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A blowup criterion for the compressible nematic liquid crystal flows in dimension two

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ABSTRACT

In this paper, we establish a blowup criterion for the two-dimensional compressible nematic liquid crystal flows. The criterion is given in terms of the density and the gradient of direction field, where the later satisfies the Serrin-type blowup criterion. For this result, we do not need the initial density to be positive.

1. Introduction

In this paper, we investigate the motion of compressible nematic liquid crystal flows, which are described by the following simplified version of the Ericksen–Leslie equation:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \nabla P = -\Delta d \cdot \nabla d, \\ d_t + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \end{cases} \tag{1.1}$$

where ρ, u, P and d stand for the density, velocity, pressure and macroscopic average of the nematic liquid crystal orientation field respectively. The direction field is conformed to $|d| = 1$. The pressure of the fluid P is a function of the density. More precisely, the equation of state is

$$P = A\rho^\gamma, \quad A > 0,\ \gamma > 1, \tag{1.2}$$

where A and γ are both constants. Without loss of generality, A is normalized to 1. The constants μ and λ are the shear viscosity and bulk viscosity coefficients of the fluid, which are assumed to satisfy the following physical conditions

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$$\mu > 0, \quad \lambda + \mu \geqslant 0. \tag{1.3}$$

In this paper, we suppose Ω to be a bounded smooth domain in $\mathbb{R}^2$ and consider an initial boundary value problem for (1.1)–(1.3) with the following conditions

$$(\rho, u, d)|_{t=0} = (\rho_0, u_0, d_0), \tag{1.4}$$

$$u = 0, \qquad \frac{\partial d}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{1.5}$$

where ν is the unit outward normal vector to $\partial\Omega$.

When d is a constant vector field, Eq. (1.1) becomes an isentropic compressible Navier–Stokes equation, which has a huge amount of literature for the three dimensional space. First of all, Choe and Kim [2] established the local existence for the strong solution in the presence of vacuum. Then it is interesting and important to investigate the possible breakdown of regularity if the maximal time of existence is finite, see [4,8,9,14,15,17,19,21] and see [5–7,18,20] for the full compressible fluids. For this aspect, Huang and Xin [15] gave a blowup criterion, analogous to the Beal–Kato–Majda criterion [1] for ideal incompressible flows, for the strong solution in three dimensional space:

$$\lim_{T \to T^*} \int_0^T \|\nabla u\|_{L^\infty}\, dt = \infty,$$

provided

$$7\mu > \lambda. \tag{1.6}$$

Recently, a Serrin-type criterion [16] was given by Huang, Li and Xin [8], which is

$$\lim_{T \to T^*} \big(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\sqrt{\rho}u\|_{L^s(0,T;L^r)}\big) = \infty, \quad \frac{2}{s} + \frac{3}{r} \leqslant 1,\ 3 < r \leqslant \infty.$$

At the same time, Sun, Wang and Zhang [17] also built up a blowup criterion, in terms of the density,

$$\lim_{T \to T^*} \|\rho\|_{L^\infty(0,T;L^\infty)} = \infty, \tag{1.7}$$

under the condition (1.6), which had been reflexed to $\frac{29\mu}{3} > \lambda$ by Wen and Zhu in [21]. Specially, in two dimensional space, Sun and Zhang [19] succeed in building up the blowup criterion (1.7) without the additional relation (1.6).

For the system (1.1), there are not so many results. It is unclear whether a global weak solution to (1.1)–(1.5) exists in dimensions greater than one. In dimension one, Ding, Wang and Wen [3] obtained an existence result. For the dimension three, Huang, Wang and Wen [12] established local existence for strong solution and had the following blowup criterion

$$\lim_{T \to T^*} \big(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\nabla d\|_{L^3(0,T;L^\infty)}\big) = \infty,$$

under the assumption

$$7\mu > 9\lambda.$$

At the same time, Huang, Wang and Wen [13] built up another blowup criterion

$$\lim_{T \to T^*} \big(\big\|\mathcal{D}(u)\big\|_{L^1(0,T;L^\infty)} + \|\nabla d\|_{L^2(0,T;L^\infty)}\big) = \infty,$$

where $\mathcal{D}(u)$ is the deformation tensor, $\mathcal{D}(u) = \frac{\nabla u + (\nabla u)^{tr}}{2}$. Recently, Huang and Wang [10] derived a Serrin blowup criterion

$$\lim_{T \to T^*} \left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^{s_1}(0,T;L^{r_1})} + \|\nabla d\|_{L^{s_2}(0,T;L^{r_2})} \right) = \infty, \quad \frac{2}{s_i} + \frac{n}{r_i} \leqslant 1,\ n < r_i \leqslant \infty,\ i = 1, 2, \tag{1.8}$$

where n is the spatial dimension. Motivated by the idea in [20] for two-dimensional space, if we can get a bound for $\|u\|_{L^2(0,T;L^\infty)}$, then (1.8) will be replaced by

$$\lim_{T \to T^*} \left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\nabla d\|_{L^s(0,T;L^r)} \right) = \infty, \quad \frac{2}{s} + \frac{2}{r} \leqslant 1,\ 2 < r \leqslant \infty.$$

Therefore, our goal in this paper is to get the uniform bound of $\|u\|_{L^2(0,T;L^\infty)}$ from the a priori energy estimate.

Before stating our main result, we first explain the notations and conventions used throughout this paper. We denote

$$\int f\,dx = \int_\Omega f\,dx.$$

Let

$$\dot{f} := f_t + u \cdot \nabla f$$

represents the material derivative of f. For $1 \leqslant q \leqslant \infty$ and integer $k \geqslant 0$, the standard Sobolev spaces are denoted by:

$$\begin{cases} L^q = L^q(\Omega), \quad W^{k,q} = W^{k,q}(\Omega), \quad H^k = W^{k,2}, \\ W_0^{1,q} = \left\{ u \in W^{1,q} \middle| u = 0 \text{ on } \partial\Omega \right\}, \quad H_0^1 = W_0^{1,2}. \end{cases}$$

For two 2×2 matrices $M = (M_{ij})$, $N = (N_{ij})$, we denote the scalar product between M and N by

$$M : N = \sum_{i,j=1}^{2} M_{ij} N_{ij}.$$

Also, we denote $M \otimes N$ by

$$M \otimes N = \sum_{k=1}^{2} M_{ik} N_{jk}, \quad 1 \leqslant i, j \leqslant 2.$$

Let

$$M(d) = \nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_2,$$

where $\mathbb{I}_2$ is the 2×2 identity matrix.

Before working with the blowup criterion of the strong solution, we need to state the result for the existence of the local strong solution, which was obtained by Huang, Wang and Wen [12]. The method there can be applied to the case in this paper, i.e. the case that Ω is a bounded smooth domain in $\mathbb{R}^2$. The corresponding result can be stated as follows:

Proposition 1.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^2$ and $q \in (2,\infty)$ be a fixed constant. Suppose that the initial data ρ_0, u_0, d_0 satisfy*

$$\rho_0 \geqslant 0, \quad \rho_0 \in W^{1,q}, \quad u_0 \in H_0^1 \cap H^2, \quad \nabla d_0 \in H^2 \quad \text{and} \quad |d_0| = 1, \tag{1.9}$$

and the compatibility condition

$$\mu\Delta u_0 + (\lambda+\mu)\nabla \operatorname{div} u_0 - \nabla\big(\rho_0^\gamma\big) - \Delta d_0 \cdot \nabla d_0 = \sqrt{\rho_0}\, g, \tag{1.10}$$

for some $g \in L^2$. Then there exist a positive time $T_0 > 0$ and a unique strong solution (ρ, u, d) of (1.1)–(1.5) such that

$$\begin{aligned}
&\rho \geqslant 0, \quad \rho \in C\big([0,T_0]; W^{1,q}\big), \quad \rho_t \in C\big([0,T_0]; L^q\big),\\
&u \in C\big([0,T_0]; H_0^1 \cap H^2\big) \cap L^2\big(0,T_0; W^{2,q}\big),\\
&u_t \in L^2\big(0,T_0; H_0^1\big), \qquad \sqrt{\rho}u_t \in L^\infty\big(0,T_0; L^2\big),\\
&\nabla d \in C\big([0,T_0]; H^2\big) \cap L^2\big(0,T_0; H^3\big),\\
&d_t \in C\big([0,T_0]; H^1\big) \cap L^2\big(0,T_0; H^2\big), \qquad |d| = 1.
\end{aligned} \tag{1.11}$$

Proposition 1.1 tells us the fact that lifespan of strong solution of (1.1)–(1.5) is positive. Now, it is time to state our main result in this paper.

Theorem 1.2. *Suppose the assumptions in Proposition 1.1 are satisfied and (ρ, u, d) is the strong solution. Let T^* be the maximal time of existence for the strong solution. If $T^* < \infty$, then*

$$\lim_{T \to T^*} \big(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\nabla d\|_{L^s(0,T;L^r)}\big) = \infty \tag{1.12}$$

with r and s satisfy

$$\frac{2}{s} + \frac{2}{r} \leqslant 1, \quad 2 < r \leqslant \infty. \tag{1.13}$$

Remark 1.3. The blowup criterion (1.12) tells the fact that the density will concentrate provided that the Serrin norm of the gradient of direction field is bounded. In other words, it is not possible for other kinds of singularities to form before the density becomes unbounded if the Serrin norm of ∇d remains bounded in two-dimensional space.

Remark 1.4. If d is a constant vector field, then Eq. (1.1) becomes an isentropic compressible Navier–Stokes equation. At this time, the blowup criterion (1.12) is the same as (1.7) mentioned in [19]. However, our method in this paper is different from the paper [19].

The rest of this paper is organized as follows: In Section 2, we state some elementary facts and inequalities; in Section 3, we give the proof of Theorem 1.2.

2. Preliminaries

In this section, we want to recall the known facts and elementary inequalities that will be used later.

Assume that $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain. Consider the following boundary value problem for the Lamé system,

$$\begin{cases} \mu\Delta U + (\lambda+\mu)\nabla \operatorname{div} U = F \quad \text{in } \Omega, \\ U = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Here and in what follows, we denote $\mathcal{L}U = \mu\Delta U + (\lambda + \mu)\nabla \operatorname{div} U$. It is well known that under the assumptions (1.3), (2.1) is a strongly elliptic system. If $F \in W^{-1,q}(\Omega)$, then there exists an unique weak solution $U \in W_0^{1,q}$. Thanks to the uniqueness of solution, we denote $U = \mathcal{L}^{-1}F$. We start recalling various estimates for this system in L^q space. For a proof, the reader refers to [17].

Proposition 2.1. *Let $q \in (1,\infty)$, then there exits some constant C depending only on λ, μ, q and Ω such that*

(1) *if $F \in L^q$, then*

$$\|U\|_{W^{2,q}} \leqslant C\|F\|_{L^q};$$

(2) *if $F \in W^{-1,q}$ (i.e., $F = \operatorname{div} f$ with $f = (f_{ij})_{2\times 2}$, $f_{ij} \in L^q$), then*

$$\|U\|_{W^{1,q}} \leqslant C\|f\|_{L^q}.$$

Moreover, for the endpoint case, if $f_{ij} \in L^\infty \cap L^2$, then $\nabla U \in BMO(\Omega)$ and there exists some constant C depending only on λ, μ, Ω such that

$$\|\nabla U\|_{BMO(\Omega)} \leqslant C\big(\|f\|_{L^\infty} + \|f\|_{L^2}\big).$$

Here $\|g\|_{BMO(\Omega)} := \|g\|_{L^2} + [g]_{BMO(\Omega)}$, with

$$[g]_{BMO(\Omega)} := \sup_{x\in\Omega,\ r\in(0,d)} \frac{1}{|\Omega_r(x)|} \int\limits_{\Omega_r(x)} \big|g(y) - g_{\Omega_r(x)}\big|\,dy,$$

$$g_{\Omega_r(x)} = \frac{1}{|\Omega_r(x)|} \int\limits_{\Omega_r(x)} g(y)\,dy,$$

where $\Omega_r(x) = B_r(x) \cap \Omega$ and $|\Omega_r(x)|$ denotes the Lebesgue measure of $\Omega_r(x)$.

The following Sobolev inequality in Lemma 2.2 is useful for us to give estimate for $\nabla\rho$. For its proof, the reader can see [17].

Lemma 2.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ and $f \in W^{1,q}$ with $q \in (2,\infty)$, then there exists a constant C depending only on q such that*

$$\|f\|_{L^\infty} \leqslant C\big(1 + \|f\|_{BMO(\Omega)} \ln\big(e + \|f\|_{W^{1,q}}\big)\big).$$

The last lemma introduced in this section will be the following logarithmic Sobolev inequality which plays an important role in the proof of Lemma 3.3. Omitting the proof for brief, one can read [11].

Lemma 2.3. *Let Ω be a bounded smooth domain in $\mathbb{R}^2$, and $f \in L^2(s,t;H^1 \cap W^{1,q})$ for $q \in (2,\infty)$. Then there exists a constant C depending only on q such that*

$$\|f\|^2_{L^2(s,t;L^\infty)} \leqslant C\big[1 + \|f\|^2_{L^2(s,t;H^1)} \ln\big(e + \|f\|_{L^2(s,t;W^{1,q})}\big)\big].$$

3. Proof of Theorem 1.2

In this section, we will give the proof of Theorem 1.2 by contradiction. More precisely, let $0 < T^* < \infty$ be the maximum time for the existence of strong solution (ρ, u, d) to (1.1)–(1.5). Suppose that (1.12) were false, that is,

$$M_0 := \lim_{T \to T^*} \left(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|\nabla d\|_{L^s(0,T;L^r)} \right) < \infty. \tag{3.1}$$

Under the condition (3.1), one will extend existence time of the strong solution to (1.1)–(1.5) beyond T^*, which contradicts with the definition of maximum existence time.

The first lemma is the basic energy inequality, which comes from [10].

Lemma 3.1. *Under the condition (3.1), it holds that for $0 \leqslant T < T^*$,*

$$\sup_{0 \leqslant t \leqslant T} \int \left(\rho |u|^2 + |\nabla d|^2 \right) dx + \int_0^T \int \left(|\nabla u|^2 + \left| \Delta d + |\nabla d|^2 d \right|^2 \right) dx\, dt \leqslant C, \tag{3.2}$$

where and in what follows, C denotes generic constants depending only on M_0, μ, λ, γ, T^*, and the initial data.

After having the above energy inequality at hand, it is easy to deduce the following lemma which will play an essential role for us to give the control of $\|\nabla^2 d\|_{L^2}$ by applying Lemma 2.3.

Lemma 3.2. *Under the condition (3.1), it holds that for $0 \leqslant T < T^*$,*

$$\int_0^T \int \left| \nabla^2 d \right|^2 dx\, dt \leqslant C. \tag{3.3}$$

Proof. Since the simple fact $|d| = 1$, we deduce that $\Delta d \cdot d = -|\nabla d|^2$. Then, we obtain

$$\begin{aligned} \int_0^T \int |\Delta d|^2 dx\, dt &= \int_0^T \int |\nabla d|^4 dx\, dt + \int_0^T \int \left| \Delta d + |\nabla d|^2 d \right|^2 dx\, dt \\ &\leqslant \int_0^T \int |\nabla d|^4 dx\, dt + C, \end{aligned} \tag{3.4}$$

where we have used (3.2).

If exploiting the interpolation, Gagliardo–Nirenberg, Young inequalities and using (3.1), we obtain

$$\begin{aligned} \int_0^T \int |\nabla d|^4 dx\, dt &\leqslant \int_0^T \|\nabla d\|_{L^r}^2 \|\nabla d\|_{L^{\frac{2r}{r-2}}}^2 dt \\ &\leqslant C \int_0^T \|\nabla d\|_{L^r}^2 \|\nabla d\|_{L^2}^{\frac{2(r-2)}{r}} \|\nabla d\|_{H^1}^{\frac{4}{r}} dt \end{aligned}$$

$$
\begin{aligned}
&\leqslant \varepsilon \int_0^T \|\nabla d\|_{H^1}^2 \,dt + C(\varepsilon) \int_0^T \|\nabla d\|_{L^r}^{\frac{2r}{r-2}} \|\nabla d\|_{L^2}^2 \,dt \\
&\leqslant \varepsilon \int_0^T \left\|\nabla^2 d\right\|_{L^2}^2 dt + C(\varepsilon).
\end{aligned}
\tag{3.5}
$$

Substituting (3.5) into (3.4), we can obtain

$$
\int_0^T \int |\Delta d|^2 \,dx\,dt \leqslant \varepsilon \int_0^T \left\|\nabla^2 d\right\|_{L^2}^2 dt + C(\varepsilon).
\tag{3.6}
$$

Taking H^2-estimate for the direction field d with the Neumann boundary, it arrives at

$$
\int_0^T \int \left|\nabla^2 d\right|^2 dx\,dt \leqslant C \int_0^T \left(\|\Delta d\|_{L^2}^2 + \|d\|_{H^1}^2\right) dt.
\tag{3.7}
$$

Plugging (3.6) into (3.7) and choosing ε small enough, we get

$$
\int_0^T \int |\nabla^2 d|^2 \,dx\,dt \leqslant C.
$$

Therefore, we complete the proof of lemma. □

Now we will derive the crucial estimate for $\|\nabla u\|_{L^2}$ and $\|\nabla^2 d\|_{L^2}$.

Lemma 3.3. *Under the condition (3.1), it holds that for $0 \leqslant T < T^*$,*

$$
\sup_{t\in[0,T]} \int \left(|\nabla u|^2 + \left|\nabla^2 d\right|^2 + |\nabla d|^4\right) dx + \int_0^T \int \left(\rho|\dot u|^2 + |\nabla d|^2\left|\nabla^2 d\right|^2 + |\nabla d_t|^2 + \left|\nabla^3 d\right|^2\right) dx\,dt < \infty.
\tag{3.8}
$$

Proof. *Step 1*: Multiplying $(1.1)_2$ by u_t, integrating over Ω and integrating by parts yield

$$
\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int \left[\mu|\nabla u|^2 + (\lambda+\mu)(\operatorname{div} u)^2 - 2P \operatorname{div} u - 2M(d)\cdot\nabla u\right] dx + \int \rho|\dot u|^2 \,dx \\
&\quad = \int \rho\dot u\cdot(u\cdot\nabla u)\,dx - \int P_t \operatorname{div} u\,dx - \int M(d)_t\cdot\nabla u\,dx.
\end{aligned}
\tag{3.9}
$$

In order to deal with the term $-\int P_t \operatorname{div} u\,dx$, following the idea in Sun, Wang and Zhang [17], we spite u into v and w. In other words, let v satisfies

$$
\begin{cases}
\mathcal{L}v = \nabla P & \text{in } \Omega,\\
v = 0 & \text{on } \partial\Omega,
\end{cases}
$$

and $w = u - v$, then w satisfies

$$
\begin{cases}
\mathcal{L}w = \rho\dot u + \Delta d\cdot\nabla d & \text{in } \Omega,\\
w = 0 & \text{on } \partial\Omega.
\end{cases}
$$

For any $r \in (1, \infty)$, the application of Proposition 2.1 and the assumption (3.1) gives

$$\|v\|_{W^{1,r}} \leqslant C\|P\|_{L^r} \tag{3.10}$$

and

$$\|w\|_{W^{2,r}} \leqslant C\|\rho\dot{u}\|_{L^r} + C\big\||\nabla^2 d||\nabla d|\big\|_{L^r}. \tag{3.11}$$

Then, we get

$$-\int P_t \operatorname{div} u \, dx = -\int P_t \operatorname{div} v \, dx - \int P_t \operatorname{div} w \, dx. \tag{3.12}$$

Integrating by parts yields

$$\begin{aligned} -\int P_t \operatorname{div} v \, dx &= \int \nabla P_t \cdot v \, dx = \int \mathcal{L} v_t \cdot v \, dx \\ &= -\frac{d}{dt}\frac{1}{2}\int \big[\mu|\nabla v|^2 + (\lambda+\mu)(\operatorname{div} v)^2\big] \, dx. \end{aligned} \tag{3.13}$$

Similarly,

$$\begin{aligned} -\int P_t \operatorname{div} w \, dx &= \int \big[\operatorname{div}(Pu) + (\gamma-1)P \operatorname{div} u\big] \operatorname{div} w \, dx \\ &= -\int Pu \cdot \nabla \operatorname{div} w \, dx + (\gamma-1)\int P \operatorname{div} u \operatorname{div} w \, dx, \end{aligned} \tag{3.14}$$

due to $(1.1)_1$.

Substituting (3.12)–(3.14) into (3.9) and applying the Young, Sobolev inequalities, (3.11), for any $\varepsilon, \delta \in (0,1)$, it arrives at

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\int \big[\mu|\nabla u|^2 + (\lambda+\mu)(\operatorname{div} u)^2 - 2P \operatorname{div} u - 2M(d)\cdot\nabla u + \mu|\nabla v|^2 + (\lambda+\mu)(\operatorname{div} v)^2\big] \, dx + \int \rho|\dot{u}|^2 \, dx \\ &\quad = \int \rho\dot{u}\cdot(u\cdot\nabla u) \, dx - \int Pu\cdot\nabla \operatorname{div} w \, dx + (\gamma-1)\int P \operatorname{div} u \operatorname{div} w \, dx - \int M(d)_t \cdot \nabla u \, dx \\ &\quad \leqslant \int \rho|\dot{u}||u||\nabla u| \, dx + C\int \rho|u||\nabla \operatorname{div} w| \, dx + C\int |\operatorname{div} u||\operatorname{div} w| \, dx + C\int |\nabla d_t||\nabla d||\nabla u| \, dx \\ &\quad \leqslant \varepsilon\int \rho|\dot{u}|^2 \, dx + C(\varepsilon)\int \rho|u|^2|\nabla u|^2 \, dx + \varepsilon\|\nabla \operatorname{div} w\|_{L^2}^2 + C(\varepsilon)\|\sqrt{\rho}u\|_{L^2}^2 + \varepsilon\|\operatorname{div} w\|_{L^2}^2 \\ &\qquad + C(\varepsilon)\|\operatorname{div} u\|_{L^2}^2 + \delta\|\nabla d_t\|_{L^2}^2 + C(\delta)\big\||\nabla u||\nabla d|\big\|_{L^2}^2 \\ &\quad \leqslant \varepsilon\int \rho|\dot{u}|^2 \, dx + \varepsilon\|w\|_{W^{2,2}}^2 + C(\varepsilon,\delta)\Big[1 + \int \big(|\nabla u|^2 + |u|^2|\nabla u|^2 + |\nabla d|^2|\nabla u|^2\big) \, dx\Big] + \delta\|\nabla d_t\|_{L^2}^2 \\ &\quad \leqslant \varepsilon\int \rho|\dot{u}|^2 \, dx + C(\varepsilon,\delta)\Big[1 + \int \big(|\nabla u|^2 + |\nabla d|^2\big|\nabla^2 d\big|^2 + |u|^2|\nabla u|^2 + |\nabla d|^2|\nabla u|^2\big) \, dx\Big] + \delta\|\nabla d_t\|_{L^2}^2. \end{aligned}$$

Choosing ε small enough, one gets

$$\begin{aligned} &\frac{d}{dt}\int \big[\mu|\nabla u|^2 + (\lambda+\mu)(\operatorname{div} u)^2 - 2P \operatorname{div} u - 2M(d)\cdot\nabla u + \mu|\nabla v|^2 + (\lambda+\mu)(\operatorname{div} v)^2\big] \, dx + \int \rho|\dot{u}|^2 \, dx \\ &\quad \leqslant C(\delta) + C(\delta)\int \big(|\nabla u|^2 + |\nabla d|^2\big|\nabla^2 d\big|^2 + |u|^2|\nabla u|^2 + |\nabla d|^2|\nabla u|^2\big) \, dx + \delta\|\nabla d_t\|_{L^2}^2. \end{aligned} \tag{3.15}$$

Step 2: Taking ∇ operator to (1.1)$_4$, then we have

$$\nabla d_t - \nabla\Delta d = -\nabla(u\cdot\nabla d) + \nabla\big(|\nabla d|^2 d\big). \tag{3.16}$$

Multiplying (3.16) by $4|\nabla d|^2\nabla d$ and integrating (by parts) over Ω, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int|\nabla d|^4\,dx + 4\int|\nabla d|^2\big|\nabla^2 d\big|^2 dx + 2\int\big|\nabla\big(|\nabla d|^2\big)\big|^2\,dx \\
&= 2\int_{\partial\Omega}|\nabla d|^2\big\langle\nabla\big(|\nabla d|^2\big),\nu\big\rangle\,d\sigma + 4\int\nabla\big(|\nabla d|^2 d\big) : |\nabla d|^2\nabla d\,dx \\
&\quad - 4\int\nabla(u\cdot\nabla d) : |\nabla d|^2\nabla d\,dx = \sum_{i=1}^{3} I_i,
\end{aligned} \tag{3.17}$$

where ν is the unit outward normal vector to $\partial\Omega$.

To estimate $I_1 = 2\int_{\partial\Omega}|\nabla d|^2\langle\nabla(|\nabla d|^2),\nu\rangle\,d\sigma$. Indeed, applying the Sobolev embedding inequality $W^{1,1}(\Omega)\hookrightarrow L^1(\partial\Omega)$, it is easy to get

$$\begin{aligned}
I_1 &\leqslant 4\int_{\partial\Omega}|\nabla d|^3\big|\nabla^2 d\big|d\sigma \leqslant C\big\||\nabla d|^3\big|\nabla^2 d\big|\big\|_{W^{1,1}(\Omega)} \\
&\leqslant C\int\big(|\nabla d|^3\big|\nabla^2 d\big| + |\nabla d|^2\big|\nabla^2 d\big|^2 + |\nabla d|^3\big|\nabla^3 d\big|\big)\,dx \\
&\leqslant C(\eta)\int\big(|\nabla d|^2\big|\nabla^2 d\big|^2 + |\nabla d|^4 + |\nabla d|^6\big)\,dx + \eta\int\big|\nabla^3 d\big|^2\,dx.
\end{aligned} \tag{3.18}$$

To estimate $I_2 = 4\int\nabla(|\nabla d|^2 d) : |\nabla d|^2\nabla d\,dx$. Indeed, since $|d| = 1$, we have $d\cdot\nabla d = 0$. Then, we get

$$\nabla\big(|\nabla d|^2 d\big) : |\nabla d|^2\nabla d = \big(\nabla\big(|\nabla d|^2\big)d + |\nabla d|^2\nabla d\big) : |\nabla d|^2\nabla d = |\nabla d|^6.$$

Hence, it arrives at

$$I_2 = 4\int\nabla\big(|\nabla d|^2 d\big) : |\nabla d|^2\nabla d\,dx = 4\int|\nabla d|^6\,dx. \tag{3.19}$$

By the Cauchy inequality, we have

$$\begin{aligned}
I_3 &= -4\int\nabla(u\cdot\nabla d) : |\nabla d|^2\nabla d\,dx \\
&\leqslant C\int\big(|\nabla u||\nabla d|^4 + |u||\nabla d|^3\big|\nabla^2 d\big|\big)\,dx \\
&\leqslant C\int\big(|\nabla d|^2|\nabla u|^2 + |\nabla d|^6 + |u|^2\big|\nabla^2 d\big|^2\big)\,dx.
\end{aligned} \tag{3.20}$$

Substituting (3.18)–(3.20) into (3.17) and choosing ε small enough, it arrives at

$$\begin{aligned}
&\frac{d}{dt}\int|\nabla d|^4\,dx + 4\int|\nabla d|^2\big|\nabla^2 d\big|^2\,dx + 2\int\big|\nabla\big(|\nabla d|^2\big)\big|^2\,dx \\
&\leqslant C(\eta)\int\big(|\nabla d|^4 + |\nabla d|^6 + |\nabla d|^2\big|\nabla^2 d\big|^2 + |\nabla d|^2|\nabla u|^2 + |u|^2\big|\nabla^2 d\big|^2\big)\,dx + \eta\int\big|\nabla^3 d\big|^2\,dx.
\end{aligned} \tag{3.21}$$

Step 3: Multiplying (3.16) by $\nabla\Delta d$, integrating (by parts) over Ω and applying Young inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int |\Delta d|^2\,dx+\int |\nabla\Delta d|^2\,dx\\
&\quad\leqslant \int \bigl(|\nabla u||\nabla d|+|u|\bigl|\nabla^2 d\bigr|+|\nabla d|^3+|\nabla d|\bigl|\nabla^2 d\bigr|\bigr)|\nabla\Delta d|\,dx\\
&\quad\leqslant C(\varepsilon)\int \bigl(|\nabla u|^2|\nabla d|^2+|u|^2\bigl|\nabla^2 d\bigr|^2+|\nabla d|^6+|\nabla d|^2\bigl|\nabla^2 d\bigr|^2\bigr)\,dx+\varepsilon\int |\nabla\Delta d|^2\,dx.
\end{aligned}$$

Choosing $\varepsilon=\frac{1}{2}$, we get

$$\frac{d}{dt}\int |\Delta d|^2\,dx+\int |\nabla\Delta d|^2\,dx\leqslant C\int \bigl(|\nabla d|^2\bigl|\nabla^2 d\bigr|^2+|\nabla d|^2|\nabla u|^2+|u|^2\bigl|\nabla^2 d\bigr|^2+|\nabla d|^6\bigr)\,dx. \tag{3.22}$$

Step 4: Multiplying (3.16) by ∇d_t, integrating (by parts) over Ω and applying Young inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int |\Delta d|^2\,dx+\int |\nabla d_t|^2\,dx=\int \bigl[\nabla\bigl(|\nabla d|^2 d\bigr)-\nabla(u\cdot\nabla d)\bigr]\cdot\nabla d_t\,dx\\
&\quad\leqslant C(\varepsilon)\int \bigl(|\nabla u|^2|\nabla d|^2+|u|^2\bigl|\nabla^2 d\bigr|^2+|\nabla d|^6+|\nabla d|^2\bigl|\nabla^2 d\bigr|^2\bigr)\,dx+\varepsilon\int |\nabla d_t|^2\,dx.
\end{aligned}$$

Choosing $\varepsilon=\frac{1}{2}$, we deduce

$$\frac{d}{dt}\int |\Delta d|^2\,dx+\int |\nabla d_t|^2\,dx\leqslant C\int \bigl(|\nabla d|^2\bigl|\nabla^2 d\bigr|^2+|\nabla u|^2|\nabla d|^2+|u|^2\bigl|\nabla^2 d\bigr|^2+|\nabla d|^6\bigr)\,dx. \tag{3.23}$$

In order to control the term $-\frac{d}{dt}\int 2(P\operatorname{div}u+M(d)\cdot\nabla u)\,dx$ on the left hand side of (3.15). One hand, choosing some constant C_1 to be suitably large, then we get

$$\mu|\nabla u|^2-2M(d)\cdot\nabla u+C_1|\nabla d|^4\geqslant \frac{\mu}{2}|\nabla u|^2+|\nabla d|^4. \tag{3.24}$$

On the other hand, choose some large constant C_2 to satisfy

$$(\lambda+\mu)(\operatorname{div}u)^2-2P\operatorname{div}u+C_2P^2\geqslant \frac{\lambda+\mu}{2}(\operatorname{div}u)^2+P^2. \tag{3.25}$$

Note that P^2 satisfies

$$\partial_t P^2+\operatorname{div}\bigl(P^2u\bigr)+(2\gamma-1)P^2\operatorname{div}u=0,$$

which implies that

$$\frac{d}{dt}\int P^2\,dx=(1-2\gamma)\int P^2\operatorname{div}u\,dx. \tag{3.26}$$

Then (3.15) + (3.21) $\times C_1$ + (3.22) + (3.23) + (3.26) $\times C_2$, after choosing δ small enough, gives immediately

$$\begin{aligned}
&\frac{d}{dt}\int\left[\mu|\nabla u|^2-2M(d)\cdot\nabla u+C_1|\nabla d|^4+(\mu+\lambda)(\operatorname{div}u)^2-2P\operatorname{div}u+C_2P^2+|\Delta d|^2\right]dx\\
&\quad+\frac{d}{dt}\int\left(\mu|\nabla v|^2+(\mu+\lambda)(\operatorname{div}v)^2\right)dx+\int\left(\rho|\dot u|^2+|\nabla d|^2\left|\nabla^2 d\right|^2+|\nabla d_t|^2+|\nabla\Delta d|^2\right)dx\\
&\leqslant C+C(\eta)\int\left(|\nabla u|^2+|\nabla d|^4+|\nabla d|^2|\nabla u|^2+|u|^2|\nabla u|^2+|\nabla d|^2\left|\nabla^2 d\right|^2+|u|^2\left|\nabla^2 d\right|^2+|\nabla d|^6\right)dx\\
&\quad+\eta\int\left|\nabla^3 d\right|^2dx.
\end{aligned}$$

For any $0\leqslant s<T<T^*$, integrating the above inequality from s to T and using (3.24)–(3.25) yield

$$\begin{aligned}
&\int\left[\frac{\mu}{2}|\nabla u|^2+|\nabla d|^4+\frac{\lambda+\mu}{2}(\operatorname{div}u)^2+P^2+|\Delta d|^2+\mu|\nabla v|^2+(\lambda+\mu)(\operatorname{div}v)^2\right]\bigg|_{t=T}dx\\
&\quad+\int_s^T\int\left(\rho|\dot u|^2+|\nabla d|^2\left|\nabla^2 d\right|^2+|\nabla d_t|^2+|\nabla\Delta d|^2\right)dx\,dt\\
&\leqslant\int\left[\mu|\nabla u|^2-2M(d)\cdot\nabla u+C_1|\nabla d|^4+(\lambda+\mu)(\operatorname{div}u)^2-2P\operatorname{div}u+C_2P^2+|\Delta d|^2\right]\big|_{t=T}dx\\
&\quad+\int\left[\mu|\nabla v|^2+(\lambda+\mu)(\operatorname{div}v)^2\right]\big|_{t=T}dx+\int_s^T\int\left(\rho|\dot u|^2+|\nabla d|^2\left|\nabla^2 d\right|^2+|\nabla d_t|^2+|\nabla\Delta d|^2\right)dx\,dt\\
&\leqslant C+\int\left[\mu|\nabla u|^2-2M(d)\cdot\nabla u+C_1|\nabla d|^4+(\lambda+\mu)(\operatorname{div}u)^2-2P\operatorname{div}u+C_2P^2+|\Delta d|^2\right]\big|_{t=s}dx\\
&\quad+\int\left[\mu|\nabla v|^2+(\lambda+\mu)(\operatorname{div}v)^2\right]\big|_{t=s}dx+C(\eta)\int_s^T\int\left(|\nabla u|^2+|\nabla d|^4+|\nabla d|^2|\nabla u|^2+|u|^2|\nabla u|^2\right.\\
&\quad\left.+|\nabla d|^2\left|\nabla^2 d\right|^2+|u|^2\left|\nabla^2 d\right|^2+|\nabla d|^6\right)dx\,dt+\eta\int_s^T\int\left|\nabla^3 d\right|^2dx\,dt\\
&\leqslant C+C\int\left(|\nabla u|^2+|\nabla d|^4+\left|\nabla^2 d\right|^2\right)(s)\,dx+C(\eta)\int_s^T\int\left(|\nabla u|^2+|\nabla d|^4+|\nabla d|^2|\nabla u|^2\right.\\
&\quad\left.+|u|^2|\nabla u|^2+|\nabla d|^2\left|\nabla^2 d\right|^2+|u|^2\left|\nabla^2 d\right|^2+|\nabla d|^6\right)dx\,dt+\eta\int_s^T\int\left|\nabla^3 d\right|^2dx\,dt.
\end{aligned}\tag{3.27}$$

Taking H^2 and H^3 regularity estimate for d with Neumann boundary condition respectively, we deduce

$$\left\|\nabla^2 d\right\|_{L^2}^2\leqslant C\left(\|\Delta d\|_{L^2}^2+\|d\|_{H^1}^2\right)\leqslant C\left(\|\Delta d\|_{L^2}^2+1\right)\tag{3.28}$$

and

$$\left\|\nabla^3 d\right\|_{L^2}^2\leqslant C\left(\|\nabla\Delta d\|_{L^2}^2+\|d\|_{H^2}^2\right)\leqslant C\left(\|\nabla\Delta d\|_{L^2}^2+\left\|\nabla^2 d\right\|_{L^2}^2+1\right).\tag{3.29}$$

Substituting (3.28) and (3.29) into (3.27) and choosing η small enough, it arrives at

$$\int\left(|\nabla u|^2+|\nabla d|^4+\left|\nabla^2 d\right|^2\right)(T)\,dx+\int_s^T\int\left(\rho|\dot u|^2+|\nabla d|^2\left|\nabla^2 d\right|^2+|\nabla d_t|^2+\left|\nabla^3 d\right|^2\right)dx\,dt$$

$$\begin{aligned}
&\leqslant C + C\int \left(|\nabla u|^2 + |\nabla d|^4 + \left|\nabla^2 d\right|^2\right)(s)\,dx + C\int_s^T \int \left(1 + |\nabla u|^2 + |\nabla d|^4 + \left|\nabla^2 d\right|^2\right.\\
&\quad \left. + |\nabla d|^2|\nabla u|^2 + |u|^2|\nabla u|^2 + |\nabla d|^2\left|\nabla^2 d\right|^2 + |u|^2\left|\nabla^2 d\right|^2 + |\nabla d|^6\right)dx\,dt\\
&\leqslant C + C\int \left(|\nabla u|^2 + |\nabla d|^4 + \left|\nabla^2 d\right|^2\right)(s)\,dx + C\int_s^T \left(1 + \|\nabla u\|_{L^2}^2 + \|\nabla d\|_{L^4}^4 + \left\|\nabla^2 d\right\|_{L^2}^2\right)\\
&\quad \times \left(1 + \|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2\right)dt. \qquad (3.30)
\end{aligned}$$

Let

$$\begin{aligned}
\Psi(t) = e + \sup_{\tau\in[0,t]} \int \left(|\nabla u|^2 + \left|\nabla^2 d\right|^2 + |\nabla d|^4\right)dx + \int_0^t \int \left(\rho|\dot{u}|^2 + |\nabla d|^2\left|\nabla^2 d\right|^2 + |\nabla d_t|^2\right.\\
\left. + \left|\nabla^3 d\right|^2\right)dx\,d\tau,
\end{aligned}$$

then applying the Grönwall inequality to (3.30), it arrives at

$$\Psi(T) \leqslant C\Psi(s)\exp\left\{C\int_s^T \left(1 + \|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2\right)d\tau\right\}. \qquad (3.31)$$

Now, let us get a proper estimate for $\|u\|_{L^2(s,T;L^\infty)}$ and $\|\nabla d\|_{L^2(s,T;L^\infty)}$. Indeed, one has, by virtue of the Lemma 2.3,

$$\begin{aligned}
&C\int_s^T \left(1 + \|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2\right)d\tau\\
&\quad \leqslant C\left[1 + \left(\|u\|_{L^2(s,T;H^1)}^2 + \|\nabla d\|_{L^2(s,T;H^1)}^2\right)\left(\ln\left(e + \|u\|_{L^2(s,T;W^{1,3})}\right) + \ln\left(e + \|\nabla d\|_{L^2(s,T;W^{1,3})}\right)\right)\right]. \qquad (3.32)
\end{aligned}$$

By (3.10), (3.11) and the Sobolev inequality, we get directly

$$\|\nabla d\|_{W^{1,3}}^2 \leqslant C\|\nabla d\|_{W^{2,2}}^2 \leqslant C\left(\left\|\nabla^2 d\right\|_{L^2}^2 + \left\|\nabla^3 d\right\|_{L^2}^2\right) \qquad (3.33)$$

and

$$\begin{aligned}
\|u\|_{W^{1,3}}^2 &\leqslant C\left(\|w\|_{W^{1,3}}^2 + \|v\|_{W^{1,3}}^2\right)\\
&\leqslant C\left(\|w\|_{W^{2,2}}^2 + \|P\|_{L^3}^2\right)\\
&\leqslant C\left(1 + \|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \left\||\nabla d|\left|\nabla^2 d\right|\right\|_{L^2}^2\right). \qquad (3.34)
\end{aligned}$$

Substituting (3.33) and (3.34) into (3.32) yields

$$\begin{aligned}
&C\int_s^T \left(1 + \|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2\right)d\tau\\
&\leqslant C\left[1 + \left(\|u\|_{L^2(s,T;H^1)}^2 + \|\nabla d\|_{L^2(s,T;H^1)}^2\right)\ln\left(C\Psi(T)\right)\right]\\
&\leqslant C + \ln\left(C\Psi(T)\right)^{C(\|u\|_{L^2(s,T;H^1)}^2 + \|\nabla d\|_{L^2(s,T;H^1)}^2)}. \qquad (3.35)
\end{aligned}$$

The combination of (3.31) and (3.35) gives

$$\Psi(T) \leqslant C\Psi(s)\big[C\Psi(T)\big]^{C(\|u\|^2_{L^2(s,T;H^1)}+\|\nabla d\|^2_{L^2(s,T;H^1)})}.$$

Lemma 3.1 and Lemma 3.2 tell us the fact one can choose some s, which is closed enough to T^*, such that

$$\lim_{T\to T^{*-}} C\big(\|u\|^2_{L^2(s,T;H^1)} + \|\nabla d\|^2_{L^2(s,T;H^1)}\big) \leqslant \frac{1}{2},$$

then

$$\Psi(T) \leqslant C\Psi^2(s) < +\infty,$$

which completes the proof of Lemma 3.3. □

Remark 3.4. Unfortunately, we cannot derive the bound, just depending on the initial data, for (3.8) owing to the technique used here. However, the bound is uniform with respect to time in (3.8) since s, which closed enough to T^*, is fixed in process of the proof for Lemma 3.3. Thus, we can rewrite (3.8) as

$$\sup_{t\in[0,T]} \int \big(|\nabla u|^2 + \big|\nabla^2 d\big|^2 + |\nabla d|^4\big)\,dx + \int_0^T \int \big(\rho|\dot{u}|^2 + |\nabla d|^2\big|\nabla^2 d\big|^2 + |\nabla d_t|^2 + \big|\nabla^3 d\big|^2\big)\,dx\,dt \leqslant C(s),$$

where and in what follows, $C(s)$ denotes generic constants depending not only on M_0, μ, λ, γ, T^*, and the initial data, but also on the data that is fixed on time s.

As a corollary of Lemma 3.3, we can derive the L^2-norm for d_t directly.

Corollary 3.5. *Under the condition (3.1), it holds that for any $0 \leqslant T < T^*$,*

$$\sup_{0\leqslant t\leqslant T} \|d_t\|^2_{L^2} + \int_0^T \|\nabla u\|^2_{L^4}\,dt \leqslant C(s). \tag{3.36}$$

Proof. By the Hölder, Sobolev inequality and Lemma 3.3, we get

$$\begin{aligned}
\|d_t\|_{L^2} &= \big\|-u\cdot\nabla d + |\nabla d|^2 d + \Delta d\big\|_{L^2} \\
&\leqslant \|u\|_{L^4}\|\nabla d\|_{L^4} + \|\nabla d\|^2_{L^4} + \big\|\nabla^2 d\big\|_{L^2} \\
&\leqslant C\|\nabla u\|_{L^2}\|\nabla d\|_{L^4} + \|\nabla d\|^2_{L^4} + \big\|\nabla^2 d\big\|_{L^2} \leqslant C(s).
\end{aligned} \tag{3.37}$$

Using (3.10)–(3.11), by splitting u into v and w, yields

$$\begin{aligned}
\|\nabla u\|_{L^4} &\leqslant \|\nabla v\|_{L^4} + \|\nabla w\|_{L^4} \\
&\leqslant C\|P\|_{L^4} + C\|\nabla w\|_{H^1} \\
&\leqslant C\|P\|_{L^4} + C\big(\|\sqrt{\rho}\dot{u}\|_{L^2} + \big\||\nabla d|\big|\nabla^2 d\big|\big\|_{L^2}\big) \\
&\leqslant C\|P\|_{L^4} + C\big(\|\sqrt{\rho}\dot{u}\|_{L^2} + \|\nabla d\|_{L^4}\big\|\nabla^2 d\big\|_{L^4}\big) \\
&\leqslant C\|P\|_{L^4} + C\big(\|\sqrt{\rho}\dot{u}\|_{L^2} + \big\|\nabla^2 d\big\|_{L^4}\big) \\
&\leqslant C(s)\big(1 + \|\sqrt{\rho}\dot{u}\|_{L^2} + \big\|\nabla^2 d\big\|_{H^1}\big),
\end{aligned} \tag{3.38}$$

where we have used the Sobolev inequality and Lemma 3.3. The combination of (3.37) and (3.38) completes the proof of the corollary. □

Next, we will deduce the L^2-norm for $\sqrt{\rho}\dot{u}$ and ∇d_t, namely:

Lemma 3.6. *Under condition (3.1), it holds for* $0 \leqslant T < T^*$,

$$\sup_{0\leqslant t\leqslant T}\int\left(\rho|\dot{u}|^2+|\nabla d_t|^2\right)dx+\int_0^T\int\left(|\nabla\dot{u}|^2+|d_{tt}|^2\right)dx\,dt\leqslant C(s). \tag{3.39}$$

Proof. *Step 1*: Applying $\dot{u}^j[\partial_t+\operatorname{div}(u\cdot)]$ to $(1.1)_2^j$ and integrating the resulting equation over Ω, we obtain after integration by parts that

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int\rho|\dot{u}|^2\,dx&=\mu\int\dot{u}^j\left[\Delta u_t^j+\operatorname{div}\left(u\Delta u^j\right)\right]dx+(\lambda+\mu)\int\dot{u}^j\left[\partial_j\operatorname{div}u_t+\operatorname{div}(u\partial_j\operatorname{div}u)\right]dx\\&\quad-\int\dot{u}^j\left[\partial_jP_t+\operatorname{div}(u\partial_jP)\right]dx-\int\dot{u}^j\partial_i\left(M^{ij}(d)\right)_t dx-\int\dot{u}^j\partial_k\left(u^k\partial_iM^{ij}(d)\right)dx\\&=\sum_{i=1}^5N_i.\end{aligned} \tag{3.40}$$

Integrating by parts and applying the Young inequality give directly

$$\begin{aligned}N_1&=\mu\int\dot{u}^j\left[\Delta u_t^j+\operatorname{div}\left(u\Delta u^j\right)\right]dx\\&=-\mu\int|\nabla\dot{u}|^2\,dx-\mu\int\left(\partial_i\dot{u}^j\partial_iu^j\partial_ku^k-\partial_i\dot{u}^j\partial_iu^k\partial_ku^j-\partial_iu^j\partial_iu^k\partial_k\dot{u}^j\right)dx\\&\leqslant-\mu\int|\nabla\dot{u}|^2\,dx+\frac{\mu}{2}\int|\nabla\dot{u}|^2\,dx+C\int|\nabla u|^4\,dx\\&\leqslant-\frac{\mu}{2}\int|\nabla\dot{u}|^2\,dx+C\int|\nabla u|^4\,dx.\end{aligned}$$

Similarly,

$$\begin{aligned}N_2&=(\lambda+\mu)\int\dot{u}^j\left[\partial_j\operatorname{div}u_t+\operatorname{div}(u\partial_j\operatorname{div}u)\right]dx\\&=-(\lambda+\mu)\int(\operatorname{div}\dot{u})^2\,dx+(\lambda+\mu)\int\left[(\operatorname{div}\dot{u})\nabla u:(\nabla u)^{tr}+(\operatorname{div}u)\nabla u:(\nabla\dot{u})^{tr}\right.\\&\quad\left.-(\operatorname{div}\dot{u})(\operatorname{div}u)^2\right]dx\\&\leqslant-(\lambda+\mu)\int(\operatorname{div}\dot{u})^2\,dx+\frac{1}{2}(\lambda+\mu)\int(\operatorname{div}\dot{u})^2\,dx+C\int|\nabla u|^4\,dx\\&\leqslant-\frac{1}{2}(\lambda+\mu)\int(\operatorname{div}\dot{u})^2\,dx+C\int|\nabla u|^4\,dx,\\N_3&=-\int\dot{u}^j\left[\partial_jP_t+\operatorname{div}(u\partial_jP)\right]dx\\&=\int P\operatorname{div}u\operatorname{div}\dot{u}\,dx-\int\left[P'(\rho)\rho\operatorname{div}u\operatorname{div}\dot{u}+P\nabla u:(\nabla\dot{u})^{tr}\right]dx\\&\leqslant\varepsilon\int|\nabla\dot{u}|^2\,dx+C(\varepsilon).\end{aligned}$$

Integrating by parts and using the Lemma 3.3, Hölder, Gagliardo–Nirenberg and Young inequalities, we have

$$\begin{aligned} N_4 &= -\int \dot{u}^j \partial_i \big(M^{ij}(d)\big)_t \, dx = \int M^{ij}(d)_t \partial_i \dot{u}^j \, dx \\ &\leqslant C\|\nabla d\|_{L^4} \|\nabla d_t\|_{L^4} \|\nabla \dot{u}\|_{L^2} \\ &\leqslant C\|\nabla d_t\|_{L^2}^{\frac{1}{2}} \|\nabla d_t\|_{H^1}^{\frac{1}{2}} \|\nabla \dot{u}\|_{L^2} \\ &\leqslant \varepsilon \|\nabla \dot{u}\|_{L^2}^2 + \delta \big\|\nabla^2 d_t\big\|_{L^2}^2 + C(\varepsilon,\delta)\|\nabla d_t\|_{L^2}^2. \end{aligned}$$

Similarly,

$$\begin{aligned} N_5 &= -\int \dot{u}^j \partial_k \big(u^k \partial_i M^{ij}(d)\big) \, dx = \int u^k \partial_k \dot{u}^j \partial_i M^{ij}(d) \, dx \\ &\leqslant C\|u\|_{L^8} \|\nabla d\|_{L^8} \big\|\nabla^2 d\big\|_{L^4} \|\nabla \dot{u}\|_{L^2} \\ &\leqslant C\|\nabla u\|_{L^2} \|\nabla d\|_{H^1} \big\|\nabla^2 d\big\|_{H^1} \|\nabla \dot{u}\|_{L^2} \\ &\leqslant \varepsilon \|\nabla \dot{u}\|_{L^2}^2 + C(\varepsilon)\big\|\nabla^3 d\big\|_{L^2}^2 + C(\varepsilon). \end{aligned}$$

Substituting N_i $(i = 1,2,3,4,5)$ into (3.40) and choosing ε suitably small, we obtain

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\int \rho|\dot{u}|^2 \, dx + \frac{1}{2}\int \big[\mu|\nabla \dot{u}|^2 + (\lambda+\mu)(\operatorname{div} \dot{u})^2\big] \, dx \\ &\quad \leqslant \delta\big\|\nabla^2 d_t\big\|_{L^2}^2 + C(\delta)\big(1 + \|\nabla d_t\|_{L^2}^2 + \big\|\nabla^3 d\big\|_{L^2}^2 + \|\sqrt{\rho}\dot{u}\|_{L^2}^4\big), \end{aligned} \tag{3.41}$$

where we have used (3.38).

Step 2: Differentiating $(1.1)_3$ with respect to t, multiplying the resulting equation by d_{tt} and integrating over Ω yield

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\int |\nabla d_t|^2 \, dx + \int |d_{tt}|^2 \, dx &\leqslant 2\int |\nabla d||\nabla d_t||d_{tt}| \, dx + \int |\nabla d|^2 |d_t||d_{tt}| \, dx \\ &\quad + \int |u_t||\nabla d||d_{tt}| \, dx + \int |u||\nabla d_t||d_{tt}| \, dx = \sum_{i=1}^4 K_i. \end{aligned} \tag{3.42}$$

Applying the Hölder, Gagliardo–Nirenberg and Young inequalities, we have

$$\begin{aligned} K_1 &= 2\int |\nabla d||\nabla d_t||d_{tt}| \, dx \leqslant C\|d_{tt}\|_{L^2} \|\nabla d\|_{L^4} \|\nabla d_t\|_{L^4} \\ &\leqslant C\|d_{tt}\|_{L^2} \|\nabla d_t\|_{L^2}^{\frac{1}{2}} \|\nabla d_t\|_{H^1}^{\frac{1}{2}} \leqslant \varepsilon\|d_{tt}\|_{L^2}^2 + \delta\|\nabla d_t\|_{H^1}^2 + C(\varepsilon,\delta)\|\nabla d_t\|_{L^2}^2, \\ K_2 &= \int |\nabla d|^2 |d_t||d_{tt}| \, dx \leqslant \|\nabla d\|_{L^8}^2 \|d_t\|_{L^4} \|d_{tt}\|_{L^2} \\ &\leqslant C\|\nabla d\|_{H^1}^2 \|d_t\|_{H^1} \|d_{tt}\|_{L^2} \leqslant \varepsilon\|d_{tt}\|_{L^2}^2 + C(\varepsilon)\big(1 + \|\nabla d_t\|_{L^2}^2\big), \\ K_3 &= \int |u_t||\nabla d||d_{tt}| \, dx \leqslant \int \big(|\dot{u}||\nabla d||d_{tt}| + |u||\nabla u||\nabla d||d_{tt}|\big) \, dx \\ &\leqslant \|d_{tt}\|_{L^2} \|\dot{u}\|_{L^4} \|\nabla d\|_{L^4} + \|d_{tt}\|_{L^2} \|u\|_{L^8} \|\nabla u\|_{L^4} \|\nabla d\|_{L^8} \\ &\leqslant \varepsilon\|d_{tt}\|_{L^2}^2 + C(\varepsilon)\big(\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^4}^2\big), \\ K_4 &= \int |u||\nabla d_t||d_{tt}| \, dx \leqslant \|d_{tt}\|_{L^2} \|u\|_{L^4} \|\nabla d_t\|_{L^4} \end{aligned}$$

$$\begin{aligned}&\leqslant C\|d_{tt}\|_{L^2}\|\nabla u\|_{L^2}\|\nabla d_t\|_{L^2}^{\frac{1}{2}}\|\nabla d_t\|_{H^1}^{\frac{1}{2}}\\&\leqslant \varepsilon\|d_{tt}\|_{L^2}^2+\delta\|\nabla^2 d_t\|_{L^2}^2+C(\varepsilon,\delta)\|\nabla d_t\|_{L^2}^2.\end{aligned}$$

Then substituting K_i $(i=1,2,3,4)$ into (3.42) and choosing ε suitably small, we get

$$\begin{aligned}&\frac{d}{dt}\int|\nabla d_t|^2\,dx+\int|d_{tt}|^2\,dx\\&\quad\leqslant 4\delta\|\nabla^2 d_t\|_{L^2}^2+C\|\nabla\dot u\|_{L^2}^2+C(\delta)\big(1+\|\nabla u\|_{L^4}^2+\|\nabla^3 d\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\big).\end{aligned}\tag{3.43}$$

Then (3.41) $\times\,\varepsilon+$ (3.43), after choosing ε suitably small, yields

$$\begin{aligned}&\frac{d}{dt}\int\big(\rho|\dot u|^2+|\nabla d_t|^2\big)\,dx+\int\big[\mu|\nabla\dot u|^2+(\lambda+\mu)(\operatorname{div}\dot u)^2+|d_{tt}|^2\big]\,dx\\&\quad\leqslant\delta\|\nabla^2 d_t\|_{L^2}^2+C(\delta)\big(\|\sqrt{\rho}\dot u\|_{L^2}^4+\|\nabla d_t\|_{L^2}^2+\|\nabla u\|_{L^4}^2+\|\nabla^3 d\|_{L^2}^2+1\big).\end{aligned}\tag{3.44}$$

Step 3: Estimate for $\|\nabla^2 d_t\|_{L^2}$. In fact, by applying the standard H^2-estimate on the equations

$$\begin{cases}-\Delta d_t=-d_{tt}+\big(|\nabla d|^2 d-u\cdot\nabla d\big)_t,\\ \left.\dfrac{\partial d_t}{\partial\nu}\right|_{\partial\Omega}=0,\end{cases}\tag{3.45}$$

Then by the Gagliardo–Nirenberg, Young inequalities and Lemma 3.3, we get

$$\begin{aligned}\|\nabla^2 d_t\|_{L^2}&\leqslant C\big[\|\nabla d_t\|_{L^2}+\|d_{tt}\|_{L^2}+\|\partial_t(u\cdot\nabla d)\|_{L^2}+\|\partial_t(|\nabla d|^2 d)\|_{L^2}\big]\\&\leqslant C\big[\|\nabla d_t\|_{L^2}+\|d_{tt}\|_{L^2}+\|\dot u\|_{L^4}\|\nabla d\|_{L^4}+\|u\|_{L^8}\|\nabla u\|_{L^4}\|\nabla d\|_{L^8}\\&\qquad+\|u\|_{L^4}\|\nabla d_t\|_{L^4}+\|\nabla d\|_{L^4}\|\nabla d_t\|_{L^4}+\|\nabla d\|_{L^8}^2\|d_t\|_{L^4}\big]\\&\leqslant C\big[\|\nabla d_t\|_{L^2}+\|d_{tt}\|_{L^2}+\|\nabla\dot u\|_{L^2}\|\nabla^2 d\|_{L^2}+\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}\|\nabla d\|_{H^1}\\&\qquad+\|\nabla u\|_{L^2}\|\nabla d_t\|_{L^2}^{\frac{1}{2}}\|\nabla d_t\|_{H^1}^{\frac{1}{2}}+\|\nabla d\|_{L^4}\|\nabla d_t\|_{L^2}^{\frac{1}{2}}\|\nabla d_t\|_{H^1}^{\frac{1}{2}}\\&\qquad+\|\nabla d\|_{H^1}^2\big(\|d_t\|_{L^2}^{\frac{1}{2}}\|d_t\|_{H^1}^{\frac{1}{2}}+\|d_t\|_{L^2}\big)\big]\\&\leqslant\frac{1}{2}\|\nabla^2 d_t\|_{L^2}+C\big(\|\nabla d_t\|_{L^2}+\|d_{tt}\|_{L^2}+\|\nabla\dot u\|_{L^2}+\|\nabla u\|_{L^4}+1\big),\end{aligned}$$

which gives immediately

$$\|\nabla^2 d_t\|_{L^2}\leqslant C\big(\|\nabla d_t\|_{L^2}+\|d_{tt}\|_{L^2}+\|\nabla\dot u\|_{L^2}+\|\nabla u\|_{L^4}+1\big).\tag{3.46}$$

Substituting (3.46) into (3.44) and choosing δ sufficiently small, one has

$$\begin{aligned}&\frac{d}{dt}\int\big(\rho|\dot u|^2+|\nabla d_t|^2\big)\,dx+\int\big(|\nabla\dot u|^2+|\operatorname{div}\dot u|^2+|d_{tt}|^2\big)\,dx\\&\quad\leqslant C\big(1+\|\sqrt{\rho}\dot u\|_{L^2}^2\big)\big(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\nabla d_t\|_{L^2}^2\big)+C\big(1+\|\nabla u\|_{L^4}^2+\|\nabla^3 d\|_{L^2}^2\big),\end{aligned}$$

which, together with the Grönwall inequality and the compatibility condition (1.10), completes the proof of Lemma 3.6. □

As a corollary of Lemma 3.6, we can easily to deduce the following corollary.

Corollary 3.7. *Under condition (3.1), it holds for* $0 \leqslant T < T^*$,

$$\sup_{0\leqslant t\leqslant T}\left(\|\nabla u\|_{L^4} + \left\|\nabla^3 d\right\|_{L^2}\right) \leqslant C(s). \tag{3.47}$$

Proof. By Gagliardo–Nirenberg and Young inequalities, we deduce from (3.38) that

$$\begin{aligned}\|\nabla u\|_{L^4} &\leqslant C\left(1 + \left\|\nabla^2 d\right\|_{L^4}\right) \\ &\leqslant C\left(1 + \left\|\nabla^2 d\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla^2 d\right\|_{H^1}^{\frac{1}{2}}\right) \leqslant \varepsilon\left\|\nabla^3 d\right\|_{L^2} + C(\varepsilon).\end{aligned} \tag{3.48}$$

Using H^3-estimate to d, (3.16), (3.48) and Gagliardo–Nirenberg inequality, it arrives at

$$\begin{aligned}\left\|\nabla^3 d\right\|_{L^2} &\leqslant C\left(\|\nabla\Delta d\|_{L^2} + \|\nabla d\|_{H^1}\right) \\ &\leqslant C\left[\|\nabla d_t\|_{L^2} + \left\|\nabla(u\cdot\nabla d)\right\|_{L^2} + \left\|\nabla\left(|\nabla d|^2 d\right)\right\|_{L^2} + \|\nabla d\|_{H^1}\right] \\ &\leqslant C\left[\|\nabla d_t\|_{L^2} + \|\nabla u\|_{L^4}\|\nabla d\|_{L^4} + \|u\|_{L^4}\left\|\nabla^2 d\right\|_{L^4} + \|\nabla d\|_{L^4}\left\|\nabla^2 d\right\|_{L^4} + \|\nabla d\|_{H^1}^3 + \|\nabla d\|_{H^1}\right] \\ &\leqslant C\left(1 + \|\nabla u\|_{L^4} + \left\|\nabla^2 d\right\|_{L^4}\right) \\ &\leqslant C\left(1 + \|\nabla u\|_{L^4} + \left\|\nabla^2 d\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla^2 d\right\|_{H^1}^{\frac{1}{2}}\right) \\ &\leqslant \varepsilon\left\|\nabla^3 d\right\|_{L^2} + C(\varepsilon).\end{aligned} \tag{3.49}$$

Then choosing ε small enough in (3.49), we obtain

$$\left\|\nabla^3 d\right\|_{L^2} \leqslant C(s),$$

which, together with (3.48), completes the proof of corollary. □

Since having (3.11), Lemma 3.6 and Corollary 3.7 at hand, it is easy to deduce the following corollary.

Corollary 3.8. *Under condition (3.1), it holds for* $0 \leqslant T < T^*$,

$$\sup_{0\leqslant t\leqslant T}\|w\|_{H^2} + \int_0^T\left(\left\|\nabla^2 w\right\|_{L^q}^2 + \|\nabla w\|_{L^\infty}^2\right)dt \leqslant C(s), \quad q\in(2,\infty). \tag{3.50}$$

Finally, we will derive the high order estimate in the next lemma.

Lemma 3.9. *Under condition (3.1), it holds for* $0 \leqslant T < T^*$,

$$\sup_{0\leqslant t\leqslant T}\left(\|\rho_t\|_{L^q} + \|\rho\|_{W^{1,q}} + \left\|\nabla^2 u\right\|_{L^2}\right) + \int_0^T\left(\|u\|_{W^{2,q}}^2 + \left\|\nabla^2 d_t\right\|_{L^2}^2 + \left\|\nabla^4 d\right\|_{L^2}^2\right)dt \leqslant C(s), \quad q\in(2,\infty). \tag{3.51}$$

Proof. For $2 \leqslant p \leqslant q$, $|\nabla\rho|^p$ satisfies the following equation

$$\left(|\nabla\rho|^p\right)_t + \operatorname{div}\left(|\nabla\rho|^p u\right) + (p-1)|\nabla\rho|^p \operatorname{div} u + p|\nabla\rho|^{p-2}(\nabla\rho)^{tr}\nabla u(\nabla\rho) + p|\nabla\rho|^{p-2}\nabla\rho\cdot\nabla\operatorname{div} u = 0. \tag{3.52}$$

Integrating (3.52) over Ω and integrating by parts, we get

$$\begin{aligned}\frac{d}{dt}\|\nabla\rho\|_{L^p} &\leqslant C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^p} + C\|\nabla^2 u\|_{L^p}\\ &\leqslant C\big(\|\nabla v\|_{L^\infty} + \|\nabla w\|_{L^\infty}\big)\|\nabla\rho\|_{L^p} + C\big(\|\nabla^2 v\|_{L^p} + \|\nabla^2 w\|_{L^p}\big)\\ &\leqslant C\big(1 + \|\nabla v\|_{L^\infty} + \|\nabla w\|_{L^\infty}\big)\|\nabla\rho\|_{L^p} + C\|\nabla^2 w\|_{L^p}. \end{aligned}\tag{3.53}$$

Now we give the estimate for $\|\nabla v\|_{L^\infty}$. Indeed, by virtue of Lemma 2.2 and Proposition 2.1, we have

$$\begin{aligned}\|\nabla v\|_{L^\infty} &\leqslant C\big[1 + \|\nabla v\|_{BMO}\ln\big(e + \|\nabla v\|_{W^{1,q}}\big)\big]\\ &\leqslant C\big[1 + \big(\|P\|_{L^\infty} + \|P\|_{L^2}\big)\ln\big(e + \|\nabla v\|_{W^{1,q}}\big)\big]\\ &\leqslant C\big[1 + \ln\big(e + \|\nabla\rho\|_{L^q}\big)\big]. \end{aligned}\tag{3.54}$$

Let $p = q$ in (3.53) and substituting (3.54) into (3.53), one gets

$$\frac{d}{dt}\|\nabla\rho\|_{L^q} \leqslant C\big[1 + \|\nabla w\|_{L^\infty} + \ln\big(e + \|\nabla\rho\|_{L^q}\big)\big]\|\nabla\rho\|_{L^q} + C\|\nabla^2 w\|_{L^q},$$

which, together with the Grönwall inequality, yields

$$\|\nabla\rho\|_{L^q} \leqslant C(s).$$

For any $1 < p < \infty$, using the standard L^p-estimate for the Lamé system, we get

$$\begin{aligned}\|\nabla^2 u\|_{L^p} &\leqslant C\big(\|\sqrt{\rho}\dot{u}\|_{L^p} + \|\nabla P\|_{L^p} + \|\Delta d\cdot\nabla d\|_{L^p}\big)\\ &\leqslant C\big(\|\nabla\dot{u}\|_{L^2} + \|\nabla\rho\|_{L^p} + \|\nabla d\|_{L^\infty}\|\nabla^2 d\|_{L^p}\big)\\ &\leqslant C\big(\|\nabla\dot{u}\|_{L^2} + \|\nabla\rho\|_{L^p} + 1\big), \end{aligned}$$

where we have used the Sobolev embedding inequality.

Specially, taking $p = q$, we obtain

$$\int_0^T \|u\|_{W^{2,q}}^2\,dt \leqslant C(s).$$

Similarly, taking $p = 2$ yields

$$\|\nabla^2 u\|_{L^2} \leqslant C + C\|\nabla\rho\|_{L^2}$$

and

$$\frac{d}{dt}\|\nabla\rho\|_{L^2} \leqslant C\big(1 + \|\nabla u\|_{L^\infty}\big)\big(1 + \|\nabla\rho\|_{L^2}\big).$$

Then, using (3.50) and applying the Grönwall inequality yield

$$\|\nabla\rho\|_{L^2} \leqslant C(s),$$

which also gives the estimate

$$\|\nabla^2 u\|_{L^2} \leqslant C(s).$$

By virtue of (3.46) and (3.39), one gets

$$\int_0^T \left\|\nabla^2 d_t\right\|_{L^2}^2 dt \leqslant C(s). \tag{3.55}$$

In view of the L^q-estimate, we have

$$\begin{aligned}\|\rho_t\|_{L^q} &\leqslant C\big(\big\||\nabla\rho||u|\big\|_{L^q} + \|\rho \operatorname{div} u\|_{L^q}\big)\\ &\leqslant C\big(\|u\|_{H^2}\|\nabla\rho\|_{L^q} + \|\nabla u\|_{H^1}\big) \leqslant C(s).\end{aligned}$$

Applying the H^4-estimate to $(1.1)_3$, we have

$$\begin{aligned}\left\|\nabla^4 d\right\|_{L^2} &\leqslant C\big(\left\|\nabla^2 d_t\right\|_{L^2} + \left\|\nabla^2(u\cdot\nabla d)\right\|_{L^2} + \left\|\nabla^2\big(|\nabla d|^2 d\big)\right\|_{L^2} + \|d\|_{H^1}\big)\\ &\leqslant C\big(\left\|\nabla^2 d_t\right\|_{L^2} + \|u\|_{L^\infty}\left\|\nabla^3 d\right\|_{L^2} + \|\nabla d\|_{L^\infty}\left\|\nabla^2 u\right\|_{L^2}\\ &\quad + \|\nabla d\|_{L^\infty}\left\|\nabla^3 d\right\|_{L^2} + \|\nabla d\|_{L^\infty}^\infty\left\|\nabla^2 d\right\|_{L^2} + \|d\|_{H^1}\big)\\ &\leqslant C\big(1 + \left\|\nabla^2 d_t\right\|_{L^2}\big),\end{aligned}$$

which, together with (3.55), completes the proof of Lemma 3.9. □

All the estimates in Lemmas 3.1–3.9 will be enough to extend the strong solution (ρ, u, d) beyond the maximal existence time T^*. More precisely, by virtue of Lemmas 3.1–3.9, the function $(\rho, u, d)|_{t=T^*} = \lim_{t\to T^*}(\rho, u, d)$ satisfies the conditions imposed on the initial data (1.9) and the compatibility condition (1.10) at $t = T^*$. Therefore, we can take $(\rho, u, d)|_{t=T^*}$ as the initial data and apply Proposition 1.1 to extend the local strong solution beyond T^*, which contradicts the maximality of T^*. Thus, we complete the proof of Theorem 1.2.

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