(\varphi_{m-1}(w))\psi(\varphi_{m-2}(w)) \cdots \psi(w)(f_m(\varphi_m(w))).$$

Thus

$$|h(w)| \le |\psi(\varphi_{m-1}(w))\psi(\varphi_{m-2}(w)) \cdots \psi(w)| \; ||f_m|| \frac{1}{\sqrt{1 - |\varphi_m(w)|^2}}. \tag{4.4}$$

Choose p so that $|\psi(\zeta)| < p < 1$, and let $U \subseteq \mathbb{D}$ be an open neighborhood of ζ such that $|\psi(z)| < p$, for $z \in U$. Now choose $r > 0$ so that $\Delta(\zeta, r) \subseteq U$, where $\Delta(\zeta, r)$ is the pseudohyperbolic disc centered at ζ with radius r; see §2.2. Then

$$\rho(\zeta, \varphi(w)) = \rho(\varphi(\zeta), \varphi(w)) \le \rho(\zeta, w), \tag{4.5}$$

by the Schwarz-Pick Lemma; see for example [11, p. 2]. Hence $\varphi(\Delta(\zeta,r)) \subset \Delta(\zeta,r)$. Therefore, for $m \geq 0$ and $w \in \Delta(\zeta,r)$, $|\psi(\varphi_m(w))| < p$. Thus from (4.4)

$$|h(w)| \leq p^m \frac{||f_m||}{\sqrt{1-|\varphi_m(w)|^2}}, \quad m \geq 0 \text{ and } w \in \Delta(\zeta,r). \tag{4.6}$$

Since $\|f_m\| = \|h\|$ and $\varphi_m(\Delta(\zeta,r)) \subset \Delta(\zeta,r)$, the right hand side of (4.6) has the limit 0 as $m \to \infty$ for each $w \in \Delta(\zeta,r)$. Therefore $h \equiv 0$ on $\Delta(\zeta,r)$, and hence on $\mathbb{D}$ as well. Thus

$$\bigcap_{m=0}^{\infty} H_m = \{\mathbf{0}\}.$$

Hence $C_{\psi,\varphi}$ is a shift operator and $W(C_{\psi,\varphi})$ is $\mathbb{D}$. □

Corollary 4.5. *Let φ be an inner function that fixes the origin. If ψ is also a nonconstant inner function, then $C_{\psi,\varphi}$ is a shift operator on H^2 and $W(C_{\psi,\varphi}) = \mathbb{D}$.*

Proof. Since $\varphi(0) = 0$ and φ is an inner function, C_φ is an isometry [4, p.123]. The multiplication operator M_ψ is also an isometry and $C_{\psi,\varphi} = M_\psi C_\varphi$. Therefore $C_{\psi,\varphi}$ is an isometry. Since $||\psi||_\infty = 1$, we have that $|\psi(0)| < 1$, and the result follows from Theorem 4.4.

□

Next we look at isometries that are not unitary operators.

Theorem 4.6. *Let $C_{\psi,\varphi}$ be an isometry on H^2. If $C_{\psi,\varphi}$ is not a unitary operator, then $W(C_{\psi,\varphi}) = \mathbb{D} \cup \sigma_p(C_{\psi,\varphi})$.*

Proof. Since $C_{\psi,\varphi}$ is an isometry, its Wold decomposition is

$$C_{\psi,\varphi} = U \oplus S,$$

where $H^2 = \mathbb{K} \oplus \mathbb{L}$, U is a unitary operator on $\mathbb{K}$, and S is a shift operator on $\mathbb{L}$; see [17, p. 2]. Since $C_{\psi,\varphi}$ is not unitary, $\mathbb{L} \neq \{\mathbf{0}\}$, and the operator $C_{\psi,\varphi}$ is the direct sum of U and S,

$$W(C_{\psi,\varphi}) = Hull(W(U) \cup W(S)). \tag{4.7}$$

Since S is a shift operator, $W(S) = \mathbb{D}$. The numerical range of any isometry is contained in the closed unit disc. Thus it follows from (4.7) that

$$\mathbb{D} \subseteq W(C_{\psi,\varphi}) \subseteq Cl(\mathbb{D}).$$

Suppose that $\lambda \in W(C_{\psi,\varphi})$ and $|\lambda| = 1$. From Lemma 2.3 it follows that λ is an eigenvalue. Since $\sigma_p(C_{\psi,\varphi}) \subseteq W(C_{\psi,\varphi})$, the desired result follows. □

4.2. **Unitary weighted composition operators.** We quote the following results about unitary operators from [3].

Theorem A. [3, Theorem 6] *The weighted composition operator $C_{\psi,\varphi}$ is unitary if and only if φ is an automorphism and $\psi = cK_\beta/||K_\beta||$, where $\varphi(\beta) = 0$ and $|c| = 1$.*

The following result describes the spectra of unitary weighted composition operators.

Theorem B. [3, Theorem 7] *Suppose that $C_{\psi,\varphi}$ is unitary.*

If φ is elliptic with the fixed point p, then $|\varphi'(p)| = 1$ and the spectrum of $C_{\psi,\varphi}$ is the closure of $\{\psi(p)\varphi'(p)^n : n = 0, 1, 2, \cdots\}$.

If φ is parabolic or hyperbolic, then the spectrum is the unit circle.

Next we compute numerical ranges of unitary operators.

Theorem 4.7. *Suppose that $C_{\psi,\varphi}$ is unitary. Then the following are true.*

(1) *If φ is hyperbolic, then $W(C_{\psi,\varphi}) = \mathbb{D}$.*
(2) *If φ is parabolic, then $W(C_{\psi,\varphi}) = \mathbb{D}$.*
(3) *If φ is elliptic and p is the fixed point for φ, then*
  (a) *If $\varphi'(p)$ is not a root of unity, then $W(C_{\psi,\varphi}) = \mathbb{D} \cup \{\psi(p)\varphi'(p)^n : n = 0, 1, 2, \cdots\}$.*
  (b) *If $\varphi'(p)$ is an n^{th} root of unity, then $W(C_{\psi,\varphi})$ is the regular polygon with the vertices $\psi(p), \psi(p)\varphi'(p), \cdots, \psi(p)\varphi'(p)^{n-1}$.*

Proof. (1) Hyperbolic φ: Clearly $W(C_{\psi,\varphi})$ is contained in the closed unit disc and $Cl(W(C_{\psi,\varphi}))$ contains $\sigma(C_{\psi,\varphi})$, which is the unit circle. Therefore $Cl(W(C_{\psi,\varphi}))$ is the closed unit disc. Since $W(C_{\psi,\varphi})$ is a convex set it must contain the open unit disc. Hence

$$\mathbb{D} \subseteq W(C_{\psi,\varphi}) \subseteq Cl(\mathbb{D}).$$

If $|\mu| = 1$ and $\mu \in W(C_{\psi,\varphi})$, then from Lemma 2.3 it follows that μ is an eigenvalue of $C_{\psi,\varphi}$. If w is an arbitrary unimodular number, then $C_{\psi,\varphi}$ is similar to $wC_{\psi,\varphi}$ (see [3, p.282]). Thus $w\mu$ is an eigenvalue of $C_{\psi,\varphi}$. Hence

$$\sigma_p(C_{\psi,\varphi}) = \mathbb{T}.$$

Since $C_{\psi,\varphi}$ is a unitary operator, eigenvectors that correspond to distinct eigenvalues are orthogonal. Since there is an uncountable collection of distinct eigenvalues, there is an uncountable collection of orthonormal eigenvectors. But this is impossible, as H^2 is separable. Therefore unimodular numbers cannot be in the numerical range, which implies

$$W(C_{\psi,\varphi}) = \mathbb{D}.$$

(2) Parabolic φ : Suppose that the unimodular number μ belongs to $W(C_{\psi,\varphi})$. From Lemma 2.3 it follows that μ is an eigenvalue of $C_{\psi,\varphi}$. When φ is parabolic, $C_{\psi,\varphi}$ is similar to $C_{\widetilde{\psi},\widetilde{\varphi}}$, where

$$\widetilde{\varphi}(z) = \frac{(1+i)z - 1}{z + i - 1}$$

or

$$\widetilde{\varphi}(z) = \frac{(1-i)z - 1}{z - i - 1}$$

[10, Lemma 3.0.6]. We only consider the first map, as the same arguments can be applied to the second map. Since $C_{\psi,\varphi}$ and $C_{\widetilde{\psi},\widetilde{\varphi}}$ are similar, μ belongs to $\sigma_p(C_{\widetilde{\psi},\widetilde{\varphi}})$. Therefore $C_{\widetilde{\psi},\widetilde{\varphi}}(g) = \mu g$ for some g in H^2. Hence

$$\widetilde{\psi}\; g \circ \widetilde{\varphi} = \mu g. \tag{4.8}$$

If $f(z) = e^{s\frac{z+1}{z-1}}$, where $s > 0$, then

$$f \circ \widetilde{\varphi} = e^{-2is} f \tag{4.9}$$

[4, p.254]. Let $h = gf$. Since f is an inner function, $h \in H^2$. We have

$$C_{\widetilde{\psi},\widetilde{\varphi}}(h) = \widetilde{\psi}\ g \circ \widetilde{\varphi}\ f \circ \widetilde{\varphi},$$

and from (4.8) and (4.9) it follows that

$$C_{\widetilde{\psi},\widetilde{\varphi}}(h) = \mu e^{-2is} h.$$

Therefore μe^{-2is} is an eigenvalue of $C_{\widetilde{\psi},\widetilde{\varphi}}$. Since s is an arbitrary positive number and μ is a unimodular number it follows that $\mathbb{T} \subseteq \sigma_p(C_{\widetilde{\psi},\widetilde{\varphi}})$. Similarity of the two operators now yields

$$\mathbb{T} \subseteq \sigma_p(C_{\psi,\varphi}).$$

Thus the unitary operator $C_{\psi,\varphi}$ possesses an uncountable collection of eigenvalues. This leads to a contradiction as shown in the proof of the hyperbolic case. Thus unimodular numbers cannot be in $W(C_{\psi,\varphi})$. The rest of the proof is also similar to the proof of the hyperbolic case.

(3) Elliptic φ :

(a) Assume that $\varphi'(p)$ is not a root of unity:
From [8, Lemma 3] it follows that $\overline{\psi(p)(\varphi'(p))^n}$ is in $\sigma(C^*_{\psi,\varphi}|_{K_n})$ where K_n is a finite dimensional invariant subspace of $C^*_{\psi,\varphi}$. Thus $\overline{\psi(p)(\varphi'(p))^n}$ is an eigenvalue of $C^*_{\psi,\varphi}$ for each nonnegative integer n. Therefore $\sigma_p(C^*_{\psi,\varphi})$ contains the set

$$\{\overline{\psi(p)(\varphi'(p))^n} : n = 0, 1, \cdots\}.$$

If $\varphi'(p)$ is not a root of unity, then since $|\psi(p)| = 1$ [3, p. 282], we have $\{\overline{\psi(p)(\varphi'(p))^n} : n = 0, 1, \cdots\}$ is a dense subset of the unit circle. Since $W(C^*_{\psi,\varphi})$ is a convex set that contains $\sigma_p(C^*_{\psi,\varphi})$, it follows that

$$\mathbb{D} \cup \{\overline{\psi(p)(\varphi'(p))^n} : n = 0, 1, \cdots\} \subseteq W(C^*_{\psi,\varphi}).$$

Also $W(C_{\psi,\varphi}) = \overline{W(C^*_{\psi,\varphi})}$, and so

$$\mathbb{D} \cup \{\psi(p)(\varphi'(p))^n : n = 0, 1, \cdots\} \subseteq W(C_{\psi,\varphi}).$$

Since $||C_{\psi,\varphi}|| = 1$, the numerical range is contained in the closed unit disc. If λ is a unimodular number that belongs to $W(C_{\psi,\varphi})$ it follows from Lemma 2.3 that λ is an eigenvalue. It is known that $\{\psi(p)(\varphi'(p))^n : n = 0, 1, \cdots\}$ contains all eigenvalues of $C_{\psi,\varphi}$ [8, p. 463]. This yields the desired result.

(b) Assume that $\varphi'(p)$ is a root of unity:
Then $\sigma(C_{\psi,\varphi}) = \{\psi(p), \psi(p)\varphi'(p), \cdots, \psi(p)\varphi'(p)^{n-1}\}$ [3, Theorem 7]. Since $C_{\psi,\varphi}$ is normal it is convexoid. Therefore $Cl(W(C_{\psi,\varphi}))$ is the polygon with vertices $\psi(p), \psi(p)\varphi'(p), \cdots, \psi(p)\varphi'(p)^{n-1}$. Because all of $\overline{\psi(p)\varphi'(p)^k}$ are eigenvalues of $C^*_{\psi,\varphi}$, it follows that these vertices are in $W(C_{\psi,\varphi})$. The result now follows from the convexity of $W(C_{\psi,\varphi})$.

□

5. Inclusion of zero

In [2] Bourdon and Shapiro prove that for any C_φ not the identity, the origin lies in the closure of $W(C_\varphi)$. They also show that 0 lies in the interior of $W(C_\varphi)$ whenever C_φ fails to have dense range; if $\varphi(0) = 0$, then 0 is in the interior of $W(C_\varphi)$ unless $\varphi(z) = tz$ for some $t \in [-1, 1]$; if φ fixes a nonzero point in $\mathbb{D}$ and is neither the identity map nor a positive conformal dilation, then 0 is in the interior of $W(C_\varphi)$.

Similar results hold for weighted composition operators, and the proofs, with minor adjustments carry over as well. Therefore some proofs are omitted.

Recall that our standing assumption is that φ is an analytic self map of $\mathbb{D}$ that is different from the identity; see (2.2).

Proposition 5.1. *If $C_{\psi,\varphi}$ is bounded on H^2, then 0 belongs to the closure of $W(C_{\psi,\varphi})$.*

Proof. Let $f_a = K_a/||K_a||$ where $a \in \mathbb{D}$ and K_a is the reproducing kernel at a. Let $w_a = \langle C_{\psi,\varphi} f_a, f_a\rangle$, so $w_a \in W(C_{\psi,\varphi})$. Now,

$$w_a = \frac{\langle C_{\psi,\varphi} K_a, K_a\rangle}{||K_a||^2}$$

and hence

$$w_a = (1 - |a|^2)\langle K_a, C^*_{\psi,\varphi} K_a\rangle.$$

But $C^*_{\psi,\varphi} K_a = \overline{\psi(a)} K_{\varphi(a)}$ from (2.1), so

$$w_a = \psi(a)(1 - |a|^2)\langle K_a, K_{\varphi(a)}\rangle = \psi(a)\frac{1 - |a|^2}{1 - \overline{a}\varphi(a)}, \quad a \in \mathbb{D}. \tag{5.1}$$

Also $C_{\psi,\varphi}(1) = \psi \in H^2$, and hence $\lim_{r\to 1^-} \psi(r\mu)$ exists for almost all $\mu \in \mathbb{D}$. Because φ is not the identity function, its boundary function differs from the identity almost everywhere. Therefore there is $\zeta \in \partial\mathbb{D}$ so that $\lim_{r\to 1^-} \psi(r\zeta)$ exists finitely and $\lim_{r\to 1^-} \varphi(r\zeta) \neq \zeta$. Equation (5.1) shows that $w_{r\zeta} \to 0$ as $r \to 1^-$, yielding the stated result. □

The next result is an immediate consequence of (5.1).

Corollary 5.2. *Suppose that $C_{\psi,\varphi}$ is bounded on H^2. If φ has a fixed point ζ in $\mathbb{D}$, then $\psi(\zeta)$ is in $W(C_{\psi,\varphi})$.*

While Proposition 5.1 shows that $0 \in Cl(W(C_{\psi,\varphi}))$ always, it is possible that $0 \in \partial W(C_{\psi,\varphi})$, as Example 5.7 below shows.

Next we look at $C_{\psi,\varphi}$ induced by a constant composition map. Recall that K_a is the reproducing kernel for the point $a \in \mathbb{D}$; see §2.1.

Proposition 5.3. *Let $\varphi \equiv a$ where $|a| < 1$. If $\psi \in H^2$, then the following are true:*

1. *If $K_a = \mu\psi$, $\mu \neq 0$, then $W(C_{\psi,\varphi})$ is the closed line segment from 0 to $\overline{\mu}$.*
2. *If $K_a \perp \psi$, then $W(C_{\psi,\varphi})$ is the closed disc centered at the origin with radius $\|K_a\|/2$.*
3. *Otherwise $W(C_{\psi,\varphi})$ is a closed ellipse with foci at 0 and $\psi(a)$.*

Proof. Since $\psi \in H^2$ and φ is constant, $C_{\psi,\varphi}$ is a bounded rank one operator. In particular, for $f \in H^2$,

$$C_{\psi,\varphi} f = \psi f(a) = \langle f, K_a \rangle \psi. \tag{5.2}$$

The stated results are now immediate from [2, Proposition 2.5], where the numerical range of a rank one operator is computed. □

We now turn to the question of when 0 is in the interior of the numerical range. Corollary 3.6 of [2] states that if φ is not one-to-one, then 0 is an interior point of $W(C_\varphi)$. A similar result holds for weighted composition operators, based on the following ideas.

It is known that if an eigenvalue λ of a bounded operator T on a Hilbert space lies on the boundary of $W(T)$, then λ is a normal eigenvalue, meaning that $\overline{\lambda}$ is an eigenvalue of T^*. In particular it follows that if T is injective and does not have dense range, then 0 is in the interior of $W(T)$. See, for example, [2, p. 419] for these facts.

Proposition 5.4. *Suppose that $C_{\psi,\varphi}$ is bounded and φ is a nonconstant analytic self map of $\mathbb{D}$. If either ψ has a zero in $\mathbb{D}$ or φ is not one-to-one, then 0 is in the interior of $W(C_{\psi,\varphi})$.*

Proof. Since φ is nonconstant, $C_{\psi,\varphi}$ is injective. If ψ has a zero at $z_0 \in \mathbb{D}$, then the range of $C_{\psi,\varphi}$ is contained in the closed subspace of functions that vanish at z_0 and hence is not dense. Thus the conclusion is a consequence of the discussion prior to the statement of the proposition. On the other hand, suppose there exist distinct points a, b in $\mathbb{D}$ such that $\varphi(a) = \varphi(b)$. If $\psi(a)\psi(b) = 0$, then we are done by the first case considered. So suppose $\psi(a)\psi(b) \neq 0$ and note that

$$C^*_{\psi,\varphi}\left(\overline{\psi(b)} K_a - \overline{\psi(a)} K_b\right) = \mathbf{0},$$

as a consequence of (2.1). Thus 0 is an eigenvalue of $C^*_{\psi,\varphi}$, but not a normal eigenvalue since $C_{\psi,\varphi}$ is one-to-one. Hence 0 does not belong to $\partial W(C_{\psi,\varphi})$ and the result follows from Proposition 5.1. □

Theorem 3.8 of [2] states that if φ fixes 0 and is neither a negative nor a positive dilation, then 0 is an interior point of $W(C_\varphi)$. The result carries over to weighted composition operators and is stated in Lemma 5.5 below. The proof carries over as well with minor modifications, and therefore is omitted.

Lemma 5.5. *Suppose that $C_{\psi,\varphi}$ is bounded and $\varphi(0) = 0$. If φ is not of the form $\varphi(z) = tz,\ -1 \leq t < 1$, then 0 is in the interior of $W(C_{\psi,\varphi})$.*

Note that if $\varphi(z) = tz, -1 \leq t < 1$, then C_φ is Hermitian. Therefore $W(C_\varphi)$ is a real line segment and hence 0 is not an interior point of $W(C_\varphi)$. However, non-trivial weighted composition operators induced by dilations are not Hermitian [6, Theorem 6] and hence the numerical range cannot be found so easily. Next we consider $C_{\psi,\varphi}$ where $\varphi(z) = tz, -1 \leq t < 1$.

Lemma 5.6. *Suppose that $\varphi(z) = tz$ where $-1 \leq t \leq 0$ and $C_{\psi,\varphi}$ is a bounded operator. Then 0 is an interior point of $W(C_{\psi,\varphi})$.*

Proof. If $\psi(0) = 0$, the result follows from Proposition 5.4 when $t < 0$, and from Proposition 5.3 when $t = 0$. Therefore it suffices to consider $\psi(0) = 1$.

We first consider the case $-1 \le t < 0$. Since ψ is not a constant, $\psi(z) = 1+\widetilde{\psi}(z)$, where $\widetilde{\psi}$ is a nonconstant analytic function and $\widetilde{\psi}(0) = 0$. If f is a unit vector in H^2, then

$$\langle C_{\psi,\varphi} f, f\rangle = \langle C_\varphi f, f\rangle + \langle C_{\widetilde{\psi},\varphi} f, f\rangle. \tag{5.3}$$

Since $\widetilde{\psi}(0) = 0$, by Proposition 5.4 the numerical range of $C_{\widetilde{\psi},\varphi}$ contains a disc of positive radius centered at the origin. Therefore there is a unit vector $f_1 \in H^2$ so that $\mathrm{Im}\langle C_{\widetilde{\psi},\varphi} f_1, f_1\rangle > 0$. But $\langle C_\varphi f_1, f_1\rangle \in [t, 1]$ (see [2, p. 416]), so by (5.3) $p_1 = \langle C_{\psi,\varphi} f_1, f_1\rangle$ is in the upper half-plane. A similar argument shows that $W(C_{\psi,\varphi})$ also contains a point p_2 in the lower half plane. It is readily verified that if $g(z) = z$, then $\langle C_{\psi,\varphi} g, g\rangle = t$ and that $\langle C_{\psi,\varphi} 1, 1\rangle = 1$. Since $-1 \le t < 0$, 0 is in the interior of the convex hull of $\{1, t, p_1, p_2\}$, and the result follows. If $t = 0$, then the result follows from Proposition 5.3, since ψ is not a constant function by (2.2). □

Now consider $\varphi(z) = tz,\ 0 < t < 1$. Then

(1) if $\psi(0) = 0$, then from Proposition 5.4 it follows that 0 is in the interior of $W(C_{\psi,\varphi})$.
(2) if $\psi(0) \ne 0$, then 0 may or may not be in the interior of $W(C_{\psi,\varphi})$. Consider the following examples.

Example 5.7. *Let $\varphi(z) = z/2$ and $\psi(z) = 1 + z/4$. Then 0 is on the boundary of $W(C_{\psi,\varphi})$.*

Proof. Suppose that $f(z) = \sum_{n=0}^{\infty} \widehat{f}_n z^n$ is a unit vector in H^2. Then,

$$\langle C_{\psi,\varphi} f, f\rangle = \sum_{m=0}^{\infty} \frac{|\widehat{f}_m|^2}{2^m} + \frac{1}{4}\sum_{n=0}^{\infty} \frac{\widehat{f}_n \overline{\widehat{f}}_{n+1}}{2^n}. \tag{5.4}$$

Let $A = \sum_{m=0}^{\infty} \frac{|\widehat{f}_m|^2}{2^m}$ and $B = \sum_{n=0}^{\infty} \frac{\widehat{f}_n \overline{\widehat{f}}_{n+1}}{2^n}$. Since $2|\widehat{f}_n \widehat{f}_{n+1}| \le |\widehat{f}_n|^2 + |\widehat{f}_{n+1}|^2$,

$$|B| \le \frac{1}{2}\sum_{n=0}^{\infty} \frac{|\widehat{f}_n|^2 + |\widehat{f}_{n+1}|^2}{2^n}. \tag{5.5}$$

Thus $|B| \le 3A/2$. Therefore $Re(A+B/4) > 0$, hence 0 is not contained in the interior of $W(C_{\psi,\varphi})$. By Proposition 5.1 it can be concluded that 0 is in the boundary of $W(C_{\psi,\varphi})$. □

Example 5.8. *Let $\varphi(z) = z/2$ and $\psi(z) = 1 + 4z$. Then 0 is in the interior of $W(C_{\psi,\varphi})$.*

Proof. If $f_\theta(z) = (1+e^{i\theta}z)/\sqrt{2}$, then $\langle C_{\psi,\varphi} f_\theta, f_\theta\rangle = 3/4+2e^{-i\theta}$. Since θ is arbitrary, 0 belongs to the interior of $W(C_{\psi,\varphi})$. □

Next consider φ with a fixed point $p \in \mathbb{D}$.

Proposition 5.9. *Suppose that $C_{\psi,\varphi}$ is bounded and φ is an analytic self map of $\mathbb{D}$ that has an interior fixed point p. If $\varphi'(p)$ is not real, then 0 is an interior point of $W(C_{\psi,\varphi})$.*

Proof. If $\psi(p) = 0$, the result is a consequence of Proposition 5.4. If $\psi(p) \neq 0$ and $\varphi'(p)$ is not real, then 0 is in the interior of the convex hull of $\{\psi(p)\varphi'(p)^j : j \geq 0\}$. Since the point spectrum of $C^*_{\psi,\varphi}$, and hence $W(C^*_{\psi,\varphi})$, contains the conjugates of the points $\{\psi(p)\varphi'(p)^j\}$ for nonnegative integers j (see [8, Lemma 3]), the result follows. □

A conformal dilation is a map that is conformally conjugate to an r-dilation, i.e. if φ is a conformal dilation then $\varphi = \alpha_p \circ \delta_r \circ \alpha_p^{-1}$, where

$$\alpha_p(z) = \frac{p-z}{1-\overline{p}z} \text{ and } \delta_r(z) = rz,$$

for $r \in \mathbb{D}$. If $0 < r < 1$, then φ is a positive conformal dilation. (see [2, p. 416]). It is easy to see that $\varphi\,'(p) = r$.

Theorem 4.4 of [2] states that if φ is not a positive conformal dilation and has a nonzero fixed point in $\mathbb{D}$, then 0 is an interior point of $W(C_\varphi)$. The result is true for weighted composition operators as well. The proof, which considers the compression of C_φ to subspaces spanned by Guyker basis vectors, carries over with minor modifications. Therefore the proof is omitted.

The next theorem summarizes our results regarding zero containment in $W(C_{\psi,\varphi})$ when φ has a fixed point in $\mathbb{D}$.

Theorem 5.10. *Suppose that $C_{\psi,\varphi}$ is a bounded operator on H^2 and that φ has a fixed point p in $\mathbb{D}$. Then the following are true.*

1. *If φ is nonconstant and not a positive conformal dilation, then 0 is an interior point of $W(C_{\psi,\varphi})$.*
2. *Suppose that $\varphi \equiv p$ is constant. If $\psi = \mu K_p$ for some $\mu \neq 0$, then $0 \in W(C_{\psi,\varphi})$ but 0 is not an interior point of $W(C_{\psi,\varphi})$. Otherwise 0 is an interior point of $W(C_{\psi,\varphi})$.*

When it is known that $W(C_{\psi,\varphi})$ contains a disc with center 0, as in Theorem 5.10, a natural problem is to estimate the radius R of the largest such disc. But $\|C_{\psi,\varphi}\|$ is always an upper bound for this radius, as a consequence of the elementary fact that $\|T\|$ is an upper bound for the numerical radius of a bounded operator T. The next proposition provides a lower estimate for the radius when $\varphi(0) = 0$ and $\psi'(0) = 0$.

Proposition 5.11. *Suppose that $C_{\psi,\varphi}$ is a bounded operator on H^2, $\varphi(0) = 0$ and ψ has a zero of order $m > 0$ at the origin. If $\widehat{\psi}_m$ denotes the m^{th} Taylor coefficient of ψ, then $W(C_{\psi,\varphi})$ contains the disc of radius $|\widehat{\psi}_m|/2$ centered at the origin.*

Proof. Let $f(z) = \dfrac{\lambda + z^m}{\sqrt{2}}$ where $|\lambda| = 1$. Let the Taylor series of φ be $\sum_{j=1}^\infty \widehat{\varphi}_j z^j$ and the Taylor series of ψ be $\sum_{k=m}^\infty \widehat{\psi}_k z^k$. Then

$$(f \circ \varphi)(z) = \frac{1}{\sqrt{2}}(\lambda + (\widehat{\varphi}_1 z + \widehat{\varphi}_2 z^2 + \cdots)^m) = \frac{1}{\sqrt{2}}(\lambda + c_1 z^m + c_2 z^{m+1} + \cdots).$$

Now

$$\begin{aligned}\langle C_{\psi,\varphi}(f), f\rangle &= \langle \psi f \circ \varphi, f\rangle \\ &= \frac{1}{2}\langle (\widehat{\psi}_m z^m + \widehat{\psi}_{m+1} z^{m+1} + \cdots)(\lambda + c_1 z^m + c_2 z^{m+1} + \cdots), \lambda + z^m\rangle \\ &= \frac{1}{2}\langle \lambda\widehat{\psi}_m z^m + \sum_{k>m} d_k z^k, \lambda + z^m\rangle \\ &= \frac{1}{2}\widehat{\psi}_m \lambda.\end{aligned}$$

Since λ is an arbitrary unimodular number the stated conclusion follows. □

We next show that if φ is assumed to be a dilation, then a different method of proof yields a similar conclusion.

Proposition 5.12. *Let $\varphi(z) = \lambda z$, where $\lambda \neq 0$, and let $\psi(z) = \hat{\psi}_1 z + \hat{\psi}_2 z^2 + \cdots$. If $C_{\psi,\varphi}$ is bounded on H^2, then $W(C_{\psi,\varphi})$ contains the closed disc centered at the origin with radius $|\lambda\hat{\psi}_m|/2$ for each $m \geq 1$.*

Proof. Let $k > 1$. Define

$$Q_k = span\{e_1, e_2\}$$

where $e_1(z) = z$ and $e_2(z) = z^k$. We next compute the matrix representation of the compression of $C_{\psi,\varphi}$ to Q_k. We have

$$C_{\psi,\varphi}(e_1)(z) = \lambda(\hat{\psi}_1 z^2 + \hat{\psi}_2 z^3 + \cdots + \hat{\psi}_{k-1} z^k + \cdots)$$

and

$$C_{\psi,\varphi}(e_2)(z) = \lambda^k(\hat{\psi}_1 z^{k+1} + \hat{\psi}_2 z^{k+2} + \cdots)$$

Thus the compression of $C_{\psi,\varphi}$ to Q_k has the matrix representation

$$\begin{bmatrix} 0 & 0 \\ \lambda\hat{\psi}_{k-1} & 0 \end{bmatrix}.$$

Therefore the numerical range of the compression is the closed disc centered at zero with radius $|\lambda\hat{\psi}_{k-1}|/2$ (see [12, p. 1]). The stated result now follows from the containment of the numerical range of the compression in the numerical range of the operator. □

We next give a simple example that shows Proposition 5.11 and Proposition 5.12 can not be extended to include the case $m = 0$.

Example 5.13. *Let $\psi(z) = 1 + \epsilon z$ and $\varphi(z) = -z/2$, with $\epsilon > 0$. Then the radius of the largest disc with center the origin and contained in $W(C_{\psi,\varphi})$ is at most ϵ.*

Proof. For a unit vector f in H^2,

$$\langle C_{\psi,\varphi} f, f\rangle = \langle (1 + \epsilon z) C_\varphi f, f\rangle = \langle C_\varphi f, f\rangle + \epsilon\langle z C_\varphi f, f\rangle.$$

It is known that $W(C_\varphi) = [-1/2, 1]$; see [2, p. 416]. Also, $||C_\varphi|| = 1$ (see for example [4, Corollary 3.7]) and hence $|\langle z C_\varphi f, f\rangle| \leq 1$. The stated conclusion is an easy consequence of these observations. □

As was discussed in section 2.3, automorphisms of $\mathbb{D}$ divide into three classes, depending upon the nature of their fixed point. It is known that the numerical range of an (unweighted) composition operator induced by a parabolic or hyperbolic automorphism of $\mathbb{D}$ is a disc; see [1, Theorem 3.1]. This result carries over to our setting of weighted composition operators, which we record as the next theorem. The proof carries over without difficulty, and so will be omitted.

Theorem 5.14. *Let φ be an automorphism of $\mathbb{D}$ that is either parabolic or hyperbolic. If $C_{\psi,\varphi}$ is a bounded operator, then $W(C_{\psi,\varphi})$ is a disc centered at 0.*

While this theorem identifies the numerical range as a disc, in general we do not know if it is open or closed, or its exact radius. This is also the case for (unweighted) composition operators; see [1, page 846]. It is known that for a bounded operator T the numerical radius is at least $||T||/2$; see [12, Theorem 1.3-1]. Therefore lower bounds for radii of discs discussed in Theorem 5.14 can be obtained using the norms of the operators. The next few results show that the spectrum can also be used to get lower bounds for invertible operators.

If $C_{\psi,\varphi}$ is invertible, then ψ is both bounded and bounded away from zero and φ is an automorphism [10, Theorem 2.0.1].

Proposition 5.15. *Suppose that $C_{\psi,\varphi}$ is invertible. Then $W(C_{\psi,\varphi})$ contains the open disc centered at the origin with radius*

$$\inf_{z\in\mathbb{D}}\{|\psi(z)|\}\sqrt{\frac{1-|\varphi^{-1}(0)|}{1+|\varphi^{-1}(0)|}}.$$

Proof. From Theorem 2.0.1 of [10] it follows that $C_{\psi,\varphi}^{-1} = C_{1/\psi\circ\varphi^{-1},\varphi^{-1}}$. Since $C_{1/\psi\circ\varphi^{-1},\varphi^{-1}} = M_{1/\psi\circ\varphi^{-1}}C_{\varphi^{-1}}$,

$$||C_{\psi,\varphi}^{-1}|| \leq ||M_{1/\psi\circ\varphi^{-1}}||\ ||C_{\varphi^{-1}}||.$$

It is known that $||M_{1/\psi\circ\varphi^{-1}}|| = ||1/\psi||_\infty$ and $||C_{\varphi^{-1}}||^2 = \dfrac{1+|\varphi^{-1}(0)|}{1-|\varphi^{-1}(0)|}$; see Theorem 3.6 of [4] and page 138 of [13]. If $\mu \in \sigma(C_{\psi,\varphi})$, then $1/\mu \in \sigma(C_{\psi,\varphi}^{-1})$. Hence $|1/\mu| \leq ||C_{\psi,\varphi}^{-1}||$. Now it follows that

$$\frac{1}{|\mu|} \leq ||1/\psi||_\infty \sqrt{\frac{1+|\varphi^{-1}(0)|}{1-|\varphi^{-1}(0)|}}.$$

Therefore the spectral radius of $C_{\psi,\varphi}$ is at least $\inf_{z\in\mathbb{D}}\{|\psi(z)|\}\sqrt{\dfrac{1-|\varphi^{-1}(0)|}{1+|\varphi^{-1}(0)|}}$. Since the closed numerical range contains the spectrum, the result follows from Theorem 5.14. $\square$

Proposition 5.16. *Suppose that $C_{\psi,\varphi}$ is invertible and φ is an elliptic automorphism with the interior fixed point a. If $\varphi'(a)$ is not 1 or -1, then the following are true.*

1. *If $\varphi'(a)$ is not a root of unity, then $W(C_{\psi,\varphi})$ contains the open disc with radius $|\psi(a)|$ centered at 0.*

(2) *If $\varphi'(a)$ is a primitive n^{th} root of unity, then $W(C_{\psi,\varphi})$ contains the open disc with radius $|\psi(a)\cos(\pi/n)|$.*

Proof. (1) If φ is elliptic then $|\varphi'(a)| = 1$. Therefore if $\varphi'(a)$ is not 1 or -1, then $\varphi'(a)$ is not a real number. The points $\psi(a)\varphi'(a)^j,\ j = 0, 1, 2, \cdots$ belong to the spectrum [10, Lemma 3.1.1]. Since $\varphi'(a)$ is non-real and unimodular, the circle of radius $|\psi(0)|$ centered at 0 is the spectrum. The result follows.

(2) If $\varphi'(a)$ is a primitive n^{th} root of unity then $W(C_{\psi,\varphi})$ is a regular polygon with vertices at $\{\psi(a)\varphi'(a)^j,\ j = 0, 1, 2, \cdots, n-1\}$. The inscribed circle has radius $|\psi(a)\cos(\pi/n)|$.

□

Proposition 5.17. *Suppose that $C_{\psi,\varphi}$ is invertible and ψ is continuous on the closed unit disc. If φ is a parabolic automorphism with the fixed point $e^{i\theta}$, then the open disc centered at 0 with radius $|\psi(e^{i\theta})|$ is contained in $W(C_{\psi,\varphi})$.*

Proof. The spectrum of $C_{\psi,\varphi}$ is the circle centered at the origin with the radius $|\psi(e^{i\theta})|$; see Theorem 3.3.1 of [10]. The result follows. □

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