



A C^* -algebra of singular integral operators with shifts admitting distinct fixed points[☆]



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ABSTRACT

Representations on Hilbert spaces for a nonlocal C^* -algebra \mathfrak{B} of singular integral operators with piecewise slowly oscillating coefficients extended by a group of unitary shift operators are constructed. The group of unitary shift operators U_g in the C^* -algebra \mathfrak{B} is associated with a discrete amenable group G of orientation-preserving piecewise smooth homeomorphisms $g: \mathbb{T} \rightarrow \mathbb{T}$ that acts topologically freely on \mathbb{T} and admits distinct fixed points for different shifts. A C^* -algebra isomorphism of the quotient C^* -algebra \mathfrak{B}/\mathcal{K} , where \mathcal{K} is the ideal of compact operators, onto a C^* -algebra of Fredholm symbols is constructed by applying the local-trajectory method, spectral measures and a lifting theorem. As a result, a Fredholm symbol calculus for the C^* -algebra \mathfrak{B} or, equivalently, a faithful representation of the quotient C^* -algebra \mathfrak{B}/\mathcal{K} on a suitable Hilbert space is constructed and a Fredholm criterion for the operators $B \in \mathfrak{B}$ is established.

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1. Introduction

In this paper we deal with a nonlocal C^* -algebra \mathfrak{B} generated by a C^* -algebra of singular integral operators with piecewise slowly oscillating coefficients and by a group of unitary shift operators U_G associated with a discrete amenable group G of orientation-preserving piecewise smooth homeomorphisms. The aim is to construct a *Fredholm symbol map* for the C^* -algebra \mathfrak{B} , that is, a representation $\Psi_{\mathfrak{B}}: \mathfrak{B} \rightarrow \mathcal{B}(\mathcal{H}_{\mathfrak{B}})$ of \mathfrak{B} on a convenient Hilbert space $\mathcal{H}_{\mathfrak{B}}$ such that an operator $B \in \mathfrak{B}$ is Fredholm if and only if $\Psi_{\mathfrak{B}}(B)$ is invertible on $\mathcal{H}_{\mathfrak{B}}$. Thus, the map $\Psi_{\mathfrak{B}}$ produces a faithful representation of the quotient C^* -algebra \mathfrak{B}/\mathcal{K} on the Hilbert space $\mathcal{H}_{\mathfrak{B}}$, where \mathcal{K} is the ideal of compact operators.

The more complicated the structure of the set of fixed points of shifts is, the harder a Fredholm symbol calculus for \mathfrak{B} is constructed and the more complicated the structure of symbols becomes. In the present paper we study the C^* -algebras \mathfrak{B} with groups G acting topologically freely. Then shifts $g \in G$ can admit both common and non-common fixed points. The typical example of such groups is the solvable group G of affine mappings

$$g: \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto k_g x + h_g \quad (k_g > 0, h_g \in \mathbb{R}),$$

where all shifts $g \in G$ have the common fixed point $x = \infty$ and, if $k_g \neq 1$, the shifts $g \in G \setminus \{e\}$ have, in general, distinct fixed points $x = h_g/(1 - k_g)$.

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The existence of non-common fixed points for shifts $g \in G \setminus \{e\}$ leads to essential difficulties in constructing a Fredholm symbol for operators $B \in \mathfrak{B}$ as compared with the case of common fixed points for all $g \in G$ studied in [4,5]. Moreover, this forced us to apply a new methodology different from those used in the previous papers [4–7] and based on the local-trajectory method and its generalizations using spectral measures (see [15,18,1,2]), which are especially effective if all shifts $g \in G \setminus \{e\}$ have the same set of fixed (or periodic) points. The new tools that allowed us to construct the desired symbol map for the C^* -algebra \mathfrak{B} are a C^* -algebra modification of the lifting theorem (cf. [14, Theorem 1.8], [21, Theorem 3.3] and [25, Section 6.3]) and ideas borrowed from [20,17,22] and related to Banach algebras of singular integral operators with piecewise continuous coefficients and discrete subexponential groups of shifts (in particular, the crucial idea of dealing with finite subsets of orbits instead of whole orbits).

Let $\mathcal{B}(L^2(\mathbb{T}))$ be the C^* -algebra of all bounded linear operators acting on the space $L^2(\mathbb{T})$, where \mathbb{T} is the unit circle in \mathbb{C} oriented anticlockwise; $PSO(\mathbb{T})$ be the C^* -algebra of piecewise slowly oscillating functions on \mathbb{T} defined in Subsection 2.1; G be a discrete amenable group of orientation-preserving piecewise smooth homeomorphisms $g: \mathbb{T} \rightarrow \mathbb{T}$ possessing derivatives g' with at most finite sets of discontinuities, and let G act on \mathbb{T} topologically freely, that is, for every finite set $G_0 \subset G \setminus \{e\}$, where e is the unit of G , and every open set $V \subset \mathbb{T}$ there exists a point $\tau \in V$ such that $g(\tau) \neq \tau$ for every $g \in G_0$ (cf. [1,18]). Suppose G acts on \mathbb{T} from the right: $(g_1 g_2)(t) = g_2(g_1(t))$ for all $g_1, g_2 \in G$ and all $t \in \mathbb{T}$.

We study the C^* -subalgebra

$$\mathfrak{B} := \text{alg}(PSO(\mathbb{T}), S_{\mathbb{T}}, U_G) \quad (1.1)$$

of $\mathcal{B}(L^2(\mathbb{T}))$ generated by all multiplication operators al with $a \in PSO(\mathbb{T})$, by the Cauchy singular integral operator $S_{\mathbb{T}}$ defined by

$$(S_{\mathbb{T}}\varphi)(t) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi i} \int_{\mathbb{T} \setminus \mathbb{T}(t, \varepsilon)} \frac{\varphi(\tau)}{\tau - t} d\tau, \quad \mathbb{T}(t, \varepsilon) = \{\tau \in \mathbb{T} : |\tau - t| < \varepsilon\}, \quad t \in \mathbb{T}, \quad (1.2)$$

and by the group $U_G := \{U_g : g \in G\}$ of unitary weighted shift operators U_g given by

$$(U_g\varphi)(t) := |g'(t)|^{1/2} \varphi(g(t)) \quad \text{for } t \in \mathbb{T}. \quad (1.3)$$

The paper is organized as follows. Section 2 contains preliminaries: definition of the commutative C^* -algebra $PSO(\mathbb{T})$ of piecewise slowly oscillating functions on \mathbb{T} and a description of its maximal ideal space $M(PSO(\mathbb{T}))$, as well as a Fredholm symbol and a Fredholm criterion for the C^* -algebra

$$\mathfrak{A} := \text{alg}(PSO(\mathbb{T}), S_{\mathbb{T}}) \subset \mathfrak{B} \quad (1.4)$$

generated by the operator $S_{\mathbb{T}}$ and all operators al with $a \in PSO(\mathbb{T})$. Section 2 also contains a description of a spectral measure associated with a central subalgebra of the C^* -algebra \mathfrak{B}/\mathcal{K} .

Section 3 contains the main results of the paper: a representation $\psi_{\mathfrak{B}}$ of the C^* -algebra \mathfrak{B} on a Hilbert space $\mathcal{H}_{\mathfrak{B}}$ with $\text{Ker } \psi_{\mathfrak{B}} = \mathcal{K}$, where

$$\psi_{\mathfrak{B}} = \left(\bigoplus_{w \in W_{\mathbb{T}}} \psi_w \right) \oplus \left(\bigoplus_{w \in W_{\mathbb{T}}^0} \psi_w^0 \right), \quad \mathcal{H}_{\mathfrak{B}} := \left(\bigoplus_{w \in W_{\mathbb{T}}} \mathcal{H}_w \right) \oplus \left(\bigoplus_{w \in W_{\mathbb{T}}^0} \mathcal{H}_w^0 \right), \quad (1.5)$$

$W_{\mathbb{T}}$ is the set of all G -orbits of points $t \in \mathbb{T}$ and $W_{\mathbb{T}}^0$ is the set of G -orbits containing more than one point. As a result, a Fredholm symbol map for \mathfrak{B} is obtained (Theorem 3.1) and a Fredholm criterion for the operators $B \in \mathfrak{B}$ in terms of their Fredholm symbols is established (Theorem 3.2).

In Section 4 we give an example of a Fredholm singular integral operator with shift, which illustrates the conditions of Theorem 3.2 in an explicit form.

Section 5 contains the main tools for studying the C^* -algebra \mathfrak{B} : a suitable version of the local-trajectory method (Theorem 5.1), a lifting theorem for C^* -algebras (Theorem 5.2) and its corollary providing sufficient Fredholm conditions (Theorem 5.3).

In Section 6, applying the local-trajectory method, we study the invertibility of functional operators that constitute the C^* -algebra

$$\mathcal{A} := \text{alg}(PSO(\mathbb{T}), U_G) \subset \mathfrak{B}. \quad (1.6)$$

In Section 7, making use of results for the C^* -algebra (1.6), we prove that every operator $B \in \mathfrak{B}$ for considered groups G is of the form

$$B = A^+ P_{\mathbb{T}}^+ + A^- P_{\mathbb{T}}^- + H_B, \quad (1.7)$$

where $A^{\pm} \in \mathcal{A}$, $P_{\mathbb{T}}^{\pm} := (I \pm S_{\mathbb{T}})/2$, $H_B \in \mathfrak{H}$, and \mathfrak{H} is the closed two-sided ideal in \mathfrak{B} generated by all commutators $[al, S_{\mathbb{T}}]$ and $[U_g, S_{\mathbb{T}}]$, where $a \in PSO(\mathbb{T})$ and $g \in G$ (Theorem 7.2). Using this decomposition we prove in Section 7 that

the $*$ -homomorphisms Ψ_w^0 ($w \in W_{\mathbb{T}}^0$) characterizing the invertibility of functional operators A^{\pm} in (1.7) and defined initially on a dense subalgebra \mathfrak{B}^0 of \mathfrak{B} are continuous (Corollary 7.3).

Finally, in Section 8, applying results of Section 2 for the C^* -algebra (1.4) and a crucial relation for the essential norms of products of operators $N \in \mathfrak{B}$ by operators with fixed singularities (Lemma 8.2), which was obtained by using finite subsets of G -orbits, we prove the continuity of the mappings $\Psi_w : \mathfrak{B} \rightarrow \mathcal{B}(\mathcal{H}_w)$ associated with G -orbits $w \in W_{\mathbb{T}}$ (Theorem 8.3), check the fulfillment of all conditions of the lifting theorem presented in Subsection 5.2 and prove the main results of the paper.

2. Preliminaries

Given a Hilbert space \mathcal{H} , we denote by $\mathcal{B}(\mathcal{H})$ the C^* -algebra of all bounded linear operators on \mathcal{H} , by $\mathcal{K}(\mathcal{H})$ the ideal of all compact operators in $\mathcal{B}(\mathcal{H})$, and by $I \in \mathcal{B}(\mathcal{H})$ the identity operator on \mathcal{H} . If $S, T \in \mathcal{B}(\mathcal{H})$ and $S - T \in \mathcal{K}(\mathcal{H})$, we write $S \simeq T$. For $A \in \mathcal{B}(\mathcal{H})$, let $A^{\pi} := A + \mathcal{K}(\mathcal{H})$ and

$$|A| = \|A^{\pi}\| = \inf\{\|A + K\| : K \in \mathcal{K}(\mathcal{H})\}.$$

Given two C^* -algebras \mathcal{A} and \mathcal{B} , we write $\mathcal{A} \cong \mathcal{B}$ if these C^* -algebras are $*$ -isomorphic and hence isometric. Given a commutative unital C^* -algebra \mathcal{A} , we denote by $M(\mathcal{A})$ the maximal ideal space of \mathcal{A} .

2.1. The C^* -algebra of $PSO(\mathbb{T})$ functions

Let $C(\mathbb{T})$, $PC(\mathbb{T})$ and $SO(\mathbb{T})$ denote the C^* -subalgebras of $L^{\infty}(\mathbb{T})$ consisting, respectively, of all continuous functions on \mathbb{T} , all piecewise continuous functions on \mathbb{T} , that is, the functions having one-sided limits at each point $t \in \mathbb{T}$, and all slowly oscillating functions on \mathbb{T} , that is (cf. [24]), the functions f that are slowly oscillating at each point $\lambda \in \mathbb{T}$:

$$\lim_{\varepsilon \rightarrow 0} \text{ess sup}\{|f(z_1) - f(z_2)| : z_1, z_2 \in \mathbb{T}_{\varepsilon}(\lambda)\} = 0,$$

where $\mathbb{T}_{\varepsilon}(\lambda) := \{z \in \mathbb{T} : \varepsilon/2 \leq |z - \lambda| \leq \varepsilon\}$. Denoting by $SO_{\lambda}(\mathbb{T})$ the C^* -subalgebra of $L^{\infty}(\mathbb{T})$ consisting of the continuous functions on $\mathbb{T} \setminus \{\lambda\}$ that are slowly oscillating at $\lambda \in \mathbb{T}$, we deduce that $SO(\mathbb{T})$ is the smallest C^* -subalgebra of $L^{\infty}(\mathbb{T})$ containing all C^* -algebras $SO_{\lambda}(\mathbb{T})$ for $\lambda \in \mathbb{T}$.

Let $PSO(\mathbb{T}) := \text{alg}(SO(\mathbb{T}), PC(\mathbb{T}))$ be the C^* -subalgebra of $L^{\infty}(\mathbb{T})$ generated by the C^* -algebras $SO(\mathbb{T})$ and $PC(\mathbb{T})$. Obviously, $PSO(\mathbb{T})$ is the closure in $L^{\infty}(\mathbb{T})$ of the set $PSO^0(\mathbb{T})$ consisting of all bounded functions on \mathbb{T} admitting piecewise slowly oscillating discontinuities at finite subsets of \mathbb{T} and being continuous at all other points of \mathbb{T} .

Since $C(\mathbb{T}) \subset SO(\mathbb{T}) \subset PSO(\mathbb{T})$, it follows from [3] that

$$M(SO(\mathbb{T})) = \bigcup_{t \in \mathbb{T}} M_t(SO(\mathbb{T})), \quad M(PSO(\mathbb{T})) = \bigcup_{\xi \in M(SO(\mathbb{T}))} M_{\xi}(PSO(\mathbb{T})), \quad (2.1)$$

where the fibers of the maximal ideal spaces $M(SO(\mathbb{T}))$ and $M(PSO(\mathbb{T}))$ are given for $t \in \mathbb{T}$ and $\xi \in M(SO(\mathbb{T}))$ by

$$\begin{aligned} M_t(SO(\mathbb{T})) &= \{\xi \in M(SO(\mathbb{T})) : \xi|_{C(\mathbb{T})} = t\}, \\ M_{\xi}(PSO(\mathbb{T})) &= \{y \in M(PSO(\mathbb{T})) : y|_{SO(\mathbb{T})} = \xi\}, \end{aligned} \quad (2.2)$$

and $t(f) = f(t)$ for $f \in C(\mathbb{T})$. The fibers $M_{\xi}(PSO(\mathbb{T}))$ for $\xi \in M(SO(\mathbb{T}))$ can be characterized as follows.

Theorem 2.1. (See [3, Theorem 4.6].) *If $\xi \in M_t(SO(\mathbb{T}))$ with $t \in \mathbb{T}$, then*

$$M_{\xi}(PSO(\mathbb{T})) = \{(\xi, 0), (\xi, 1)\}, \quad (2.3)$$

where, for $\mu \in \{0, 1\}$, $(\xi, \mu)|_{SO(\mathbb{T})} = \xi$, $(\xi, \mu)|_{C(\mathbb{T})} = t$, $(\xi, \mu)|_{PC(\mathbb{T})} = (t, \mu)$; and $(t, 0)a = a(t - 0)$ and $(t, 1)a = a(t + 0)$ are the left and right one-sided limits of a function $a \in PC(\mathbb{T})$ at the point $t \in \mathbb{T}$.

By (2.1) and (2.3), $M(PSO(\mathbb{T})) = M(SO(\mathbb{T})) \times \{0, 1\}$. The Gelfand topology on $M(PSO(\mathbb{T}))$ is defined as follows. If $t \in \mathbb{T}$ and $\xi \in M_t(SO(\mathbb{T}))$, a base of neighborhoods of $(\xi, \mu) \in M(PSO(\mathbb{T}))$ consists of all open sets

$$U_{(\xi, \mu)} = \begin{cases} (U_{\xi, t} \times \{0\}) \cup (U_{\xi, t}^{-} \times \{0, 1\}) & \text{if } \mu = 0, \\ (U_{\xi, t} \times \{1\}) \cup (U_{\xi, t}^{+} \times \{0, 1\}) & \text{if } \mu = 1, \end{cases} \quad (2.4)$$

where $U_{\xi, t} = U_{\xi} \cap M_t(SO(\mathbb{T}))$, U_{ξ} is an open neighborhood of $\xi \in M(SO(\mathbb{T}))$, and $U_{\xi, t}^{-}$, $U_{\xi, t}^{+}$ consists of all $\zeta \in U_{\xi}$ such that $\tau = \zeta|_{C(\mathbb{T})}$ belong, respectively, to the open arcs $(te^{-i\varepsilon}, t)$ and $(t, te^{i\varepsilon})$ of \mathbb{T} for some $\varepsilon \in (0, 2\pi)$.

2.2. The C^* -algebra \mathfrak{A}

Consider the C^* -algebra \mathfrak{A} of singular integral operators on $L^2(\mathbb{T})$ with $PSO(\mathbb{T})$ coefficients, which is given by (1.4). With \mathfrak{A} we associate the set

$$\mathfrak{M} := M(SO(\mathbb{T})) \times \bar{\mathbb{R}}, \quad (2.5)$$

where $\bar{\mathbb{R}} = [-\infty, +\infty]$. Let $B(\mathfrak{M}, \mathbb{C}^{2 \times 2})$ be the C^* -algebra of all bounded matrix functions $f : \mathfrak{M} \rightarrow \mathbb{C}^{2 \times 2}$. According to [9, Section 7] and [4, Theorem 5.1] we have the following symbol calculus for the C^* -algebra \mathfrak{A} .

Theorem 2.2. *The map $\text{Sym} : \{aI : a \in PSO(\mathbb{T})\} \cup \{S_{\mathbb{T}}\} \rightarrow B(\mathfrak{M}, \mathbb{C}^{2 \times 2})$ given by the matrix functions*

$$[\text{Sym}(aI)](\xi, x) = \begin{pmatrix} a(\xi, 1) & 0 \\ 0 & a(\xi, 0) \end{pmatrix}, \quad [\text{Sym } S_{\mathbb{T}}](\xi, x) = \begin{pmatrix} u(x) & -v(x) \\ v(x) & -u(x) \end{pmatrix}, \quad (2.6)$$

where $a(\xi, \mu)$ is the Gelfand transform of a function $a \in PSO(\mathbb{T})$ at the point $(\xi, \mu) \in M(PSO(\mathbb{T}))$ and

$$u(x) := \tanh(\pi x), \quad v(x) := -i / \cosh(\pi x) \quad \text{for } x \in \bar{\mathbb{R}}, \quad (2.7)$$

extends to a C^* -algebra homomorphism $\text{Sym} : \mathfrak{A} \rightarrow B(\mathfrak{M}, \mathbb{C}^{2 \times 2})$ whose kernel consists of all compact operators on $L^2(\mathbb{T})$. An operator $A \in \mathfrak{A}$ is Fredholm on the space $L^2(\mathbb{T})$ if and only if $\det([\text{Sym } A](\xi, x)) \neq 0$ for all $(\xi, x) \in \mathfrak{M}$.

To each point $t \in \mathbb{T}$ we assign the operator $V_t \in \mathcal{B}(L^2(\mathbb{T}))$ with fixed singularity at t , which is given for $z \in \mathbb{T}$ by

$$(V_t \varphi)(z) := \frac{\chi_t^+(z)}{\pi i} \int_{\mathbb{T}} \frac{\varphi(y) \chi_t^+(y)}{y + z - 2t} dy - \frac{\chi_t^-(z)}{\pi i} \int_{\mathbb{T}} \frac{\varphi(y) \chi_t^-(y)}{y + z - 2t} dy, \quad (2.8)$$

where χ_t^\pm are the characteristic functions of arcs γ_t^\pm such that $\gamma_t := \gamma_t^+ \cup \gamma_t^-$ is a neighborhood of t separated from $-t$, $\gamma_t^+ \cap \gamma_t^- = \{t\}$, and $\gamma_t^+ \cap (-t, t) = \emptyset$, $\gamma_t^- \cap (t, -t) = \emptyset$. The operators V_t for all $t \in \mathbb{T}$ belong to the C^* -algebra \mathfrak{A} (see, e.g., [4, Lemma 5.3]).

Let \mathcal{P} consist of all polynomials $\sum_{k=0}^n a_k u^k$ ($a_k \in \mathbb{C}$, $n = 0, 1, \dots$), and

$$\mathcal{Z} := \text{alg}\{aI, H_{P,t} : a \in SO(\mathbb{T}), P \in \mathcal{P}, t \in \mathbb{T}\} \subset \mathcal{B}(L^2(\mathbb{T})) \quad (2.9)$$

be the C^* -subalgebra of \mathfrak{A} generated by the operators aI ($a \in SO(\mathbb{T})$) and

$$H_{P,t} := P(\chi_t^+ S_{\mathbb{T}} \chi_t^+ I - \chi_t^- S_{\mathbb{T}} \chi_t^- I) V_t \in \mathfrak{A} \quad (P \in \mathcal{P}, t \in \mathbb{T}).$$

By [4, (4.10)–(4.11)] and [22, (5.24)], we get

$$aH_{P,t} \simeq H_{P,t}aI, \quad S_{\mathbb{T}}H_{P,t} \simeq H_{P,t}S_{\mathbb{T}}, \quad U_g H_{P,t} \simeq H_{P,g^{-1}(t)} U_g \quad (2.10)$$

for all $a \in PSO(\mathbb{T})$, $t \in \mathbb{T}$ and $g \in G$. Moreover, because

$$bS_{\mathbb{T}} \simeq S_{\mathbb{T}}bI \quad \text{for all } b \in SO(\mathbb{T}), \quad (2.11)$$

we conclude that $\mathcal{Z}^\pi := (\mathcal{Z} + \mathcal{K})/\mathcal{K}$ is a central C^* -subalgebra of the C^* -algebra $\mathfrak{A}^\pi := \mathfrak{A}/\mathcal{K}$, where $\mathcal{K} := \mathcal{K}(L^2(\mathbb{T}))$. Given the set

$$\dot{\mathfrak{M}} := M(SO(\mathbb{T})) \times \dot{\mathbb{R}} \quad (2.12)$$

with $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, we infer from [4, Theorem 6.3] that $\mathcal{Z}^\pi \cong C(\dot{\mathfrak{M}})$, where $\dot{\mathfrak{M}}$ is the compact Hausdorff space equipped with the Gelfand topology whose neighborhood base of a point $(\xi, x) \in \dot{\mathfrak{M}}$ consists of all open sets of the form

$$W_{(\xi, x)} = \begin{cases} U_{\xi, t} \times (x - \varepsilon, x + \varepsilon) & \text{if } (\xi, x) \in M(SO(\mathbb{T})) \times \mathbb{R}, \\ (U_{\xi} \times \dot{\mathbb{R}}) \setminus (U_{\xi, t} \times [-\varepsilon, \varepsilon]) & \text{if } (\xi, x) \in M(SO(\mathbb{T})) \times \{\infty\}, \end{cases}$$

$\varepsilon > 0$, U_{ξ} is an open neighborhood of a point $\xi \in M(SO(\mathbb{T}))$, and $U_{\xi, t} = U_{\xi} \cap M_t(SO(\mathbb{T}))$ for $t = \xi|_{C(\mathbb{T})} \in \mathbb{T}$. As usual, $a(\xi) := \xi(a)$ for $a \in SO(\mathbb{T})$.

Below we need the next result obtained by analogy with [4, Lemma 5.4].

Lemma 2.3. *If g is an orientation-preserving piecewise smooth homeomorphism on \mathbb{T} with a fixed point $t \in \mathbb{T}$, then $U_g V_t \simeq V_t U_g \in \mathfrak{A}$ and*

$$[\text{Sym}(U_g V_t)](\xi, x) := \begin{cases} \text{diag}\{e^{ix \ln g'(t+0)} v(x), e^{ix \ln g'(t-0)} v(x)\} & \text{if } (\xi, x) \in M_t(SO(\mathbb{T})) \times \bar{\mathbb{R}}, \\ 0_{2 \times 2} & \text{if } (\xi, x) \in \mathfrak{M} \setminus (M_t(SO(\mathbb{T})) \times \bar{\mathbb{R}}), \end{cases}$$

where $v(x) = -i / \cosh(\pi x)$ for $x \in \bar{\mathbb{R}}$ and \mathfrak{M} is given by (2.5).

2.3. The spectral measure associated with the C^* -algebra \mathcal{Z}^π

Let $\mathfrak{B} = \text{alg}(PSO(\mathbb{T}), S_{\mathbb{T}}, U_G)$ be the C^* -subalgebra of $\mathcal{B}(L^2(\mathbb{T}))$ defined by (1.1)–(1.3), and let $\mathfrak{B}^\pi := \mathfrak{B}/\mathcal{K}$. Consider an isometric representation

$$\varphi : \mathfrak{B}^\pi \rightarrow \mathcal{B}(\mathcal{H}_\varphi), \quad B^\pi \mapsto \varphi(B^\pi) \quad (2.13)$$

of the C^* -algebra \mathfrak{B}^π on an abstract Hilbert space \mathcal{H}_φ . Let $\mathfrak{R}(\mathfrak{M})$ be the σ -algebra of all Borel subsets of the compact \mathfrak{M} given by (2.12), and let

$$P_\varphi : \mathfrak{R}(\mathfrak{M}) \rightarrow \mathcal{B}(\mathcal{H}_\varphi) \quad (2.14)$$

be the spectral measure associated to the representation (2.13) restricted to the commutative unital C^* -algebra \mathcal{Z}^π , where \mathcal{Z} is defined by (2.9). Since all shifts $g \in G$ are piecewise smooth homeomorphisms, we have the relations

$$U_g a U_g^* = (a \circ g)I \in \mathfrak{A}, \quad U_g S_{\mathbb{T}} U_g^* \in \mathfrak{A} \quad \text{for all } a \in PSO(\mathbb{T}) \text{ and } g \in G \quad (2.15)$$

(see, e.g., [3, Lemma 4.2] and [7, Theorem 2.4]). Thus, for all $g \in G$ the mappings $\alpha_g : A^\pi \mapsto U_g^\pi A^\pi (U_g^\pi)^{-1}$ are $*$ -automorphisms of the C^* -algebras \mathfrak{A}^π and \mathcal{Z}^π . These $*$ -automorphisms induce on $M(\mathcal{Z}^\pi) = \mathfrak{M}$ the group of homeomorphisms $\beta_g : \mathfrak{M} \rightarrow \mathfrak{M}$, $(\xi, x) \mapsto (g(\xi), x)$, where $\xi \mapsto g(\xi)$ is the homeomorphism on $M(SO(\mathbb{T}))$ given by

$$a(g(\xi)) = (a \circ g)(\xi) \quad \text{for all } a \in SO(\mathbb{T}) \text{ and all } \xi \in M(SO(\mathbb{T})). \quad (2.16)$$

Taking the G -invariant subset of $\mathfrak{R}(\mathfrak{M})$ given by

$$\mathfrak{R}_G(\mathfrak{M}) := \{\Delta \in \mathfrak{R}(\mathfrak{M}) : \beta_g(\Delta) = \Delta \text{ for all } g \in G\},$$

we conclude from [18] that, for each $\Delta \in \mathfrak{R}_G(\mathfrak{M})$ and each operator $B \in \mathfrak{B}$,

$$P_\varphi(\Delta)\varphi(B^\pi) = \varphi(B^\pi)P_\varphi(\Delta).$$

For every point $t \in \mathbb{T}$ we introduce the open subset of \mathfrak{M} given by

$$\mathfrak{M}_t^\circ := M_t(SO(\mathbb{T})) \times \mathbb{R}. \quad (2.17)$$

If t is a fixed point for all $g \in G$, for every function $a \in SO(\mathbb{T})$ it follows that $a(g(\xi)) = a(\xi)$ for all $\xi \in M_t(SO(\mathbb{T}))$ (see, e.g., [4, Theorem 6.4]), and therefore \mathfrak{M}_t° is a set of fixed points for all homeomorphisms β_g ($g \in G$). Consequently, $\mathfrak{M}_t^\circ \in \mathfrak{R}_G(\mathfrak{M})$, while $\mathfrak{M}_t^\circ \notin \mathfrak{R}_G(\mathfrak{M})$ if $g(t) \neq t$ for some $g \in G$.

For each $g \in G$, the homeomorphism $\xi \mapsto g(\xi)$ defined by (2.16) sends the fibers $M_t(SO(\mathbb{T}))$ onto the fibers $M_{g(t)}(SO(\mathbb{T}))$. Hence, similarly to [7, Lemma 4.2], we obtain the following.

Lemma 2.4. *For every $t \in \mathbb{T}$ and every $g \in G$,*

$$P_\varphi(\mathfrak{M}_t^\circ)\varphi(U_g^\pi) = \varphi(U_g^\pi)P_\varphi(\mathfrak{M}_{g(t)}^\circ). \quad (2.18)$$

Below we will also use the following result.

Lemma 2.5. (See [4, Corollary 9.3].) *For any $t \in \mathbb{T}$, the map*

$$\text{Sym}_t^\circ : P_\varphi(\mathfrak{M}_t^\circ)\varphi(\mathfrak{A}^\pi) \rightarrow \mathcal{B}(l^2(\mathfrak{M}_t^\circ, \mathbb{C}^2)), \quad P_\varphi(\mathfrak{M}_t^\circ)\varphi(A^\pi) \mapsto (\text{Sym } A)|_{\mathfrak{M}_t^\circ}$$

is an isometric C^ -algebra homomorphism. An operator $P_\varphi(\mathfrak{M}_t^\circ)\varphi(A^\pi)$ for $A \in \mathfrak{A}$ is invertible on the space $P_\varphi(\mathfrak{M}_t^\circ)\mathcal{H}_\varphi$ if and only if*

$$\det([\text{Sym } A](\xi, x)) \neq 0 \quad \text{for all } (\xi, x) \in M_t(SO(\mathbb{T})) \times \overline{\mathbb{R}}.$$

3. Main results

Let \mathbb{T}_G denote the set of common fixed points for all $g \in G$. The set $W_{\mathbb{T}}$ of all G -orbits $G(t) = \{g(t) : g \in G\}$ of points $t \in \mathbb{T}$ has the form

$$W_{\mathbb{T}} = W_{\mathbb{T}_G} \cup W_{\mathbb{T}}^0, \quad (3.1)$$

where $W_{\mathbb{T}_G}$ is the set of all one-point G -orbits on \mathbb{T} and $W_{\mathbb{T}}^0 := W_{\mathbb{T}} \setminus W_{\mathbb{T}_G}$.

Given $t, \tau \in \mathbb{T}$, we define the set

$$Y_{t,\tau} := \{g \in G : g(t) = \tau\}. \quad (3.2)$$

Fix $t_w \in w$ for every $w \in W_{\mathbb{T}}$ and, for each $\tau \in w$, fix $g_{\tau} \in Y_{t_w,\tau}$ such that $g_{t_w} = e$, the unit of G . For every $g \in Y_{t,\tau}$ with $t, \tau \in w$, we deduce that

$$\tilde{g}_{t,\tau} := g_t g g_{\tau}^{-1} \in Y_{t_w,t_w} \quad (3.3)$$

because

$$\tilde{g}_{t,\tau}(t_w) = (g_t g g_{\tau}^{-1})(t_w) = g_{\tau}^{-1}[g(g_t(t_w))] = t_w.$$

If $t \in \mathbb{T}_G$, then $G(t) = \{t\}$, $Y_{t,t} = G$, and $\tilde{g}_{t,t} = g$ for all $g \in Y_{t,t}$.

With the C^* -algebra \mathfrak{B} we associate the Hilbert space

$$\mathcal{H}_{\mathfrak{B}} := \left(\bigoplus_{w \in W_{\mathbb{T}}} \mathcal{H}_w \right) \oplus \left(\bigoplus_{w \in W_{\mathbb{T}}^0} \mathcal{H}_w^0 \right) \quad (3.4)$$

where the Hilbert spaces

$$\begin{aligned} \mathcal{H}_w &:= l^2(M_{t_w}(SO(\mathbb{T})) \times \mathbb{R}, l^2(w, \mathbb{C}^2)) \quad (w \in W_{\mathbb{T}}), \\ \mathcal{H}_w^0 &:= l^2(M_{t_w}(SO(\mathbb{T})) \times \{\pm\infty\}, l^2(G, \mathbb{C}^2)) \quad (w \in W_{\mathbb{T}}^0) \end{aligned} \quad (3.5)$$

consist of $l^2(w, \mathbb{C}^2)$ -valued functions defined on the set $M_{t_w}(SO(\mathbb{T})) \times \mathbb{R}$ and of $l^2(G, \mathbb{C}^2)$ -valued functions defined on the set $M_{t_w}(SO(\mathbb{T})) \times \{\pm\infty\}$, where all these functions have at most countable sets of non-zero values. In its turn, $l^2(X, \mathbb{C}^2)$ for $X \in W_{\mathbb{T}} \cup \{G\}$ is the Hilbert space of vectors $f = (f_{\tau})_{\tau \in X}$ with at most countable sets of non-zero entries $f_{\tau} = (f_{\tau,i})_{i=1}^2 \in \mathbb{C}^2$ and the norm

$$\|f\|_{l^2(X, \mathbb{C}^2)} := \left(\sum_{\tau \in X} \|f_{\tau}\|_{\mathbb{C}^2}^2 \right)^{1/2}, \quad \|f_{\tau}\|_{\mathbb{C}^2}^2 := |f_{\tau,1}|^2 + |f_{\tau,2}|^2.$$

Thus, the norms of vector functions

$$\begin{aligned} F : M_{t_w}(SO(\mathbb{T})) \times \mathbb{R} &\rightarrow l^2(w, \mathbb{C}^2), & (\xi, x) &\mapsto F(\xi, x) = (F_{\tau}(\xi, x))_{\tau \in w}, \\ \Phi : M_{t_w}(SO(\mathbb{T})) \times \{\pm\infty\} &\rightarrow l^2(G, \mathbb{C}^2), & (\xi, x) &\mapsto \Phi(\xi, x) = (\Phi_g(\xi, x))_{g \in G} \end{aligned}$$

in the Hilbert spaces \mathcal{H}_w and \mathcal{H}_w^0 are given, respectively, by

$$\begin{aligned} \|F\|_{\mathcal{H}_w} &:= \left(\sum_{(\xi, x) \in M_{t_w}(SO(\mathbb{T})) \times \mathbb{R}} \sum_{\tau \in w} \|F_{\tau}(\xi, x)\|_{\mathbb{C}^2}^2 \right)^{1/2}, \\ \|\Phi\|_{\mathcal{H}_w^0} &:= \left(\sum_{(\xi, x) \in M_{t_w}(SO(\mathbb{T})) \times \{\pm\infty\}} \sum_{g \in G} \|\Phi_g(\xi, x)\|_{\mathbb{C}^2}^2 \right)^{1/2}. \end{aligned}$$

We now construct a representation

$$\psi_{\mathfrak{B}} : \mathfrak{B} \rightarrow \mathcal{B}(\mathcal{H}_{\mathfrak{B}}), \quad B \mapsto \left(\bigoplus_{w \in W_{\mathbb{T}}} \psi_w(B) \right) \oplus \left(\bigoplus_{w \in W_{\mathbb{T}}^0} \psi_w^0(B) \right) \quad (3.6)$$

of the C^* -algebra \mathfrak{B} on the Hilbert space (3.4). A Fredholm criterion for the operators $B \in \mathfrak{B}$ will be described in terms of invertibility of the operators $\psi_{\mathfrak{B}}(B)$ on the space $\mathcal{H}_{\mathfrak{B}}$. Hence, the representation $\psi_{\mathfrak{B}} : \mathfrak{B} \rightarrow \mathcal{B}(\mathcal{H}_{\mathfrak{B}})$ can be referred to as the *Fredholm symbol map* for the C^* -algebra \mathfrak{B} , and $\psi_{\mathfrak{B}}$ can be considered as the direct sum of the following C^* -algebra homomorphisms

$$\begin{aligned} \psi_w : \mathfrak{B} &\rightarrow \mathcal{B}(\mathcal{H}_w), & B &\mapsto \psi_w(B) = \text{Sym}_w(B)I \quad (w \in W_{\mathbb{T}}), \\ \psi_w^0 : \mathfrak{B} &\rightarrow \mathcal{B}(\mathcal{H}_w^0), & B &\mapsto \psi_w^0(B) = \text{Sym}_w^0(B)I \quad (w \in W_{\mathbb{T}}^0), \end{aligned} \quad (3.7)$$

defined initially on the generators of the C^* -algebra \mathfrak{B} .

In (3.7), $\psi_w(B)$ are operators of multiplication by finite or infinite matrix functions $\text{Sym}_w(B) : M_{t_w}(SO(\mathbb{T})) \times \mathbb{R} \rightarrow l^2(w, \mathbb{C}^2)$ whose values at the points $(\xi, x) \in M_{t_w}(SO(\mathbb{T})) \times \mathbb{R}$ define bounded linear operators on the Hilbert space $l^2(w, \mathbb{C}^2)$ and are given on the generators of the C^* -algebra \mathfrak{B} by

$$\begin{aligned}
[\text{Sym}_w(aI)](\xi, x) &:= \text{diag} \left\{ \begin{pmatrix} (a \circ g_t)(\xi, 1) & 0 \\ 0 & (a \circ g_t)(\xi, 0) \end{pmatrix} \right\}_{t \in w} \quad (a \in \text{PSO}(\mathbb{T})), \\
[\text{Sym}_w(S_{\mathbb{T}})](\xi, x) &:= \text{diag} \left\{ \begin{pmatrix} \tanh(\pi x) & i/\cosh(\pi x) \\ -i/\cosh(\pi x) & -\tanh(\pi x) \end{pmatrix} \right\}_{t \in w}, \\
[\text{Sym}_w(U_g)](\xi, x) &:= \left(\begin{pmatrix} \delta_g(t, \tau) e^{ixk_{g,t,\tau}^+} & 0 \\ 0 & \delta_g(t, \tau) e^{ixk_{g,t,\tau}^-} \end{pmatrix} \right)_{t, \tau \in w} \quad (g \in G),
\end{aligned} \tag{3.8}$$

where $\delta_g(t, \tau) = 1$ if $g \in Y_{t,\tau}$ and $\delta_g(t, \tau) = 0$ if $g \notin Y_{t,\tau}$, $k_{g,t,\tau}^{\pm} := \ln[\tilde{g}'_{t,\tau}(t_w \pm 0)]$ and $\tilde{g}'_{t,\tau}(t_w \pm 0) > 0$.

Further, $\Psi_w^0(B)$ are operators of multiplication by finite or infinite matrix functions $\text{Sym}_w^0(B)$ given on $M_{t_w}(\text{SO}(\mathbb{T})) \times \{\pm\infty\}$, where the values of these matrix functions at the points $(\xi, x) \in M_{t_w}(\text{SO}(\mathbb{T})) \times \{\pm\infty\}$ define bounded linear operators on the space $l^2(G, \mathbb{C}^2)$ and are given on the generators of the C^* -algebra \mathfrak{B} as follows:

$$\begin{aligned}
[\text{Sym}_w^0(aI)](\xi, x) &:= \text{diag} \left\{ \begin{pmatrix} (a \circ h)(\xi, 1) & 0 \\ 0 & (a \circ h)(\xi, 0) \end{pmatrix} \right\}_{h \in G} \quad (a \in \text{PSO}(\mathbb{T})), \\
[\text{Sym}_w^0(S_{\mathbb{T}})](\xi, x) &:= \text{diag} \left\{ \begin{pmatrix} \tanh(\pi x) & i/\cosh(\pi x) \\ -i/\cosh(\pi x) & -\tanh(\pi x) \end{pmatrix} \right\}_{h \in G}, \\
[\text{Sym}_w^0(U_g)](\xi, x) &:= \left(\begin{pmatrix} \delta_{hg,s} & 0 \\ 0 & \delta_{hg,s} \end{pmatrix} \right)_{h,s \in G} \quad (g \in G),
\end{aligned} \tag{3.9}$$

where $\delta_{h,s} = 1$ if $h = s$ and $\delta_{h,s} = 0$ if $h \neq s$.

Identifying the Hilbert space $l^2(w, \mathbb{C}^2)$ with \mathbb{C}^2 for every G -orbit $w \in W_{\mathbb{T}_G}$, we will prove below the following main results of the paper.

Theorem 3.1. *The map $\Psi_{\mathfrak{B}}$ defined on the generators of the C^* -algebra \mathfrak{B} by formulas (3.6)–(3.9) extends to a C^* -algebra homomorphism $\Psi_{\mathfrak{B}}$ of \mathfrak{B} into the C^* -algebra $\mathcal{B}(\mathcal{H}_{\mathfrak{B}})$ such that $\|\Psi_{\mathfrak{B}}(B)\| \leq \|B\|$ for all $B \in \mathfrak{B}$, and $\text{Ker } \Psi_{\mathfrak{B}}$ coincides with the ideal of all compact operators in the C^* -algebra $\mathcal{B}(L^2(\mathbb{T}))$.*

Theorem 3.2. *An operator $B \in \mathfrak{B}$ is Fredholm on the space $L^2(\mathbb{T})$ if and only if the operator $\Psi_{\mathfrak{B}}(B)$ is invertible on the space $\mathcal{H}_{\mathfrak{B}}$, that is, if the following three conditions hold:*

(i) *for every $w \in W_{\mathbb{T}_G}$ and every $(\xi, x) \in M_{t_w}(\text{SO}(\mathbb{T})) \times \mathbb{R}$ the 2×2 matrix $[\text{Sym}_w(B)](\xi, x)$ is invertible and*

$$\inf_{w \in W_{\mathbb{T}_G}} \inf_{(\xi, x) \in M_{t_w}(\text{SO}(\mathbb{T})) \times \mathbb{R}} |\det([\text{Sym}_w(B)](\xi, x))| > 0;$$

(ii) *for every $w \in W_{\mathbb{T}}^0$ and every $(\xi, x) \in M_{t_w}(\text{SO}(\mathbb{T})) \times \mathbb{R}$ the operator $[\text{Sym}_w(B)](\xi, x)I$ is invertible on the Hilbert space $l^2(w, \mathbb{C}^2)$ and*

$$\sup_{w \in W_{\mathbb{T}}^0} \sup_{(\xi, x) \in M_{t_w}(\text{SO}(\mathbb{T})) \times \mathbb{R}} \|([\text{Sym}_w(B)](\xi, x)I)^{-1}\|_{\mathcal{B}(l^2(w, \mathbb{C}^2))} < \infty;$$

(iii) *for every $w \in W_{\mathbb{T}}^0$ and every $(\xi, x) \in M_{t_w}(\text{SO}(\mathbb{T})) \times \{\pm\infty\}$ the operator $[\text{Sym}_w^0(B)](\xi, x)I$ is invertible on the Hilbert space $l^2(G, \mathbb{C}^2)$ and*

$$\sup_{w \in W_{\mathbb{T}}^0} \sup_{(\xi, x) \in M_{t_w}(\text{SO}(\mathbb{T})) \times \{\pm\infty\}} \|([\text{Sym}_w^0(B)](\xi, x)I)^{-1}\|_{\mathcal{B}(l^2(G, \mathbb{C}^2))} < \infty.$$

4. Example of a Fredholm singular integral operator with shift

Consider an example that illustrates the conditions of Theorem 3.2 in an explicit form. Let the group G consist of the shifts

$$g_{k,h} : \mathbb{T} \rightarrow \mathbb{T}, \quad t \mapsto \frac{ki(1+t) + (h-i)(1-t)}{ki(1+t) + (h+i)(1-t)} \quad (k > 0, h \in \mathbb{R}),$$

which are similar to affine mappings $x \mapsto kx + h$ on \mathbb{R} . Then, according to (3.1), $\mathbb{T}_G = \{1\}$, the sets $W_{\mathbb{T}_G}$ and $W_{\mathbb{T}}^0$ consist of the G -orbits $w_1 = \{1\}$ and $w_{-1} = \mathbb{T} \setminus \{1\}$, respectively. If $k > 0$ and $k \neq 1$, then for every $h \in \mathbb{R}$ the shift $g_{k,h}$ has two fixed points on \mathbb{T} : the point $t_{k,h} = [h - i(1-k)]/[h + i(1-k)]$ and the point 1, at which $g'_{k,h}(t_{k,h}) = k$ and $g'_{k,h}(1) = k^{-1}$.

Fix a shift $g = g_{k,h}$ with $k \neq 1$ and put $t_0 := t_{k,h}$. Consider the operator

$$B := (aI - bU_g)P_{\mathbb{T}}^+ + (cI - dU_g)P_{\mathbb{T}}^- \in \mathfrak{B} \subset \mathcal{B}(L^2(\mathbb{T})), \tag{4.1}$$

where $a, b, c, d \in \text{PSO}(\mathbb{T})$, the unitary weighted shift operator U_g is given by (1.3) and $P_{\mathbb{T}}^{\pm} = (I \pm S_{\mathbb{T}})/2$. Let us establish explicit Fredholm conditions for the operator (4.1) by applying Theorem 3.2.

By (3.8)–(3.9) and (4.1), we obtain

$$[\text{Sym}_{w_1}(B)](\xi, x) = (B_{n,m}(\xi, x))_{n,m=1}^2 \quad \text{for } (\xi, x) \in M_1(\text{SO}(\mathbb{T})) \times \mathbb{R},$$

where the entries of the 2×2 matrix function $\text{Sym}_{w_1}(B)$ are given by

$$\begin{aligned} B_{1,1}(\xi, x) &= [a(\xi, 1) - b(\xi, 1)k^{-ix}]\mathcal{P}_+(x) + [c(\xi, 1) - d(\xi, 1)k^{-ix}]\mathcal{P}_-(x), \\ B_{1,2}(\xi, x) &= -([a(\xi, 1) - b(\xi, 1)k^{-ix}] - [c(\xi, 1) - d(\xi, 1)k^{-ix}])v(x)/2, \\ B_{2,1}(\xi, x) &= ([a(\xi, 0) - b(\xi, 0)k^{-ix}] - [c(\xi, 0) - d(\xi, 0)k^{-ix}])v(x)/2, \\ B_{2,2}(\xi, x) &= [a(\xi, 0) - b(\xi, 0)k^{-ix}]\mathcal{P}_-(x) + [c(\xi, 0) - d(\xi, 0)k^{-ix}]\mathcal{P}_+(x), \end{aligned}$$

and $\mathcal{P}_{\pm}(x) = [1 \pm \tanh(\pi x)]/2$, $v(x) = -i/\cosh(\pi x)$.

Hence, condition (i) of Theorem 3.2 takes the form:

$$\inf_{(\xi, x) \in M_1(\text{SO}(\mathbb{T})) \times \mathbb{R}} |\det([\text{Sym}_{w_1}(B)](\xi, x))| > 0, \quad (4.2)$$

where

$$\begin{aligned} \det([\text{Sym}_{w_1}(B)](\xi, x)) &= [a(\xi, 1) - b(\xi, 1)k^{-ix}][c(\xi, 0) - d(\xi, 0)k^{-ix}]\mathcal{P}_+(x) \\ &\quad + [a(\xi, 0) - b(\xi, 0)k^{-ix}][c(\xi, 1) - d(\xi, 1)k^{-ix}]\mathcal{P}_-(x). \end{aligned}$$

Condition (iii) of Theorem 3.2 takes the form: for every $\xi \in M_{-1}(\text{SO}(\mathbb{T}))$ the operators $[\text{Sym}_{w_{-1}}^0(B)](\xi, \pm\infty)I$ are invertible on the space $\ell^2(G, \mathbb{C}^2)$ and

$$\sup_{\xi \in M_{-1}(\text{SO}(\mathbb{T}))} \|([\text{Sym}_{w_{-1}}^0(B)](\xi, \pm\infty)I)^{-1}\|_{\mathcal{B}(\ell^2(G, \mathbb{C}^2))} < \infty, \quad (4.3)$$

where $[\text{Sym}_{w_{-1}}^0(B)](\xi, \pm\infty) = (\text{diag}\{B_{h,s}^+(\xi, \pm\infty), B_{h,s}^-(\xi, \pm\infty)\})_{h,s \in G}$,

$$\begin{aligned} B_{h,s}^+(\xi, +\infty) &= (a \circ h)(\xi, 1)\delta_{h,s} - (b \circ h)(\xi, 1)\delta_{hg,s}, \\ B_{h,s}^+(\xi, -\infty) &= (c \circ h)(\xi, 1)\delta_{h,s} - (d \circ h)(\xi, 1)\delta_{hg,s}, \\ B_{h,s}^-(\xi, +\infty) &= (c \circ h)(\xi, 0)\delta_{h,s} - (d \circ h)(\xi, 0)\delta_{hg,s}, \\ B_{h,s}^-(\xi, -\infty) &= (a \circ h)(\xi, 0)\delta_{h,s} - (b \circ h)(\xi, 0)\delta_{hg,s}, \end{aligned}$$

with $\delta_{h,s} = 1$ for $h = s$ and $\delta_{h,s} = 0$ for $h \neq s$.

In fact (see Theorem 6.2 and relations (7.24) below), condition (iii) of Theorem 3.2 is equivalent to the invertibility of both the functional operators $A^+ = aI - bU_g$ and $A^- = cI - dU_g$ on the space $L^2(\mathbb{T})$, which in its turn is equivalent to the invertibility of these operators on the spaces $L^2(I_1)$ and $L^2(I_2)$, where $I_1 = [1, t_0] \subset \mathbb{T}$, $I_2 = [t_0, 1] \subset \mathbb{T}$ and $\{1, t_0\}$ is the set of fixed points of g on \mathbb{T} . Hence, by analogy with [19, Theorem 1.2] one can prove that the operator A^+ is invertible on the space $L^2(I_1)$ if and only if either

$$(C1) \quad \min_{\xi \in M_1(\text{SO}(\mathbb{T}))} (|a(\xi, 1)| - |b(\xi, 1)|) > 0, \quad \min_{\xi \in M_{t_0}(\text{SO}(\mathbb{T}))} (|a(\xi, 0)| - |b(\xi, 0)|) > 0,$$

and $a(\xi, \mu) \neq 0$ for every $(\xi, \mu) \in \bigcup_{t \in (1, t_0)} M_t(\text{SO}(\mathbb{T})) \times \{0, 1\}$; or

$$(C2) \quad \max_{\xi \in M_1(\text{SO}(\mathbb{T}))} (|a(\xi, 1)| - |b(\xi, 1)|) < 0, \quad \max_{\xi \in M_{t_0}(\text{SO}(\mathbb{T}))} (|a(\xi, 0)| - |b(\xi, 0)|) < 0,$$

and $b(\xi, \mu) \neq 0$ for every $(\xi, \mu) \in \bigcup_{t \in (1, t_0)} M_t(\text{SO}(\mathbb{T})) \times \{0, 1\}$. Analogously, the operator A^+ is invertible on the space $L^2(I_2)$ if and only if either

$$(C3) \quad \min_{\xi \in M_{t_0}(\text{SO}(\mathbb{T}))} (|a(\xi, 1)| - |b(\xi, 1)|) > 0, \quad \min_{\xi \in M_1(\text{SO}(\mathbb{T}))} (|a(\xi, 0)| - |b(\xi, 0)|) > 0,$$

and $a(\xi, \mu) \neq 0$ for every $(\xi, \mu) \in \bigcup_{t \in (t_0, 1)} M_t(\text{SO}(\mathbb{T})) \times \{0, 1\}$; or

$$(C4) \quad \max_{\xi \in M_{t_0}(\text{SO}(\mathbb{T}))} (|a(\xi, 1)| - |b(\xi, 1)|) < 0, \quad \max_{\xi \in M_1(\text{SO}(\mathbb{T}))} (|a(\xi, 0)| - |b(\xi, 0)|) < 0,$$

and $b(\xi, \mu) \neq 0$ for every $(\xi, \mu) \in \bigcup_{t \in (t_0, 1)} M_t(\text{SO}(\mathbb{T})) \times \{0, 1\}$.

Similarly, the operator $A^- = cI - dU_g$ is invertible on the space $L^2(\mathbb{T})$ if and only if one of the conditions (C1) or (C2) and one of the conditions (C3) or (C4) hold, with a and b replaced by c and d , respectively.

Fix $t_{w_{-1}} = -1$. Then condition (ii) of Theorem 3.2 takes the form: for every $(\xi, x) \in M_{-1}(SO(\mathbb{T})) \times \mathbb{R}$ the operator $[\text{Sym}_{w_{-1}}(B)](\xi, x)I$ is invertible on the space $l^2(w_{-1}, \mathbb{C}^2)$ and

$$\sup_{(\xi, x) \in M_{-1}(SO(\mathbb{T})) \times \mathbb{R}} \|([\text{Sym}_{w_{-1}}(B)](\xi, x)I)^{-1}\|_{\mathcal{B}(l^2(w_{-1}, \mathbb{C}^2))} < \infty, \quad (4.4)$$

where $[\text{Sym}_{w_{-1}}(B)](\xi, x) = \mathcal{C}(\xi, x) - \mathcal{D}(\xi, x)$,

$$\mathcal{C}(\xi, x) = \text{diag}\{\mathcal{C}_t(\xi, x)\}_{t \in w_{-1}}, \quad \mathcal{C}_t(\xi, x) = (C_{t,n,m}(\xi, x))_{n,m=1}^2, \quad (4.5)$$

$$C_{t,1,1}(\xi, x) = (a \circ g_t)(\xi, 1)\mathcal{P}_+(x) + (c \circ g_t)(\xi, 1)\mathcal{P}_-(x),$$

$$C_{t,1,2}(\xi, x) = -[(a \circ g_t)(\xi, 1) - (c \circ g_t)(\xi, 1)]v(x)/2,$$

$$C_{t,2,1}(\xi, x) = [(a \circ g_t)(\xi, 0) - (c \circ g_t)(\xi, 0)]v(x)/2,$$

$$C_{t,2,2}(\xi, x) = (a \circ g_t)(\xi, 0)\mathcal{P}_-(x) + (c \circ g_t)(\xi, 0)\mathcal{P}_+(x);$$

$$\mathcal{D}(\xi, x) = (\mathcal{D}_{t,\tau}(\xi, x))_{t,\tau \in w_{-1}}, \quad \mathcal{D}_{t,\tau}(\xi, x) = (D_{t,\tau,n,m}(\xi, x))_{n,m=1}^2,$$

$$D_{t,\tau,1,1}(\xi, x) = [(b \circ g_t)(\xi, 1)\mathcal{P}_+(x) + (d \circ g_t)(\xi, 1)\mathcal{P}_-(x)]\delta_g(t, \tau)e^{ixk_{g,t,\tau}^+},$$

$$D_{t,\tau,1,2}(\xi, x) = -[(b \circ g_t)(\xi, 1) - (d \circ g_t)(\xi, 1)]v(x)\delta_g(t, \tau)e^{ixk_{g,t,\tau}^+}/2,$$

$$D_{t,\tau,2,1}(\xi, x) = [(b \circ g_t)(\xi, 0) - (d \circ g_t)(\xi, 0)]v(x)\delta_g(t, \tau)e^{ixk_{g,t,\tau}^-}/2,$$

$$D_{t,\tau,2,2}(\xi, x) = [(b \circ g_t)(\xi, 0)\mathcal{P}_-(x) + (d \circ g_t)(\xi, 0)\mathcal{P}_+(x)]\delta_g(t, \tau)e^{ixk_{g,t,\tau}^-}.$$

Changing the order of rows and columns, the matrix function $\mathcal{C} - \mathcal{D}$ can be represented as a block diagonal matrix function, where the blocks correspond to different \mathcal{G} -orbits of points $t \in w_{-1}$ and $\mathcal{G} = \{g^n: n \in \mathbb{Z}\}$ is a cyclic subgroup of G . If, for example, conditions (C1) and (C3) hold for the operators A^\pm , then condition (ii) of Theorem 3.2 in view of (4.4) can be rewritten in the following equivalent form by analogy with conditions (iii) and (i):

$$\inf_{t \in w_{-1} \setminus \{t_0\}} \inf_{(\xi, x) \in M_{-1}(SO(\mathbb{T})) \times \mathbb{R}} |\det \mathcal{C}_t(\xi, x)| > 0, \quad (4.6)$$

$$\inf_{(\xi, x) \in M_{t_0}(SO(\mathbb{T})) \times \mathbb{R}} |[a(\xi, 1) - b(\xi, 1)k^{ix}][c(\xi, 0) - d(\xi, 0)k^{ix}]\mathcal{P}_+(x) + [a(\xi, 0) - b(\xi, 0)k^{ix}][c(\xi, 1) - d(\xi, 1)k^{ix}]\mathcal{P}_-(x)| > 0, \quad (4.7)$$

where from (4.5) it follows that

$$\det \mathcal{C}_t(\xi, x) = (a \circ g_t)(\xi, 1)(c \circ g_t)(\xi, 0)\mathcal{P}_+(x) + (a \circ g_t)(\xi, 0)(c \circ g_t)(\xi, 1)\mathcal{P}_-(x).$$

Thus, for example, the operator (4.1) is Fredholm on the space $L^2(\mathbb{T})$ if all the conditions (4.2), (C1) and (C3) for A^\pm , (4.6) and (4.7) are fulfilled.

Since for every real-valued function $f \in P(SO(\mathbb{T}))$ and every $t \in \mathbb{T}$ we have

$$\min_{\xi \in M_t(SO(\mathbb{T}))} f(\xi, 1) = \liminf_{\varepsilon \rightarrow +0} f(te^{i\varepsilon}), \quad \min_{\xi \in M_t(SO(\mathbb{T}))} f(\xi, 0) = \liminf_{\varepsilon \rightarrow +0} f(te^{-i\varepsilon}),$$

we can rewrite conditions (C1) and (C3) for A^\pm in the next equivalent form:

$$\begin{aligned} a^{-1} \in L^\infty(\mathbb{T}), \quad \liminf_{\varepsilon \rightarrow \pm 0} (|a(e^{i\varepsilon})| - |b(e^{i\varepsilon})|) &> 0, & \liminf_{\varepsilon \rightarrow \pm 0} (|a(t_0 e^{i\varepsilon})| - |b(t_0 e^{i\varepsilon})|) &> 0, \\ c^{-1} \in L^\infty(\mathbb{T}), \quad \liminf_{\varepsilon \rightarrow \pm 0} (|c(e^{i\varepsilon})| - |d(e^{i\varepsilon})|) &> 0, & \liminf_{\varepsilon \rightarrow \pm 0} (|c(t_0 e^{i\varepsilon})| - |d(t_0 e^{i\varepsilon})|) &> 0. \end{aligned} \quad (4.8)$$

Analogously, conditions (4.2), (4.7) and (4.6) can be rewritten, respectively, in the form:

$$\begin{aligned} \inf_{x \in \mathbb{R}} \liminf_{\varepsilon \rightarrow +0} &|[a(e^{i\varepsilon}) - b(e^{i\varepsilon})k^{-ix}][c(e^{-i\varepsilon}) - d(e^{-i\varepsilon})k^{-ix}]\mathcal{P}_+(x) \\ &+ [a(e^{-i\varepsilon}) - b(e^{-i\varepsilon})k^{-ix}][c(e^{i\varepsilon}) - d(e^{i\varepsilon})k^{-ix}]\mathcal{P}_-(x)| > 0, \end{aligned} \quad (4.9)$$

$$\begin{aligned} \inf_{x \in \mathbb{R}} \liminf_{\varepsilon \rightarrow +0} &|[a(t_0 e^{i\varepsilon}) - b(t_0 e^{i\varepsilon})k^{ix}][c(t_0 e^{-i\varepsilon}) - d(t_0 e^{-i\varepsilon})k^{ix}]\mathcal{P}_+(x) \\ &+ [a(t_0 e^{-i\varepsilon}) - b(t_0 e^{-i\varepsilon})k^{ix}][c(t_0 e^{i\varepsilon}) - d(t_0 e^{i\varepsilon})k^{ix}]\mathcal{P}_-(x)| > 0, \end{aligned} \quad (4.10)$$

$$\inf_{t \in w_{-1} \setminus \{t_0\}} \min_{\mu \in [0,1]} \liminf_{\varepsilon \rightarrow +0} |a(te^{i\varepsilon})c(te^{-i\varepsilon})\mu + a(te^{-i\varepsilon})c(te^{i\varepsilon})(1-\mu)| > 0. \quad (4.11)$$

Suppose now that the coefficients a, b, c, d of the operator (4.1) are continuous functions on the set $\mathbb{T} \setminus \{1, t_0, t_1, \dots, t_n\}$, where t_1, t_2, \dots, t_n are distinct points in $\mathbb{T} \setminus \{1, t_0\}$, and coincide at sufficiently small semi-neighborhoods u_τ^\pm of the points $\tau \in \mathcal{T} := \{1, t_0, t_1, \dots, t_n\}$ with functions which are slowly oscillating at τ and given for $\varepsilon > 0$, respectively, by

$$\begin{aligned} a(\tau e^{\pm i\varepsilon}) &= \lambda_{a,\tau}^\pm + \delta_{a,\tau}^\pm \sin(\ln(-\ln \varepsilon)), & b(\tau e^{\pm i\varepsilon}) &= \lambda_{b,\tau}^\pm + \delta_{b,\tau}^\pm \cos(\ln(-\ln \varepsilon)), \\ c(\tau e^{\pm i\varepsilon}) &= \lambda_{c,\tau}^\pm + \delta_{c,\tau}^\pm \cos(\ln(-\ln \varepsilon)), & d(\tau e^{\pm i\varepsilon}) &= \lambda_{d,\tau}^\pm + \delta_{d,\tau}^\pm \sin(\ln(-\ln \varepsilon)), \end{aligned} \quad (4.12)$$

where $\lambda_{f,\tau}^\pm$ and $\delta_{f,\tau}^\pm$ are complex constants for all $f \in \{a, b, c, d\}$.

Observe that for any differentiable function $f: \mathbb{R} \rightarrow \mathbb{C}$ bounded with its first derivative, the function $h(x) = f(\ln(-\ln|x|))$ is continuous on the set $(-1, 1) \setminus \{0\}$ and slowly oscillating at 0 because $\lim_{x \rightarrow 0} xh'(x) = 0$, which ensures according to (4.12) that the functions a, b, c, d are in fact in $PSO(\mathbb{T})$.

With the functions $a, b, c, d \in PSO(\mathbb{T})$ chosen as above, one can see that conditions (4.8) are fulfilled if the parameters in (4.12) are such that

$$|\lambda_{a,\tau}^\pm| > |\delta_{a,\tau}^\pm|, \quad |\lambda_{c,\tau}^\pm| > |\delta_{c,\tau}^\pm| \quad \text{for all } \tau \in \{t_1, \dots, t_n\}, \quad (4.13)$$

$$|\lambda_{a,\tau}^\pm| > |\lambda_{b,\tau}^\pm| + |\delta_{a,\tau}^\pm| + |\delta_{b,\tau}^\pm|, \quad |\lambda_{c,\tau}^\pm| > |\lambda_{d,\tau}^\pm| + |\delta_{c,\tau}^\pm| + |\delta_{d,\tau}^\pm| \quad (4.14)$$

for all $\tau \in \{1, t_0\}$, and

$$a(t)c(t) \neq 0 \quad \text{for all } t \in \mathbb{T} \setminus \bigcup_{\tau \in \mathcal{T}} (u_\tau^- \cup \{\tau\} \cup u_\tau^+). \quad (4.15)$$

Indeed, for every $\tau \in \mathcal{T}$ and every $f \in \{a, b, c, d\}$, the graphs of the functions $\varepsilon \mapsto f(\tau e^{\pm i\varepsilon})$ for all sufficiently small $\varepsilon > 0$ lie in the closed discs

$$D_{f,\tau}^\pm := \{z \in \mathbb{C}: |z - \lambda_{f,\tau}^\pm| \leq |\delta_{f,\tau}^\pm|\},$$

respectively. Inequalities (4.13) and (4.14) imply that the closed discs $D_{a,\tau}^\pm$ and $D_{c,\tau}^\pm$ are separated from the origin for all $\tau \in \mathcal{T}$, which together with (4.15) ensures the invertibility of the functions a and b in $L^\infty(\mathbb{T})$. Inequalities (4.14) mean that the closed discs $D_{a,\tau}^\pm$ and $D_{c,\tau}^\pm$ are separated, respectively, from the closed discs $D_{b,\tau}^\pm$ and $D_{d,\tau}^\pm$ for each $\tau \in \{1, t_0\}$, and hence, by (4.14),

$$\begin{aligned} |a(\tau e^{\pm i\varepsilon})| - |b(\tau e^{\pm i\varepsilon})| &\geq (|\lambda_{a,\tau}^\pm| - |\delta_{a,\tau}^\pm|) - (|\lambda_{b,\tau}^\pm| + |\delta_{b,\tau}^\pm|) > 0, \\ |c(\tau e^{\pm i\varepsilon})| - |d(\tau e^{\pm i\varepsilon})| &\geq (|\lambda_{c,\tau}^\pm| - |\delta_{c,\tau}^\pm|) - (|\lambda_{d,\tau}^\pm| + |\delta_{d,\tau}^\pm|) > 0, \end{aligned}$$

which completes the proof of (4.8) on the basis of (4.13)–(4.15).

Conditions (4.9)–(4.11) are fulfilled if, in addition to the fulfillment of (4.8), for every $\tau \in \{1, t_0\}$ both the sectors

$$\begin{aligned} \mathcal{S}_{\tau,1} &:= \{z = re^{i\varphi}: r > 0, |\varphi - \arg(\lambda_{a,\tau}^+ \lambda_{c,\tau}^-)| \leq \nu_{a,\tau}^+ + \nu_{c,\tau}^-\}, \\ \mathcal{S}_{\tau,2} &:= \{z = re^{i\varphi}: r > 0, |\varphi - \arg(\lambda_{a,\tau}^- \lambda_{c,\tau}^+)| \leq \nu_{a,\tau}^- + \nu_{c,\tau}^+\} \end{aligned}$$

with excluded vertices lie in the same open complex half-plane whose boundary contains the origin, where, by (4.14),

$$\begin{aligned} \nu_{a,\tau}^\pm &:= \arcsin((|\lambda_{b,\tau}^\pm| + |\delta_{a,\tau}^\pm| + |\delta_{b,\tau}^\pm|)/|\lambda_{a,\tau}^\pm|) \in (0, \pi/2), \\ \nu_{c,\tau}^\pm &:= \arcsin((|\lambda_{d,\tau}^\pm| + |\delta_{c,\tau}^\pm| + |\delta_{d,\tau}^\pm|)/|\lambda_{c,\tau}^\pm|) \in (0, \pi/2); \end{aligned}$$

and for every $\tau \in \{t_1, \dots, t_n\}$ both the sectors

$$\begin{aligned} \mathcal{S}_{\tau,1}^\circ &:= \{z = re^{i\varphi}: r > 0, |\varphi - \arg(\lambda_{a,\tau}^+ \lambda_{c,\tau}^-)| \leq \mu_{a,\tau}^+ + \mu_{c,\tau}^-\}, \\ \mathcal{S}_{\tau,2}^\circ &:= \{z = re^{i\varphi}: r > 0, |\varphi - \arg(\lambda_{a,\tau}^- \lambda_{c,\tau}^+)| \leq \mu_{a,\tau}^- + \mu_{c,\tau}^+\} \end{aligned}$$

with excluded vertices lie in the same open complex half-plane whose boundary contains the origin, where, in view of (4.13),

$$\mu_{a,\tau}^\pm := \arcsin(|\delta_{a,\tau}^\pm|/|\lambda_{a,\tau}^\pm|) \in [0, \pi/2), \quad \mu_{c,\tau}^\pm := \arcsin(|\delta_{c,\tau}^\pm|/|\lambda_{c,\tau}^\pm|) \in [0, \pi/2).$$

Indeed, if the mentioned conditions are fulfilled, then for every $\tau \in \{1, t_0\}$ the graphs of the functions $(\varepsilon, x) \mapsto a(\tau e^{\pm i\varepsilon}) - b(\tau e^{\pm i\varepsilon})k^{ix}$ and $(\varepsilon, x) \mapsto c(\tau e^{\pm i\varepsilon}) - d(\tau e^{\pm i\varepsilon})k^{ix}$ for all sufficiently small $\varepsilon > 0$ and all $x \in \mathbb{R}$ lie, respectively, in the closed discs

$$\begin{aligned}\tilde{D}_{a,\tau}^{\pm} &:= \{z \in \mathbb{C}: |z - \lambda_{a,\tau}^{\pm}| \leq |\lambda_{b,\tau}^{\pm}| + |\delta_{a,\tau}^{\pm}| + |\delta_{b,\tau}^{\pm}|\}, \\ \tilde{D}_{c,\tau}^{\pm} &:= \{z \in \mathbb{C}: |z - \lambda_{c,\tau}^{\pm}| \leq |\lambda_{d,\tau}^{\pm}| + |\delta_{c,\tau}^{\pm}| + |\delta_{d,\tau}^{\pm}|\},\end{aligned}$$

which are separated from the origin due to (4.14). Then for every $\tau \in \{1, t_0\}$ the closed sets

$$\{z_1 z_2 \in \mathbb{C}: z_1 \in \tilde{D}_{a,\tau}^+, z_2 \in \tilde{D}_{c,\tau}^-\} \quad \text{and} \quad \{z_1 z_2 \in \mathbb{C}: z_1 \in \tilde{D}_{a,\tau}^-, z_2 \in \tilde{D}_{c,\tau}^+\}$$

lie in the sectors $\mathcal{S}_{\tau,1}$ and $\mathcal{S}_{\tau,2}$, respectively. Analogously, we infer that for every $\tau \in \{t_1, \dots, t_n\}$ the closed sets

$$\{z_1 z_2 \in \mathbb{C}: z_1 \in D_{a,\tau}^+, z_2 \in D_{c,\tau}^-\} \quad \text{and} \quad \{z_1 z_2 \in \mathbb{C}: z_1 \in D_{a,\tau}^-, z_2 \in D_{c,\tau}^+\}$$

lie in the sectors $\mathcal{S}_{\tau,1}^{\circ}$ and $\mathcal{S}_{\tau,2}^{\circ}$, respectively. Since the sets

$$\begin{aligned}\{z_1 \mu + z_2(1 - \mu): z_1 \in \mathcal{S}_{\tau,1}, z_2 \in \mathcal{S}_{\tau,2}, \mu \in [0, 1]\} &\quad \text{for } \tau \in \{1, t_0\}, \\ \{z_1 \mu + z_2(1 - \mu): z_1 \in \mathcal{S}_{\tau,1}^{\circ}, z_2 \in \mathcal{S}_{\tau,2}^{\circ}, \mu \in [0, 1]\} &\quad \text{for } \tau \in \{t_1, \dots, t_n\}\end{aligned}$$

lie in open half-planes whose boundaries contains the origin, we conclude that conditions (4.9)–(4.11) are fulfilled.

In particular, the operator (4.1) with coefficients $a, b, c, d \in \text{PSO}(\mathbb{T})$ described above and satisfying (4.12) is Fredholm on the space $L^2(\mathbb{T})$ if

$$\begin{aligned}a^{-1}, c^{-1} &\in L^{\infty}(\mathbb{T}), \quad \arg((\lambda_{a,\tau}^+ \lambda_{c,\tau}^-) / (\lambda_{a,\tau}^- \lambda_{c,\tau}^+)) < \pi/3 \quad \text{for all } \tau \in \mathcal{T}, \\ |\delta_{a,\tau}^{\pm}| &< |\lambda_{a,\tau}^{\pm}|/2, \quad |\delta_{c,\tau}^{\pm}| < |\lambda_{c,\tau}^{\pm}|/2 \quad \text{for all } \tau \in \{t_1, \dots, t_n\}, \\ (|\lambda_{b,\tau}^{\pm}| + |\delta_{a,\tau}^{\pm}| + |\delta_{b,\tau}^{\pm}|) &< |\lambda_{a,\tau}^{\pm}|/2, \quad (|\lambda_{d,\tau}^{\pm}| + |\delta_{c,\tau}^{\pm}| + |\delta_{d,\tau}^{\pm}|) < |\lambda_{c,\tau}^{\pm}|/2 \quad \text{for } \tau \in \{1, t_0\},\end{aligned}$$

which implies that $\mu_{a,\tau}^{\pm}, \mu_{c,\tau}^{\pm}, \nu_{a,\tau}^{\pm}, \nu_{c,\tau}^{\pm} \in [0, \pi/6)$.

5. Local methods for nonlocal C^* -algebras

5.1. The local-trajectory method

To study the C^* -algebra (1.6) of functional operators we will use the following simplified version of the local-trajectory method (cf. [18,4]) based on the Allan–Douglas local principle [11,10].

Let \mathcal{D} be a commutative C^* -algebra with unit I , G a discrete group with unit e , $u: g \mapsto u_g$ a homomorphism of the group G onto a group $u_G = \{u_g: g \in G\}$ of unitary elements such that $u_{g_1 g_2} = u_{g_1} u_{g_2}$ and $u_e = I$. Suppose \mathcal{D} and u_G are contained in a C^* -algebra \mathcal{Y} and assume that

- (A1) for every $g \in G$, the mapping $\alpha_g: d \mapsto u_g d u_g^*$ is a $*$ -automorphism of the commutative C^* -algebra \mathcal{D} ;
 (A2) G is an amenable discrete group.

By [13, §1.2], a discrete group G is called *amenable* if the C^* -algebra $l^{\infty}(G)$ of all bounded complex-valued functions on G with sup-norm has an invariant mean, that is, a positive linear functional ρ of norm 1 such that

$$\rho(f) = \rho({}_s f) = \rho(f_s) \quad \text{for all } s \in G \text{ and all } f \in l^{\infty}(G),$$

where $({}_s f)(g) = f(s^{-1}g)$, $(f_s)(g) = f(gs)$, $g \in G$.

Let $\mathcal{Q} := \text{alg}(\mathcal{D}, u_G)$ be the minimal C^* -algebra containing the C^* -algebra \mathcal{D} and the group u_G . By virtue of (A1), \mathcal{Q} is the closure of the set \mathcal{Q}^0 of elements $q = \sum d_g u_g$, where $d_g \in \mathcal{D}$ and g runs through finite subsets of G .

Let $M := M(\mathcal{D})$ be the maximal ideal space of the commutative C^* -algebra \mathcal{D} . By the Gelfand–Naimark theorem [23, §16], $\mathcal{D} \cong C(M)$. Under assumption (A1), identifying characters φ_m of the C^* -algebra \mathcal{D} and the maximal ideals $m = \text{Ker } \varphi_m \in M$, we obtain the homomorphism $g \mapsto \beta_g(\cdot)$ of the group G into the homeomorphism group of M according to the rule

$$d(\beta_g(m)) = (\alpha_g(d))(m), \quad d \in \mathcal{D}, m \in M, g \in G,$$

where $d(\cdot) \in C(M)$ is the Gelfand transform of the element $d \in \mathcal{D}$. The set $G(m) := \{\beta_g(m): g \in G\}$ is called the G -orbit of a point $m \in M$.

Let the next version of topologically free action of G hold (see [15,4]):

- (A3) there is a set $M_0 \subset M(\mathcal{D})$ such that for every finite set $G_0 \subset G \setminus \{e\}$ and every nonempty open set $V \subset M(\mathcal{D})$ there exists a point $m_0 \in V \cap G(M_0)$ such that $\beta_g(m_0) \neq m_0$ for all $g \in G_0$.

For every $m \in M$, we take the representation $\pi_m : \mathcal{D} \rightarrow \mathcal{B}(\mathbb{C})$, $d \mapsto d(m)$. Given $M_0 \subset M$, let $\Omega(M_0)$ be the set of G -orbits of all points $m \in M_0$. Fix a point $m = m_\omega$ in each G -orbit $\omega \in \Omega(M_0)$, and let $\ell^2(G)$ be the Hilbert space of all functions $f : G \rightarrow \mathbb{C}$ such that $f(g) \neq 0$ for at most countable set of points $g \in G$ and $\|f\| := (\sum |f(g)|^2)^{1/2} < \infty$. For every $\omega \in \Omega(M_0)$ we consider the representation $\pi_\omega : \mathcal{Q} \rightarrow \mathcal{B}(\ell^2(G))$ defined by

$$[\pi_\omega(d)f](g) = \pi_m(\alpha_g(d))f(g), \quad [\pi_\omega(u_h)f](g) = f(gh)$$

for all $d \in \mathcal{D}$, all $g, h \in G$ and all $f \in \ell^2(G)$.

We infer the following nonlocal version of the Allan–Douglas local principle from [18, Theorems 4.1, 4.12] and [4, Theorem 3.1].

Theorem 5.1. *If assumptions (A1)–(A3) are satisfied, then an element $q \in \mathcal{Q}$ is invertible in \mathcal{Q} if and only if for every orbit $\omega \in \Omega(M_0)$ the operator $\pi_\omega(q)$ is invertible on the space $\ell^2(G)$ and, in the case of infinite set $\Omega(M_0)$,*

$$\sup\{\|(\pi_\omega(q))^{-1}\|_{\mathcal{B}(\ell^2(G))} : \omega \in \Omega(M_0)\} < \infty.$$

5.2. Lifting theorem

Let $\mathcal{B} := \mathcal{B}(\mathcal{H})$ be the C^* -algebra of all bounded linear operators on a Hilbert space \mathcal{H} and let \mathfrak{B} be a C^* -subalgebra of \mathcal{B} containing the identity operator $I \in \mathcal{B}$. Suppose the ideal $\mathcal{K} := \mathcal{K}(\mathcal{H})$ is contained in \mathfrak{B} .

To investigate the Fredholmness of operators $B \in \mathfrak{B}$, we will apply the following analogue of the lifting theorem from [14, Theorem 1.8] (see also [25, Section 6.3]), which is a C^* -algebra modification of [21, Theorem 3.3].

Theorem 5.2. *Let Λ be an index set and suppose that, for each $\lambda \in \Lambda$, we are given a unital C^* -algebra \mathcal{L}_λ , a $*$ -homomorphism $\Psi_\lambda : \mathfrak{B} \rightarrow \mathcal{L}_\lambda$, and a closed two-sided ideal $\mathfrak{H}_\lambda \subset \mathfrak{B}$ such that:*

- (i) $\mathcal{K} \subset \mathfrak{H}_\lambda \cap \text{Ker } \Psi_\lambda$ and $\mathfrak{H}_\mu \subset \text{Ker } \Psi_\lambda$ for all $\mu \in \Lambda \setminus \{\lambda\}$;
- (ii) *the restriction of the quotient homomorphism*

$$\mathfrak{B}/\mathcal{K} \rightarrow \mathcal{L}_\lambda, \quad B + \mathcal{K} \mapsto \Psi_\lambda(B)$$

(which is well defined by (i) and denoted again by Ψ_λ) onto the ideal $\mathfrak{H}_\lambda/\mathcal{K}$ is a $$ -isomorphism of $\mathfrak{H}_\lambda/\mathcal{K}$ onto the closed two-sided ideal $\mathcal{R}_\lambda := \Psi_\lambda(\mathfrak{H}_\lambda)$ of the C^* -algebra $\mathfrak{B}_\lambda := \Psi_\lambda(\mathfrak{B}) \subset \mathcal{L}_\lambda$.*

Let \mathfrak{H} be the smallest closed two-sided ideal of \mathfrak{B} containing all ideals \mathfrak{H}_λ ($\lambda \in \Lambda$). Then an operator $B \in \mathfrak{B}$ is Fredholm if and only if the coset $B + \mathfrak{H}$ is invertible in $\mathfrak{B}/\mathfrak{H}$ and for every $\lambda \in \Lambda$ the element $\Psi_\lambda(B)$ is invertible in \mathcal{L}_λ .

In addition to the previous conditions of this subsection on the C^* -algebra $\mathfrak{B} \subset \mathcal{B} = \mathcal{B}(\mathcal{H})$, we assume that \mathfrak{B} contains a unital C^* -algebra \mathcal{A} (with unit $I \in \mathcal{B}$). Let $P^\pm \in \mathfrak{B} \setminus \mathcal{A}$, $P^+ + P^- = I$, and let \mathfrak{H}_0 be the smallest closed two-sided ideal in \mathfrak{B} containing the ideal \mathcal{K} , the operator P^+P^- and all commutators $[A, P^\pm] = AP^\pm - P^\pm A$ for $A \in \mathcal{A}$. Consider an operator

$$B = A^+P^+ + A^-P^- + H_0 \in \mathfrak{B} \tag{5.1}$$

where $A^\pm \in \mathcal{A}$ and $H_0 \in \mathfrak{H}_0$. If the operators A^\pm are invertible in \mathcal{B} and therefore $(A^\pm)^{-1} \in \mathcal{A}$, then the coset $B + \mathfrak{H}_0$ is invertible in the C^* -algebra $\mathfrak{B}/\mathfrak{H}_0$ with $(B + \mathfrak{H}_0)^{-1} = (A^+)^{-1}P^+ + (A^-)^{-1}P^- + \mathfrak{H}_0$. Hence, Theorem 5.2 implies the following sufficient Fredholm conditions for the operator (5.1).

Theorem 5.3. *If the conditions of Theorem 5.2 are fulfilled with $\mathfrak{H} = \mathfrak{H}_0$, then an operator $B \in \mathfrak{B}$ of the form (5.1) is Fredholm on the space \mathcal{H} if the operators A^\pm are invertible in the C^* -algebra \mathcal{B} and for every $\lambda \in \Lambda$ the element $\Psi_\lambda(B)$ is invertible in the C^* -algebra \mathcal{L}_λ .*

Theorem 5.3 becomes a Fredholm criterion for the operator (5.1) under an addition condition on \mathcal{A} , P^\pm and \mathfrak{H} which implies the invertibility of A^\pm from the Fredholmness of B (see, e.g., [20,16] and [21, Theorems 2.1, 3.5]).

6. The C^* -algebra \mathcal{A} of functional operators

Using the local-trajectory method we will obtain here an invertibility criterion for the operators in the C^* -algebra $\mathcal{A} = \text{alg}(PSO(\mathbb{T}), U_G) \subset \mathcal{B}(L^2(\mathbb{T}))$.

Since $U_g a U_g^{-1} = (a \circ g)I$ where $a \circ g \in PSO(\mathbb{T})$ for each $g \in G$ and each $a \in PSO(\mathbb{T})$ (see, e.g., [3, Lemma 4.2]), the C^* -algebra \mathcal{A} is the closure of the algebra \mathcal{A}^0 consisting of the functional operators $\sum_{g \in F} a_g U_g$, where $a_g \in PSO^0(\mathbb{T})$ and g runs through finite subsets $F \subset G$.

Consider the central C^* -subalgebra $\tilde{\mathcal{Z}} := \{aI : a \in PSO(\mathbb{T})\}$ of \mathcal{A} . Then $\tilde{\mathcal{Z}} \cong C(M(PSO(\mathbb{T})))$. For each $g \in G$, the $*$ -automorphism $\tilde{\alpha}_g : aI \mapsto (a \circ g)I$ of $\tilde{\mathcal{Z}}$ induces on $M(PSO(\mathbb{T})) = M(SO(\mathbb{T})) \times \{0, 1\}$ the homeomorphism

$$\tilde{\beta}_g : (\xi, \mu) \mapsto (g(\xi), \mu) \quad (6.1)$$

where $\xi \mapsto g(\xi)$ is given by (2.16). For any $g \in G$, let \mathbb{T}_g denote the set of all fixed points of g on \mathbb{T} . If $t \in \mathbb{T}_g$ and $\xi \in M_t(SO(\mathbb{T}))$, then $g(\xi) = \xi$ by the proof of [4, Theorem 6.4], which in view of (6.1) gives the following.

Lemma 6.1. *For each $g \in G$, the set $M_g := \bigcup_{t \in \mathbb{T}_g} (M_t(SO(\mathbb{T})) \times \{0, 1\})$ is the set of all fixed points of $\tilde{\beta}_g$ on $M(PSO(\mathbb{T}))$.*

Since G acts topologically freely on \mathbb{T} , we easily deduce from Lemma 6.1 and the Gelfand topology (2.4) on $M(PSO(\mathbb{T}))$ that the group G acts topologically freely on $M(PSO(\mathbb{T}))$ as well. Moreover, since the set

$$M_0 := \bigcup_{t \in \mathbb{T} \setminus \mathbb{T}_G} (M_t(SO(\mathbb{T})) \times \{0, 1\})$$

is dense in $M(PSO(\mathbb{T}))$, we see that for every nonempty open set $V \subset M(PSO(\mathbb{T}))$ and every finite set $G_0 \subset G$ there exists a point $(\xi_0, \mu_0) \in V \cap M_0$ such that $\tilde{\beta}_g(\xi_0, \mu_0) \neq (\xi_0, \mu_0)$ for all $g \in G_0 \setminus \{e\}$. Due to this fact and the amenability of the group G , we infer that all conditions of the local-trajectory method (see Subsection 5.1) for the C^* -algebra \mathcal{A} are fulfilled.

Clearly, from (6.1) and (2.16) it follows that the set

$$\Omega := \bigcup_{w \in W_{\mathbb{T}}^0} (M_{t_w}(SO(\mathbb{T})) \times \{0, 1\}) \quad (6.2)$$

contains exactly one point in each G -orbit of every point in M_0 . Consider the Hilbert space $l^2(G)$ consisting of all complex-valued functions defined on G and having at most countable sets of non-zero values, and with every point $(\xi, \mu) \in \Omega$ we associate the representation

$$\Pi_{\xi, \mu} : \mathcal{A} \rightarrow \mathcal{B}(l^2(G)), \quad A \mapsto A_{\xi, \mu} \quad (6.3)$$

given on the operator $A = \sum_{g \in F} a_g U_g \in \mathcal{A}^0$ by

$$(A_{\xi, \mu} f)(h) = \sum_{g \in F} [(a_g \circ h)(\xi, \mu)] f(hg) \quad \text{for all } f \in l^2(G) \text{ and } h \in G. \quad (6.4)$$

Then we immediately obtain an invertibility criterion from Theorem 5.1.

Theorem 6.2. *A functional operator $A \in \mathcal{A}$ is invertible on the space $L^2(\mathbb{T})$ if and only if the operators $A_{\xi, \mu}$ for all $(\xi, \mu) \in \Omega$ are invertible on the space $l^2(G)$ and $\sup_{(\xi, \mu) \in \Omega} \|A_{\xi, \mu}^{-1}\| < \infty$.*

Applying Theorem 6.2 to the operator $AA^* \in \mathcal{A}$ and using the relation $\|A\| = \|AA^*\|^{1/2} = [r(AA^*)]^{1/2}$, where $r(AA^*)$ is the spectral radius of the operator AA^* , we conclude that

$$\|A\|_{\mathcal{B}(L^2(\mathbb{T}))} = [r(AA^*)]^{1/2} = \sup_{(\xi, \mu) \in \Omega} [r(A_{\xi, \mu} A_{\xi, \mu}^*)]^{1/2} = \sup_{(\xi, \mu) \in \Omega} \|A_{\xi, \mu}\|_{\mathcal{B}(l^2(G))}, \quad (6.5)$$

whence we obtain the following.

Corollary 6.3. *The representation*

$$\mathcal{A} \rightarrow \mathcal{B}\left(\bigoplus_{(\xi, \mu) \in \Omega} l^2(G)\right), \quad A \mapsto \bigoplus_{(\xi, \mu) \in \Omega} \Pi_{\xi, \mu}(A),$$

where Ω and $\Pi_{\xi, \mu}$ are given by (6.2) and (6.3), respectively, is an isometric C^* -algebra homomorphism.

7. General form of singular integral operators with shifts

We establish here a general form of the operators in the C^* -algebra

$$\mathfrak{B} = \text{alg}(PSO(\mathbb{T}), S_{\mathbb{T}}, U_G) \subset \mathcal{B}(L^2(\mathbb{T})).$$

Obviously, $\mathfrak{B} = \text{alg}(\mathfrak{A}, U_G)$, where the C^* -algebra \mathfrak{A} is given by (1.4).

Let \mathfrak{A}^0 and \mathfrak{B}^0 be the non-closed algebras of operators of the form

$$\sum_{i=1}^n T_{i,1} T_{i,2} \dots T_{i,j_i} \quad (n, j_i \in \mathbb{N}) \quad (7.1)$$

where $T_{i,k}$ are, respectively, the generators aI ($a \in PSO^0(\mathbb{T})$) and $S_{\mathbb{T}}$ of the C^* -algebra \mathfrak{A} and the generators aI ($a \in PSO^0(\mathbb{T})$), $S_{\mathbb{T}}$ and U_g ($g \in G$) of the C^* -algebra \mathfrak{B} . Clearly, \mathfrak{A}^0 is a dense subalgebra of \mathfrak{A} , \mathfrak{B}^0 is a dense subalgebra of \mathfrak{B} , and the non-closed algebra $\mathcal{A}^0 \subset \mathfrak{B}^0$ consisting of all operators of the form $A = \sum_{g \in F} a_g U_g$ with coefficients $a_g \in PSO^0(\mathbb{T})$ and finite sets $F \subset G$ is dense in the C^* -algebra \mathcal{A} of functional operators.

Let \mathfrak{H} be the closed two-sided ideal of \mathfrak{B} generated by all commutators $[aI, S_{\mathbb{T}}] = aS_{\mathbb{T}} - S_{\mathbb{T}}aI$, where $a \in PSO^0(\mathbb{T})$, and by all operators $U_g S_{\mathbb{T}} U_g^* - S_{\mathbb{T}}$, where $g \in G$. Then \mathfrak{H} is the closure of the set

$$\mathfrak{H}^0 := \left\{ \sum_{i=1}^n B_i H_i C_i : B_i, C_i \in \mathfrak{B}^0, H_i \in \mathbb{H}, n \in \mathbb{N} \right\}, \quad (7.2)$$

where, in view of (2.15),

$$\mathbb{H} := \{[aI, S_{\mathbb{T}}], U_g S_{\mathbb{T}} U_g^* - S_{\mathbb{T}} : a \in PSO^0(\mathbb{T}), g \in G\} \subset \mathfrak{A}. \quad (7.3)$$

The ideal \mathcal{K} of all compact operators on the space $L^2(\mathbb{T})$ is contained in \mathfrak{H} (see, e.g., [12]). Furthermore, by (7.2) and (7.3), $[A, S_{\mathbb{T}}] \in \mathfrak{H}$ for all $A \in \mathcal{A}$. Thus, \mathfrak{H} can be viewed as the ideal \mathfrak{H}_0 defined in Subsection 5.2.

Given $w \in W_{\mathbb{T}}$, let \mathfrak{H}_w be the closed two-sided ideal of the C^* -algebra \mathfrak{B} generated by V_{t_w} and \mathcal{K} , and let $\mathfrak{H}_w^\pi := \mathfrak{H}_w / \mathcal{K}$. Applying Lemma 2.3 and (2.10), we obtain the following characterization of the ideal $\mathfrak{H}^\pi := \mathfrak{H} / \mathcal{K}$ of the C^* -algebra \mathfrak{B}^π by analogy with [22, Lemma 5.4] and [4, Lemma 10.4].

Lemma 7.1. Every coset $H^\pi \in \mathfrak{H}^\pi$ is represented in the form

$$H^\pi = \lim_{m \rightarrow \infty} \sum_{\omega \in \Lambda_m} H_{\omega, m}^\pi \quad (7.4)$$

where Λ_m are finite subsets of $W_{\mathbb{T}}$ and $H_{\omega, m}^\pi \in \mathfrak{H}_{\omega, m}^\pi$. For every $w \in W_{\mathbb{T}}$, the cosets $H_w^\pi \in \mathfrak{H}_w^\pi$ have the form

$$H_w^\pi = \begin{cases} \lim_{n \rightarrow \infty} (A_{n, w}^\pi V_{t_w}^\pi) & \text{if } w \in W_{\mathbb{T}_G}, \\ \lim_{n \rightarrow \infty} (\sum_{t \in \mathbb{T}_n} \sum_{g \in F_n} A_{t, g, n}^\pi V_t^\pi U_g^\pi) & \text{if } w \in W_{\mathbb{T}}^0, \end{cases} \quad (7.5)$$

$A_{n, w}, A_{t, g, n} \in \mathfrak{A}^0$, and \mathbb{T}_n and F_n are finite subsets of w and G , respectively.

Let $\tilde{\mathcal{A}}$ be the C^* -algebra of 2×2 diagonal matrices with \mathcal{A} -valued entries, where \mathcal{A} is the C^* -algebra of functional operators considered in Section 6. As in [22, Theorem 5.1] and [4, Theorem 10.3], for \mathfrak{B} we obtain the following.

Theorem 7.2. Every operator $B \in \mathfrak{B}$ is uniquely represented in the form

$$B = A^+ P_{\mathbb{T}}^+ + A^- P_{\mathbb{T}}^- + H_B, \quad (7.6)$$

where A^\pm are functional operators in the C^* -algebra \mathcal{A} , $P_{\mathbb{T}}^\pm = (I \pm S_{\mathbb{T}})/2$ are the orthogonal projections associated with the Cauchy singular integral operator $S_{\mathbb{T}}$, and $H_B \in \mathfrak{H}$. Moreover, the mapping $B \mapsto \text{diag}\{A^+, A^-\}$ is a C^* -algebra homomorphism of \mathfrak{B} onto $\tilde{\mathcal{A}}$ whose kernel is \mathfrak{H} , and

$$\|A^\pm\| \leq \inf_{H \in \mathfrak{H}} \|B + H\| \leq |B| = \inf_{K \in \mathcal{K}} \|B + K\|. \quad (7.7)$$

Proof. Since $U_g a U_g^* = (a \circ g)I$ and $U_g S_{\mathbb{T}} U_g^* - S_{\mathbb{T}} \in \mathfrak{H}^0$ for every $a \in PSO^0(\mathbb{T})$ and every $g \in G$, we infer from (2.11) and (2.15) that every operator $B \in \mathfrak{B}^0$ has the form (7.6) where $A^\pm \in \mathcal{A}^0$ and $H_B \in \mathfrak{H}^0$. Moreover, the mapping $B \mapsto \text{diag}\{A^+, A^-\}$ is an algebraic $*$ -homomorphism of the non-closed algebra \mathfrak{B}^0 into $\tilde{\mathcal{A}}$ whose kernel is \mathfrak{H}^0 . This map is given on the generators of \mathfrak{B} by

$$aI \mapsto \text{diag}\{aI, aI\}, \quad U_g \mapsto \text{diag}\{U_g, U_g\}, \quad S \mapsto \text{diag}\{I, -I\}.$$

It remains to prove (7.7) for all $B \in \mathfrak{B}^0$. Then, by continuity, we will obtain (7.7) for all operators $B \in \mathfrak{B}$ and hence decomposition (7.6) for these B .

Fix $B = A^+ P_{\mathbb{T}}^+ + A^- P_{\mathbb{T}}^- + H_B \in \mathfrak{B}^0$, where

$$A^\pm := \sum_{g \in F} a_g^\pm U_g \in \mathcal{A}^0, \quad a_g^\pm \in PSO^0(\mathbb{T}), \quad H_B \in \mathfrak{H}^0,$$

and F is a finite subset of G . Suppose without loss of generality that the set F is symmetric, that is, $g^{-1} \in F$ for each $g \in F$. In view of (6.5) we have

$$\|A^\pm\|_{\mathcal{B}(l^2(\mathbb{T}))} = \sup_{(\xi, \mu) \in \Omega} \|A_{\xi, \mu}^\pm\|_{\mathcal{B}(l^2(G))}, \quad (7.8)$$

where Ω is given by (6.2) and the operators $A_{\xi, \mu}^\pm$ are defined as in (6.4). Thus, by (7.8), it suffices to prove that, for all $H \in \mathfrak{H}$ and all $(\xi, \mu) \in \Omega$,

$$\|A_{\xi, \mu}^\pm\|_{\mathcal{B}(l^2(G))} \leq \|B + H\|_{\mathcal{B}(l^2(\mathbb{T}))}. \quad (7.9)$$

Fix $H \in \mathfrak{H}^0$. Then we get

$$B + H = A^+ P_{\mathbb{T}}^+ + A^- P_{\mathbb{T}}^- + H_0, \quad \text{with } H_0 = H_B + H \in \mathfrak{H}^0. \quad (7.10)$$

For every function $a \in PSO^0(\mathbb{T})$ and every function $c \in C(\mathbb{T})$ whose support $\text{supp } c$ does not contain the discontinuities of a , it follows that $c[aI, S_{\mathbb{T}}] \simeq 0$. Further, since the functions $g' \in PC(\mathbb{T})$ have finite sets $\Lambda_{g'}$ of discontinuities on \mathbb{T} , that is, $g' \in PC^0(\mathbb{T})$, it follows that, by analogy with (7.5),

$$(U_g S_{\mathbb{T}} U_g^* - S_{\mathbb{T}})^\pi = \sum_{t \in \Lambda_g} \lim_{n \rightarrow \infty} (A_{n,t}^\pi V_t^\pi) \quad (A_{n,t} \in \mathfrak{A}^0).$$

Hence, for any operator $H_0 \in \mathfrak{H}^0$ there is a finite set $\Lambda(H_0) \subset \mathbb{T}$ such that

$$\chi H_0 \chi I \simeq 0 \quad (7.11)$$

for all characteristic functions χ of Borel sets on \mathbb{T} with $\text{supp } \chi \subset \mathbb{T} \setminus \Lambda(H_0)$.

Consider the subgroup F^∞ of G generated by the finite set F and choose a subset G_∞ of G containing exactly one element in each left coset gF^∞ of G with respect to the subgroup F^∞ . Since for every $g \in G_\infty$ the Hilbert spaces $l^2(gF^\infty)$ are invariant with respect to all operators $A_{\xi, \mu}^\pm$ with $(\xi, \mu) \in \Omega$, we infer that every such operator $A_{\xi, \mu}^\pm$ can be represented as the direct sum of its restrictions to the subspaces $l^2(gF^\infty)$, where g runs the set G_∞ . Thus,

$$\|A_{\xi, \mu}^\pm\|_{\mathcal{B}(l^2(G))} = \sup_{g \in G_\infty} \|A_{\xi, \mu}^\pm\|_{\mathcal{B}(l^2(gF^\infty))}. \quad (7.12)$$

Fix $w \in W_{\mathbb{T}}^0$, $(\xi, \mu) \in M_{t_w}(SO(\mathbb{T})) \times \{0, 1\} \subset \Omega$ and $\varepsilon > 0$. Then, from (7.12) it follows that there exists a shift $g \in G_\infty$ such that

$$\|A_{\xi, \mu}^\pm\|_{\mathcal{B}(l^2(G))} \leq \|A_{\xi, \mu}^\pm\|_{\mathcal{B}(l^2(gF^\infty))} + \varepsilon. \quad (7.13)$$

Without loss of generality assume that (7.13) holds for $g = e$.

For each $n \in \mathbb{N}$, let $F^n \subset F^\infty$ be the finite set of words of the length $\leq n$ which are constituted by the elements $g \in F$ and let $\Pi_n \in \mathcal{B}(l^2(F^\infty))$ be the multiplication operator by the characteristic function of the set F^n . Obviously, $s\text{-}\lim_{n \rightarrow \infty} \Pi_n = I$ on $l^2(F^\infty)$, and consequently

$$\|A_{\xi, \mu}^\pm\|_{\mathcal{B}(l^2(F^\infty))} \leq \liminf_{n \rightarrow \infty} \|\Pi_n A_{\xi, \mu}^\pm \Pi_n\|_{\mathcal{B}(l^2(F^\infty))}. \quad (7.14)$$

Since for every $g \in F$ the functions a_g^\pm and g' belong to $PSO^0(\mathbb{T})$ and $PC^0(\mathbb{T})$, respectively, and therefore have finite sets of discontinuities, we conclude that in each neighborhood of the point $t_w \in \mathbb{T}$ there exists an open arc $\gamma \subset \mathbb{T} \setminus \{t_w\}$ such that all functions a_g^\pm and g' ($g \in F$) are continuous on the sets $h(\gamma)$ for all $h \in F^{n+1}$. In view of the topologically free action of group G on \mathbb{T} and due to the continuity of the shifts $h \in G$ on \mathbb{T} , there exists an open arc $u \subset \gamma$ such that the closures $\bar{h}(u)$ of $h(u)$ are disjoint for all $h \in F^{n+1}$, do not intersect the set $\Lambda(H_0)$ and

$$\begin{aligned} \|\Pi_n A_{\xi, \mu}^\pm \Pi_n\|_{\mathcal{B}(l^2(F^\infty))} &\leq \|\Pi_n A_{\xi, \tilde{\mu}}^\pm \Pi_n\|_{\mathcal{B}(l^2(F^\infty))} + \varepsilon \\ &= \|\Pi_n A_{\xi, \tilde{\mu}}^\pm \Pi_n\|_{\mathcal{B}(l^2(F^n))} + \varepsilon \end{aligned} \quad (7.15)$$

for every $(\zeta, \tilde{\mu}) \in M_u(PSO(\mathbb{T})) := \bigcup_{\tau \in u} M_\tau(SO(\mathbb{T})) \times \{0, 1\}$.

Let $m = m_n$ be the cardinality of the set F^n . Since the Hilbert space $l^2(F^n)$ is isometrically isomorphic to \mathbb{C}^m , the Hilbert space of m -dimensional complex vectors, we infer that for all $(\zeta, \tilde{\mu}) \in M_u(PSO(\mathbb{T}))$ the operators $\Pi_n A_{\xi, \tilde{\mu}}^\pm \Pi_n \in \mathcal{B}(l^2(F^n))$ admit matrix representations of the form

$$A_{(n)}^\pm(\zeta, \tilde{\mu}) := ((a_{h^{-1}s}^\pm \circ h)(\zeta, \tilde{\mu}))_{h,s \in F^n} = ((a_{h^{-1}s}^\pm \circ h)(t))_{h,s \in F^n},$$

for all $(\zeta, \tilde{\mu}) \in M_t(SO(\mathbb{T})) \times \{0, 1\}$ with $t \in u$.

Let χ_n be the characteristic function of the open set $u_n := \bigcup_{h \in F^n} h(u)$. Then $\text{supp } \chi_n \cap \Lambda(H_0) = \emptyset$, and therefore from (7.11) we obtain

$$\chi_n H_0 \chi_n I \simeq 0. \quad (7.16)$$

Furthermore, taking a function $\tilde{\chi}_n \in C(\mathbb{T})$ such that $\tilde{\chi}_n = 1$ on $\overline{u_n}$ and

$$\bigcup_{g \in F} (\text{supp}(\chi_n \circ g^{-1})) \cap (\text{supp} \tilde{\chi}_n \setminus \text{supp} \chi_n) = \emptyset,$$

and applying the equality $(\chi_n \circ g^{-1})(\tilde{\chi}_n - \chi_n)I = 0$, we deduce that

$$(\chi_n \circ g^{-1})P_{\mathbb{T}}^{\pm} \chi_n I \simeq (\chi_n \circ g^{-1})\tilde{\chi}_n P_{\mathbb{T}}^{\pm} \chi_n I = (\chi_n \circ g^{-1})\chi_n P_{\mathbb{T}}^{\pm} \chi_n I. \quad (7.17)$$

Consequently, we infer from (7.10), (7.16) and (7.17) that

$$\begin{aligned} \chi_n(B+H)\chi_n I &\simeq \chi_n A^+ P_{\mathbb{T}}^+ \chi_n I + \chi_n A^- P_{\mathbb{T}}^- \chi_n I \\ &= \chi_n \sum_{g \in F} [a_g^+ \chi_n U_g P_{\mathbb{T}}^+ \chi_n + a_g^- \chi_n U_g P_{\mathbb{T}}^- \chi_n] I \\ &= \chi_n \sum_{g \in F} [a_g^+ U_g (\chi_n \circ g^{-1}) P_{\mathbb{T}}^+ \chi_n + a_g^- U_g (\chi_n \circ g^{-1}) P_{\mathbb{T}}^- \chi_n] I \\ &\simeq \chi_n \sum_{g \in F} [a_g^+ U_g (\chi_n \circ g^{-1}) \chi_n P_{\mathbb{T}}^+ \chi_n + a_g^- U_g (\chi_n \circ g^{-1}) \chi_n P_{\mathbb{T}}^- \chi_n] I \\ &= \chi_n \sum_{g \in F} [a_g^+ U_g \chi_n P_{\mathbb{T}}^+ \chi_n + a_g^- U_g \chi_n P_{\mathbb{T}}^- \chi_n] I \\ &= \chi_n A^+ \chi_n P_{\mathbb{T}}^+ \chi_n I + \chi_n A^- \chi_n P_{\mathbb{T}}^- \chi_n I. \end{aligned} \quad (7.18)$$

The continuity of the derivatives g' on the sets $h(\gamma)$ for all $h \in F^{n+1}$ implies the continuity of the derivatives h' on $\gamma \supset u$ for all $h \in F^n$. Hence, applying the isometric isomorphism

$$\sigma_n : L^2(u_n) \rightarrow L_m^2(u), \quad (\sigma_n \varphi)(t) = (U_h \varphi)(t)_{h \in F^n}, \quad t \in u,$$

we infer that $\sigma_n(\chi_n P_{\mathbb{T}}^{\pm} \chi_n I) \sigma_n^{-1} \simeq \text{diag}\{P_u^{\pm}\}_{n=1}^m$, where $P_u^{\pm} = (I \pm S_u)/2$ and S_u is given by (1.2) with \mathbb{T} replaced by $u \subset \mathbb{T}$. Therefore, we obtain

$$\sigma_n(\chi_n A^+ \chi_n P_{\mathbb{T}}^+ \chi_n I + \chi_n A^- \chi_n P_{\mathbb{T}}^- \chi_n I) \sigma_n^{-1} \simeq \mathcal{A}_{(n)}^+ P_u^+ + \mathcal{A}_{(n)}^- P_u^- \in \mathcal{B}(L_m^2(u)). \quad (7.19)$$

Assuming without loss of generality that u is an interval of the real line \mathbb{R} , with operator (7.19) we associate the operator

$$T := \chi \mathcal{A}_{(n)}^+ \chi P_{\mathbb{R}}^+ \chi I + \chi \mathcal{A}_{(n)}^- \chi P_{\mathbb{R}}^- \chi I \in \mathcal{B}(L_m^2(\mathbb{R})),$$

where χ is the characteristic function of $u \subset \mathbb{R}$, $P_{\mathbb{R}}^{\pm} = (I \pm S_{\mathbb{R}})/2$, $S_{\mathbb{R}} = \mathcal{F}^{-1} \text{sign}(\cdot) \mathcal{F}$, and \mathcal{F} is the Fourier transform, $(\mathcal{F}f)(x) = \int_{\mathbb{R}} e^{-ixt} f(t) dt$, $x \in \mathbb{R}$.

Thus, from (7.19) and (7.18) it follows that

$$|T| = |\chi_n(B+H)\chi_n I|. \quad (7.20)$$

Applying [8, Lemma 10.1], we obtain

$$\begin{aligned} \text{s-lim}_{t \rightarrow \pm\infty} (e_{-t} K e_t I) &= 0 \quad \text{for all } K \in \mathcal{K}(L_m^2(\mathbb{R})), \\ \text{s-lim}_{t \rightarrow \pm\infty} (e_{-t} S_{\mathbb{R}} e_t I) &= \text{s-lim}_{t \rightarrow \pm\infty} (\mathcal{F}^{-1} \text{sign}(\cdot + t) \mathcal{F}) = \pm I, \end{aligned} \quad (7.21)$$

where $e_{\pm t}(x) = e^{\pm itx}$ for $x, t \in \mathbb{R}$. Then we infer from (7.21) that

$$\text{s-lim}_{t \rightarrow \pm\infty} (e_{-t}(T+K)e_t I) = \text{s-lim}_{t \rightarrow \pm\infty} (e_{-t} T e_t I) + \text{s-lim}_{t \rightarrow \pm\infty} (e_{-t} K e_t I) = \chi \mathcal{A}_{(n)}^{\pm} \chi I,$$

for every $K \in \mathcal{K}(L_m^2(\mathbb{R}))$, and hence

$$\|\chi \mathcal{A}_{(n)}^{\pm} \chi I\| \leq \liminf_{t \rightarrow \pm\infty} \|e_{-t}(T+K)e_t I\| \leq \|T+K\|.$$

Consequently, $\|\chi \mathcal{A}_{(n)}^{\pm} \chi I\| \leq |T|$ and, due to the fact that

$$\sup_{(\zeta, \tilde{\mu}) \in M_u(\text{PSO}(\mathbb{T}))} \|\Pi_n A_{\zeta, \tilde{\mu}}^{\pm} \Pi_n\|_{\mathcal{B}(l^2(F^{\infty}))} = \|\mathcal{A}_{(n)}^{\pm} I\|_{\mathcal{B}(L_m^2(u))} = \|\chi \mathcal{A}_{(n)}^{\pm} \chi I\|_{\mathcal{B}(L^2(\mathbb{R}))},$$

from (7.20) it follows that, for all $(\zeta, \tilde{\mu}) \in M_u(PSO(\mathbb{T}))$,

$$\|\Pi_n A_{\zeta, \tilde{\mu}}^{\pm} \Pi_n\|_{\mathcal{B}(l^2(F^\infty))} \leq |T| = |\chi_n(B+H)\chi_n I| \leq \|B+H\|. \quad (7.22)$$

Finally, taking into account (7.13) with $g = e$, (7.14), (7.15) and (7.22), we infer that, for every $(\xi, \mu) \in M_{t_w}(SO(\mathbb{T})) \times \{0, 1\} \subset \Omega$ and every $w \in W_{\mathbb{T}}^0$,

$$\|A_{\xi, \mu}^{\pm}\|_{\mathcal{B}(l^2(G))} \leq \liminf_{n \rightarrow \infty} \|\Pi_n A_{\zeta, \tilde{\mu}}^{\pm} \Pi_n\|_{\mathcal{B}(l^2(F^\infty))} + 2\varepsilon \leq \|B+H\| + 2\varepsilon.$$

This implies (7.9) for all $(\xi, \mu) \in \Omega$ and all $H \in \mathfrak{H}$ due to the arbitrariness of $\varepsilon > 0$, and hence completes the proof of (7.7). \square

For every $w \in W_{\mathbb{T}}^0$, we consider the algebraic $*$ -homomorphism

$$\Psi_w^0 : \mathfrak{B}^0 \rightarrow \mathcal{B}(\mathcal{H}_w^0), \quad B \mapsto \Psi_w^0(B) = \text{Sym}_w^0(B)I \quad (7.23)$$

on the dense subalgebra \mathfrak{B}^0 of the C^* -algebra \mathfrak{B} , where the Hilbert space \mathcal{H}_w^0 is defined in (3.5) and the matrix functions $(\xi, x) \mapsto [\text{Sym}_w^0(B)](\xi, x)$ for $(\xi, x) \in M_{t_w}(SO(\mathbb{T})) \times \{\pm\infty\}$ are defined for operators $B \in \mathfrak{B}^0$ by (3.9).

It is easily seen that for every $w \in W_{\mathbb{T}}^0$ there exists a matrix D_w such that $D_w I \in \mathcal{B}(l^2(G, \mathbb{C}^2))$, D_w has exactly one non-zero entry in each row and each column, all these entries equal 1, and the similarity transform $\text{Sym}_w^0(B) \mapsto D_w \text{Sym}_w^0(B) D_w^{-1} := (A_{i,j})_{i,j=1}^2$ changing positions of rows and columns sends odd diagonal entries into $A_{1,1}$ and even diagonal entries into $A_{2,2}$. Then for any operator $B = A^+ P_{\mathbb{T}}^+ + A^- P_{\mathbb{T}}^- + H_B \in \mathfrak{B}^0$, every $w \in W_{\mathbb{T}}^0$ and every point $\xi \in M_{t_w}(SO(\mathbb{T}))$, we infer from (3.9) and (6.4) that

$$\begin{aligned} D_w([\text{Sym}_w^0(B)](\xi, +\infty)) D_w^{-1} I &= \text{diag}\{A_{\xi,1}^+, A_{\xi,0}^-\}, \\ D_w([\text{Sym}_w^0(B)](\xi, -\infty)) D_w^{-1} I &= \text{diag}\{A_{\xi,1}^-, A_{\xi,0}^+\}. \end{aligned} \quad (7.24)$$

Hence, applying (7.8) and estimate (7.7), we deduce from (7.24) that

$$\begin{aligned} \|[\text{Sym}_w^0(B)](\xi, +\infty)\|_{\mathcal{B}(l^2(G, \mathbb{C}^2))} &= \max\{\|A_{\xi,1}^+\|_{\mathcal{B}(l^2(G))}, \|A_{\xi,0}^-\|_{\mathcal{B}(l^2(G))}\} \leq \max\{\|A^{\pm}\|_{\mathcal{B}(l^2(\mathbb{T}))}\} \leq |B|, \\ \|[\text{Sym}_w^0(B)](\xi, -\infty)\|_{\mathcal{B}(l^2(G, \mathbb{C}^2))} &= \max\{\|A_{\xi,1}^-\|_{\mathcal{B}(l^2(G))}, \|A_{\xi,0}^+\|_{\mathcal{B}(l^2(G))}\} \leq \max\{\|A^{\pm}\|_{\mathcal{B}(l^2(\mathbb{T}))}\} \leq |B|. \end{aligned} \quad (7.25)$$

Relations (7.25) immediately imply the following.

Corollary 7.3. For every $w \in W_{\mathbb{T}}^0$, the algebraic $*$ -homomorphisms $\Psi_w^0 : \mathfrak{B}^0 \rightarrow \mathcal{B}(\mathcal{H}_w^0)$ defined on the generators of the C^* -algebra \mathfrak{B} by formulas (3.7) and (3.9), extends by continuity to representations $\Psi_w^0 : \mathfrak{B} \rightarrow \mathcal{B}(\mathcal{H}_w^0)$ such that $\|\Psi_w^0(B)\| \leq |B|$ for all $B \in \mathfrak{B}$ and hence $\text{Ker } \Psi_w^0 \supset \mathcal{K}$.

8. Homomorphisms into local algebras

8.1. Algebraic homomorphisms Ψ_w

For every G -orbit $w \in W_{\mathbb{T}}$, taking the dense subalgebra \mathfrak{B}^0 of \mathfrak{B} composed by the operators of the form (7.1), we consider the algebraic $*$ -homomorphism

$$\Psi_w : \mathfrak{B}^0 \rightarrow \mathcal{B}(\mathcal{H}_w), \quad \Psi_w(B) = \text{Sym}_w(B)I, \quad (8.1)$$

where the Hilbert space \mathcal{H}_w is given in (3.5) and the matrix function $(\xi, x) \mapsto [\text{Sym}_w(B)](\xi, x)$ for $(\xi, x) \in M_{t_w}(SO(\mathbb{T})) \times \mathbb{R}$ is defined for operators $B \in \mathfrak{B}^0$ by formulas (3.8).

Given any set $\alpha \subset \mathbb{T}$, we define the sets

$$M_{\alpha}(SO(\mathbb{T})) := \bigcup_{t \in \alpha} M_t(SO(\mathbb{T})), \quad \mathfrak{M}_{\alpha}^{\circ} := M_{\alpha}(SO(\mathbb{T})) \times \mathbb{R}. \quad (8.2)$$

By (2.17) and (8.2), $\mathfrak{M}_{\{t\}}^{\circ} = \mathfrak{M}_t^{\circ}$ for all $t \in \mathbb{T}$. For any finite set $\alpha \subset \mathbb{T}$ we introduce the operator

$$V_{\alpha} := \sum_{t \in \alpha} V_t \in \mathfrak{H}, \quad (8.3)$$

where the operators V_t for $t \in \mathbb{T}$ are given by (2.8) and \mathfrak{H} is the closed two-sided ideal being the closure of (7.2). For any set $Y \subset \mathbb{T} \setminus \mathbb{T}_G$, let

$$\Pi_Y := \text{diag}\{E_2 \chi_Y(t)\}_{t \in \mathbb{T} \setminus \mathbb{T}_G}, \quad (8.4)$$

where $E_2 := \text{diag}\{1, 1\}$ and χ_Y is the characteristic function of the set Y .

Lemma 8.1. If $N \in \mathfrak{B}^0$ and $w \in W_{\mathbb{T}_G}$, then

$$|NV_{t_w}| = \|\Psi_w(N)vI\|_{\mathcal{B}(\mathcal{H}_w)}, \quad (8.5)$$

where the function v is given by (2.7).

Proof. Fix a G -orbit $w = \{t_w\} \in W_{\mathbb{T}_G}$. Since $U_g \mathfrak{A} U_g^{-1} = \mathfrak{A}$ for all $g \in G$, we infer that every operator $N \in \mathfrak{B}^0$ has the form

$$N = \sum_{g \in F} A_g U_g \quad (8.6)$$

where $A_g \in \mathfrak{A}$ and F is a finite subset of G . Let φ be the isometric representation of the C^* -algebra \mathfrak{B}^π on a Hilbert space \mathcal{H}_φ considered in (2.13). Then

$$|NV_{t_w}| = \|\varphi([NV_{t_w}]^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)}. \quad (8.7)$$

It follows from (8.6), Lemma 2.3 and formulas (3.8) that the operator $NV_{t_w} = \sum_{g \in F} A_g U_g V_{t_w}$ belongs to the C^* -algebra \mathfrak{A} , and

$$\begin{aligned} [\text{Sym}(NV_{t_w})](\xi, x) &= \sum_{g \in F} ([\text{Sym} A_g](\xi, x)) \text{diag}\{e^{ixk_{g,w}^+}, e^{ixk_{g,w}^-}\} v(x) = [\Psi_w(N)](\xi, x) v(x) \\ &\quad \text{if } (\xi, x) \in M_{t_w}(SO(\mathbb{T})) \times \bar{\mathbb{R}}, \\ [\text{Sym}(NV_{t_w})](\xi, x) &= 0_{2 \times 2} \quad \text{if } (\xi, x) \in \mathfrak{M} \setminus (M_{t_w}(SO(\mathbb{T})) \times \bar{\mathbb{R}}), \end{aligned} \quad (8.8)$$

where $k_{g,w}^\pm = \ln g'(t_w \pm 0)$ and $g'(t_w \pm 0) > 0$. Hence, applying the spectral projection given by (2.14), we infer from Lemma 2.5 that

$$\|P_\varphi(\mathfrak{M}_{t_w}^\circ) \varphi([NV_{t_w}]^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = \|(\text{Sym}(NV_{t_w}))|_{\mathfrak{M}_{t_w}^\circ} I\|_{\mathcal{B}(\mathcal{H}_w)} = \|\Psi_w(N)vI\|_{\mathcal{B}(\mathcal{H}_w)}. \quad (8.9)$$

Taking the open set $M_{\mathbb{T} \setminus \{t_w\}}(SO(\mathbb{T})) \times \dot{\mathbb{R}}$ in $\dot{\mathfrak{M}}$, we infer from (8.8) by analogy with [4, Subsection 8.1] that

$$P_\varphi(M_{\mathbb{T} \setminus \{t_w\}}(SO(\mathbb{T})) \times \dot{\mathbb{R}}) \varphi([NV_{t_w}]^\pi) = 0, \quad (8.10)$$

and, by [4, Lemma 10.5], for the closed set $M_{t_w}(SO(\mathbb{T})) \times \{\infty\} \subset \dot{\mathfrak{M}}$,

$$P_\varphi(M_{t_w}(SO(\mathbb{T})) \times \{\infty\}) \varphi([NV_{t_w}]^\pi) = 0. \quad (8.11)$$

Applying the equalities $P_\varphi(\dot{\mathfrak{M}}) = I$ and

$$P_\varphi(\dot{\mathfrak{M}}) = P_\varphi(M_{\mathbb{T} \setminus \{t_w\}}(SO(\mathbb{T})) \times \dot{\mathbb{R}}) + P_\varphi(\mathfrak{M}_{t_w}^\circ) + P_\varphi(M_{t_w}(SO(\mathbb{T})) \times \{\infty\}),$$

we infer from (8.10)–(8.11) that

$$\begin{aligned} \|\varphi([NV_{t_w}]^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} &= \|P_\varphi(\dot{\mathfrak{M}}) \varphi([NV_{t_w}]^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} \\ &= \|P_\varphi(\mathfrak{M}_{t_w}^\circ) \varphi([NV_{t_w}]^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)}. \end{aligned} \quad (8.12)$$

Combining (8.7), (8.12) and (8.9), we obtain (8.5). \square

Lemma 8.2. If $N \in \mathfrak{B}^0$, $w \in W_{\mathbb{T}}^0$ and the operator V_α is given by (8.3), then for every finite set $\alpha \subset w$,

$$|NV_\alpha| = \|\Psi_w(N)\Pi_\alpha vI\|_{\mathcal{B}(\mathcal{H}_w)}. \quad (8.13)$$

Proof. Fix a G -orbit $w \in W_{\mathbb{T}}^0$ and a finite set $\alpha \subset w$ and take an operator $N \in \mathfrak{B}^0$ written in the form (8.6) where $A_g \in \mathfrak{A}$ and F is a finite subset of G . The set $\beta := \{g^{-1}(t) : t \in \alpha, g \in F\}$ is a finite subset of w along with $\alpha \subset w$ and $F \subset G$. Taking the isometric $*$ -isomorphism φ of the quotient C^* -algebra \mathfrak{B}^π on a Hilbert space \mathcal{H}_φ from (2.13), we conclude that

$$|NV_\alpha| = \|\varphi([NV_\alpha]^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)}. \quad (8.14)$$

Applying (8.6) and the relations $U_g V_t \simeq V_{g^{-1}(t)} U_g$ and $A_g V_t \simeq V_t A_g$ for $t \in \mathbb{T}$ and $A_g \in \mathfrak{A}$, we can represent the operator NV_α in the form

$$NV_\alpha = \sum_{g \in F} \sum_{t \in \alpha} A_g U_g V_t = \sum_{g \in F} \sum_{t \in \alpha} V_{g^{-1}(t)} A_g U_g + K, \quad (8.15)$$

where $K \in \mathcal{K}$. Taking the symbol

$$[\text{Sym } V_\alpha](\xi, x) = \begin{cases} \text{diag}\{v(x), v(x)\} & \text{if } (\xi, x) \in M_\alpha(SO(\mathbb{T})) \times \bar{\mathbb{R}}, \\ 0 & \text{if } (\xi, x) \in \mathfrak{M} \setminus (M_\alpha(SO(\mathbb{T})) \times \bar{\mathbb{R}}), \end{cases}$$

of the operator $V_\alpha \in \mathfrak{A}$ (see Lemma 2.3), we infer from the second equality in (8.15) by analogy with (8.10) and (8.11) that

$$P_\varphi(M_{\mathbb{T} \setminus \beta}(SO(\mathbb{T})) \times \dot{\mathbb{R}}) \varphi([NV_\alpha]^\pi) = 0$$

for the open set $M_{\mathbb{T} \setminus \beta}(SO(\mathbb{T})) \times \dot{\mathbb{R}} \subset \dot{\mathfrak{M}}$, and

$$P_\varphi(M_\beta(SO(\mathbb{T})) \times \{\infty\}) \varphi([NV_\alpha]^\pi) = 0$$

for the closed set $M_\beta(SO(\mathbb{T})) \times \{\infty\} \subset \dot{\mathfrak{M}}$. Applying then the partition

$$\dot{\mathfrak{M}} = \mathfrak{M}_\beta^\circ \cup (M_{\mathbb{T} \setminus \beta}(SO(\mathbb{T})) \times \dot{\mathbb{R}}) \cup (M_\beta(SO(\mathbb{T})) \times \{\infty\})$$

we conclude, using (2.18), that

$$\begin{aligned} \|\varphi([NV_\alpha]^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} &= \|P_\varphi(\mathfrak{M}_\beta^\circ) \varphi([NV_\alpha]^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} \\ &= \|P_\varphi(\mathfrak{M}_\beta^\circ) \varphi([NV_\alpha]^\pi) P_\varphi(\mathfrak{M}_\alpha^\circ)\|_{\mathcal{B}(\mathcal{H}_\varphi)}. \end{aligned} \quad (8.16)$$

Let G_N be the subgroup of G generated by the finite set F and let $\Omega_{N,\alpha}$ be the finite set of G_N -orbits λ of all points $t \in \alpha$. Then $\alpha_\lambda := \alpha \cap \lambda$ is a finite subset of $\lambda \in \Omega_{N,\alpha}$. Since $V_\alpha = \sum_{\lambda \in \Omega_{N,\alpha}} V_{\alpha_\lambda}$, $V_{\alpha_\lambda} = \sum_{t \in \alpha_\lambda} V_t$ and $\Pi_\alpha = \sum_{\lambda \in \Omega_{N,\alpha}} \Pi_{\alpha_\lambda}$, and therefore

$$|NV_\alpha| = \max_{\lambda \in \Omega_{N,\alpha}} |NV_{\alpha_\lambda}|, \quad \|\Psi_w(N) \Pi_\alpha v I\|_{\mathcal{B}(\mathcal{H}_w)} = \max_{\lambda \in \Omega_{N,\alpha}} \|\Psi_w(N) \Pi_{\alpha_\lambda} v I\|_{\mathcal{B}(\mathcal{H}_w)},$$

it remains to prove (8.13) for V_α replaced by any V_{α_λ} . Hence, in what follows we assume without loss of generality that $\alpha \subset \lambda$ and $\lambda = G_N(t_w)$. Since the group G_N for $F \neq \{e\}$ is at most countable, so is the G_N -orbit λ .

Let $\tilde{\mathcal{H}}_\lambda := \bigoplus_{t \in \lambda} P_\varphi(\mathfrak{M}_{t_w}^\circ) \mathcal{H}_\varphi$. Obviously, this space is isometrically isomorphic to the Hilbert space $l^2(\mathfrak{M}_{t_w}^\circ, l^2(\lambda, \mathbb{C}^2))$. Consider the isomorphism

$$\eta_\lambda : P_\varphi(\mathfrak{M}_\lambda^\circ) \mathcal{H}_\varphi \rightarrow \bigoplus_{t \in \lambda} P_\varphi(\mathfrak{M}_{t_w}^\circ) \mathcal{H}_\varphi, \quad P_\varphi(\mathfrak{M}_\lambda^\circ) f \mapsto (P_\varphi(\mathfrak{M}_{t_w}^\circ) \varphi(U_{g_t}^\pi) f)_{t \in \lambda}, \quad (8.17)$$

where $f \in \mathcal{H}_\varphi$ and g_t for every $t \in \lambda$ is a fixed shift in $Y_{t_w, t}$ given by (3.2). Clearly, for every $\beta \subset \lambda$ and every $f \in \mathcal{H}_\varphi$ we get

$$\eta_\lambda(P_\varphi(\mathfrak{M}_\beta^\circ) f) = \Pi_\lambda^\beta (P_\varphi(\mathfrak{M}_{t_w}^\circ) \varphi(U_{g_t}^\pi) f)_{t \in \lambda}, \quad (8.18)$$

where

$$\Pi_\lambda^\beta = \text{diag}\{\chi_\beta(t)\}_{t \in \lambda} I. \quad (8.19)$$

Taking now the isometric C^* -algebra homomorphism

$$\Upsilon_\lambda : \mathcal{B}(P_\varphi(\mathfrak{M}_\lambda^\circ) \mathcal{H}_\varphi) \rightarrow \mathcal{B}\left(\bigoplus_{t \in \lambda} P_\varphi(\mathfrak{M}_{t_w}^\circ) \mathcal{H}_\varphi\right), \quad T \mapsto \eta_\lambda T \eta_\lambda^{-1}, \quad (8.20)$$

where η_λ is given by (8.17), and applying the relations

$$A_{g,t} := U_{g_t} A_g U_{g_t}^{-1} \in \mathfrak{A}, \quad U_{g_t} U_g U_{g_t}^{-1} = U_{\tilde{g}_{t,\tau}}, \quad U_{g_\tau} V_s U_{g_\tau}^{-1} \simeq V_{g_\tau^{-1}(s)} \quad (s \in \alpha)$$

for $t, \tau \in w$, where $\tilde{g}_{t,\tau} = g_t g g_\tau^{-1} \in Y_{t_w, t_w}$ by (3.3), we infer from (8.15), (8.18) and (8.19) that

$$\begin{aligned} &\Upsilon_\lambda(P_\varphi(\mathfrak{M}_\beta^\circ) \varphi([NV_\alpha]^\pi) P_\varphi(\mathfrak{M}_\alpha^\circ)) \\ &= \Upsilon_\lambda\left(P_\varphi(\mathfrak{M}_\beta^\circ) \sum_{g \in F} \sum_{s \in \alpha} \varphi([A_g U_g V_s]^\pi) P_\varphi(\mathfrak{M}_\alpha^\circ)\right) \\ &= \Pi_\lambda^\beta \left(P_\varphi(\mathfrak{M}_{t_w}^\circ) \sum_{g \in F} \sum_{s \in \alpha} \varphi([U_{g_t} A_g U_{g_t}^{-1}]^\pi [U_{g_t} U_g U_{g_t}^{-1}]^\pi [U_{g_\tau} V_s U_{g_\tau}^{-1}]^\pi) P_\varphi(\mathfrak{M}_{t_w}^\circ)\right)_{t, \tau \in \lambda} \Pi_\omega^\alpha \\ &= \Pi_\lambda^\beta \left(\sum_{g \in F} \delta_g(t, \tau) P_\varphi(\mathfrak{M}_{t_w}^\circ) \varphi([A_{g,t} U_{\tilde{g}_{t,\tau}} V_{t_w}]^\pi)\right)_{t, \tau \in \lambda} \Pi_\lambda^\alpha. \end{aligned} \quad (8.21)$$

On the other hand, from (8.20) and (8.21) it follows that

$$\begin{aligned} & \|P_\varphi(\mathfrak{M}_\beta^\circ)\varphi([NV_\alpha]^\pi)P_\varphi(\mathfrak{M}_\alpha^\circ)\|_{\mathcal{B}(\mathcal{H}_\varphi)} \\ &= \|\gamma_\lambda(P_\varphi(\mathfrak{M}_\beta^\circ)\varphi([NV_\alpha]^\pi)P_\varphi(\mathfrak{M}_\alpha^\circ))\|_{\mathcal{B}(\tilde{\mathcal{H}}_\lambda)} \\ &= \left\| \Pi_\lambda^\beta \left(\sum_{g \in F} \delta_g(t, \tau) P_\varphi(\mathfrak{M}_{t_w}^\circ) \varphi([A_{g,t} U_{\tilde{g},\tau} V_{t_w}]^\pi) \right) \right\|_{t, \tau \in \lambda} \Pi_\lambda^\alpha \Big\|_{\mathcal{B}(\tilde{\mathcal{H}}_\lambda)}. \end{aligned} \quad (8.22)$$

Since the coset $[A_{g,t} U_{\tilde{g},\tau} V_{t_w}]^\pi$ belongs to the quotient C^* -algebra \mathfrak{A}^π , we have

$$\delta_g(t, \tau) P_\varphi(\mathfrak{M}_{t_w}^\circ) \varphi([A_{g,t} U_{\tilde{g},\tau} V_{t_w}]^\pi) \in P_\varphi(\mathfrak{M}_{t_w}^\circ) \varphi(\mathfrak{A}^\pi) \quad \text{for all } t, \tau \in \lambda.$$

Hence, taking into account the finiteness of the sets $\alpha, \beta \subset \lambda$ in (8.22) and applying entry-wise the isometric C^* -algebra homomorphism

$$\text{Sym}_{t_w}^\circ : P_\varphi(\mathfrak{M}_{t_w}^\circ) \varphi(\mathfrak{A}^\pi) \rightarrow \mathcal{B}(l^2(\mathfrak{M}_{t_w}^\circ, \mathbb{C}^2))$$

from Lemma 2.5, we infer from (3.8), (8.1), (8.4), (8.22) and Lemma 2.3 that

$$\begin{aligned} & \left\| \Pi_\lambda^\beta \left(\sum_{g \in F} \delta_g(t, \tau) P_\varphi(\mathfrak{M}_{t_w}^\circ) \varphi([A_{g,t} U_{\tilde{g},\tau} V_{t_w}]^\pi) \right) \right\|_{t, \tau \in \lambda} \Pi_\lambda^\alpha \Big\|_{\mathcal{B}(\tilde{\mathcal{H}}_\lambda)} \\ &= \left\| \Pi_\lambda^\beta \left(\sum_{g \in F} \delta_g(t, \tau) (\text{Sym}(A_{g,t} U_{\tilde{g},\tau} V_{t_w}))|_{\mathfrak{M}_{t_w}^\circ} I \right) \right\|_{t, \tau \in \lambda} \Pi_\lambda^\alpha \Big\|_{\mathcal{B}(l^2(\mathfrak{M}_{t_w}^\circ, l^2(\lambda, \mathbb{C}^2)))} \\ &= \left\| \Pi_\beta \left(\sum_{g \in F} \delta_g(t, \tau) (\text{Sym}(A_{g,t} (U_{\tilde{g},\tau} V_{t_w})))|_{\mathfrak{M}_{t_w}^\circ} I \right) \right\|_{t, \tau \in w} \Pi_\alpha I \Big\|_{\mathcal{B}(\mathcal{H}_w)} \\ &= \|\Psi_w(N) \Pi_\alpha \vee I\|_{\mathcal{B}(\mathcal{H}_w)}. \end{aligned} \quad (8.23)$$

Finally, combining (8.14), (8.16), (8.22) and (8.23), we obtain (8.13). \square

8.2. Continuity of homomorphisms Ψ_w

Theorem 8.3. *If $N \in \mathfrak{B}^0$, then for every $w \in W_{\mathbb{T}}$,*

$$\|\Psi_w(N)\|_{\mathcal{B}(\mathcal{H}_w)} \leq |N|. \quad (8.24)$$

Proof. Let $N \in \mathfrak{B}^0$. If $w \in W_{\mathbb{T}_G}$, we deduce from [4, Theorem 9.5] that

$$\|\Psi_w(N)\|_{\mathcal{B}(\mathcal{H}_w)} = \|P_\varphi(\mathfrak{M}_{t_w}^\circ) \varphi(N^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} \leq \|\varphi(N^\pi)\|_{\mathcal{B}(\mathcal{H}_\varphi)} = |N|,$$

which proves (8.24) for $w \in W_{\mathbb{T}_G}$.

Fix $w \in W_{\mathbb{T}}^0$. Take an operator $N \in \mathfrak{B}^0$ of the form (8.6) and consider the at most countable subgroup G_N of G generated by the finite set $F \subset G$. Let Ω_N be the set of all G_N -orbits λ of points $t \in w$. Using the algebraic $*$ -homomorphism $\Psi_w : \mathfrak{B}^0 \rightarrow \mathcal{B}(\mathcal{H}_w)$ given by (8.1) and (3.8), we obtain

$$\|\Psi_w(N)\|_{\mathcal{B}(\mathcal{H}_w)} = \sup_{\lambda \in \Omega_N} \|\Pi_\lambda \Psi_w(N) \Pi_\lambda I\|_{\mathcal{B}(\mathcal{H}_w)}. \quad (8.25)$$

Fix $\lambda \in \Omega_N$. Then the G_N -orbit λ is at most countable. Consider a sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of finite subsets of λ such that $\alpha_n \subset \alpha_{n+1}$ and $\bigcup_{n \in \mathbb{N}} \alpha_n = \lambda$. Since $s\text{-}\lim_{n \rightarrow \infty} (\Pi_{\alpha_n} I) = \Pi_\lambda I$ on the space \mathcal{H}_w , we conclude that

$$\Pi_\lambda \Psi_w(N) \Pi_\lambda I = s\text{-}\lim_{n \rightarrow \infty} (\Pi_\lambda \Psi_w(N) \Pi_{\alpha_n} I).$$

Let χ_k be the characteristic function of the segment $[-k, k] \subset \mathbb{R}$. Because $s\text{-}\lim_{k \rightarrow \infty} (\chi_k I) = I$ on the space \mathcal{H}_w , we infer that, for every $n \in \mathbb{N}$,

$$\Pi_\lambda \Psi_w(N) \Pi_{\alpha_n} I = s\text{-}\lim_{k \rightarrow \infty} (\Pi_\lambda \Psi_w(N) \chi_k \Pi_{\alpha_n} I)$$

on the space $\Pi_\lambda \mathcal{H}_w$. Hence

$$\begin{aligned} \|\Pi_\lambda \Psi_w(N) \Pi_\lambda I\|_{\mathcal{B}(\mathcal{H}_w)} &\leq \liminf_{n \rightarrow \infty} \|\Pi_\lambda \Psi_w(N) \Pi_{\alpha_n} I\|_{\mathcal{B}(\mathcal{H}_w)} \\ &\leq \liminf_{n \rightarrow \infty} \liminf_{k \rightarrow \infty} \|\Pi_\lambda \Psi_w(N) \chi_k \Pi_{\alpha_n} I\|_{\mathcal{B}(\mathcal{H}_w)}. \end{aligned} \quad (8.26)$$

Given $k \in \mathbb{N}$, let $\tilde{\chi}_k$ be a function in $C(\overline{\mathbb{R}})$ such that $\tilde{\chi}_k(x) = 1$ for $|x| \leq k$, $\tilde{\chi}_k(x) = 0$ for $|x| \geq k+1$, and $\tilde{\chi}_k(x) \in [0, 1]$ for $|x| \in [k, k+1]$. Further, we infer from [4, Lemmas 6.1 and 6.2] that for every $t \in w$, every $k \in \mathbb{N}$ and every $m \in \mathbb{N}$ there is an operator $B_{t,k,m} \in \mathfrak{A}^0$ such that

$$\begin{aligned} [\text{Sym}(B_{t,k,m} V_t)](\xi, x) &= \begin{cases} \text{diag}\{P_{k,m}(u(x))v(x), P_{k,m}(u(x))v(x)\} & \text{if } (\xi, x) \in M_t(SO(\mathbb{T})) \times \overline{\mathbb{R}}, \\ 0_{2 \times 2} & \text{if } (\xi, x) \in \mathfrak{M} \setminus (M_t(SO(\mathbb{T})) \times \overline{\mathbb{R}}), \end{cases} \\ \max_{x \in \overline{\mathbb{R}}} |P_{k,m}(u(x))v(x) - \tilde{\chi}_k(x)| &< 1/m, \end{aligned} \quad (8.27)$$

where $P_{k,m}(u)$ is a polynomial in $u \in [0, 1]$, and the functions $u(\cdot)$ and $v(\cdot)$ are given by (2.7). Taking a finite set $\alpha \subset w$, pairwise disjoint closed neighborhoods γ_t of points $t \in \alpha$, and the operator $B_{\alpha,k,m} = \sum_{t \in \alpha} \chi_{\gamma_t} B_{t,k,m} \in \mathfrak{A}^0$, where χ_{γ_t} are the characteristic functions of γ_t , we conclude that

$$[\text{Sym}(B_{\alpha,k,m} V_\alpha)](\xi, x) = \begin{cases} \text{diag}\{P_{k,m}(u(x))v(x), P_{k,m}(u(x))v(x)\} & \text{if } (\xi, x) \in M_\alpha(SO(\mathbb{T})) \times \overline{\mathbb{R}}, \\ 0_{2 \times 2} & \text{if } (\xi, x) \in \mathfrak{M} \setminus M_\alpha(SO(\mathbb{T})) \times \overline{\mathbb{R}}. \end{cases}$$

Taking $\alpha = \alpha_n$ and applying the previous formula, we obtain

$$[\text{Sym}_w(B_{\alpha_n,k,m})](\xi, x) v(x) \Pi_{\alpha_n} = (P_{k,m}(u(x))v(x) E_2)_{t \in w} \Pi_{\alpha_n}. \quad (8.28)$$

Taking into account (8.27), we define the function $v_{k,m} \in C(\overline{\mathbb{R}})$ for $m > 1$ by

$$v_{k,m}(x) = [P_{k,m}(u(x))v(x)]^{-1} \quad \text{for } x \in [-k, k], \quad v_{k,m}(\pm x) = v_{k,m}(\pm k) \quad \text{for } x > k. \quad (8.29)$$

Making use of (8.28) and (8.29), we infer that

$$\Pi_\lambda \Psi_w(N) \chi_k \Pi_{\alpha_n} I = \Pi_\lambda \Psi_w(N B_{\alpha_n,k,m}) \Pi_{\alpha_n} v_{k,m} \chi_k \Pi_{\alpha_n} I.$$

Consequently, applying Lemma 8.2, we get

$$\begin{aligned} \|\Pi_\lambda \Psi_w(N) \chi_k \Pi_{\alpha_n} I\|_{\mathcal{B}(\mathcal{H}_w)} &= \|\Pi_\lambda \Psi_w(N B_{\alpha_n,k,m}) \Pi_{\alpha_n} v_{k,m} \chi_k \Pi_{\alpha_n} I\|_{\mathcal{B}(\mathcal{H}_w)} \\ &\leq \|\Psi_w(N B_{\alpha_n,k,m}) \Pi_{\alpha_n} v I\|_{\mathcal{B}(\mathcal{H}_w)} \|v_{k,m} \chi_k \Pi_{\alpha_n} I\|_{\mathcal{B}(\mathcal{H}_w)} \\ &\leq \|N B_{\alpha_n,k,m} V_{\alpha_n}\| \|v_{k,m} \chi_k \Pi_{\alpha_n} I\|_{\mathcal{B}(\mathcal{H}_w)}. \end{aligned} \quad (8.30)$$

By (8.27) and (8.29), we obtain

$$\|v_{k,m} \chi_k \Pi_{\alpha_n} I\|_{\mathcal{B}(\mathcal{H}_w)} = \max_{x \in [-k, k]} |[P_{k,m}(u(x))v(x)]^{-1}| \leq 1/(1 - 1/m). \quad (8.31)$$

On the other hand, from Lemma 8.2, (8.28) and (8.27) it follows that

$$\|B_{\alpha_n,k,m} V_{\alpha_n}\| = \|\Psi_w(B_{\alpha_n,k,m}) \Pi_{\alpha_n} v I\|_{\mathcal{B}(\mathcal{H}_w)} = \max_{x \in \overline{\mathbb{R}}} |P_{k,m}(u(x))v(x)| \leq 1 + 1/m. \quad (8.32)$$

Thus, according to (8.30)–(8.32), we get

$$\|\Pi_\lambda \Psi_w(N) \chi_k \Pi_{\alpha_n} I\|_{\mathcal{B}(\mathcal{H}_w)} \leq |N|(1 + 1/m)/(1 - 1/m). \quad (8.33)$$

Combining (8.25), (8.26) and (8.33), we conclude that

$$\|\Psi_w(N)\|_{\mathcal{B}(\mathcal{H}_w)} \leq |N|(1 - 1/m^2),$$

which in view of the arbitrariness of $m \in \mathbb{N}$ implies the estimate (8.24) for all $w \in W_{\mathbb{T}}^0$ and hence for all $w \in W_{\mathbb{T}}$. \square

By continuity, estimate (8.24) holds for every $N \in \mathfrak{B}$ and every $w \in W_{\mathbb{T}}$. Hence, Theorem 8.3 implies the following corollary.

Corollary 8.4. For every $w \in W_{\mathbb{T}}$, the algebraic homomorphisms Ψ_w given by (8.1) and (3.8) extend to representations $\Psi_w : \mathfrak{B} \rightarrow \mathcal{B}(\mathcal{H}_w)$ such that (8.24) holds for every $N \in \mathfrak{B}$ and $\text{Ker } \Psi_w \supset \mathcal{K}$.

Lemma 8.5. For every $w \in W_{\mathbb{T}}$, the restriction of the quotient homomorphism $\Psi_w : \mathfrak{B}^{\pi} \rightarrow \mathcal{B}(\mathcal{H}_w)$, $B^{\pi} \mapsto \Psi_w(B)$ onto the ideal \mathfrak{H}_w^{π} of \mathfrak{B}^{π} is an isometric $*$ -isomorphism of \mathfrak{H}_w^{π} onto the closed two-sided ideal $\Psi_w(\mathfrak{H}_w)$ of the C^* -algebra $\Psi_w(\mathfrak{B})$.

Proof. Since the set $\{NV_{tw} : N \in \mathfrak{B}^0\}$ for $w \in W_{\mathbb{T}_G}$ is dense in the ideal \mathfrak{H}_w and $\Psi_w(V_{tw}) = vI$, and as the set $\{NV_{\alpha} : N \in \mathfrak{B}^0, \alpha \text{ runs finite subsets of } w\}$ is dense in the ideal \mathfrak{H}_w for $w \in W_{\mathbb{T}}^0$ and $\Psi_w(V_{\alpha}) = \Pi_{\alpha}vI$, we infer from [Lemmas 8.1 and 8.2](#) that for every $w \in W_{\mathbb{T}}$ the restriction $\Psi_w|_{\mathfrak{H}_w}$ is a $*$ -homomorphism of \mathfrak{H}_w into $\mathcal{L}_w := \mathcal{B}(\mathcal{H}_w)$ such that $\|\Psi_w(H_w)\| = |H_w|$ for every $H_w \in \mathfrak{H}_w$. Consequently, $\text{Ker}(\Psi_w|_{\mathfrak{H}_w}) = \mathcal{K}$ and Ψ_w is a C^* -algebra isomorphism of $\mathfrak{H}_w/\mathcal{K}$ onto $\Psi_w(\mathfrak{H}_w)$. \square

8.3. Proof of [Theorem 3.1](#)

[Corollaries 8.4 and 7.3](#) imply that the map $\Psi_{\mathfrak{B}}$ defined on the generators of the C^* -algebra \mathfrak{B} by formulas (3.6)–(3.9) extends to a C^* -algebra homomorphism of \mathfrak{B} into the C^* -algebra $\mathcal{B}(\mathcal{H}_{\mathfrak{B}})$, and for all $B \in \mathfrak{B}$,

$$\|\Psi_{\mathfrak{B}}(B)\| = \max \left\{ \sup_{w \in W_{\mathbb{T}}} \|\Psi_w(B)\|, \sup_{w \in W_{\mathbb{T}}^0} \|\Psi_w^0(B)\| \right\} \leq |B|. \quad (8.34)$$

By (8.34), $\text{Ker} \Psi_{\mathfrak{B}} \supset \mathcal{K}$. On the other hand, if an operator $B \in \mathfrak{B}$ is of the form (7.6) and $\Psi_{\mathfrak{B}}(B) = 0$, then $\Psi_w^0(B) = \text{Sym}_w^0(B)I = 0$ for all $w \in W_{\mathbb{T}}^0$, and from (7.24) and (7.8) it follows that $A^{\pm} = 0$, and hence $B = H_B \in \mathfrak{H}$. Thus,

$$\Psi_w(H_B) = \Psi_w(B) = 0 \quad \text{for all } w \in W_{\mathbb{T}}. \quad (8.35)$$

By [Lemma 7.1](#), there is a sequence of finite subsets $\Lambda_m \subset W_{\mathbb{T}}$ such that

$$H_B^{\pi} = \lim_{m \rightarrow \infty} \sum_{\omega \in \Lambda_m} H_{\omega, m}^{\pi} \quad (H_{\omega, m} \in \mathfrak{H}_{\omega}). \quad (8.36)$$

By [\[22, Lemma 6.9\]](#), for every finite set $\Lambda \subset W_{\mathbb{T}}$ and any $\{H_{\omega}^{\pi} \in \mathfrak{H}_{\omega}^{\pi} : \omega \in \Lambda\}$,

$$\left\| \sum_{\omega \in \Lambda} H_{\omega}^{\pi} \right\| = \sup_{\omega \in \Lambda} \|H_{\omega}^{\pi}\|. \quad (8.37)$$

Fix $w \in W_{\mathbb{T}}$. Setting $H_{w, m} = H_{\omega, m}$ if $w \in \Lambda_m$ and $w = \omega$, and setting $H_{w, m} = 0$ otherwise, we infer from (8.36) and (8.37) that for every $w \in W_{\mathbb{T}}$ there exists a limit $H_{B, w}^{\pi} = \lim_{m \rightarrow \infty} H_{w, m}^{\pi} \in \mathfrak{H}_w$, $H_{B, w}^{\pi} \neq 0$ for at most countable set of $w \in W_{\mathbb{T}}$, and

$$|H_B| = \sup_{w \in W_{\mathbb{T}}} \|H_{B, w}^{\pi}\|. \quad (8.38)$$

Applying (8.1), (3.8) and [Lemma 2.3](#), we infer that $\Psi_w(V_t) = 0$ for all $w \in W_{\mathbb{T}}$ and all $t \in \mathbb{T} \setminus w$. This in view of (7.5) implies that

$$\Psi_w(\mathfrak{H}_{\omega}) = \{0\} \quad \text{for all } w \in W_{\mathbb{T}} \text{ and all } \omega \in W_{\mathbb{T}} \setminus \{w\}. \quad (8.39)$$

Hence, from (8.36), (8.39) and (8.35) it follows that $\Psi_w(H_{B, w}^{\pi}) = \Psi_w(H_B) = 0$ for all $w \in W_{\mathbb{T}}$. Thus, we conclude from [Lemma 8.5](#) that $H_{B, w}^{\pi} = \mathcal{K}$ for all $w \in W_{\mathbb{T}}$. Consequently, we infer from (8.38) that

$$|B| = |H_B| = \sup_{w \in W_{\mathbb{T}}} \|H_{B, w}^{\pi}\| = 0,$$

whence $\text{Ker} \Psi_{\mathfrak{B}} = \mathcal{K}$, which completes the proof of [Theorem 3.1](#).

8.4. Proof of [Theorem 3.2](#)

Sufficiency. Let $\Lambda = W_{\mathbb{T}}$ and $\mathcal{L}_w = \mathcal{B}(\mathcal{H}_w)$ for all $w \in W_{\mathbb{T}}$. It follows from [Corollary 8.4](#), [Lemma 8.5](#) and relations (8.39) that all the conditions of [Theorem 5.2](#) are fulfilled for the C^* -algebra \mathfrak{B} defined by (1.1), for the ideal \mathfrak{H}_0 of \mathfrak{B} generated by all commutators $[aI, S_{\mathbb{T}}]$ ($a \in \text{PSO}^0(\mathbb{T})$) and by all operators $U_g S_{\mathbb{T}} U_g^* - S_{\mathbb{T}}$ ($g \in G$), which coincides with the ideal \mathfrak{H} generated by the ideals \mathfrak{H}_w ($w \in W_{\mathbb{T}}$), and for the representations $\Psi_w : \mathfrak{B} \rightarrow \mathcal{B}(\mathcal{H}_w)$ given by (8.1) for all $w \in W_{\mathbb{T}}$, where the Hilbert spaces \mathcal{H}_w are defined in (3.5). Hence, by [Theorems 7.2 and 5.3](#), an operator $B \in \mathfrak{B}$ is Fredholm on the space $L^2(\mathbb{T})$ if the functional operators A^{\pm} in (7.6) are invertible in the C^* -algebra $\mathcal{B}(L^2(\mathbb{T}))$ and for every $w \in W_{\mathbb{T}}$ the operator $\Psi_w(B)$ is invertible in the C^* -algebra $\mathcal{B}(\mathcal{H}_w)$.

Fix an operator $B = A^+ P_{\mathbb{T}}^+ + A^- P_{\mathbb{T}}^- + H_B$ in \mathfrak{B} . If conditions (iii) of [Theorem 3.2](#) are fulfilled, then by relations (7.24) and [Theorem 6.2](#) the functional operators A^{\pm} are invertible on the space $L^2(\mathbb{T})$. On the other hand, the fulfillment of conditions (i) and (ii) of [Theorem 3.2](#) implies the invertibility of operators $\Psi_w(B) \in \mathcal{B}(\mathcal{H}_w)$ for all $w \in W_{\mathbb{T}_G}$ and all $w \in W_{\mathbb{T}}^0$, respectively. Hence, we conclude from [Theorem 5.3](#) that the operator $B \in \mathfrak{B}$ is Fredholm on the space $L^2(\mathbb{T})$.

Necessity. Let an operator $B \in \mathfrak{B}$ be Fredholm on the space $L^2(\mathbb{T})$ or, equivalently, the coset B^π be invertible in the quotient C^* -algebra \mathfrak{B}^π . Since $\text{Ker } \Psi_{\mathfrak{B}} = \mathcal{K}$ by Theorem 3.1, the quotient $*$ -homomorphism

$$\Psi_{\mathfrak{B}} : \mathfrak{B}/\mathcal{K} \rightarrow \mathcal{B}(\mathcal{H}_{\mathfrak{B}}), \quad N + \mathcal{K} \mapsto \Psi_{\mathfrak{B}}(N),$$

is a C^* -algebra isomorphism. Consequently, $\Psi_{\mathfrak{B}}(B)$ is invertible on the space $\mathcal{H}_{\mathfrak{B}}$ and then, according to (3.6), the operators $\Psi_w(B)$ are invertible on the spaces \mathcal{H}_w for all $w \in W_{\mathbb{T}}$, the operators $\Psi_w^0(B)$ are invertible on the spaces \mathcal{H}_w^0 for all $w \in W_{\mathbb{T}}^0$, and the norms of their inverses are uniformly bounded. This, in view of (3.7)–(3.9), immediately implies parts (i)–(iii) of Theorem 3.2, which completes the proof.

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